Properties of Virtual Resolutions over Toric Varieties

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Dedication

To my wife, Megan.

Abstract

Minimal free resolutions enable researchers to use homological algebra to study geometric invariants of subvarieties of projective space. Virtual resolutions for smooth projective toric varieties are a generalization of minimal free resolutions for projective space. They enable researchers to use homological algebra to study a wider range of geometric objects. This thesis investigates what light can be shed by virtual resolutions on toric varieties and also explores properties of virtual resolutions themselves. For example, two algebraic conditions are identified that completely determine when a graded chain complex is a virtual resolution. The invariance of saturated Fitting ideals is also proved and a connection between saturated Fitting ideals and locally free sheaves is shown. After studying virtual resolutions in general, sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ are explored. In particular, both necessary and sufficient combinatorial conditions are proved for when a given set of points is exactly the solution set to two bihomogeneous polynomials. These sets of points are called virtual complete intersections. Finally, the `VirtualResolutions` software package for *Macaulay2* is introduced. This package contains tools for computing examples and testing conjectures.

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Chapter 1

Introduction

In n-dimensional projective space, there is a correspondence between certain sheaves and finitely generated modules over the standard graded polynomial ring in $n+1$ variables over a field. Because of this, the geometry of varieties can be investigated through modules over the standard graded polynomial ring over a field $\Bbbk$. Homological tools can be used to study the modules and therefore understand properties of the geometry of the varieties. The main tool is the minimal free resolution of the module which encodes important numerical invariants of both the module and the corresponding sheaf such as the Hilbert function, regularity, multiplicity, and dimension.

There is an abundance of homological tools that can be used to study minimal free resolutions, but complications arise when moving from projective space to more general toric varieties. The minimal free resolution does not reflect the geometry as closely, even in the case when the toric variety is a product of projective spaces. These more general toric varieties have applications as far reaching as computer vision, computational biology and chemistry, statistics, and physics. For instance, chemists use toric geometry when modeling behavior of chemical reactions to investigate steady states of reactions that are modeled using polynomial differential equations [CDSS09]. The research in this dissertation is motivated by the desire to understand the anatomy of toric varieties, the homological tools used to study them, and their inherent combinatorial structure.

In order to arrange a tighter connection between the geometry and homological algebra for smooth projective toric varieties, Berkesch, Erman, and Smith introduce virtual resolutions in [BES17]. A virtual resolution is a chain complex over the Cox ring

of a smooth projective toric variety that admits homology so long as it is annihilated by a power of the irrelevant ideal of the toric variety.

The background of syzygies, minimal free resolutions, toric varieties, and virtual resolutions are outlined in Chapter 2. The section on syzygies includes an introduction to Betti numbers, the Hilbert function and Castelnuovo–Mumford regularity of a module over the standard graded polynomial ring. In the next section, toric varieties are discussed, including toric actions, the Cox ring or total coordinate ring S, the irrelevant ideal B, the connection of normal toric varieties with fans, the quotient construction of a smooth projective toric variety, and two examples. Chapter 2 ends with the definition of a virtual resolution, multigraded Castelnuovo–Mumford regularity, and three ways to construct a virtual resolution.

After the groundwork of Chapter 2 has been covered, Chapter 3 contains newer results. This chapter leads with an exposition of the structure of the B-saturated prime ideals inside the Cox ring of a smooth projective toric variety. Secondly, two algebraic conditions are named that completely determine when a graded chain complex is a virtual resolution without computing the homology. The benefit of this theorem is that the conditions can be restated as conditions on the minors of the chain complex maps. This allows an alternative, possibly more computational, procedure of checking when a complex is a virtual resolution. This theorem has been implemented in the *Macaulay2* package `VirtualResolutions` which is available in the most current version of *Macaulay2*. After the proof of what makes a complex virtual, some results regarding the B-saturation of Fitting ideals are proved. In particular, the B-saturated Fitting ideals are an invariant of a coherent sheaf over a smooth projective toric variety, and there is a connection between the saturated Fitting ideals and locally free sheaves.

In Chapter 4, the focus is shifted to to points inside the product of two projective lines $\mathbb{P}^1 \times \mathbb{P}^1$. More specifically, we examine when sets of reduced points (points of multiplicity one) are the solution set of exactly two polynomials inside the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$, $\Bbbk[x_0, x_1, y_0, y_1]$. We call these points virtual complete intersections. Our first result is that if we consider the points set-theoretically, rather than scheme-theoretically, then every set of points is a virtual complete intersection. When we move to the richer theory of schemes, this is not the case. We prove a multitude of results and name conditions that determine when points are a virtual complete intersection and also

obstructions to virtual complete intersections. The chapter culminates with a complete classification of which the sets of points form a virtual complete intersection, provided the points lie on at most three horizontal or vertical rulings of $\mathbb{P}^1 \times \mathbb{P}^1$.

The final chapter, Chapter 5, introduces the package `VirtualResolutions` for the computer algebra software *Macaulay2* [GS]. This package was instrumental in creating examples and formulating conjectures that led to propositions and theorems in the previous two chapters. This chapter contains example example code on how to use the package and explains how certain functions are implemented. The contents of the actual code is included in Appendix A. It is the hope that this package will be of use to other researchers to further the study of toric varieties via virtual resolutions.

Chapter 2

Background and Preliminaries

2.1 Chapter Overview

This chapter establishes the foundation for the results to come in the following few chapters. In particular, this chapter contains important notions and definitions such as syzygies and minimal resolutions. These algebraic tools can be used to study the geometric objects of varieties and sheaves over varieties. The one-to-one correspondence between coherent sheaves over projective space and graded modules over the standard graded polynomial ring enable researchers to translate geometric questions into algebraic ones and then use homological tools to investigate. There is a similar correspondence between sheaves over a toric variety and multigraded modules over the Cox ring.

Section 2.2 introduces tools, such as minimal free resolutions, that are useful in studying modules. The Hilbert function and polynomial are defined, and invariants such as the Betti numbers of a module are discussed. The connection between Betti numbers and properties of the module are explained in this section. Finally, Castelnuovo–Mumford regularity is described. This section lays the groundwork necessary for attempting to generalize many of these notions to the more general toric variety case.

Toric varieties are defined and discussed in Section 2.3. In this section, the Cox ring is defined and the correspondence between subvarieties of a toric variety and certain ideals of the Cox ring is discussed. Just as subvarieties of projective space can be understood by studying modules of a polynomial ring, subvarieties of toric varieties can be better understood by studying certain modules over the Cox ring. An emphasis is

placed on understanding the multigrading of the Cox ring through the group action of a torus.

In a product of projective spaces or, more generally, in a toric variety, the tools introduced in Section 2.2 do not as closely reflect the geometry of subvarieties. A proposed replacement of minimal free resolutions called virtual resolutions is given in Section 2.4. We define virtual resolutions and give two different ways of constructing them. A couple of examples are also provided in this section.

2.2 Syzygies

In this section we will discuss some results and techniques well known for investigating syzygies of modules in projective space $\mathbb{P}^n$. Throughout this section, let $S = \Bbbk[x_0, x_1, \ldots, x_n]$ be the standard graded polynomial ring in $n+1$ variables over an algebraically closed field $\Bbbk$ and let $\mathfrak{m} = \langle x_0, ..., x_n \rangle$ be the maximal ideal.

We are able to use modules over S to investigate subvarieties of $\mathbb{P}^n$ because of the following theorem.

Theorem 2.2.1. *There is a bijective correspondence*

$$\{\textit{nonempty closed subvarieties of } \mathbb{P}^n\} \longleftrightarrow \{\mathfrak{m}\textit{-saturated radical homogeneous ideals}\}.$$

2.2.1 Minimal Free Resolutions

Free resolutions were originally studied by David Hilbert in the 19th century. We start by defining a minimal resolution and providing a few examples. This discussion is largely following [Eis95] and [Eis05].

Definition 2.2.2. *Let R be a ring. A **projective resolution** of an R-module M is a complex*

$$F_\bullet := \left[F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_n} F_n \xleftarrow{\varphi_{n+1}} \cdots \right]$$

*of projective R-modules F_i such that $H_0(F) = M$ and otherwise F has no homology. We say F is a **free resolution** if all the F_i are free and is a **graded free resolution** if R is a graded ring, all the F_i are graded free modules, and the maps are all homogeneous*

*maps of degree 0. If, for some $n < \infty$, we have $F_{n+1} = 0$, but $F_i \neq 0$ for $0 \leq i \leq n$, then we shall say that F is a **finite** resolution of **length** n.*

It is straightforward to see that every finitely generated module M over S has a free resolution. To construct one, simply let F_0 be the free module generated by a set of generators on M. The kernel of the map $F_0 \to M$ is called the **first syzygy module** of M. Next let F_i be the free module on a set of generators of the ith syzygy of M and repeat this process. The ith syzygy module is the image of the map $F_i \to F_{i-1}$ and the elements of these syzygy modules are called syzygies.

If M is a graded S-module, then define $M(a)$ to be the module M shifted by a. That is, $M(a)_n$, the nth graded component of $M(a)$ is M_{a+n}, the $(a+n)$th graded component of M.

Example 2.2.3. Let $S = \Bbbk[x, y]$ and $M = S/\langle x, y\rangle = \Bbbk$. Following the procedure above, $F_0 = S$. Now let $K_0 = \ker(F_0 \to M)$. Then $K_0 = \langle x, y\rangle$ and thus F_1 is free of rank 2.

Let $\varphi_1 \colon S^2(-1) \to S$ be defined by $\varphi_1(a, b) = ax + by$. Notice that this is a degree zero map. $K_1 = \ker(\varphi_1) = \langle (y, -x)\rangle$ is generated by a single element so F_2 has rank 1.

In order for φ_2 to be a degree zero map, $F_2 = S(-2)$ and $\varphi_2 \colon 1 \mapsto (y, -x)$. This map is injective so $K_3 = 0$ which means that $F_3 = 0$ so our free resolution has length 2. The free resolution is

$$S \xleftarrow{\begin{bmatrix} x & y \end{bmatrix}} S^2(-1) \xleftarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} S(-2) \longleftarrow 0,$$

where the maps over the arrows are denoted by the corresponding matrices. This particular example is the Koszul complex of length 2. ◇

It is not true in general that free resolutions are unique. Of course, we could always take the direct sum of a complex $F_\bullet$ with the shifted trivial complex $0 \leftarrow S \xleftarrow{\text{id}} S \leftarrow 0$ to yield a new free resolution. If S is a graded ring (such as a polynomial ring), there is however, one "smallest" free resolution, called the minimal resolution, in the following sense. If $F_\bullet$ and $G_\bullet$ are two minimal resolutions of M, then there is a graded isomorphism of complexes $F_\bullet \to G_\bullet$ that induces the identity on M.

Definition 2.2.4. *A free resolution $F_\bullet$ of an S-module M is **minimal** if the image of each $\varphi_n \colon F_n \to F_{n-1}$ is contained in $\mathfrak{m}F_{n-1}$. In other words, a free resolution $F_\bullet$ is minimal if all the maps of the complex $\Bbbk \otimes_S F$ are zero.*

In fact, it turns out that every free resolution contains the minimal resolution as a direct summand. Looking at Example 2.2.3 above, it is easy to see that $F_\bullet \otimes_S \Bbbk$ has all zero maps (because all entries in the matrices are positively graded) so $F_\bullet$ is the minimal resolution of $\Bbbk$. Next is a slightly more interesting example of a minimal free resolution.

Example 2.2.5. Let $S = \Bbbk[x, y]$. Let $\mathfrak{m}$ be the maximal ideal $\langle x, y\rangle$ in S. Following the exact same procedure as in Example 2.2.3, a resolution of $S/\mathfrak{m}^2 = S/\langle x^2, xy, y^2\rangle$ is

$$S \xleftarrow{\begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}} S(-2)^3 \xleftarrow{\begin{bmatrix} y & 0 \\ -x & y \\ 0 & -x \end{bmatrix}} S(-3)^2 \longleftarrow 0.$$

Notice that all entries of the matrices are contained in $\mathfrak{m}$ so this resolution is indeed minimal. ◇

Up to this point, all examples have had finite minimal free resolutions. It turns out that for finitely generated modules over S, this will always be the case.

Theorem 2.2.6 (Hilbert Syzygy Theorem)**.** *If $S = \Bbbk[x_0, ..., x_n]$, then every finitely generated graded S-module has a finite graded resolution of length $\leq n + 1$, by finitely generated free modules.*

2.2.2 The Hilbert Function

Hilbert also defined the Hilbert function which is a way to measure invariants of a module.

Definition 2.2.7. *Let M be a finitely generated module over $S = \Bbbk[x_0, x_1, ..., x_n]$. Recall the degree d component of M be denoted by M_d. The* ***Hilbert function*** *of M is the numerical function*

$$H_M(d) := \dim_{\Bbbk}(M_d).$$

Let $F_\bullet$ be a finite free resolution of M of length r. Since each φ_i is a degree zero map, extracting the dth degree component of M and the F_i, there is an exact sequence of vector spaces

$$(F_\bullet)_d := [(F_0)_d \longleftarrow (F_1)_d \longleftarrow \cdots \longleftarrow (F_r)_d \longleftarrow 0]\,.$$

So the Hilbert function can be computed as

$$H_M(d) = \sum_{i=0}^{r} (-1)^i \dim_{\Bbbk}(F_i)_d. \tag{2.1}$$

Example 2.2.8. Revisiting Example 2.2.5 where $S = \Bbbk[x, y]$ and $M = S/\mathfrak{m}^2$, notice that the zero degree component of M is $M_0 = \Bbbk$, so $H_M(0) = 1$. Moving to the degree 1 component of M, since $M_1 = \operatorname{span}_{\Bbbk}\langle x, y\rangle$, then $H_M(1) = 2$. Finally, $M_n = 0$ for $n \geq 2$ so $H_M(n) = 0$ for $n \geq 2$. This is summarized below:

$$H_M(d) = \begin{cases} 1 & \text{for } d = 0, \\ 2 & \text{for } d = 1, \\ 0 & \text{for } d \geq 2. \end{cases}$$

$\diamond$

In particular, in Example 2.2.8, the Hilbert function stabilized to zero, however this does not always happen. Though the general situation can be more complicated, Hilbert discovered that for d large enough, the values of the Hilbert function $H_M(d)$ can be encoded in a polynomial.

