

**DIFFUSION OF PENETRANT IN A POLYMER:
A FREE BOUNDARY PROBLEM**

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IMA Preprint Series # 608

February 1990

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Abstract. In this paper, a free boundary problem arising in modeling diffusion of a penetrant in a polymer is studied. The asymptotic behavior of the solution for short time, long time and for small physical parameter ϵ are proved. Some explicit error estimates are also given.

Key Words. free boundary, variational inequality, asymptotic behavior.

AMS(MOS) subject classifications. 35B40, 35R35.

1. Introduction. A model describing the diffusion of a penetrant in a glassy polymer is given by:

$$(1.1) \quad \epsilon u_t = u_{xx} \quad \text{for } 0 < x < s(t), \quad t > 0,$$

$$(1.2) \quad u(0, t) = 1,$$

$$(1.3) \quad [1 + \epsilon u(s(t), t)] \cdot s'(t) = -u_x(s(t), t),$$

$$(1.4) \quad s'(t) = u^n(s(t), t),$$

$$(1.5) \quad s(0) = 0$$

where u is the penetrant concentration over its equilibrium value, s is the penetrant front driven by u , n , ϵ are positive physical parameters. In special examples in [2], n takes values varying from 10^{-2} to 10^2 ; ϵ need not be small; however it is an interesting problem to study the dependence of the solution on the parameter ϵ when ϵ is small (see [2] for details).

The problem with $\epsilon = 1$, (1.3) replaced by $[q + u(s(t), t)] \cdot s'(t) = -u_x(s(t), t)$ ($q \geq 0$) and (1.4) replaced by $s'(t) = f(u(s(t), t))$ was studied in [3] by Fasano, Meyer, Primicerio. Their results imply, in particular:

THEOREM 1.1. *The problem (1.1)–(1.5) has a unique classical solution (u, s) such that $s \in C^2[0, \infty) \cap C^\infty(0, \infty)$, $u \in C^\infty(D_\infty) \cap C^{2,1}(\overline{D_\infty})$ where $D_\infty = \{(x, t) : 0 < x < s(t), 0 < t < \infty\}$; further,*

$$(1.6) \quad 0 < u(x, t) < 1 \quad \text{for } (x, t) \in D_\infty,$$

$$(1.7) \quad -1 - \epsilon < u_x(x, t) < 0 \quad \text{for } (x, t) \in D_\infty,$$

$$(1.8) \quad u_t(x, t) > 0 \quad \text{for } (x, t) \in D_\infty,$$

$$(1.9) \quad s''(t) < 0 \quad \text{for } 0 \leq t < \infty.$$

They also studied the long time behavior of $s(t)$ ($t \rightarrow +\infty$), and proved (for $\epsilon = 1$)

* The author is partially supported by Graduate School dissertation fellowship

that:

$$\sqrt{\frac{3}{2}}(1 - o(t)) \leq \frac{s(t)}{\sqrt{t}} \leq \sqrt{2} \quad \text{for } t \rightarrow \infty.$$

More recently, Cohen and Erneux [2] obtained more precise asymptotic behavior:

$$(1.10) \quad \lim_{t \rightarrow \infty} s(t)s'(t) = \frac{M^2}{2}, \quad \lim_{t \rightarrow \infty} u(ys(t), t) = 1 - \frac{\int_0^{My} \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi}{\int_0^M \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi},$$

where M is given by

$$\exp\left(-\frac{\epsilon}{4}M^2\right) = \frac{1}{2}M \int_0^M \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi;$$

notice that (1.10) implies that

$$(1.11) \quad \lim_{t \rightarrow \infty} \frac{s(t)}{\sqrt{t}} = M.$$

They also obtained similar formulas for the asymptotic behavior of $u(x, t)$ and $s(t)$ as $t \rightarrow 0$ and as $\epsilon \rightarrow 0$:

$$(1.12) \quad u_0(x, t) - C_1\epsilon \leq u_\epsilon(x, t) \leq u_0(x, t) + C_2\epsilon \quad \text{for } 0 < x < s_\epsilon(t), \quad 0 < t < \infty,$$

$$(1.13) \quad 1 - C_3\epsilon < \frac{s_0(t)}{s_\epsilon(t)} < 1 + C_4\epsilon \quad \text{for } 0 < t < \infty,$$

where (u_ϵ, s_ϵ) denotes the solution of (1.1)–(1.5); u_0, s_0 are independent of ϵ . However, all these results were obtained by formal power series expansions.

In this paper, we shall give rigorous justification to these facts. In §2 we consider the short time behavior for (u, s) and give explicit error estimates. In §3 we study the long time behavior and prove (1.10) and (1.11). In §4 we establish (1.12) and (1.13) with $C_1 = 5 \max(n, 1)$, $C_2 = 0$, $C_3 = 0$ and $C_4 = 4 \max(n, 1)$.