Theorem 2.2.9. *If M is a finitely generated graded module over S, then for $d \gg 0$, $H_M(d)$ agrees with a polynomial of degree $\leq n + 1$.*

Definition 2.2.10. *The polynomial in Theorem 2.2.9, denoted $P_M(d)$, is called the* ***Hilbert polynomial*** *of M.*

2.2.3 Betti Numbers

Because minimal free resolutions are unique up to graded isomorphism, the twists of the free modules at each step in the resolution are independent of the choice of minimal free resolution. These data are called the **Betti numbers**. If $F_\bullet$ is a free complex of R-modules, where

$$F_\bullet := [F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_n \longleftarrow \cdots]$$

with $F_i = \bigoplus_j R(-j)^{\beta_{i,j}}$, then the $\beta_{i,j}$ are called the ith graded Betti numbers of $F_\bullet$. The ith total Betti number is $\beta_i = \sum_j \beta_{i,j}$. If $F_\bullet$ is a free resolution of a module M, then this means that M is generated in degrees $\beta_{0,j}$. In Example 2.2.5, the only nonzero Betti numbers are

$$\beta_{0,0} = 1, \quad \beta_{1,2} = 3, \quad \beta_{2,3} = 2.$$

However, simply listing Betti numbers can be cumbersome. We can better encode the Betti numbers in a Betti diagram or Betti table where the entry in column i and row j is $\beta_{i,i+j}$. The Betti table of Example 2.2.5 is:

	0	1	2
0	1	-	-
1	-	3	2

where, for clarity, the zeros in the table have been replaced with dashes. Also notice that this table could be extended infinitely to the right and down with dashes, but it makes more sense to only include the nonzero part of the Betti table. The projective dimension of a module can immediately be seen by looking at the width of the Betti table of the module. In this example, the projective dimension is 2. We will see in Section 2.2.4 that the height of the Betti table has to do with the regularity of the module.

An easy combinatorial argument gives $\dim_{\Bbbk} S_d = \binom{n+d}{d}$. Combining this with the above discussion of Betti numbers and Equation (2.1) provides the following proposition.

Proposition 2.2.11. *Suppose $S = \Bbbk[x_0, ..., x_n]$ and M is a graded S-module. If M has a finite free resolution*

$$F_\bullet := [F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_r]$$

where $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$ are all finitely generated, then

$$H_M(d) = \sum_{i=0}^{r} (-1)^i \sum_j \binom{n+d-\beta_{i,j}}{d}.$$

Notice that we will always be able to find such a finite free resolution by Theorem 2.2.6. This proposition says exactly what the Hilbert polynomial of M is. An interesting question to ask is if there is an invariant of M that tells us when $H_M(d) = P_M(d)$. We will answer this question in Section 2.2.4.

Notice that the Hilbert Syzygy Theorem (Theorem 2.2.6) only gives a bound on the length of a free resolution of a module. Another natural question to ask is whether there are invariants of a module that determine the length of a resolution. In the graded case, where projective modules are free modules, the depth of the module and ring can be chosen to play the role of these invariants.

Theorem 2.2.12 (Auslander–Buchsbaum formula)**.** *Let R be a graded ring with maximal ideal $\mathfrak{m}$ and M be a finitely generated R-module. If M is of finite projective dimension, then*

$$\operatorname{pdim}(M) = \operatorname{depth}(\mathfrak{m}, R) - \operatorname{depth}(\mathfrak{m}, M).$$

Consider the ideal I of a finite number of points in $\mathbb{P}^2$. If $S = \Bbbk[x, y, z]$, then the Auslander–Buchsbaum formula tells us that the projective dimension of S/I is 2 so the projective dimension of I is 1. We also have the following useful tool for ideals of projective dimension 1.

Theorem 2.2.13 (Hilbert–Burch)**.** *Suppose that an ideal I in a Noetherian ring R admits a free resolution of length 1:*

$$0 \longleftarrow I \longleftarrow G \xleftarrow{M} F \longleftarrow 0.$$

If the rank of the free module F is t, then the rank of G is $t+1$, and there exists a nonzerodivisor a such that $I = aI_t(M)$, where $I_t(M)$ is the ideal generated by the $t+1$

maximal minors of the matrix representation of M with respect to given bases of F and G. In fact, the generator of I that is the image of ith basis vector of G is a times the ith minor of M. Furthermore, the depth of $I_t(M)$ is 2.

Conversely, given a nonzerodivisor $a \in R$ and given a $(t+1) \times t$ matrix M with entries in R such that $\operatorname{depth}(I_t(M)) \geq 2$*, the ideal $I = aI_t(M)$ admits a free resolution of length one as above. The ideal I has depth 2 precisely when the element a is a unit.*

One way to prove the converse of this theorem is to construct the complex

$$0 \leftarrow I \leftarrow F \leftarrow G \leftarrow 0$$

and then check that it has no homology. In general, though, it is not always easy to check that a given free complex is a resolution. However, by the following theorem, it suffices to count the ranks of free modules and depth of ideals generated by maximal minors. If φ is a map of free modules, denote by $I(\varphi)$ the ideal generated by the maximal minors of the matrix representation of φ with respect to some given basis.

Theorem 2.2.14 (What Makes a Complex Exact? [BE73])**.** *Let R be a Noetherian ring. A complex of free R-modules*

$$F_\bullet := \left[F_0 \xleftarrow{\varphi_1} F_1 \longleftarrow \cdots \longleftarrow F_{r-1} \xleftarrow{\varphi_r} F_r \longleftarrow 0 \right].$$

is exact if and only if $\operatorname{rank} \varphi_{i+1} + \operatorname{rank} \varphi_i = \operatorname{rank} F_i$ *and* $\operatorname{depth} I(\varphi_i) \geq i$ *for every i.*

An important component of the proof of the theorem above relies on the Peskine–Szpiro Acyclicity Lemma below.

Lemma 2.2.15 (Peskine–Szpiro Acyclicity Lemma [PS73])**.** *Suppose*

$$F_\bullet := [F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_n \longleftarrow 0]$$

is a complex of finitely generated (but not necessarily free) modules over a graded ring R such that $\operatorname{depth}(F_i) \geq i$*. If the homology module $H_i(F_\bullet)$ has positive depth for some $i > 0$, then for the largest such i, it must be true that* $\operatorname{depth}(H_i(F_\bullet)) \geq 1$*.*

This theorem and lemma will play a vital role in Chapter 3.

2.2.4 Castelnuovo–Mumford Regularity

Since Proposition 2.2.11 shows the Hilbert function always becomes polynomial eventually, it is natural to investigate when this occurs. Here is it necessary to introduce the notion of Castelnuovo–Mumford regularity. From now on, regularity means Castelnuovo–Mumford regularity. There is more than one way to define regularity for S-modules. As seen below, regularity is not really a property of a module, but rather a property of a complex.

Definition 2.2.16. *Let*

$$F_\bullet := [F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_i \longleftarrow \cdots]$$

be a graded complex of free S-modules with $F_i = \bigoplus_j S(-a_{i,j})$. The regularity of F is

$$\operatorname{reg} F := \sup_{i,j}\{a_{i,j} - i\}.$$

If M is an S-module, and F is the minimal resolution of M we define $\operatorname{reg} M := \operatorname{reg} F_\bullet$.

If the Betti diagram of a module is known, then we can instantly find the regularity of the module. The regularity is simply the height of the Betti diagram. This is because the entry in column i and row j in the Betti diagram is defined to be $\beta_{i,j-i}$.

To answer the question of when the Hilbert function becomes polynomial, we have the following.

Theorem 2.2.17. *Let M be a finitely generated graded module over the polynomial ring $S = \Bbbk[x_0, ..., x_n]$.*

(1) The Hilbert function $H_M(d)$ agrees with the Hilbert polynomial $P_M(d)$ for $d \geq \operatorname{reg} M + 1$.

(2) More precisely, if M is a module of projective dimension δ, then $H_M(d) = P_M(d)$ for $d \geq \operatorname{reg} M + \delta - n$.

It is also possible to define the regularity in terms of local cohomology. Following Brodmann and Sharp's introduction to local cohomology [BS13], for an R-module M

let

$$\Gamma_I(M) = \bigcup_{n\in\mathbb{N}} (0 :_M I^n)\,.$$

That is, $\Gamma_I(M)$ is the set of elements of M that are annihilated by some power of I. This is a submodule of M and if f is a homomorphism of R-modules $M \to N$, then $f(\Gamma_i(M)) \subset \Gamma_i(N)$ so f induces a homomorphism $\Gamma_I(M) \to \Gamma_I(N)$ and it is easy to check that $\Gamma_I(-)$ is functorial. In fact, Γ_I is a left-exact covariant functor. So it makes sense to give the ith right derived functor of Γ_I a name.

Definition 2.2.18. *For $i \in \mathbb{N}_0$, the ith right derived functor of Γ_I is denoted by H^i_I and is called the **ith local cohomology functor** with respect to I.*

The regularity of a module can be defined in the following equivalent ways.

Theorem 2.2.19. *Let M be a finitely generated graded S-module and let d be an integer. The following conditions are equivalent:*

(i) $d \geq \operatorname{reg} M$

(ii) $d \geq \max\{e|\, \mathrm{H}^i_{\mathfrak{m}}(M)_e \neq 0\} + i$ *for all* $i \geq 0$.

(iii) $d \geq \max\{e|\, \mathrm{H}^0_{\mathfrak{m}}(M)_e \neq 0\}$ *and* $\mathrm{H}^i_{\mathfrak{m}}(M)_{d-i+1} = 0$ *for all* $i \geq 0$.

2.3 Toric Varieties

Toric varieties are among the most concrete varieties since they have much more combinatorial structure than other abstract varieties. In this section we will concentrate on the correspondence between the geometry of a toric variety and algebra. Specifically we will look at the Cox ring. Much of the content of this section can be found in [CLS11].

Definition 2.3.1. *A **toric variety** is an irreducible variety V that contains a torus $T \cong (\Bbbk^*)^m$ as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on V.*

Affine toric varieties can be constructed from strongly convex rational cones. General toric varieties can be glued together from affine toric varieties and their combinatorial

data is encoded in a fan Σ of cones. Let $\Sigma(1)$ denote the set of one-dimensional cones (rays) in Σ. Following [Cox95], define the **total coordinate ring** or **Cox ring** as

$$S = \Bbbk[x_\rho | \rho \in \Sigma(1)].$$

In fact, given a fan, the toric variety of the fan can be constructed using the quotient construction as outlined in [CLS11]. If the cardinality of $\Sigma(1)$ is n, then the toric variety will be $(\Bbbk^n \backslash Z)/\sim$ where Z is some exceptional set in $\Bbbk^n$ and points in $\Bbbk^n \backslash Z$ are identified if they are in the same orbit of a specific group action $G \times \Bbbk^n \to \Bbbk^n$ where $G \subset (\Bbbk^*)^n$. The exceptional set can be defined as the vanishing set of a specific ideal in the Cox ring. Namely, define

$$x^{\hat{\sigma}} := \prod_{\rho \in \Sigma(1) \backslash \sigma} x_\rho.$$

Then $Z = V(B) = \{\vec{a} \in \Bbbk^n \mid b(a) = 0 \text{ for every } b \in B\}$, where

$$B := \langle x^{\hat{\sigma}} | \sigma \in \Sigma \rangle \subset S.$$

This is called the **irrelevant ideal** of the Cox ring. If τ is a face of σ, then $x^{\hat{\sigma}}$ divides $x^{\hat{\tau}}$ so in computing B, it suffices to index over all maximal faces of Σ. Additionally, the group action will induce a multigrading on the Cox ring. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_k)$ and $\boldsymbol{\lambda}^{\boldsymbol{a}} = \lambda_1^{a_1} \lambda_2^{a_2} \cdots \lambda_k^{a_k}$. Then if $G \cong (\Bbbk^*)^k$ and $\boldsymbol{\lambda} \in G$ acts on $(t_1, ..., t_n) \in (\Bbbk^*)^n$ by

$$\boldsymbol{\lambda} \cdot (t_1, ..., t_n) = (\boldsymbol{\lambda}^{\boldsymbol{a_1}} t_1, \boldsymbol{\lambda}^{\boldsymbol{a_2}} t_2, ..., \boldsymbol{\lambda}^{\boldsymbol{a_n}} t_n),$$

then we say

$$\deg(x_i) = \boldsymbol{a_i}. \tag{2.2}$$

In this way the geometry can be changed into algebra and tools from commutative algebra (such as those described in Section 2.2) can be used to explore properties of the toric variety.

Theorem 2.3.2. *Let X_Σ be a simplicial toric variety. Then there is a bijective correspondence*

$$\{\textit{nonempty closed subvarieties of } X_\Sigma\} \longleftrightarrow \{B\textit{-saturated radical homogeneous ideals}\}.$$

Another way to interpret the multigrading of S is through the class group of S. The class group $\mathrm{Cl}(X)$ is the free abelian group generated by the codimension one subvarieties of X. Under the Orbit-Cone Correspondence [CLS11, Theorem 3.2.6], one dimensional cones in the fan of a toric variety X_Σ correspond to codimension one orbits. The closure of this codimension one orbit is a prime divisor so each ray $\rho \in \Sigma$ corresponds to a prime divisor D_ρ. Defining $\deg(x_\rho) = D_\rho$, the Cox ring is graded by Weil divisors.

The grading by the class group matches with the grading given by the weights of the G action. If M is the group of characters for the toric variety X_Σ, then there is a short exact sequence (for toric varieties without torus factors)

$$0 \longrightarrow M \xrightarrow{\mathrm{div}} \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

where if $\chi^m \in M$ and u_ρ is the minimal generator of the ray ρ, then

$$\mathrm{div}(m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

The grading of x_ρ is given by the image of e_ρ (the basis vector that ρ corresponds to in $\mathbb{Z}^{\Sigma(1)}$) in $\mathrm{Cl}(X_\Sigma)$. This grading matches the grading from the weights of the $(\Bbbk^*)^k \cong G$ action as described in (2.2) above. In fact, the group G that acts on $\Bbbk^n \backslash Z$ is the group $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \Bbbk^*)$. In the examples that follow we will simply state G for concreteness. However, it is not hard to find G from the following lemma from [CLS11].

Lemma 2.3.3. *Let X_Σ be the toric variety of a fan Σ. Let $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \Bbbk^*) \subset (\Bbbk^*)^{\Sigma(1)}$. Then given a basis $e_1, e_2, ..., e_n$ of M, and minimal generators μ_ρ of $\rho \in \Sigma(1)$,*

$$G = \left\{ (t_\rho) \in (\Bbbk^*)^{\Sigma(1)} \,\middle|\, \prod_{\rho \in \Sigma(1)} t_\rho^{\langle e_i, \mu_\rho \rangle} = 1 \textit{ for } 1 \leq i \leq n \right\}.$$

As a first example, the toric variety $\mathbb{P}^n$ can be constructed using the quotient construction.

Example 2.3.4. Let $e_1, e_2, ..., e_n$ be the standard basis vectors of the lattice $\mathbb{Z}^n$. Define $e_0 = -e_1 - e_1 - \cdots - e_n$. The fan corresponding to $\mathbb{P}^n$ consists of cones generated by all proper subsets of $\{e_0, e_1, ..., e_n\}$. So in this case, the Cox ring is $\Bbbk[x_0, x_1, ..., x_n]$. To recover $\mathbb{P}^n$, first start with $\Bbbk^{n+1}$. Now remove the exceptional set and then quotient out by the group action. Notice that every maximal face will not contain exactly one ray. The irrelevant ideal B is then $\langle x_0, x_1, ..., x_n\rangle$ and the exceptional set $Z = V(B)$ is the origin 0. The group

$$G = \{(\lambda, \lambda, ..., \lambda) \in \Bbbk^{n+1} | \lambda \in \Bbbk^*\} \cong \Bbbk^*$$

acts on $\Bbbk^{n+1}$ by scalar multiplication. So the toric variety is

$$(\Bbbk^{n+1} \backslash 0)/\Bbbk^* = \mathbb{P}^n_{\Bbbk}.$$

Since $\Bbbk^*$ acts on $\Bbbk^{n+1}$ by

$$\lambda \cdot (t_0, t_1, ..., t_n) = (\lambda t_0, \lambda t_1, ..., \lambda t_n),$$

then $\deg(x_i) = 1$ for every $0 \leq i \leq n$. This is the standard grading on $\Bbbk[x_0, ..., x_n]$ and the irrelevant ideal is the maximal ideal. $\diamond$

Example 2.3.4 makes it clear that Theorem 2.2.1 is a special case of Theorem 2.3.2 where $X_\Sigma = \mathbb{P}^n$ and $B = \mathfrak{m}$.

In Example 2.3.4, the grading on the Cox ring is the standard grading, but in general this does not always happen. Next we examine the Hirzebruch surface $\mathbb{F}_a$.

Example 2.3.5. The fan $\Sigma \subset \mathbb{R}^2$ of the Hirzebruch surface $\mathbb{F}_a$ consists of the cones $\text{Cone}(e_1, e_2)$, $\text{Cone}(e_2, -e_1 + ae_2)$, $\text{Cone}(-e_1 + ae_2, -e_2)$ and $\text{Cone}(-e_2, e_1)$ where e_1, e_2 are the standard basis vectors of $\mathbb{R}^2$.

Let $u_1 = -e_1 + ae_2$, $u_2 = e_1$, $u_3 = -e_2$, and $u_4 = e_2$. These are the rays of Σ. The Cox ring of $\mathbb{F}_a$ is $S = \Bbbk[x_1, x_2, x_3, x_4]$. The irrelevant ideal is

$$B = \langle x_1x_4, x_1x_3, x_2x_3, x_2x_4\rangle = \langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle$$

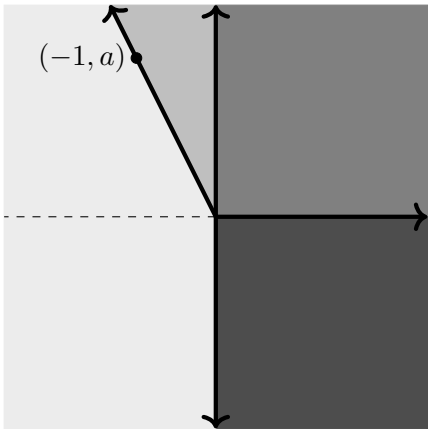


Figure 2.1: The fan of the Hirzebruch surface $\mathbb{F}_a$.

so $Z = V(B) = \Bbbk^2 \times \{0,0\} \cup \{0,0\} \times \Bbbk^2$. The group

$$G = \{(\lambda, \lambda, \mu, \lambda^{-a}\mu) | \lambda, \mu \in \Bbbk^*\} \cong (\Bbbk^*)^2$$

acts on $\Bbbk^4$ via coordinate-wise multiplication. In this case the grading of S is

$$\begin{aligned} \deg(x_1) &= (1,0) \\ \deg(x_2) &= (1,0) \\ \deg(x_3) &= (0,1) \\ \deg(x_4) &= (-a,1). \end{aligned}$$

As a final remark to this example, we note that if $a = 0$, then $\mathbb{F}_a \cong \mathbb{P}^1 \times \mathbb{P}^1$. This implies the irrelevant ideal of $\mathbb{P}^1 \times \mathbb{P}^1$ is $\langle x_1x_4, x_1x_3, x_2x_3, x_2x_4 \rangle$, which has primary decomposition $\langle x_1, x_2 \rangle \cap \langle x_3, x_4 \rangle$. $\diamond$

As mentioned above, the Cox ring is graded by the class group of X_Σ. All Weil divisors in a smooth projective toric variety are Cartier so the class group of X_Σ is the same as the Picard group $\mathrm{Pic}(X_\Sigma)$. For the rest of this document we will refer the Cox ring as being $\mathrm{Pic}(X)$-graded.

2.4 Virtual Resolutions

As shown in Section 2.2, minimal resolutions are a good way to study modules over projective space. However, in products of projective space, it turns out that minimal resolutions contain too much algebraic structure. In particular, they contain information that does not matter geometrically. This results in minimal resolutions being longer and the ranks of the free modules being larger than needed. The reason for this difference is that in $\mathbb{P}^n$, the vanishing set of the irrelevant ideal is the origin. Recall from Example 2.3.4 that in this case, the irrelevant ideal is the maximal ideal.

On the other hand, consider the simplest case of a product of projective spaces $\mathbb{P}^1 \times \mathbb{P}^1$ where the first coordinate is parametrized by x_1, x_2 and the second is parametrized by y_1, y_2. The irrelevant ideal is no longer the maximal ideal. Instead, as noted at the end of Example 2.3.5, the irrelevant ideal is $(x_1, x_2) \cap (y_1, y_2) = (x_1y_1, x_1y_2, x_2y_1, x_2y_2)$. The vanishing set of this ideal is more complicated than just the origin. Recall that the definition of the minimal free resolution required the image of φ_i to be contained in $\mathfrak{m}F_{i-1}$. The irrelevant ideal will play a similar role in the definition of a virtual resolution.

Prior to defining a virtual resolution, notational conventions will be established. For the remainder of this thesis, $\boldsymbol{n} = (n_1, n_2, ..., n_k)$ and $\mathbb{P}^{\boldsymbol{n}} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$. Also denote the Cox ring by S and the irrelevant ideal by B. As mentioned in Section 2.3, the Cox ring is graded by the class group. In a product of projective spaces, all Weil divisors are Cartier so the class group coincides with the Picard group. Therefore S is graded by the Picard group $\mathrm{Pic}(X)$.

Definition 2.4.1. *Let $X = \mathbb{P}^{\boldsymbol{n}}$. A **virtual resolution** of a $\mathrm{Pic}(X)$-graded S-module M is a complex of $\mathrm{Pic}(X)$-graded free S-modules*

$$F_\bullet := [F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_n \longleftarrow \cdots]$$

such that the associated sheaf $\widetilde{H_0(F_\bullet)}$ is isomorphic to the associated sheaf $\widetilde{M}$ and the higher homology groups are supported on B.

An equivalent definition of a virtual resolution of M is a complex of free S-modules that when sheafified becomes a locally free vector bundle resolution of the sheaf $\widetilde{M}$.

2.4.1 Multigraded Regularity

In $\mathbb{P}^{\boldsymbol{n}}$, the Hilbert function and polynomial of modules are no longer functions of a single variable, but rather a function of k variables. So it is harder to talk about regularity in the previous sense. Instead, [MS04] defines a multigraded regularity.

Definition 2.4.2. *Let X be a smooth projective toric variety and let S denote the Cox ring of X. Let $\mathbb{N}\mathcal{C}$ be the semigroup generated by a finite subset $\mathcal{C} = \{\mathbf{c_1}, ..., \mathbf{c_l}\}$ of* $\mathrm{Pic}(X)$. *For $\boldsymbol{m} \in \mathrm{Pic}(X)$, a $\mathrm{Pic}(X)$-graded S-module M is $\boldsymbol{m}$-regular if the following conditions are satisfied:*

(a) $H^i_B(M)_{\boldsymbol{p}} = 0$ *for all* $i \geq 1$ *and all* $\boldsymbol{p} \in \bigcup(\boldsymbol{m} - \lambda_1 \boldsymbol{c}_1 - \cdots - \lambda_l \boldsymbol{c}_l + \mathbb{N}\mathcal{C})$ *where the union is over all* $\lambda_1, ..., \lambda_l \in \mathbb{N}$ *such that* $\lambda_1 + \cdots \lambda_l = i - 1$.

(b) $H^0_B(M)_{\boldsymbol{p}} = 0$ *for all* $\boldsymbol{p} \in \bigcup_{1 \leq j \leq l}(\boldsymbol{m} + \boldsymbol{c}_j + \mathbb{N}\mathcal{C})$.

The ***regularity*** *of M is* $\mathrm{reg}(M) := \{\boldsymbol{p} \in \mathrm{Pic}(X) | M$ *is $\boldsymbol{p}$-regular*$\}$.

Notice that if $X \subset \mathbb{P}^n$ so that $\mathrm{Pic}(X) = \mathbb{Z}$, then by Theorem 2.2.19 this definition is the same as the single-graded regularity defined in Section 2.2.4.

A difference between the single-graded and multigraded regularity is that the multigraded regularity is not a single element, but rather a set of elements. This difference is not so large because single-graded regularity can be regarded as a set as well. Indeed, if M is p-regular, then M is q-regular for all $q \geq p$. In general, the multigraded regularity cannot be defined by a single minimal element, but we can still talk about the minimal elements of the multigraded regularity. This is because it follows from Definition 2.4.2 that if M is $\boldsymbol{p}$-regular, then M is $\boldsymbol{q}$-regular for all $\boldsymbol{q} \succeq \boldsymbol{p}$ where $\succeq$ denotes the component-wise partial ordering on $\mathbb{Z}^k$.

The multigraded regularity acts like the single graded regularity in another useful way. If $\boldsymbol{p} \in \mathrm{reg}(M)$, then the Hilbert function matches the Hilbert polynomial for all $\boldsymbol{q} \succeq \boldsymbol{p}$.

2.4.2 Virtual Resolutions of a Pair

Berkesch, Erman, and Smith showed in [BES17] that there are different ways of constructing virtual resolutions. One way is to use the multigraded Castelnuovo–Mumford regularity.

Let M be an S-module and let $\boldsymbol{d}$ be an element in the regularity of M. Then the **virtual resolution of the pair** $(M, \boldsymbol{d})$ is obtained from the minimal resolution of M by finding the minimal free resolution of M and removing all summands generated in degree not less than $\boldsymbol{n} + \boldsymbol{d}$. Removing these summands can only decrease the length of the minimal resolution so the virtual resolution of a pair is always at least as short as the minimal resolution. Also notice that the phrase "the" virtual resolution of a pair is justified; the minimal resolution of the module M is unique up to isomorphism, so the virtual resolution of the pair $(M, \boldsymbol{d})$ is as well.