Consider next the case where condition (1.2) for maintaining constant concentration at $x = 0$ is replaced by flux control, namely:

$$(1.14) \quad u_x(0, t) = g(t).$$

THEOREM 1.2 (Anderucci, Ricci [1]). *If*

$$(1.15) \quad g \in C^1[0, \infty), \quad g(t) \leq 0 \quad \text{for } t > 0, \quad g(0) < 0,$$

then the problem (1.1), (1.14), (1.3)–(1.5) has a unique classical solution (u, s) with $s \in C^1[0, \infty)$, $u \in C^{2,1}(D_\infty) \cap C^{1,0}(\overline{D_\infty})$ where $D_\infty = \{(x, t) : 0 < x < s(t), 0 < t < \infty\}$; furthermore,

$$(1.16) \quad u(x, t) > 0 \quad \text{for } (x, t) \in \overline{D_\infty},$$

$$(1.17) \quad u_x(x, t) < 0 \quad \text{for } (x, t) \in D_\infty.$$

If in addition to (1.15), $g'(t) \geq 0$ for $t \geq 0$, then

$$(1.18) \quad s''(t) \leq 0 \quad \text{for } 0 \leq t < \infty.$$

It was also proved in [1] that if $\int_0^\infty g(t)dt > -\infty$, then $\lim_{t \rightarrow \infty} s(t)$ exists; the limit was also computed.

In §5 of this paper, we shall study the long time behavior of $s(t)$ when the condition $\int_0^\infty g(t)dt > -\infty$ is dropped. We shall prove that $x = s(t)$ approaches the free boundary of a Stefan problem; in particular, it follows that $\lim_{t \rightarrow \infty} s(t) = \infty$ in the case $\int_0^\infty g(t)dt = -\infty$, in contrast to the case $\int_0^\infty g(t)dt > -\infty$, where $\lim_{t \rightarrow \infty} s(t)$ is finite.

2. Short time behavior.

THEOREM 2.1. *For the solution of (1.1)-(1.5), we have:*

$$(2.1) \quad 0 \leq u(x, t) - [1 - (1 + \epsilon)x] \leq Ct^2 \quad \text{for } 0 \leq x \leq s(t), \quad 0 \leq t \leq \frac{1}{2(1 + \epsilon)}$$

$$(2.2) \quad -C_1(1 + \epsilon)^2 t^3 \leq s(t) - \left(t - \frac{n}{2}(1 + \epsilon)t^2 \right) \leq C_2(1 + \epsilon)^2 t^3 \quad \text{for } 0 \leq t \leq \frac{1}{2(1 + \epsilon)}$$

where $C = (1 + \epsilon)(2n\epsilon + \epsilon + 2n)$, C_1 and C_2 are constants depending only on n and we can take $C_1 = 0$ if $n \geq 1$.

Proof. Let $w(x, t) = 1 - (1 + \epsilon)x$, then obviously $\epsilon w_t - w_{xx} = 0$, $w(0, t) = u(0, t)$ and

$$(2.3) \quad w_x(s(t), t) + [1 + \epsilon w(s(t), t)]w^n(s(t), t) \leq -(1 + \epsilon) + (1 + \epsilon) = 0.$$

Hence, by maximum principle, $w(x, t) \leq u(x, t)$ for $0 \leq x \leq s(t)$, $t \geq 0$.

Next, let $\tilde{w}(x, t) = 1 - (1 + \epsilon)x + Cxt$, then $\epsilon \tilde{w}_t - \tilde{w}_{xx} = C\epsilon x \geq 0$ for $x \geq 0$ and $\tilde{w}(0, t) = u(0, t)$. Since $s'(t) = u^n(s(t), t) \in (0, 1)$, we have $0 < s(t) < t$ and hence

$$\begin{aligned} \left. \{ \tilde{w}_x + (1 + \epsilon \tilde{w}) \tilde{w}^n \} \right|_{x=s(t)} &\geq -1 - \epsilon + Ct + [1 + \epsilon - \epsilon(1 + \epsilon)s(t)][1 - (1 + \epsilon)s(t)]^n \\ &\geq -1 - \epsilon + Ct + [1 + \epsilon - \epsilon(1 + \epsilon)t][1 - (1 + \epsilon)t]^n \\ &\equiv a(t) \quad \text{for } 0 \leq x \leq s(t), \quad 0 \leq t \leq \frac{1}{2(1 + \epsilon)}. \end{aligned}$$

For $0 \leq t \leq 1/[2(1 + \epsilon)]$,

$$\begin{aligned} a'(t) &= C - \epsilon(1 + \epsilon)[1 - (1 + \epsilon)t]^n - n(1 + \epsilon)[1 + \epsilon - \epsilon(1 + \epsilon)t][1 - (1 + \epsilon)t]^{n-1} \\ &\geq C - \epsilon(1 + \epsilon) - n(1 + \epsilon)^2 \cdot 2 \\ &= 0, \end{aligned}$$

and thus $a(t) \geq a(0) = 0$ for $0 \leq t \leq 1/[2(1 + \epsilon)]$. Now (2.1) follows by maximum principle.