Example 2.4.3. Consider an ideal I of three points $([1:1],[1:4])$, $([1:2],[1:5])$, and $([1:3],[1:6])$ (where $[a:b]$ is a set of homogeneous coordinates in $\mathbb{P}^1$) in $\mathbb{P}^1 \times \mathbb{P}^1$. *Macaulay2* [GS] shows that the B-saturation of I, $J := (I : B^\infty)$, is

$$\langle 3x_0x_2 + x_1x_2 - x_0x_3, 40x_1^2x_2 + 6x_0^2x_3 - 13x_0x_1x_3 - 3x_1^2x_3, 6x_0^3 - 11x_0^2x_1 + 6x_0x_1^2 - x_1^3,$$
$$120x_1x_2^2 - 34x_1x_2x_3 - 2x_0x_3^2 + 3x_1x_3^2, 120x_2^3 - 74x_2^2x_3 + 15x_2x_3^2 - x_3^3\rangle.$$

The minimal resolution of S/J is

$$S \longleftarrow \begin{matrix} S(0,-3) \\ \oplus \\ S(-1,-2) \\ \oplus \\ S(-2,-1) \\ \oplus \\ S(-3,0) \\ \oplus \\ S(-1,-1) \end{matrix} \longleftarrow \begin{matrix} S(-1,-3)^2 \\ \oplus \\ S(-2,-2)^2 \\ \oplus \\ S(-3,-1)^2 \end{matrix} \longleftarrow \begin{matrix} S(-2,-3) \\ \oplus \\ S(-3,-2) \end{matrix} \longleftarrow 0.$$

By examining the Hilbert function and Hilbert polynomial, it can be seen that for all $(a,b) \succeq (2,0)$, $H_{S/J}(a,b) = P_{S/J}(a,b)$. Indeed, (2,0) is in the regularity of S/J. The

virtual resolution of the pair $(S/J, (2,0))$ is

$$S \longleftarrow \begin{matrix} S(-1,-1) \\ \oplus \\ S(-2,-1) \\ \oplus \\ S(-3,0) \end{matrix} \longleftarrow S(-3,-1)^2 \longleftarrow 0.$$

The above virtual resolution of the pair has shorter length and is thinner than the minimal resolution. $\diamond$

If the minimal resolution of a module M is known, then given an element $\boldsymbol{d}$ in the regularity of M, it is very easy to find the virtual resolution of the pair $(M, \boldsymbol{d})$. However, the full minimal resolution need not be computed in order to find the virtual resolution of the pair $(M, \boldsymbol{d})$. Rather the same strategy as outlined in Section 2, used to find the minimal resolution of M, can be used. The only difference is that any summands generated in degree $\boldsymbol{n} + \boldsymbol{d}$ or higher are omitted. This is a computationally easier way to compute the virtual resolution of the pair and is the content of Algorithm 4.4 in [BES17].

In fact, it appears that the multigraded regularity of a module and the module's set of virtual resolutions are intimately related. Not only can we construct a virtual resolution if we know an element in the regularity of a module, but [BES17] showed that the set of virtual resolutions of a module determine the regularity of the module as well.

2.4.3 Another Way of Constructing a Virtual Resolution

In the case where the variety is zero-dimensional (i.e. a set of points in $\mathbb{P}^{\boldsymbol{n}}$), there is another way of finding a virtual resolution. First, for $\mathbb{P}^{\boldsymbol{n}}$, write the Cox ring as

$$S = \Bbbk[x_{1,0}, ..., x_{1,n_1}, x_{2,0}, ..., x_{2,n_2}, ..., x_{k,n_k}].$$

Then for a vector $\boldsymbol{a} \in \mathbb{N}^k$ let

$$B^{\boldsymbol{a}} = \bigcap_{i=1}^{k} \langle x_{i,0}, ..., x_{i,n_i} \rangle^{a_i}.$$

We will call $B^{\boldsymbol{a}}$ a power of the irrelevant ideal.

Theorem 2.4.4 ([BES17, Theorem 4.1])**.** *If $X \subset \mathbb{P}^{\boldsymbol{n}}$ is a set of points and I is the corresponding B-saturated S-ideal, then there exists $\boldsymbol{a} \in \mathbb{N}^k$ such that the minimal free resolution of $S/(I \cap B^{\boldsymbol{a}})$ is a virtual resolution of S/I.*

Example 2.4.5. Consider the B-saturated ideal J from Example 2.4.3. Recall that the variety of this ideal is a set of three points in general position in $\mathbb{P}^1 \times \mathbb{P}^1$. Finding the minimal free resolution of $S/\left(J \cap B^{(0,1)}\right)$ recovers the same virtual resolution as in Example 2.4.3. $\diamond$

A possible research direction is to determine when a virtual resolution of an ideal I can be obtained from a minimal free resolution of a strictly smaller ideal created by taking a subset of the generators of I.

Example 2.4.6. Consider the ideal K generated by the first and second generators of the B-saturated ideal mentioned in Example 2.4.3,

$$K = \langle 3x_0x_2 + x_1x_2 - x_0x_3, 40x_1^2x_2 + 6x_0^2x_3 - 13x_0x_1x_3 - 3x_1^2x_3 \rangle.$$

The minimal resolution of the module S/K recovers the virtual resolution of S/J from Example 2.4.3

$$S \xleftarrow{\varphi_1} \begin{matrix} S(-1,-1) \\ \oplus \\ S(-1,2) \end{matrix} \xleftarrow{\varphi_2} S(-2,-3) \longleftarrow 0.$$

Macaulay2 [GS] shows that the maps between these resolutions are given by the matrices

$$\varphi_1 = \begin{bmatrix} x_0x_2 + (1/3)x_1x_2 - (1/3)x_0x_3 \\ x_1^2x_2 + (3/20)x_0^2x_3 - (13/40)x_0x_1x_3 - (3/40)x_1^2x_3 \end{bmatrix}^T$$

$$\varphi_2 = \begin{bmatrix} -(20/3)x_1^2x_2 - x_0^2x_3 + (13/6)x_0x_1x_3 + (1/2)x_1^2x_3 \\ (20/3)x_0x_2 + (20/9)x_1x_2 - (20/9)x_0x_3 \end{bmatrix}.$$

This complex is a Koszul complex of length two. Points in $\mathbb{P}^1 \times \mathbb{P}^1$ that have virtual resolutions of this form are investigated more in Chapter 4. $\diamond$

Chapter 3

What Makes a Complex Virtual

3.1 Chapter Overview

This chapter is an adaptation of a paper by the same name [Lop19]. The name of both this chapter and [Lop19] pays homage to a famous paper, in which Buchsbaum and Eisenbud present two criteria that completely determine whether or not a chain complex is exact over a Noetherian ring without computing the homology of the complex [BE73]. These criteria are useful in investigating a module by examining the minimal free resolution.

Before stating the main theorem from [BE73], we reintroduce some notation that was briefly covered in Chapter 2. A map of free S-modules $\varphi\colon F \to G$ can be expressed as a matrix with entries in S by choosing bases of F and G. Denote by $I_r(\varphi)$ the ideal generated by the $r \times r$ minors of φ. Then $\operatorname{rank}(\varphi)$ will be the largest r such that $I_r(\varphi) \neq 0$. The ideal $I_{\operatorname{rank}(\varphi)}(\varphi)$ will be the most important of these ideals of minors, and we set $I(\varphi) := I_{\operatorname{rank}(\varphi)}(\varphi)$. By convention, we define $I_k(\varphi) = S$ for every integer $k \leq 0$.

In fact, these ideals of minors can be extended to projective modules (that may not be finitely generated). Indeed φ gives rise to a map $\bigwedge^r \varphi\colon \bigwedge^r F \to \bigwedge^r G$, and the rank of φ is the largest r such that $\bigwedge^r \varphi \neq 0$. In this context, $I_r(\varphi)$ is the image of $\bigwedge^r F \otimes (\bigwedge^r G)^* \to S$. Recall Theorem 3.1.1 below.

Theorem 3.1.1 ([BE73]). *Let R be a Noetherian ring. Suppose*

$$F_\bullet := \left[F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \xleftarrow{\varphi_3} \cdots \xleftarrow{\varphi_n} F_n \longleftarrow 0\right]$$

is a chain complex of free R-modules. Then F is exact if and only if both of the following conditions are satisfied:

(a) $\operatorname{rank}(\varphi_i) + \operatorname{rank}(\varphi_{i+1}) = \operatorname{rank}(F_i)$ *for each* $i = 1, 2, ..., n$ *(taking* $\varphi_{n+1} = 0$*),*

(b) $\operatorname{depth}(I(\varphi_i)) \geq i$.

Unfortunately when studying coherent sheaves over more general smooth projective toric varieties, the situation is not as well-behaved. Locally free resolutions of a coherent sheaf are often shorter and thinner than the corresponding minimal free resolutions of the modules. Tying these concepts more closely together, Berkesch, Erman, and Smith introduced the notion of virtual resolutions in [BES17]. The main theorem in the present chapter (Theorem 3.1.2) is the virtual analogue to the main theorem of [BE73] (Theorem 3.1.1).

Notation

Throughout this chapter, X will be a smooth projective toric variety and S will denote the Cox ring of X over an algebraically closed field $\Bbbk$. The Cox ring S is graded by the Picard group of X, which we denote by $\operatorname{Pic}(X)$ [Cox95, §1]. In particular, S is a polynomial ring with a multigrading by $\mathbb{Z}^r$ for $r = \operatorname{rank}\ \operatorname{Pic}(X)$. Let B denote the irrelevant ideal of S, which is radical; M will be a finitely generated $\operatorname{Pic}(X)$-graded module over S and $\widetilde{M}$ denotes the sheaf of M over X, as constructed in [Cox95, §3]. Given an ideal I of S, we denote the set of all homogenous prime ideals containing I by $V(I)$.

The main result of this chapter is the following theorem which identifies when a chain complex is a virtual resolution without computing homology.

Theorem 3.1.2. *Let X be a smooth projective toric variety with $S = \mathrm{Cox}(X)$. Suppose*

$$F_\bullet := \left[F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \xleftarrow{\varphi_3} \cdots \xleftarrow{\varphi_n} F_n \longleftarrow 0\right]$$

is a $\mathrm{Pic}(X)$-graded complex of free S-modules. Then F is a virtual resolution if and only if both of the following conditions are satisfied:

(a) $\mathrm{rank}(\varphi_i) + \mathrm{rank}(\varphi_{i+1}) = \mathrm{rank}(F_i)$ *for each* $i = 1, 2, ..., n$ *(with* $\varphi_{n+1} = 0$*),*

(b) $\mathrm{depth}(I(\varphi_i) : B^\infty) \geq i$.

As in [BE73], we assign the unit ideal infinite depth, so that condition (b) holds if $I(\varphi_i) : B = S$ (i.e. $I(\varphi_i)$ is irrelevant). The difference between Theorem 3.1.1 and Theorem 3.1.2 is the replacement of exactness with virtuality and the addition of the saturation of ideals of minors by the irrelevant ideal B. Of course, any complex that is exact will also be a virtual resolution. Further, if a complex is exact, then the conditions of Theorem 3.1.2 will be satisfied by Theorem 3.1.1. On the other hand, below is an example of a complex that is a virtual resolution but is not exact.

Example 3.1.3. Let $X = \mathbb{P}^1 \times \mathbb{P}^2$ so that $S = \Bbbk[x_0, x_1, y_0, y_1, y_2]$ with $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Then the irrelevant ideal B is $\langle x_0, x_1\rangle \cap \langle y_0, y_1, y_2\rangle$. Let I be the following B-saturated ideal of 4 points:

$$\begin{gathered}\langle y_0y_1 - 22y_1^2 - 20y_0y_2 - 26y_1y_2 - 6y_2^2,\\ x_0y_0 + 43x_0y_2 - 13x_1y_0 + 50x_1y_1 + 20x_1y_2,\\ x_0y_1 + 10x_0y_2 - x_1y_0 - 19x_1y_1 + 47x_1y_2,\\ y_0^2 - 20y_1^2 - 14y_0y_2 - 22y_1y_2 + 33y_2^2,\\ x_0^2y_2 - 42x_0x_1y_2 + 42x_1^2y_0 - 48x_1^2y_1 + 13x_1^2y_2,\\ x_0^4 + 27x_0^3x_1 + 25x_0^2x_1^2 - 29x_0x_1^3 - 40x_1^4\rangle.\end{gathered}$$

Macaulay2 [GS] calculates the minimal free resolution of S/I as

$$
\begin{array}{ccccccccc}
 & & & & S(-1,-2)^2 & & & & \\
 & & S(-1,-1)^2 & & \oplus & & & & \\
 & & \oplus & & S(-2,-2)^3 & & S(-2,-3)^3 & & \\
 & & S(0,-2)^2 & & \oplus & & \oplus & & S(-2,-4) \\
S & \longleftarrow & \oplus & \longleftarrow & S(-1,-3)^2 & \longleftarrow & S(-1,-4)^2 & \longleftarrow & \oplus & \longleftarrow 0. \\
 & & S(-2,-1) & & \oplus & & \oplus & & S(-4,-3) & \\
 & & \oplus & & S(0,-4) & & S(-4,-2)^3 & & & \\
 & & S(-4,0) & & \oplus & & & & & \\
 & & & & S(-1,-4)^3 & & & & &
\end{array}
$$

[BES17, Theorem 1.3] states that if $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$, the $S = \mathrm{Cox}(X)$-ideal I is saturated (as it is in this example), and $\vec{d}$ is an element in the multigraded Castelnuovo–Mumford regularity of S/I (see [MS04] for a detailed introduction to multigraded regularity), then the chain complex consisting of all twists less than or equal to $\vec{d} + (n_1, n_2, ..., n_r)$ is a virtual resolution. This is the **virtual resolution of the pair** $(S/I, \vec{d}\,)$. In this example, $(1,1)$ is in the multigraded regularity of S/I and the virtual resolution of the pair $(S/I, (1,1))$ is

$$
\begin{array}{ccccccccc}
 & & S(-1,-1)^2 & & S(-1,-2)^2 & & & & \\
 & & \oplus & & \oplus & & & & \\
S & \xleftarrow{\varphi_1} & S(0,-2)^2 & \xleftarrow{\varphi_2} & S(-2,-2)^3 & \xleftarrow{\varphi_3} & S(-2,-3)^3 & \longleftarrow & 0. \\
 & & \oplus & & \oplus & & & & \\
 & & S(-2,-1) & & S(-1,-3)^2 & & & &
\end{array}
$$

Notice that this virtual resolution is both shorter and thinner than the minimal free resolution, as is often the case. *Macaulay2* [GS] calculates that this virtual resolution has nonzero first homology module $S/\langle x_0, x_1\rangle$, which is annihilated by B. Consequently, this virtual resolution is not exact. Calculating the depth of the (non-B-saturated) ideal of minors of each differential yields

$$\mathrm{depth}(I(\varphi_1)) = 3, \qquad \mathrm{depth}(I(\varphi_2)) = 2, \qquad \mathrm{depth}(I(\varphi_3)) = 2,$$

indicating again that this virtual resolution is not exact because the depth of $I(\varphi_3)$ is less than three. Though the complex is not exact, $\operatorname{depth}(I(\varphi_3) : B^\infty) = 3$, so it is indeed a virtual resolution. $\diamond$

Remark 3.1.4. In the special case of the smooth projective toric variety $\mathbb{P}^n_{\Bbbk}$, the Cox ring is $S = \Bbbk[x_0, x_1, ..., x_n]$, and the irrelevant ideal is the maximal homogeneous ideal $B = \mathfrak{m} = \langle x_0, x_1, ..., x_n \rangle$. If the length of the complex F is not longer than $n+1$ (the bound from the Hilbert Syzygy Theorem (Theorem 2.2.6)), then Theorem 3.1.2 recovers Theorem 3.1.1. Each $I(\varphi_i)$ is homogeneous (in fact, every minor of φ_i is homogeneous), and for any homogeneous ideal $I \subsetneq \mathfrak{m}$,

$$\operatorname{depth}(I) \leq \operatorname{depth}(I : \mathfrak{m}^\infty) \leq \operatorname{depth}(\sqrt{I} : \mathfrak{m}^\infty) = \operatorname{depth}(\sqrt{I}) = \operatorname{depth}(I).$$

The second to last equality follows from the fact that any homogeneous radical ideal other than $\mathfrak{m}$ is $\mathfrak{m}$-saturated. Therefore, condition (b) in Theorem 3.1.2 becomes

$$i \leq \operatorname{depth}(I(\varphi_i) : B^\infty) = \operatorname{depth}(I(\varphi_i)),$$

which exactly matches condition (b) in Theorem 3.1.1.

As condition (a) is the same in the two theorems, we have thus proved the following proposition. A slightly different proof is offered below.

Proposition 3.1.5. *Let $S = \operatorname{Cox}(\mathbb{P}^N_{\Bbbk}) = \Bbbk[x_0, x_1, .., x_N]$. If*

$$F_\bullet := \left[F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \xleftarrow{\varphi_3} \cdots \xleftarrow{\varphi_m} F_m \longleftarrow 0 \right]$$

is a virtual resolution with $m \leq N+1$, then F is exact.

Proof. The irrelevant ideal is $B = \langle x_0, x_1, ..., x_N \rangle$. Since F is a virtual resolution, all of its homology modules are annihilated by a power of B. Assume F has a nonzero homology module and let $H := H_i(F)$ where i is the largest index with $H_i(F)$ nonzero. The Peskine–Szpiro Acyclicity Lemma (Lemma 2.2.15) guarantees $\operatorname{depth}(H) \geq 1$ and hence B has a nonzerodivisor on H, a contradiction to the assumption that F is a virtual resolution. Therefore, in this setting, if F is a virtual resolution, it must be exact. $\square$

If the length of $F_\bullet$ is larger than $N+1$, then virtual resolutions do not need to be exact. An example is illustrated below.

Example 3.1.6. Let $X = \mathbb{P}^2$ and $S = \Bbbk[x,y,z]$. Let $G_\bullet$ be the Koszul complex on (x,y,z).

$$G_\bullet := \left[S \longleftarrow S(-1)^3 \longleftarrow S(-2)^3 \longleftarrow S(-3) \longleftarrow 0\right].$$

Set $F_\bullet = G_\bullet \oplus G_\bullet[-3]$. That is,

$$F_\bullet := \begin{bmatrix} & & & & & & S(-3) & & \\ S \longleftarrow S(-1)^3 \longleftarrow S(-2)^3 \longleftarrow & \oplus & \longleftarrow S(-1)^3 \longleftarrow S(-2)^3 \longleftarrow S(-3) \longleftarrow 0 \\ & & & & & & S & & \end{bmatrix}.$$

Then the length of $F_\bullet$ is $6 > 3$ and $F_\bullet$ is a virtual resolution as the only nonzero higher homology module is $H_3(F_\bullet) = \Bbbk$ which is annihilated by the irrelevant ideal $\langle x,y,z\rangle$. $\diamond$

Outline

Section 3.2 lays the groundwork for the rest of the chapter. This includes notation, relevant definitions, and some preliminary facts about B-saturated prime ideals. These primes are important because the homogeneous localization of a module at a B-saturated homogeneous prime corresponds to taking the stalk of a sheaf over the toric variety X. Section 3.3 contains the proof of Theorem 3.1.2. In Section 3.4, the invariance of saturated Fitting ideals of virtual presentations is presented along with an obstruction to the number of generators of an ideal up to saturation. Finally, Section 3.5 contains a connection between saturated Fitting ideals and locally free sheaves. It ends with a useful result concerning unbounded virtual resolutions. The results in Sections 3.4 and 3.5 can be used to prove Theorem 3.1.2 in the same way that results of Fitting ideals can be used to prove Theorem 3.1.1.

3.2 B-Saturated Prime Ideals

In this section the structure of B-saturated prime ideals in the Cox ring S is discussed, which will aid in later proofs. Indeed, in order to show that a complex is a virtual resolution, we will show that after sheafifying, the complex of vector bundles is acyclic

by showing it is exact in each place. The latter will be true if and only if the stalk at each B-saturated homogeneous prime is exact.

Recall the **saturation of an ideal I by B** is

$$I : B^\infty = \bigcup_{n \geq 0} (I : B^n) = \{s \in S \mid sB^n \subset I \text{ for some } n\}.$$

There is a correspondence between B-saturated ideals of S and closed subschemes of X [CLS11, Proposition 6.A.7].

Throughout the chapter we will often be concerned with the structure of the homogeneous B-saturated prime ideals of S, which form a proper subset of the homogeneous prime ideals of S. It will be important to see which prime ideals are B-saturated.

Proposition 3.2.1 ([AM16, Exercise 1.12])**.** *Suppose P is a homogenous prime ideal of S. Then either P is B-saturated or a prime component of B is contained in P, in which case $P : B^\infty = S$.*

Proof. We first prove the second statement. Let $B = \bigcap_{i=1}^m Q_i$ with each Q_i a homogeneous prime ideal. Then

$$P : B^\infty = (\cdots((P : Q_1^\infty) : Q_2^\infty) : \cdots : Q_m^\infty).$$

Now if $P \supset Q_i$, then $P : Q_i^\infty = P : Q_i = S$, proving the second part of the proposition.

In order to show the first part, it suffices to show that if P and Q are prime ideals so that P does not contain Q, then $P : Q^\infty = P$. Suppose $s \in P : Q^\infty$ and let $\ell \in \mathbb{Z}_{\geq 0}$ be such that $sQ^\ell \subset P$. Since $Q \not\subset P$, there is a $q \in Q - P$. Thus $q^\ell \in Q^\ell - P$, but $sq^\ell \in P$ so $s \in P$. □

This first proposition says that every homogenous prime ideal of S is either B-saturated or irrelevant. The lemma below implies we need only consider prime ideals of small enough height.

Lemma 3.2.2. *If I is a homogeneous ideal of codimension greater than $\dim X$, then $I : B^\infty = S$.*

Proof. Homogeneous ideals that saturate to all of S are ideals that correspond to the empty subvariety of X. It is enough to show that if I corresponds to a nonempty

subvariety of X, then the height of I is less than or equal to the dimension of X. Let $d := \dim X$, $c := \operatorname{codim} I$, and n be the number of variables of S. Suppose x is in the subvariety of X corresponding to I. Considering the quotient construction of X, there is a torus G that acts on n-dimensional affine space $\mathbb{A}^n$. As G acts freely on $\mathbb{A}^n \backslash V(B)$ [CLS11, Exercise 5.1.11], the dimension of both G and the orbit Gx in $\mathbb{A}^n$ is $n - d$. Then $Gx \subset V(I) \subset \mathbb{A}^n$ implies that $c \leq d$, since

$$n - d = \dim(Gx) \leq \dim V(I) = n - c. \qquad \square$$

Therefore, when considering homogeneous B-saturated primes of the Cox ring S of X, we need only consider the homogeneous primes of codimension at most $\dim(X)$ that do not contain any prime components of the irrelevant ideal B.

3.3 When Complexes are Virtual Resolutions

Lemma 3.3.1. *Suppose* $0 \to \mathbb{Z}^r \xrightarrow{A} \mathbb{Z}^n \xrightarrow{B} \mathbb{Z}^{n-r} \to 0$ *is a short exact sequence with* A *an* $n \times r$ *matrix and* B *an* $(n-r) \times n$ *matrix with integer coefficients. If the* $r \times r$ *minor of* A *consisting of rows* $i_1, i_2, ..., i_r$ *has determinant* ± 1*, then the* $(n-r) \times (n-r)$ *minor of* B *obtained by omitting columns* $i_1, i_2, ..., i_r$ *has determinant* ± 1*.*

Proof. We will prove this by changing the basis of $\mathbb{Z}^n$. Let C be the $n \times n$ matrix so that

$$CA = \left[\begin{array}{c} \mathrm{id}_r \\ \hline \mathbf{0} \end{array}\right],$$

where id_r denotes the $r \times r$ identity matrix and $\mathbf{0}$ is the $(n-r) \times r$ zero matrix. Notice that C is invertible. Then as the sequence is exact, the columns of BC^{-1} must form a $\mathbb{Z}$-basis of $\mathbb{Z}^{n-r}$. In fact, the sequence being exact combined with the form of CA implies that the first r rows for BC^{-1} must be zero. Thus, $BC^{-1} = \left[\begin{array}{c|c} 0 & Q \end{array}\right]$, where Q has determinant ± 1. Now the subgroup of $\mathbb{Z}^{n-r}$ spanned by the columns of Q are the same as the subgroup spanned by the columns of N after removing the columns $i_1, i_2, ..., i_r$. This subspace is all of $\mathbb{Z}^{n-r}$ so the determinant of the $(n-r) \times (n-r)$ minor of N obtained by omitting columns $i_1, i_2, ..., i_r$ is ± 1. $\square$

Let σ be a cone of the complete fan Σ of the smooth projective toric variety X, and

Σ_1 denote the rays of Σ. Then as in [CLS11], let

$$x^{\hat{\sigma}} = \prod_{\rho \in \Sigma_1 \setminus \sigma} x_\rho.$$

The irrelevant ideal B is generated by $x^{\hat{\sigma}}$ as σ ranges over the maximal cones of Σ.

Proposition 3.3.2. *Let X be a smooth projective toric variety. Let σ be a maximal cone of the fan of X. The elements of* $\mathrm{Pic}(X)$, $\{\deg(x_\rho) | \rho \notin \sigma\}$ *generate the Picard group.*

Proof. Consider the short exact sequence $0 \to L \to \mathrm{Div}_{T_N}(X) \to \mathrm{Pic}(X) \to 0$ of [Cox95, Theorem 4.3], where L is the lattice of characters of X and $\mathrm{Div}_{T_N}(X)$ is the group of torus invariant Weil divisors of X. Since L, Div_{T_N}, and $\mathrm{Pic}(X)$ are free abelian groups, the conditions of Lemma 3.3.1. Here the matrix A contain the rays of Σ as rows. Since σ is a smooth cone, the determinant of the rows corresponding to the rays of σ is ± 1. By Lemma 3.3.1, the determinant of B obtained by choosing the columns corresponding to rays not in σ is also ± 1. Therefore, the degrees of the x_ρ generate $\mathrm{Pic}(X)$. □

Lemma 3.3.3. *If I is a homogeneous ideal of S with $I : B^\infty \neq S$, then*

$$\mathrm{depth}(I : B^\infty) = \min\left\{\mathrm{depth}(I_{(P)}) \mid P \text{ is a } B\text{-saturated prime ideal of } S\right\}.$$

Proof. First, we claim $\mathrm{depth}(I_P) = \mathrm{depth}(I_{(P)})$ for B-saturated homogeneous primes P. As P is B-saturated, it does not contain $x^{\hat{\sigma}}$ for some maximal cone σ. By Proposition 3.3.2, the set $\{\deg(x_\rho)\}_{\rho \notin \sigma}$ generate $\mathrm{Pic}(X)$. So if f is a homogeneous element of S_P, then there is a unit $u \in S_P$ so that $fu \in S_{(P)}$. As multiplying by a unit does not change whether or not an element is a zero divisor, this proves that $\mathrm{depth}(I_P) = \mathrm{depth}(I_{(P)})$.

Then as S is Cohen–Macaulay,

$$\begin{aligned}\min\{\operatorname{depth}(I_P | P \text{ is a } B\text{-saturated prime ideal of } S\} \\ &= \min\{\operatorname{codim}(I_P | P \text{ is a } B\text{-saturated prime ideal of } S\} \\ &= \min\{\operatorname{codim} P | I \subset P\} \\ &= \min\{\operatorname{codim} P | I : B^\infty \subset P\} \\ &= \operatorname{codim}(I : B^\infty) \\ &= \operatorname{depth}(I : B^\infty).\end{aligned}$$ □

Proof of Theorem 3.1.2. Suppose first that $F_\bullet$ is a virtual resolution. Then as $\widetilde{F_\bullet}$ is exact, it is also true that $(F_\bullet)_{(P)}$ is exact for every B-saturated prime ideal P. In particular, $(F_\bullet)_{(\langle 0 \rangle)}$ is an exact sequence of vector spaces which implies condition (a) is satisfied. Further,

$$\operatorname{depth}(I(\varphi_i) : B^\infty) = \min\{\operatorname{depth}(I(\varphi_i)_{(P)} | P \text{ is a } B\text{-saturated prime ideal of } S\}$$

by Lemma 3.3.3 and $\min\{\operatorname{depth}(I(\varphi)_{(P)} | P \text{ is a } B\text{-saturated prime ideal of } S\} \geq i$ by Theorem 3.1.1.