By (2.1),

$$\begin{aligned} s'(t) &\geq [1 - (1 + \epsilon)s(t)]^n \\ &\geq 1 - n(1 + \epsilon)s(t) - C_1[(1 + \epsilon)s(t)]^2 \\ &\geq 1 - n(1 + \epsilon)t - C_1[(1 + \epsilon)t]^2 \end{aligned}$$

where we can take $C_1 = 0$ if $n \geq 1$. Hence

$$s(t) \geq t - \frac{n}{2}(1 + \epsilon)t - C_1(1 + \epsilon)^2 t^2.$$

Next, using (2.1) again, we get

$$\begin{aligned} s'(t) &\leq [1 - (1 + \epsilon)s(t) + C[(1 + \epsilon)t]^2]^n \\ &\leq 1 - n(1 + \epsilon)s(t) + C[(1 + \epsilon)t]^2. \end{aligned}$$

Solving this inequality we obtain

$$\begin{aligned} s(t) &\leq \frac{1 - \exp[-n(1 + \epsilon)t]}{n(1 + \epsilon)} + C(1 + \epsilon)^2 \int_0^t \exp[-n(1 + \epsilon)(t - \tau)] \tau^2 d\tau \\ &\leq t - \frac{n}{2}(1 + \epsilon)t^2 + C(1 + \epsilon)^2 t^3. \quad \square \end{aligned}$$

3. Long time behavior. Define $\phi(x, t)$ to be the solution of the corresponding solution of the Stefan problem, i.e.,

$$(3.1) \quad \phi(x, t) = 1 - \frac{\int_0^{x/\sqrt{t}} \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi}{\int_0^M \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi},$$

where $M = M(\epsilon)$ is such that

$$(3.2) \quad \exp\left(-\frac{\epsilon}{4}M^2\right) = \frac{1}{2}M \int_0^M \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi.$$

The main result of this section is:

THEOREM 3.1. *Suppose (u, s) is a solution of (1.1)-(1.5). Then*

$$(3.3) \quad \lim_{t \rightarrow \infty} s(t)s'(t) = \frac{M^2}{2},$$

which implies that

$$(3.4) \quad \lim_{t \rightarrow \infty} \frac{s(t)}{\sqrt{t}} = M.$$

Furthermore,

$$(3.5) \quad \lim_{t \rightarrow \infty} \sup_{0 \leq x \leq s(t)} |u(x, t) - \phi(x, t)| = 0$$

where ϕ is given by (3.1). \square

In order to prove (3.3), we need to prove (3.4) first. We shall compare the solution (u, s) of (1.1)–(1.5) to the solution ϕ of the Stefan problem. Let us first transform our problem into a variational inequality. Set

$$(3.6) \quad v(x, t) = \int_{s^{-1}(x)}^t [u(x, \tau) - (s'(\tau))^{1/n}] d\tau$$

where $s^{-1}(x)$ is the inverse function of $s(t)$; it is C^1 since $s'(t) > 0$. A calculation shows that v satisfies the following variational inequality.

$$(3.7) \quad \epsilon v_t - v_{xx} = -1 - \epsilon (s'(t))^{1/n} \quad \text{for } 0 < x < s(t), t > 0,$$

$$(3.8) \quad v = v_x = 0 \quad \text{on } x = s(t), t > 0,$$

$$(3.9) \quad v = \int_0^t [1 - (s'(\tau))^{1/n}] d\tau \quad \text{on } x = 0, t > 0,$$

$$(3.10) \quad v = 0 > -1 - \epsilon (s'(t))^{1/n} \quad \text{for } x > s(t), t > 0,$$

$$(3.11) \quad v > 0 \quad \text{for } x < s(t), t > 0 \quad (\text{by using } v_x < 0 \text{ and (3.8)}).$$

LEMMA 3.2.

$$(3.12) \quad s(t) \leq M\sqrt{t} \quad \text{for all } t > 0.$$

Proof. Let $w(x, t) = \int_{x^2/M}^t \phi(x, \tau) d\tau$. Then w satisfies the variational inequality:

$$(3.13) \quad \epsilon w_t - w_{xx} = -1 \quad \text{for } 0 < x < M\sqrt{t}, t > 0,$$

$$(3.14) \quad w = w_x = 0 \quad \text{on } x = M\sqrt{t}, t > 0,$$

$$(3.15) \quad w = t \quad \text{on } x = 0, t > 0,$$

$$(3.16) \quad w = 0 > -1 \quad \text{for } x > M\sqrt{t}, t > 0,$$

$$(3.17) \quad w > 0 \quad \text{for } x < M\sqrt{t}, t > 0.$$

Therefore, by comparison principle for variational inequalities we get $v(x, t) \leq w(x, t)$ for $0 < x < \infty, t > 0$, and hence $s(t) \leq M\sqrt{t}$. \square

Next, we prove:

LEMMA 3.3.

$$(3.18) \quad \lim_{t \rightarrow \infty} \frac{s(t)}{\sqrt{t}} = M.$$

Proof. By [3, Theorem 5.2], we have

$$(3.19) \quad \lim_{t \rightarrow \infty} s'(t) = 0;$$

therefore for any $\eta > 0$, there exist $T > 0$ such that

$$(3.20) \quad 0 < (s'(t))^{1/n} \leq \eta \quad \text{for } t \geq T.$$

Let N_η be the solution of

$$(3.21) \quad (1 - \eta) \exp\left(-\frac{\epsilon}{4}N^2\right) = (1 + \eta)\frac{1}{2}N \int_0^N \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi;$$

it is then clear that $N_\eta < M$ and $N_\eta \rightarrow M$ as $\eta \rightarrow 0$. Next, set

$$(3.22) \quad \psi(x, t) = (1 - \eta) \left(1 - \frac{\int_0^{x/\sqrt{t-T}} \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi}{\int_0^{N_\eta} \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi} \right).$$

Then

$$\begin{aligned} \epsilon\psi_t &= \psi_{xx} \quad \text{for } 0 < x < N_\eta\sqrt{t-T}, \quad t > T \\ \psi(0, t) &= 1 - \eta \quad \text{for } t \geq T, \\ \psi &= 0, \quad -\psi_x = (1 + \eta)\frac{d}{dt}(N_\eta\sqrt{t-T}) \quad \text{on } x = N_\eta\sqrt{t-T}. \end{aligned}$$

Repeating the proof of Lemma 3.2 we find that

$$(3.23) \quad s(t) \geq N_\eta\sqrt{t-T} \quad \text{for } t \geq T;$$

it follows that

$$(3.24) \quad \liminf_{t \rightarrow \infty} \frac{s(t)}{\sqrt{t}} \geq N_\eta$$

and we conclude the lemma by letting $\eta \rightarrow 0$. \square

LEMMA 3.4.

$$(3.25) \quad \lim_{t \rightarrow \infty} \sup_{0 \leq x \leq s(t)} |u(x, t) - \phi(x, t)| = 0$$

where ϕ is given by (3.1).

Proof. Set

$$k(x, t) = \frac{1}{t^\alpha} \left(2 - \frac{x^2}{s^2(t)} \right),$$

where $\alpha > 0$ is to be determined. Then

$$\begin{aligned} \epsilon k_t - k_{xx} &= -\frac{\epsilon\alpha}{t^{\alpha+1}} \left(2 - \frac{x^2}{s^2(t)} \right) + \frac{\epsilon}{t^\alpha} \frac{2x^2 s'(t)}{s^3(t)} + \frac{1}{t^\alpha} \frac{2}{s^2(t)} \\ &\geq \frac{2}{t^{\alpha+1}} \left(-\epsilon\alpha + \frac{t}{s^2(t)} \right) \\ &\geq \frac{2}{t^{\alpha+1}} \left(-\epsilon\alpha + \frac{1}{M^2} \right) \quad (\text{by Lemma 3.2}) \\ &> 0 \quad \text{for } t > 0, 0 < x < s(t) \end{aligned}$$

if α is small enough. For $T > 0$, we set

$$w(x, t) = \phi(x, t) + \sup_{t \geq T} (s'(t))^{1/n} + T^\alpha k(x, t).$$

Then $\epsilon w_t - w_{xx} > 0$ for $0 < x < s(t), t > T$; $w(0, t) \geq 1 = u(0, t)$ for $t \geq T$; $w(s(t), t) \geq \sup_{t \geq T} (s'(t))^{1/n} \geq u(s(t), t)$ for $t \geq T$ and $w(x, T) \geq 1 \geq u(x, T)$ for $0 \leq x \leq s(T)$. Therefore by maximum principle

$$w(x, t) \geq u(x, t) \quad \text{for } 0 \leq x \leq s(t), t \geq T.$$

Hence

$$\limsup_{t \rightarrow \infty} \sup_{0 \leq x \leq s(t)} [u(x, t) - \phi(x, t)] \leq \sup_{t \geq T} (s'(t))^{1/n}.$$

Letting $T \rightarrow \infty$, we obtain

$$(3.26) \quad \limsup_{t \rightarrow \infty} \sup_{0 \leq x \leq s(t)} [u(x, t) - \phi(x, t)] \leq 0.$$

Next, by Lemma 3.3,

$$\lim_{t \rightarrow \infty} \phi(s(t), t) = 0.$$

Thus by using the subsolution

$$\tilde{w}(x, t) = \phi(x, t) - \sup_{t \geq T} \phi(s(t), t) - T^\alpha k(x, t),$$

to estimate u from below and letting $T \rightarrow \infty$, we get the complement of (3.26), which completes the proof. \square

LEMMA 3.5.

$$(3.27) \quad \limsup_{t \rightarrow \infty} s(t)s'(t) \leq \frac{M^2}{2}.$$

Proof. By (1.8), $u_{xx} = \epsilon u_t \geq 0$ for $0 \leq x \leq s(t), t > 0$. Hence (as in [3, Proposition 5.1])

$$\begin{aligned} u(x, t) &\geq u(s(t), t) + u_x(s(t), t)(x - s(t)) \\ &\geq -(1 + \epsilon u(s(t), t))s'(t)(x - s(t)). \end{aligned}$$

Letting $x = \gamma s(t)$ ($0 < \gamma < 1$) in the above inequality and letting $t \rightarrow \infty$, we obtain:

$$\begin{aligned} (1 - \gamma) \limsup_{t \rightarrow \infty} s(t)s'(t) &\leq \limsup_{t \rightarrow \infty} \frac{u(\gamma s(t), t)}{1 + \epsilon u(s(t), t)} \\ &\leq \limsup_{t \rightarrow \infty} \phi(\gamma s(t), t) \quad (\text{by Lemma 3.4}) \\ &= 1 - \frac{\int_0^{\gamma M} \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi}{\int_0^M \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi} \quad (\text{by Lemma 3.3}). \end{aligned}$$