Conversely, suppose conditions (a) and (b) are satisfied. We use a similar strategy and show that $\widetilde{F_\bullet}$ is exact by showing that $(F_\bullet)_{(P)}$ is exact for every B-saturated prime ideal P. Proposition 3.3.2 implies that after homogeneous localization, $(F_\bullet)_{(P)}$ has the same ranks as $F_\bullet$ in every homological degree so condition (a) of Theorem 3.1.1 is satisfied. By Lemma 3.3.3, $\operatorname{depth}(I(\varphi_i)_{(P)} \geq \operatorname{depth}(I(\varphi) : B^\infty) \geq i$, so condition (b) is also satisfied, proving the $(F_\bullet)_{(P)}$ is exact. Therefore, $F_\bullet$ is a virtual resolution. □

Remark 3.3.4. The same proof shows that if R is a Cohen–Macaulay domain and $F_\bullet$ is a bounded free complex of R-modules with homology only possibly supported on a radical ideal J, then the complex of $\mathcal{O}_{\operatorname{Spec} R}$-modules $\widetilde{F_\bullet}$ is exact on $\operatorname{Spec} R - V(J)$.

Below is an alternative proof of the forward direction of Theorem 3.1.2. This proof, while similar to the proof of Theorem 3.1.1 is independent of Theorem 3.1.1. It relies on the Peskine–Szpiro Acyclicity Lemma.

Proof that conditions (a) and (b) imply the complex is virtual. It is enough to show for each B-saturated homogeneous prime ideal P of S, the complex $\widetilde{F}_P$ is exact. Since S is a polynomial ring, it is an integral domain, so each $I(\varphi_i)$ contains a nonzerodivisor. Therefore, as localization commutes with taking Fitting ideals [Eis95, Corollary 20.5], the hypotheses are not weakened. We will show that $\widetilde{F}_P$ is exact by induction on the codimension of P. The unique minimal prime ideal of S is $\langle 0 \rangle$, and localizing at this ideal gives a complex of vector spaces. The complex $F_{\langle 0 \rangle}$ therefore becomes exact by assumption (a).

Now suppose P is a B-saturated prime of codimension $c > 0$. By assumption (b), $\operatorname{depth}(I(\varphi_{c+1}) : B^\infty) \geq c+1$, so $I(\varphi_{c+1}) : B^\infty$ is not contained in P. Therefore, $I(\varphi_{c+1})$ is also not contained in P. Indeed, if $I(\varphi_{c+1}) \subset P$, then $I(\varphi_{c+1}) : B^\infty \subset P : B^\infty = P$. Hence $I(\varphi_{c+1})_P = S_P$, so F_P can be broken into two complexes:

$$G_P := \left[0 \longleftarrow (\operatorname{Coker} \varphi_{c+1})_P \longleftarrow (F_c)_P \xleftarrow{(\varphi_{c+1})_P} (F_{c+1})_P \longleftarrow \cdots \longleftarrow (F_n)_P \longleftarrow 0 \right],$$

$$H_P := \left[(F_0)_P \longleftarrow \cdots \longleftarrow (F_{c-2})_P \xleftarrow{(\varphi_{c-1})_P} (F_{c-1})_P \xleftarrow{(\varphi'_c)_P} (\operatorname{Coker} \varphi_{c+1})_P \longleftarrow 0 \right],$$

where φ'_c is induced from φ_c. Now since $I((\varphi_i)_P) = S_P$ for every $i \geq c+1$, the complex G_P is exact. All that is left to show is that H_P is exact. Since $I((\varphi_{c+1})_P) = S_P$, the S_P-module $(\operatorname{Coker} \varphi_{c+1})_P$ is projective and hence free. Also notice that $I((\varphi'_c)_P) = I((\varphi_c)_P)$.

By induction, H_P becomes exact when localizing at any homogeneous B-saturated prime ideal Q properly contained in P. By Proposition 3.2.1, it becomes exact when localizing at any prime ideal contained in P (if a smaller prime contains a prime component of B, then P does as well, which means that P would not be B-saturated). Therefore, the homology modules are only supported on the maximal ideal P_P and thus have depth 0. The depth of each free S_P-module is equal to the codimension of P (since S is Cohen–Macaulay), which is strictly positive. Lemma 2.2.15 implies there cannot be nonzero homology modules, thus completing the proof. $\square$

3.4 Invariance of Saturated Fitting Ideals

Between this section and Section 3.5, we record the groundwork of tools for an alternative proof of Theorem 3.1.2, using B-saturated Fitting ideals in the Cox ring S of a smooth projective toric variety X. Here, we present some facts about the B-saturation of Fitting ideals of an S-module. The Fitting ideals of a finitely-generated module M over a Noetherian ring can be calculated by a free presentation

$$0 \longleftarrow M \longleftarrow G \xleftarrow{\varphi} F.$$

Let r denote the rank of G. The **ith Fitting ideal of** M, $\mathrm{Fitt}_i(M)$, is defined to be $I_{r-i}(\varphi)$. This ideal is independent of the free presentation [Fit36].

We now adapt this idea of Fitting ideals to virtual presentations of a $\mathrm{Pic}(X)$-graded S-module M, beginning with the invariance of the B-saturated Fitting ideals. We then produce further facts about saturated Fitting ideals that mirror the classical theory of Fitting ideals.

Let a chain complex $0 \longleftarrow M \longleftarrow G \xleftarrow{\varphi} F$ be called a **virtual presentation** if both F and G are free S-modules and $\widetilde{\mathrm{Coker}(\varphi)} \cong \widetilde{M}$.

Lemma 3.4.1 (Saturated Fitting's Lemma)**.** *Suppose X is a smooth projective toric variety and S is the Cox ring of X with irrelevant ideal B. Let*

$$0 \longleftarrow M \longleftarrow G \xleftarrow{\varphi} F \qquad \text{and} \qquad 0 \longleftarrow M' \longleftarrow G' \xleftarrow{\varphi'} G'$$

be finite virtual presentations of M and M', respectively, with G and G' of ranks r and r'. If $\widetilde{M} \cong \widetilde{M'}$, then $I_{r-i}(\varphi) : B^\infty = I_{r'-i}(\varphi') : B^\infty$ for every $i \in \mathbb{Z}_{\geq 0}$.

Proof. We may assume that M is already B-saturated and that

$$0 \longleftarrow M \longleftarrow G \xleftarrow{\varphi} F$$

is the truncation of the minimal free resolution of M. By replacing M' with $\mathrm{Coker}\,\varphi'$ if necessary, we may also assume that

$$0 \longleftarrow M' \longleftarrow G' \xleftarrow{\varphi} F'$$

is a free presentation of M'.

Now $I_{r-i}(\varphi) : B^\infty = I_{r'-i}(\varphi') : B^\infty$ if and only if the sheaves $\widetilde{I_{r-i}(\varphi)}$ and $\widetilde{I_{r'-i}(\varphi')}$ are equal. We will show the equality of the sheaves by showing that they agree on an open cover of X by some distinguished open affines $D(f)$. The cocycle condition is satisfied because taking Fitting ideals commutes with base change. Given $f \in S$, $D(f)$ can be written (by abusing notation) as $\{p \in X | f(p) \neq 0\}$.

The open cover will be by the generators $\{x^{\hat{\sigma}}\}_{\sigma \in \Sigma}$ of the irrelevant ideal. This forms an open cover as if some toric subvariety V does not intersect any $x^{\hat{\sigma}}$, then $x^{\hat{\sigma}} \in I(V)$ for every $\sigma \in \Sigma$, which implies that $I(V)$ is irrelevant and hence V is empty.

By Proposition 3.3.2, homogeneous localization by $x^{\hat{\sigma}}$ preserves free modules and so

$$\widetilde{I_{r-i}(\varphi)}(D(x^{\hat{\sigma}})) \cong I_{r-i}(\varphi)_{(x^{\hat{\sigma}})} = I_{r-i}(\varphi_{(x^{\hat{\sigma}})}),$$

where the last equality holds because taking Fitting ideals commutes with base change. Because localization is exact, and Proposition 3.3.2 implies that every element of $F_{x^{\hat{\sigma}}}$ is equal to a product of a unit and an element of $F_{(x^{\hat{\sigma}})}$,

$$0 \longleftarrow M_{(x^{\hat{\sigma}})} \longleftarrow G_{(x^{\hat{\sigma}})} \xleftarrow{\varphi_{(x^{\hat{\sigma}})}} F_{(x^{\hat{\sigma}})}$$

is a free presentation of $M_{(x^{\hat{\sigma}})}$.

Furthermore, since $\widetilde{M} = \widetilde{M'}$, we have

$$M_{(x^{\hat{\sigma}})} \cong \widetilde{M}(D(x^{\hat{\sigma}})) = \widetilde{M'}(D(x^{\hat{\sigma}})) \cong M'_{(x^{\hat{\sigma}})}.$$

Fitting's Lemma [Fit36] then implies $I_{r-i}(\varphi_{(x^{\hat{\sigma}})}) = I_{r'-i}(\varphi'_{(x^{\hat{\sigma}})})$.

Thus $\widetilde{I_{r-i}(\varphi)}$ and $\widetilde{I_{r'-i}(\varphi')}$ agree on an open cover of X, and glue in the same way so the sheaves are equal. □

We will call $I_{r-i}(\varphi) : B^\infty$ the **ith saturated Fitting ideal of M**. Lemma 3.4.1 allows us to prove that the jth saturated Fitting ideal not equaling S is an obstruction to generating a module up to saturation by j elements. It is analogous to the fact that the jth Fitting ideal not equaling S is an obstruction to generating a module by j elements in the classical theory of Fitting ideals of Noetherian rings (see [Eis95,

Proposition 20.6]).

Define $\mathbb{V}(I)$ to be the set of homogeneous B-saturated prime ideals P of S such that $P \supset I$. In particular,

$$\mathbb{V}(I) := \{P \in V(I) \mid P \text{ is } B\text{-saturated}\}.$$

Proposition 3.4.2. *The set $\mathbb{V}(\mathrm{Fitt}_j(M) : B^\infty)$ consists of exactly the homogeneous B-saturated primes P such that there does not exist an S-module N where N_P can be generated by j elements and M and N have the same saturation.*

Proof. Suppose $P \in \mathbb{V}(\mathrm{Fitt}_j(M) : B^\infty)$ and N is a module such that the sheaves $\widetilde{M}$ and $\widetilde{N}$ are isomorphic. Then by Lemma 3.4.1, for every $j \in \mathbb{Z}_{\geq 0}$, it is true that $\mathrm{Fitt}_j(M) : B^\infty = \mathrm{Fitt}_j(N) : B^\infty$. So

$$\mathrm{Fitt}_j(N) \subset \mathrm{Fitt}_j(N) : B^\infty \subset P,$$

and therefore N_P cannot be generated by j elements [Eis95, Proposition 20.6].

On the other hand, suppose P is a homogeneous B-saturated prime not belonging to $\mathbb{V}(\mathrm{Fitt}_j(M) : B^\infty)$. In this case $\mathrm{Fitt}_j(M)$ cannot be contained in P. For if $\mathrm{Fitt}_j(M)$ is contained in P, then

$$\mathrm{Fitt}_j(M) : B^\infty \subset P : B^\infty = P.$$

Thus M_P can be generated by j elements. □

As mentioned above, the previous proposition can be rephrased as saying the $\mathrm{Fitt}_j(M)$ is an obstruction to generating the sheaf $\widetilde{M}$ by j elements. One may hope that the converse is true as well, that is if $\mathrm{Fitt}_j(M) : B^\infty = S$, then there is a module N generated by j elements such that $\widetilde{M} \cong \widetilde{N}$. Unfortunately, this fails to be true as shown by the following example.

Example 3.4.3. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ so that $S = \Bbbk[x_0, x_1, y_0, y_1]$ with $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Consider the B-saturated ideal $I = \langle y_0y_1, x_0y_0, x_0x_1\rangle$ of three points lying in the following way on a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$. Then S/I has minimal resolution

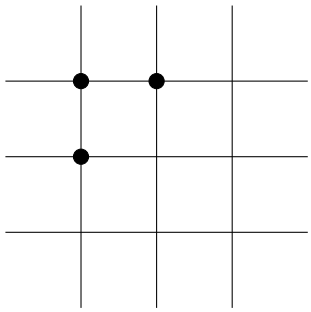


Figure 3.1: Example of a subscheme whose ideal has irrelevant second saturated Fitting ideal, but the ideal cannot be generated by two elements.

$$S \longleftarrow \begin{matrix} S(-2,0) \\ \oplus \\ S(-1,-1) \\ \oplus \\ S(0,-2) \end{matrix} \xleftarrow{\begin{bmatrix} -y_0 & 0 \\ x_1 & -y_1 \\ 0 & x_0 \end{bmatrix}} \begin{matrix} S(-2,-1) \\ \oplus \\ S(-1,-2) \end{matrix} \longleftarrow 0.$$

The ideal $\mathrm{Fitt}_2(S/I)$ is generated by the entries of the matrix above, so $\mathrm{Fitt}_2(S/I) : B^\infty = S$.

However, there cannot be two homogenous polynomials of S whose intersection vanishes exactly at the above three points. For if f and g are homogeneous forms of degree (a, b) and (c, d) respectively, the multigraded version of Bézout's Theorem (see example 4.9 in [Sha13]) says that the intersection multiplicity of f and g is $ad + bc$. For concreteness, the leftmost vertical line is $V(x_0)$, the middle vertical line is $V(x_1)$, the topmost horizontal line is $V(y_0)$, and the middle horizontal line is $V(y_1)$. Then there are two cases for which $ad + bc = 3$.