Dividing by $1 - \gamma$ and then letting $\gamma \rightarrow 1^-$, we immediately obtain (upon recalling (3.2)) the estimates (3.27). \square

LEMMA 3.6. *There exist positive constants C and T such that*

$$(3.28) \quad u_t(x, t) \leq \frac{C}{t} \quad \text{for } 0 \leq x \leq s(t), \quad T \leq t < \infty.$$

Proof. It is clear that $\epsilon u_{tt} - u_{txx} = 0$ for $0 < x < s(t)$, $t > 0$, and $u_t(0, t) = 0$ for $t > 0$. Differentiating (1.4) and using $s''(t) < 0$ we get that $u_t(s(t), t) + u_x(s(t), t)s'(t) < 0$. Hence

$$\begin{aligned} u_t(s(t), t) &\leq -u_x(s(t), t)s'(t) \\ &= (1 + \epsilon u(s(t), t))(s'(t))^2 \quad (\text{by (1.3)}) \\ &\leq \frac{C^*}{t} \quad \text{for } 0 < t < \infty \quad (\text{by Lemmas 3.3 and 3.5}). \end{aligned}$$

Next, let

$$\theta(x, t) = - \int_0^{(x+1)/\sqrt{t}} \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi.$$

Then $\epsilon\theta_t = \theta_{xx}$ for $t > 0$; and by Lemma 3.3, there exist $T, c_0 > 0$ such that

$$\begin{aligned} \theta_t(s(t), t) &= \exp\left(-\frac{\epsilon}{4} \frac{(s(t)+1)^2}{t}\right) \frac{s(t)+1}{2t\sqrt{t}} \\ &\geq \frac{c_0}{t} \quad \text{for } t \geq T. \end{aligned}$$

It is obvious that $\theta_t(0, t) > 0$ and

$$\inf_{0 \leq x \leq s(T)} \theta_t(x, T) = \inf_{0 \leq x \leq s(T)} \exp\left(-\frac{\epsilon}{4} \frac{(x+1)^2}{T}\right) \frac{x+1}{2T\sqrt{T}} \equiv c_1 > 0.$$

Therefore by maximum principle,

$$u_t(x, t) \leq C\theta_t(x, t) \quad \text{for } 0 \leq x \leq s(t), \quad t \geq T$$

if C is large enough so that $Cc_0 \geq C^*$ and $Cc_1 \geq \max_{0 \leq x \leq s(T)} u_t(x, T)$. It follows that (3.28) holds. \square

Proof of Theorem 3.1. Since we have already proved (3.9), (3.5) and (3.27), it remains only to show that

$$(3.29) \quad \liminf_{t \rightarrow \infty} s(t)s'(t) \geq \frac{M^2}{2}.$$

By Lemma 3.6,

$$\begin{aligned} u(x, t) &= u(s(t), t) + u_x(s(t), t)(x - s(t)) + \int_x^{s(t)} (\xi - x)u_{xx}(\xi, t)d\xi \\ &\leq u(s(t), t) + u_x(s(t), t)(x - s(t)) + \frac{C}{t} \frac{(s(t) - x)^2}{2} \quad \text{for } T \leq t < \infty. \end{aligned}$$

Similarly to the proof of Lemma 3.5, we can now obtain, for $0 < \gamma < 1$, (notice that $\lim_{t \rightarrow \infty} u(s(t), t) = 0$)

$$(1 - \gamma) \liminf_{t \rightarrow \infty} \left(s(t)s'(t) + \frac{C}{2}(1 - \gamma) \frac{s^2(t)}{t} \right) \geq 1 - \frac{\int_0^{\gamma M} \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi}{\int_0^M \exp\left(-\frac{\epsilon}{4}\xi^2\right) d\xi}.$$

Since $s^2(t)/t$ is bounded from above, the above inequality implies, after dividing by $1 - \gamma$ and letting $\gamma \rightarrow 1^-$, that (3.29) holds. \square

4. Small ϵ . As in [2], by formally letting $\epsilon \rightarrow 0$ in equations (1.1)–(1.5), one obtains equations for the limit functions (u_0, s_0) :

$$(4.1) \quad u_0(x, t) = 1 - B(t)x,$$

$$(4.2) \quad s'_0(t) = (1 - B(t)s_0(t))^n = B(t),$$

$$(4.3) \quad B(0) = 1, \quad s_0(0) = 0.$$

From this, one can uniquely determine $B(t)$ and $s_0(t)$; in particular $B(t)$ satisfies:

$$(4.4) \quad t + \frac{1}{2} + \frac{n-1}{1-2n} = \frac{1}{2}B^{-2} + \frac{n-1}{1-2n}B^{(1/n)-2} \quad \text{if } n \neq \frac{1}{2},$$

$$(4.5) \quad t + \frac{1}{2} = \frac{1}{2}B^{-2} + \log \frac{1}{B} \quad \text{if } n = \frac{1}{2}.$$

It easily follows that

$$(4.6) \quad 0 < B(t) < 1, \quad 0 < s_0(t)B(t) < 1,$$

$$(4.7) \quad \frac{dB}{dt} = -\frac{B^3}{1 + [(1-n)/n]B^{1/n}} < 0.$$

It is also clear that $\lim_{t \rightarrow \infty} s_0(t)/\sqrt{t} = \sqrt{2}$. Now we shall prove that the solution (u_ϵ, s_ϵ) of (1.1)–(1.5) converges to (u_0, s_0) .

THEOREM 4.1. *We have, for $\epsilon > 0$,*

$$(4.8) \quad u_\epsilon(x, t) \leq 1 - B(t)x \quad \text{for } 0 \leq x \leq s_\epsilon(t), \quad 0 \leq t < \infty,$$

$$(4.9) \quad s_\epsilon(t) < s_0(t), \quad \text{for } 0 \leq t < \infty.$$

Proof. i) First we show that if (4.9) holds true for $0 < t < t_0$, then (4.8) must be true for $0 \leq t \leq t_0$. In fact, the function $w = 1 - B(t)x$ satisfies $w_t - \epsilon^{-1}w_{xx} = -B'(t)x \geq 0$ for $0 < x < s_\epsilon(t)$, $t > 0$, and $w(0, t) = u_\epsilon(0, t)$ for $t \geq 0$. Since $1 - B(t)s_\epsilon(t) > 1 - B(t)s_0(t) > 0$ for $0 < t < t_0$,

$$(4.10) \quad \begin{aligned} \{w_x + (1 + \epsilon w)w^n\} \Big|_{x=s_\epsilon(t)} &\geq -B(t) + (1 - B(t)s_\epsilon(t))^n \\ &\geq -B(t) + (1 - B(t)s_0(t))^n \\ &= 0 \quad \text{for } 0 \leq t \leq t_0. \end{aligned}$$

Therefore, by maximum principle, $u(x, t) \leq w(x, t)$ for $0 \leq t \leq t_0$, i.e., (4.8) holds for $0 \leq t \leq t_0$.

ii) Next, we prove that (4.9) holds for small t . In fact, from (4.1)–(4.6) it follows that

$$\begin{aligned} s'_0(t) &\geq 1 - nB(t)s_0(t) - C[B(t)s_0(t)]^2 \\ &\geq 1 - nt - Ct^2; \end{aligned}$$

therefore $s_0(t) \geq t - (n/2)t^2 - Ct^3$. Hence, by (2.2), (4.9) holds for small $t > 0$.

iii) It now follows that if the theorem is not true, then there exists a $T > 0$ such that

$$(4.11) \quad s_\epsilon(t) < s_0(t) \quad \text{for } 0 < t < T$$

$$(4.12) \quad s_\epsilon(T) = s_0(T).$$

From i), we obtain

$$(4.13) \quad s'_\epsilon(t) = u_\epsilon^n(s_\epsilon(t), t) \leq [1 - B(t)s_\epsilon(t)]^n \quad \text{for } 0 \leq t \leq T.$$

Notice that $1 - B(t)s_\epsilon(t) \geq 1 - B(t)s_0(t) > c_0 > 0$ for $0 \leq t \leq T$, and the function $f(u) = u^n$ is Lipschitz continuous for $u \geq c_0$. Therefore, by using (4.2), (4.13) and ii), we can apply the comparison principle of ODE and get

$$(4.14) \quad s_\epsilon(t) < s_0(t) \quad \text{for } 0 < t \leq T,$$

which contradicts (4.12) at $t = T$. \square

THEOREM 4.2. *There exists a $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$,*

$$(4.15) \quad u_\epsilon(x, t) \geq 1 - B(t)x - 5 \max(n, 1)\epsilon \quad \text{for } 0 \leq x \leq s_\epsilon(t), \quad 0 \leq t < \infty,$$

$$(4.16) \quad [1 + 4 \max(n, 1)\epsilon]s_\epsilon(t) > s_0(t) \quad \text{for } 0 \leq t < \infty.$$

Proof. i) Set

$$(4.17) \quad w = 1 - (1 + C^*\epsilon)B(t)x - \frac{C^*\epsilon}{4}B(t) \left(s_\epsilon(t) - \frac{x^2}{s_\epsilon(t)} \right)$$

where $C^* = 4 \max(n, 1)$, then

$$\begin{aligned} &w_t - \epsilon^{-1}w_{xx} \\ &= -(1 + C^*\epsilon)B'(t)x - \frac{C^*\epsilon}{4}B'(t) \left(s_\epsilon(t) - \frac{x^2}{s_\epsilon(t)} \right) \\ &\quad - \frac{C^*\epsilon}{4}B(t) \left(s'_\epsilon(t) + \frac{x^2}{s_\epsilon^2(t)}s'_\epsilon(t) \right) - \epsilon^{-1} \frac{C^*\epsilon B(t)}{2 s_\epsilon(t)} \\ &\leq -(1 + C^*\epsilon)B'(t)s_\epsilon(t) - \frac{C^*\epsilon}{4}B'(t)s_\epsilon(t) - \frac{C^* B(t)}{2 s_\epsilon(t)} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{s_\epsilon(t)} \left[-(1 + C^*\epsilon + C^*\epsilon/4)B'(t)s_\epsilon^2(t) - \frac{C^*}{2}B(t) \right] \\
&\leq \frac{1}{s_\epsilon(t)} \left[(1 + C^*\epsilon + C^*\epsilon/4) \frac{B^3(t)}{1 + [(1-n)/n]B^{1/n}} s_0^2(t) - \frac{C^*}{2}B(t) \right] \quad (\text{by (4.7), (4.9)}) \\
&\leq \frac{1}{s_\epsilon(t)} \left[(1 + C^*\epsilon + C^*\epsilon/4) \max(n, 1)B(t) - \frac{C^*}{2}B(t) \right] \quad (\text{by (4.6)}) \\
&< 0
\end{aligned}$$