Case 1: One of the terms (ad or bc) is 3 and one is 0. Without loss of generality, assume $a = 3$, $d = 1$. If $c = 0$, then $\deg(g) = (0, 1)$, which cannot vanish at all three points. Hence, $b = 0$ and $\deg(f) = (3, 0)$. Plugging in $x_0 = 0$, $x_1 = 1$ gives a degree 0 form that must vanish at $[0 : 1]$ and $[1 : 0]$ which means f must vanish on all of $V(x_0)$. Similarly, plugging in $x_0 = 0$, $x_1 = 1$ to g yields a degree 1 form that also vanishes at these two points, implying g vanishes on all of $V(x_0)$. Hence, $V(f) \cap V(g) \neq X$.

Case 2: One of the terms is 2, and the other is equal to 1. It suffices to assume $a = 2$ and $b = c = d = 1$. Then both f and g restricted to $V(x_0)$ are degree one polynomials in y_0 and y_1 that vanish at two points so again $V(x_0) \subset V(f) \cap V(g)$. Therefore, no ideal generated by two elements will saturate to the ideal I. $\diamond$

3.5 Saturated Fitting Ideals and Locally Free Sheaves

In the classical theory of Fitting ideals, a module is projective if and only if its first nonzero Fitting ideal is the whole ring. Sheafifying a projective module yields a locally free sheaf. The situation is similar for modules over the Cox ring and saturated Fitting ideals.

Proposition 3.5.1. *If M is a $\mathrm{Pic}(X)$-graded S-module, then $\mathrm{Fitt}_r(M) : B^\infty = S$ and $\mathrm{Fitt}_{r-1}(M) : B^\infty = 0$ if and only if the sheaf $\widetilde{M}$ is locally free of constant rank r.*

Proof. First, if $\widetilde{M}$ is locally free of rank r, then for any B-saturated prime P, $M_{(P)} \cong \widetilde{M}_P$ is a free $S_{(P)}$-module of rank r. Since P is B-saturated, there is some $x^{\hat{\sigma}}$ so that $x^{\hat{\sigma}} \notin P$, which implies $M_P = S_P M_{(P)}$. Therefore, M_P has the free presentation $0 \longrightarrow S_P^r$, so $\mathrm{Fitt}_r(M_P) = S_P$ and $\mathrm{Fitt}_{r-1}(M_P) = 0$. Since taking Fitting ideals commutes with localization, $\mathrm{Fitt}_{r-1}(M) = 0$ and for every B-saturated prime P, $\mathrm{Fitt}_r(M) \not\subset P$. By definition,

$$\operatorname{codim} \mathrm{Fitt}_r(M) = \sup_{P \in \operatorname{Spec} S} \{\operatorname{codim} P : \mathrm{Fitt}_r(M) \subset P\}$$

is strictly greater than the codimension of any B-saturated prime (see Section 3.2). The maximal codimension of any B-saturated prime of S is $\dim X$ by Lemma 3.2.2. Utilizing Lemma 3.2.2 again, $\mathrm{Fitt}_r(M) : B^\infty = S$ as desired.

Now suppose $\mathrm{Fitt}_r(M) : B^\infty = S$ and $\mathrm{Fitt}_{r-1}(M) : B^\infty = 0$, and let P be a B-saturated prime ideal of S. By Proposition 3.4.2, there is an S-module N with $\widetilde{N} \cong \widetilde{M}$ such that N_P can be generated by r elements over S_P. Let $\varphi_P \colon G_P \to S_P^r$ be a free presentation of N_P. Then

$$I_1(\varphi_P) = \mathrm{Fitt}_{r-1}(N_P) \subset \mathrm{Fitt}_{r-1}(N_P) : B^\infty = \mathrm{Fitt}_{r-1}(M_P) : B^\infty = 0.$$

Thus $\varphi_P = 0$, which implies $N_P \cong S_P^r$ and $N_{(P)} \cong S_{(P)}^r$. Since the stalk at each point

of the sheaf $\widetilde{N}$ is free of rank r, this means $\widetilde{N}$ is locally free of rank r, as desired. $\square$

Notice that in everything up to this point, we have required that the complex be bounded. This may seem unsatisfying as Definition 2.4.1 does not require the length of a virtual resolution to be finite. The following result does not require this hypothesis, which mirrors the classical theory of Fitting ideals and exactness.

Proposition 3.5.2. *A complex of free S-modules*

$$F \xrightarrow{\varphi} G \xrightarrow{\psi} H$$

with $I(\varphi) : B^\infty = S = I(\psi) : B^\infty$ has irrelevant homology if and only if $\operatorname{rank}(\varphi) + \operatorname{rank}(\psi) = \operatorname{rank}(G)$.

Proof. The sequence

$$F \xrightarrow{\varphi} G \to \operatorname{Coker}\varphi \to 0$$

is right exact and a free presentation of $\operatorname{Coker}\varphi$. Letting $\operatorname{rank}(G) = n$ and $\operatorname{rank}(\varphi) = r$,

$$\operatorname{Fitt}_{n-r}(\operatorname{Coker}\varphi) : B^\infty = I_r(\varphi) : B^\infty = S, \text{ and}$$

$$\operatorname{Fitt}_{n-r-1}(\operatorname{Coker}\varphi) : B^\infty = I_{r+1}(\varphi) : B^\infty = 0.$$

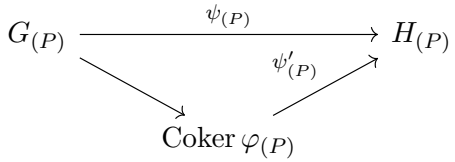
By Proposition 3.5.1, $\widetilde{\operatorname{Coker}\varphi}$ is a locally free sheaf of constant rank $n-r$. As localization is exact, homogeneous localization at a B-saturated prime P gives

$$F_{(P)} \to G_{(P)} \to \operatorname{Coker}(\varphi_{(P)}) \to 0.$$

The rank of $\operatorname{Coker}\varphi_{(P)}$ is equal to $\operatorname{rank}(G_{(P)}) - \operatorname{rank}(\varphi_{(P)})$. Notice that $\operatorname{rank}(\varphi_{(P)})$ is equal to $\operatorname{rank}(\varphi)$ because $I(\varphi)$ contains a nonzerodivisor. Now since

$$F_{(P)} \xrightarrow{\varphi_{(P)}} G_{(P)} \xrightarrow{\psi_{(P)}} H_{(P)}$$

is a complex, $\psi_{(P)}$ factors through $\operatorname{Coker}\varphi_{(P)}$.



As $\operatorname{Coker}\varphi_{(P)}$ is free of rank $n-r$, $\operatorname{rank}(\psi_{(P)}) = \operatorname{rank}(\psi'_{(P)})$ and $I_j(\psi_{(P)}) = I_j(\psi'_{(P)})$ for every $j \in \mathbb{Z}_{\geq 0}$.

Now $F_{(P)} \to G_{(P)} \to H_{(P)}$ is exact if and only $G_{(P)} \xrightarrow{0} \operatorname{Coker}\varphi_{(P)} \xrightarrow{\psi'_{(P)}} H_{(P)}$ is exact in which case $\operatorname{rank}(\psi'_{(P)}) = \operatorname{rank}(\operatorname{Coker}\varphi_{(P)})$. Assembling the equalities shows

$$\begin{aligned}\operatorname{rank}(\varphi) + \operatorname{rank}(\psi) &= \operatorname{rank}(\varphi_{(P)}) + \operatorname{rank}(\psi'_{(P)}) \\ &= \operatorname{rank}(\varphi_{(P)}) + \operatorname{rank}(G_{(P)}) - \operatorname{rank}(\varphi_{(P)}) \\ &= \operatorname{rank}(G).\end{aligned}$$

□

Proposition 3.5.3. *Let I be an ideal of S. Then $\sqrt{I} : B^\infty = \sqrt{I : B^\infty}$.*

Proof. First we show that $\sqrt{I} : B^\infty$ is a radical ideal. Suppose $s^n \in \sqrt{I} : B^\infty$. Then there is some m such that $B^m s^n \subset \sqrt{I}$. There is some N such that $B^{mN} s^{nN} \subset I$ which implies $(Bs)^{\max\{mN,nN\}} \subset I$ so $Bs \subset \sqrt{I}$. Thus $s \in \sqrt{I} : B^\infty$, hence $\sqrt{I} : B^\infty$ is radical.

Now let $J = \sqrt{I}$. We will show that $J : B^\infty$ and $I : B^\infty$ have the same radical. There is an n such that $J^n \subset I \subset J$. Therefore,

$$J^n : B^\infty \subset I : B^\infty \subset J : B^\infty.$$

It suffices to show that $(J : B^\infty)^n \subset J^n : B^\infty$. To this end, suppose the n ring elements $s_1, s_2, ..., s_n$ lie in $J : B^\infty$. Then there is an N such that $B^N s_i \subset J$ for every $i = 1, 2, ..., n$. So

$$B^{nN} s_1 s_2 \cdots s_n = B^N s_1 B^N s_2 \cdots B^N s_n \subset J^n.$$

□

In the general theory, the zeroth Fitting ideal of a module has the same radical as the annihilator of the module. The above proposition gives the analogous fact for saturated Fitting ideals as a corollary.

Corollary 3.5.4. $\sqrt{\operatorname{Fitt}_0(M) : B^\infty} = \sqrt{\operatorname{ann}(M) : B^\infty}$

Proof. We use the previous proposition and the fact that the annihilator of a module has the same radical as the zeroth Fitting ideal to obtain

$$\sqrt{\mathrm{Fitt}_0(M) : B^\infty} = \sqrt{\mathrm{Fitt}_0(M)} : B^\infty = \sqrt{\mathrm{ann}(M)} : B^\infty = \sqrt{\mathrm{ann}(M) : B^\infty}. \qquad \square$$

Chapter 4

Virtual Complete Intersections in $\mathbb{P}^1 \times \mathbb{P}^1$

4.1 Chapter Overview

This chapter is an adaptation of a paper with the same title [GLLM19]. Complete intersections are a fundamental object of study in commutative algebra and algebraic geometry. In projective space $\mathbb{P}^r_{\Bbbk}$, a complete intersection Y of codimension t is defined by an ideal of codimension t which can be generated by exactly t elements of the ring $\Bbbk[x_0, \ldots, x_r]$. In this case, there are hypersurfaces $H_1, \ldots, H_t$ such that Y is the scheme-theoretic intersection of the H_i's. Complete intersections have coordinate rings that are Cohen–Macaulay. The defining ideal of a complete intersection in $\mathbb{P}^r$ is generated by a regular sequence and therefore the minimal free resolution of the coordinate ring is a Koszul complex [Pee11, Theorem 14.7].

Unfortunately, in a product of projective spaces, the nice properties of complete intersections in $\mathbb{P}^r$ are not completely captured homologically. A zero dimensional scheme $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a scheme-theoretic complete intersection or virtual complete intersection if there are two polynomials f and g such that the ideal sheaf generated by f and g equals the ideal sheaf of X. On the other hand, X is an ideal-theoretic complete intersection (in this chapter we will just say complete intersection) if I_X, the set of all functions in the Cox ring vanishing on X, is generated by two elements. In $\mathbb{P}^r$, these two notions are the same, but in $\mathbb{P}^1 \times \mathbb{P}^1$, they are not (see Example 4.1.1 below). This

is where the virtual resolutions introduced by Berkesch, Erman, and Smith in [BES17] help. By allowing some irrelevant homology in a free complex, we expand the notion of a complete intersection via a virtual resolution, while still reaping the benefits of many properties of complete intersections. The goal of this chapter is to state conditions on whether a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ do or do not form a virtual complete intersection (see Definition 4.1.3).

Example 4.1.1. Consider the zero-dimensional reduced scheme $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ consisting of the two points $([1:0],[0:1])$ and $([0:1],[1:0])$. The set of all functions vanishing on X is

$$I_X = \langle x_0x_1, x_0y_0, x_1y_1, y_0y_1 \rangle.$$

However, the two polynomials x_0y_0 and x_1y_1 generate the same ideal sheaf that I_X generates. Therefore, X is a scheme-theoretic complete intersection or virtual complete intersection, but not an ideal-theoretic complete intersection. $\diamond$

Points in $\mathbb{P}^1 \times \mathbb{P}^1$ have been studied in the past, but often from a point of view of studying the saturated defining ideals of these points in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$. Some results include several classifications of when both reduced and fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ are arithmetically Cohen–Macaulay [GMR92, GMR94, Gua01, GVT08a, GVT08b, GVT04, GVT07, GVT12]. Further characterizations of points in more general products of projective spaces can be found in [GVT08a, GVT08b, FGM18, GVT07, GVT12, VT03] . In [GMR92], Giuffrida, Maggioni, and Rugusa prove that points in $\mathbb{P}^1 \times \mathbb{P}^1$ are defined by the ideal generated by two forms of bidegree $(a,0)$ and $(0,b)$, and further, if f and g are two forms of any bidegree in $\mathbb{P}^1 \times \mathbb{P}^1$, then the ideal $\langle f, g \rangle$ is not saturated, except in this case. In this chapter, we study when points have a complete intersection ideal that saturates to the defining ideal of the set of points, which is equivalent to being a virtual complete intersection. Surprisingly, all sets of points that are virtual complete intersections are not arithmetically Cohen–Macaulay, with the exception of points that are complete intersections (Corollary 4.3.7). While the results in this chapter concentrate on points in $\mathbb{P}^1 \times \mathbb{P}^1$, perhaps recent results of [FGM18], which use techniques of liaison, could help find VCIs in any product of projective spaces.