if ϵ is small enough so that $C^*\epsilon + C^*\epsilon/4 < 1$.

Now we show that if (4.16) holds for $0 < t < t_0$, then

$$(4.18) \quad u_\epsilon(x, t) \geq w(x, t) \quad \text{for } 0 \leq x \leq s_\epsilon(t), 0 \leq t \leq t_0,$$

which implies that (4.15) is valid for $0 \leq t \leq t_0$. Clearly $w(0, t) = u(0, t)$ and

$$\begin{aligned}
&\{w_x + (1 + \epsilon w)w^n\} \Big|_{x=s_\epsilon(t)} \\
&\leq -(1 + C^*\epsilon)B(t) + \frac{C^*\epsilon}{2}B(t) + (1 + \epsilon)\{[1 - (1 + C^*\epsilon)B(t)s_\epsilon(t)]^+\}^n \\
&\leq -\left(1 + \frac{C^*\epsilon}{2}\right)B(t) + (1 + \epsilon)[1 - B(t)s_0(t)]^n \quad (\text{by (4.16)}) \\
&= -\left(1 + \frac{C^*\epsilon}{2}\right)B(t) + (1 + \epsilon)B(t) \\
&< 0 \quad \text{for } 0 \leq t \leq t_0.
\end{aligned}$$

It follows by maximum principle that (4.19) holds.

ii) It is clear that $(1 + C^*\epsilon)s_\epsilon(t) = (1 + C^*\epsilon)t + O(t^2)$ as $t \rightarrow 0$, and therefore (4.16) holds for small t . As in the proof of Theorem 4.1, if the theorem is not true, then there exists a $T > 0$ such that

$$(4.19) \quad (1 + C^*\epsilon)s_\epsilon(t) > s_0(t) \quad \text{for } 0 < t < T$$

$$(4.20) \quad (1 + C^*\epsilon)s_\epsilon(T) = s_0(T).$$

From (4.18), we obtain

$$(4.21) \quad (1 + C^*\epsilon)s'_\epsilon(t) > s'_\epsilon(t) = u'_\epsilon(s_\epsilon(t), t) \geq \{[1 - (1 + C^*\epsilon)B(t)s_\epsilon(t)]^+\}^n.$$

Clearly

$$\max_{0 \leq t \leq T} B(t)s_0(t) \equiv \lambda < 1;$$

therefore, we can take $\lambda < \bar{\lambda} < 1$. Set

$$(4.22) \quad G = \{t \in [0, T] : (1 + C^*\epsilon)B(t)s_\epsilon(t) \geq \bar{\lambda}\}.$$

Then for $t \in G$, we have

$$(1 + C^*\epsilon)s_\epsilon(t) \geq \frac{\bar{\lambda}}{B(t)} > s_0(t),$$

whereas for $t \in [0, T] \setminus G$ we have

$$(4.23) \quad 1 - (1 + C^*\epsilon)B(t)s_\epsilon(t) \geq 1 - \bar{\lambda} > 0.$$

The set $[0, T] \setminus G$ is open and hence consists of open intervals with end points in G . The function $f(u) = u^n$ is Lipschitz continuous for $u \geq 1 - \bar{\lambda}$. Therefore by (4.2) and (4.21), we can apply the comparison principle of ODE and get that (4.19) holds also for $t \in [0, T] \setminus G$, which is a contradiction to (4.20) at $t = T$. \square

5. Neumann boundary condition. In this section we consider the Neumann boundary condition at $x = 0$. Taking for simplicity $\epsilon = 1$, the system become

$$(5.1) \quad u_t = u_{xx} \quad \text{for } 0 < x < s(t), t > 0,$$

$$(5.2) \quad u_x(0, t) = g(t),$$

$$(5.3) \quad [1 + u(s(t), t)] \cdot s'(t) = -u_x(s(t), t),$$

$$(5.4) \quad s'(t) = u^n(s(t), t),$$

$$(5.5) \quad s(0) = 0.$$

It is shown in [1] that if

$$(5.6) \quad g \in C^2[0, \infty), \quad g(t) \leq 0 \quad \text{for } t > 0, \quad g(0) < 0; \quad g'(t) \geq 0 \quad \text{for } t \leq 0,$$

then (1.16)–(1.18) hold. The asymptotic behavior of $s(t)$ as $t \rightarrow \infty$ is also studied in the case $\int_0^\infty g(t)dt > -\infty$.

We shall study the asymptotic behavior in the general case. Define $\psi_\eta = \psi_\eta(x, t)$, $h_\eta = h_\eta(t)$ to be the solution for the Stefan problem:

$$(5.7) \quad \psi_t = \psi_{xx} \quad \text{for } 0 < x < h(t), t > 0,$$

$$(5.8) \quad \psi_x(0, t) = g(t),$$

$$(5.9) \quad \psi = 0, \quad -\psi_x = (1 + \eta)h'(t) \quad \text{on } x = h(t),$$

$$(5.10) \quad h(0) = 0$$

where g satisfies (5.6) and $\eta \geq 0$.

THEOREM 5.1. *Assume in addition to assumption (5.6) that*

$$(5.11) \quad \lim_{t \rightarrow \infty} g(t) = 0.$$

Then

$$(5.12) \quad \lim_{t \rightarrow \infty} \frac{s(t)}{h_0(t)} = 1$$

where $x = h_0(t)$ corresponds to the free boundary of the Stefan problem (5.7)–(5.13) with $\eta = 0$. \square

We first establish several lemmas.

LEMMA 5.2. For ψ_η and h_η defined in (5.7)–(5.10), we have

$$(5.13) \quad (1 + \eta)h_\eta(t) \geq h_0(t) \quad \text{for } t > 0.$$

Proof. Let $w_\eta(x, t)$ be defined as

$$(5.14) \quad w_\eta(x, t) = \int_{h_\eta^{-1}(x)}^t \psi_\eta(x, \tau) d\tau$$

($h_\eta^{-1}(x)$ exists since $h'(t) > 0$). Then

$$(5.15) \quad (w_\eta)_t - (w_\eta)_{xx} = -1 - \eta \quad \text{for } 0 < x < h_\eta(t), t > 0,$$

$$(5.16) \quad w_\eta = (w_\eta)_x = 0 \quad \text{on } x = h_\eta(t), t > 0,$$

$$(5.17) \quad (w_\eta)_t = \int_0^t g(\tau) d\tau \quad \text{on } x = 0, t > 0,$$

$$(5.18) \quad w_\eta = 0 > -1 - \eta \quad \text{for } x > s(t), t > 0.$$

Introducing the change of variables $y = (1 + \eta)x$ and $\tilde{w}_\eta(y, t) = (1 + \eta)w_\eta(x, t)$, we find that (5.15) is equivalent to

$$(5.19) \quad (\tilde{w}_\eta)_t - (\tilde{w}_\eta)_{xx} = -1 + \left((1 + \eta) - \frac{1}{1 + \eta} \right) (w_\eta)_t(x, t) \\ \text{for } 0 < y < (1 + \eta)h_\eta(t), t > 0.$$

Since $(w_\eta)_t(x, t) = \psi_\eta(x, t) \geq 0$, it follows from the comparison principle for variational inequalities that

$$(5.20) \quad \tilde{w}_\eta(y, t) \geq w_0(y, t) \quad \text{for all } y > 0, t > 0.$$

Hence (5.13) holds. \square

LEMMA 5.3. For any $T > 0$

$$(5.21) \quad \lim_{t \rightarrow \infty} \frac{h_\eta(t - T)}{h_\eta(t)} = 1.$$

Proof. We have $h_\eta(t) \geq h_\eta(t - T)$ since $h'_\eta \geq 0$. (5.21) is obviously valid if $h_\eta(t)$ has a finite limit as $t \rightarrow \infty$. If, however, $\lim_{t \rightarrow \infty} h_\eta(t) = \infty$, then

$$(5.22) \quad \left| \frac{h_\eta(t - T)}{h_\eta(t)} - 1 \right| \leq \frac{T|h'_\eta(\xi)|}{h_\eta(t)}.$$

The right-hand side of the above inequality converges to zero since $h'_\eta(t)$ is bounded as $t \rightarrow \infty$. \square

Proof of Theorem 5.1. First we prove that

$$(5.23) \quad \lim_{t \rightarrow \infty} s'(t) = 0.$$

Since $s''(t) \leq 0$, $\lim_{t \rightarrow \infty} s'(t) = c_0$ exists. If $c_0 > 0$, then

$$\lim_{t \rightarrow \infty} \frac{s(t)}{t} = c_0 > 0.$$

However, by the mass balance

$$(5.24) \quad s(t) = - \int_0^{s(t)} u(x, t) dx - \int_0^t g(\tau) d\tau,$$

and hence

$$\frac{s(t)}{t} \leq -\frac{1}{t} \int_0^t g(\tau) d\tau \longrightarrow 0 \quad \text{as } t \rightarrow \infty$$

by (5.11), which is a contradiction.

Now, by using (5.23), Lemmas 5.2 and 5.3, we can proceed in the same proof as in Lemma 3.3 to finish the proof of this theorem. \square

Having proved Theorem 5.1, we now reduce the asymptotic behavior of $s(t)$ near ∞ to that of the corresponding Stefan problem, which is well known. Therefore, let us use, for example, [5, Chapter 8, Theorem 3], we immediately get the behavior of $s(t)$ near ∞ if $g(t)$ is like $t^{-\delta}$ ($1/2 \leq \delta < 1$) near ∞ .

Acknowledgements. The author thanks Professor Avner Friedman for his directions and help.

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