Setup

This chapter is mostly concerned with reduced zero-dimensional schemes in the product of projective spaces $\mathbb{P}^1 \times \mathbb{P}^1$ over an algebraically closed field $\Bbbk$. The Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is the $\mathbb{Z}^2$-graded ring $S := \Bbbk[x_0, x_1, y_0, y_1]$ where $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. The irrelevant ideal of S is $B := \langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle = \langle x_0y_0, x_0y_1, x_1y_0, x_1y_1 \rangle$. In this setting, closed subschemes are in one-to-one correspondence with B-saturated bihomogeneous ideals [CLS11, Proposition 6.A.7]. The B-saturation of an ideal I is

$$I : B^\infty = \bigcup_{k \geq 0} I : B^k = \{s \in S | sB^k \subset I \text{ for some } k\}.$$

If $I \subset S$ is an ideal, then $\mathbb{V}(I)$ denotes the subscheme of X consisting of all B-saturated bihomogeneous prime ideals that contain I. On the other hand, if X is a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$, then I_X denotes the B-saturated bihomogeneous ideal of polynomials in S that vanish at every point in X.

We will call reduced subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$ "sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$." The maximum number of points on a single horizontal ruling in a set of points X is denoted as m, and the maximum number of points on a single vertical ruling is denoted as n.

Definition 4.1.2 ([BES17, Definition 1.1])**.** *A* **virtual resolution** *for a module S/I in the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ is a $\mathbb{Z}^2$-graded complex of free S-modules*

$$F_\bullet := [F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \xleftarrow{\varphi_3} \cdots]$$

such that for every homology module $H_i(F_\bullet)$, it is true that $\operatorname{ann}(H_i(F)) \supseteq B^\ell$ *for some $\ell > 0$, and the associated sheaves $\widetilde{S/I}$ and $\widetilde{\operatorname{Coker}(\varphi_1)}$ are isomorphic.*

Definition 4.1.3. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with defining ideal I_X. We say X is a* **virtual complete intersection (VCI)** *if S/I_X has a virtual resolution that is a Koszul complex $K(f, g)$ of bihomogeneous forms f and g, where bihomogeneous means every term in the polynomial has the same x-degree and y-degree.*

In other words, a set of points $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ is called a virtual complete intersection or scheme-theoretic complete intersection generated by 2 forms $f, g \in S$ with $\deg(f) =$

(a, b) and $\deg(g) = (c, d)$ if there exists a sheaf surjection

$$0 \longleftarrow \mathcal{I}_X \longleftarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-a, -b) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-c, -d).$$

Given two curves of $\mathbb{P}^1 \times \mathbb{P}^1$ having no common component, C of bidegree (a, b) defined by equation $f = 0$ and D of bidegree (c, d) of equation $g = 0$, let $X = C \cap D$ be their scheme-theoretic complete intersection. The ideal $\langle f, g\rangle \subset S$ is not saturated (except in the cases $b = c = 0$ or $a = d = 0$), but we have the exact (Koszul) sequence of sheaves

$$0 \longleftarrow \mathcal{I}_X \longleftarrow \mathcal{O}(-a, -b) \oplus \mathcal{O}(-c, -d) \longleftarrow \mathcal{O}(-a - c, -b - d) \longleftarrow 0.$$

Next, we review the notion of configurations as introduced in [GVT15, §3.2] and show that the property of being a VCI is not a combinatorial invariant. Points in $\mathbb{P}^1 \times \mathbb{P}^1$ may be placed on a grid, according to their coordinates in each copy of $\mathbb{P}^1$, in the following way. There are two projections $\pi_i \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$:

$$\pi_1(a, b) = a \quad \text{and} \quad \pi_2(a, b) = b.$$

Making a grid of horizontal and vertical lines, the vertical lines correspond to the first copy of $\mathbb{P}^1$ and the horizontal lines correspond to the second copy of $\mathbb{P}^1$. Two points $p, q \in \mathbb{P}^1 \times \mathbb{P}^1$ lie on the same vertical line if $\pi_1(p) = \pi_1(q)$ and lie on the same horizontal line if $\pi_2(p) = \pi_2(q)$. By permuting the horizontal and vertical lines, we arrange the points so that the number of points on each horizontal and vertical line decreases from top to bottom and from left to right, forming a ***configuration***.

For example, letting a_i denote a point in the first copy of $\mathbb{P}^1$ and b_i denote a point in the second copy of $\mathbb{P}^1$, the set of points

$$\{(a_1, b_1), (a_2, b_1), (a_3, b_1), (a_1, b_2), (a_4, b_2), (a_2, b_3), (a_5, b_4)\}$$

in $\mathbb{P}^1 \times \mathbb{P}^1$, can be represented as in Figure 4.1 (here the points are labeled, but in what follows they will not be labeled). Note that the configuration of a set of points is not unique: in Figure 4.1, switching the horizontal rulings with coordinates b_2 and b_3 also yields a valid configuration. Thus, we consider two sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$

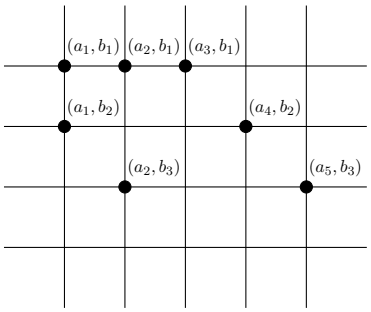


Figure 4.1: An example of a configuration

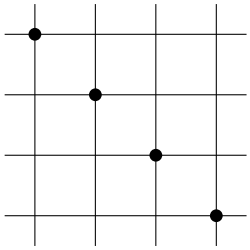


Figure 4.2: A four-point configuration whose the minimal free resolution depends on the coordinates.

to be ***equivalent up to configuration*** if they have the same configurations after permutation and relabeling of the rulings.

Unfortunately, the property of being a VCI depends on the coordinates of the points, not just on the configuration. This is not so surprising as the Betti numbers of points in $\mathbb{P}^1 \times \mathbb{P}^1$ also depend on more than just the configuration. To illustrate this point, we use the cross ratio.

Definition 4.1.4 ([Ahl78, Section 3.2 Definition 12])**.** *If four points in* $\mathbb{P}^1$ *have homogeneous coordinates* $[a : a'], [b : b'], [c : c'], [d : d']$*, their* **cross ratio** *is:*

$$\frac{(ca' - ac')(db' - bd')}{(da' - ad')(cb' - bc')}.$$

If a point is $[1 : 0]$ or $[0 : 1]$, then the terms involving this point are dropped from both the numerator and the denominator.

In Figure 4.2, the total Betti numbers of the minimal free resolution depends on the

cross ratio. Let I be the ideal of bihomogeneous forms vanishing at the points. When the cross ratios of the coordinates are the same after projection to each copy of $\mathbb{P}^1$, the minimal free resolution of S/I (omitting the twists of the free modules) is

$$S^1 \longleftarrow S^6 \longleftarrow S^8 \longleftarrow S^3 \longleftarrow 0.$$

When the cross ratios of the two copies of $\mathbb{P}^1$ are different, the minimal free resolution is

$$S^1 \longleftarrow S^6 \longleftarrow S^7 \longleftarrow S^2 \longleftarrow 0.$$

Moreover, for any collection of points with a subconfiguration of this kind, the minimal free resolution will depend on the value of the coordinates. By contrast, this configuration is always a VCI, regardless of the cross ratios (by Theorem 4.3.1).

Proposition 4.1.5. *Given the configuration of four points in Figure 4.2, the minimal resolution of these points depends on whether or not the cross ratios are equal after projection to each copy of* $\mathbb{P}^1$.

Proof. We may change coordinates so that three of the four points are $[0:1], [1:1], [1:0]$ and the last point is $[1:c]$, where c is the cross ratio [Ahl78, Section 3.2 Definition 12]. Now consider the form $x_0y_1 - x_1y_0$. If the cross ratios on both copies of $\mathbb{P}^1$ are the same, the form $x_0y_1 - x_1y_0$ vanishes, which explains the degree differences in the minimal free resolutions in the two cases – one has a $(1,1)$ graded piece whereas the other has two $(1,2)$ forms. Since the Hilbert function is recoverable from the minimal free resolution, the minimal free resolution changes accordingly. □

Virtual resolutions are also not invariant under configurations when the total number of points is large relative to the maximum number of points lying on a single horizontal ruling and a single vertical ruling, which we respectively denote by m and n.

Remark 4.1.6. When $|X| \geq mn$, VCIs are not always determined by configuration. That is, the same configuration may be a VCI with some coordinates, but not with others. For example, the configuration below is a VCI when either the four rightmost points or the four bottommost points lie on a $(1,1)$-form. If these points do not lie on such a conic, the configuration is not a VCI. This example is explored in more detail in Theorem 4.3.6, after the necessary machinery has been developed.

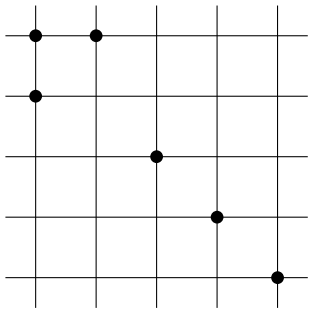


Figure 4.3: This configuration is a VCI with some coordinates, but is not a VCI with others.

However, the configuration below is far from being an arithmetically Cohen–Macaulay set of points or a complete intersection in [GVT15, Theorem 4.11, Theorem 5.13], whose criteria depend only on the combinatorial configuration and not the actual coordinates of the points. Hence the question of when sets of points form VCIs is another interesting and subtle question.

Summary of Main Results

Complete intersections are always VCI, however, VCIs form a strictly larger set of points than complete intersections. Our main results are summarized below.

1. A set of points is a VCI when it has the same number of points in each vertical (or each horizontal) ruling (Theorem 4.3.1).

2. A set of points X is not a VCI when

   (a) $|X| < mn$, and there is at least one point in X that is on a horizontal ruling with m points and a vertical ruling with n points (Theorem 4.3.2).

   (b) $|X| < mn$ and $\gcd(m, n)$ does not divide $|X|$ (Theorem 4.4.1).

   (c) The degrees of two forms that intersect at X are known and one of the conditions in Proposition 4.4.4 holds.

3. VCIs are not solely determined by the configuration of the points, which is a characterization of where points lie in relation to each other: when $|X| > mn$, the

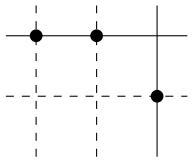


Figure 4.4: A 3-point variety that is generated by two forms geometrically but not a complete intersection.

actual coordinates of points can play a role in determining whether or not X is a VCI (Remark 4.1.6, Theorem 4.3.6).

4. When all points lie on at most three vertical or horizontal rulings, we provide a complete classification of VCIs (Section 4.5).

Example 4.1.7. Consider Figure 4.4 of the three points

$$X = \{([1:0],[0:1]),([1:1],[0:1]),([0:1],[1:1])\} \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

Letting I_X be the B-saturated ideal of bihomogeneous polynomials vanishing at X, the minimal free resolution of S/I_X is

$$0 \longleftarrow S \longleftarrow \begin{matrix} S(0,-2) \\ \oplus \\ S(-1,-1) \\ \oplus \\ S(-2,-1) \\ \oplus \\ S(-3,0) \end{matrix} \longleftarrow \begin{matrix} S(-1,-2) \\ \oplus \\ S(-2,-2) \\ \oplus \\ S(-3,-1)^2 \end{matrix} \longleftarrow S(-3,-2) \longleftarrow 0.$$

It is well known that X is not arithmetically Cohen–Macaulay by the criterion in [GVT15, Theorem 4.11]. Therefore X does not form a complete intersection. As the picture indicates, however, X is a VCI as it is the intersection of the varieties of two forms

$$f = x_1y_1 \quad \text{and} \quad g = x_0(x_1 - x_0)(y_1 - y_0).$$

Therefore, $K(f,g)$ is a virtual resolution of S/I_X, where

$$K(f,g) := [0 \longleftarrow S \xleftarrow{\begin{bmatrix} f & g \end{bmatrix}} \begin{matrix} S(-1,-1) \\ \oplus \\ S(-2,-1) \end{matrix} \xleftarrow{\begin{bmatrix} -g \\ f \end{bmatrix}} S(-3,-2) \longleftarrow 0].$$

Although the points do not form a complete intersection, they nonetheless share similar properties with complete intersections. The saturation of $\langle f, g\rangle$ by the irrelevant ideal B is equal to I_X so $\mathbb{V}(f) \cap \mathbb{V}(g) = X$ scheme-theoretically. $\diamond$

All theorems proved in this chapter are from the virtual viewpoint. This differs from the viewpoint of [GVT15], which does not consider complete intersections up to irrelevance. A set of points is a complete intersection exactly when all horizontal and vertical rulings contain the same number of points [GVT15, Theorem 5.13]. That is, the set of points looks like a rectangle when put on a grid.

Guardo and Van Tuyl not only classified when points in $\mathbb{P}^1 \times \mathbb{P}^1$ are complete intersections, they also classified when points are arithmetically Cohen–Macaulay (i.e. the coordinate ring S/I_X is Cohen–Macaulay). This occurs exactly when X can be written in a configuration that resembles a Ferrers diagram [GVT15, Theorem 4.11]. By the Auslander–Buchsbaum formula, points in $\mathbb{P}^1 \times \mathbb{P}^1$ are arithmetically Cohen–Macaulay exactly when the minimal free resolution of S/I_X is of length 2. All sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ have virtual resolutions of length 2 [BES17, Theorem 1.5]. Hence in this chapter, we concentrate only on when sets of points are VCIs.

Outline

In Section 4.2, it is proved that every set of points forms a VCI when considered set-theoretically. Then, in Sections 4.3 and 4.4, we examine the scheme-theoretic case of reduced points. The majority of the proofs of our main theorems are in these sections. We name many conditions which are guaranteed to either give rise to or never give rise to VCIs. Finally, Section 4.5 is an application of these results, giving a complete classification of VCIs that lie on at most three horizontal (or vertical) rulings.

4.2 Set-Theoretic VCIs

In this section, we consider zero-dimensional subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$. We show that set-theoretically, all such subschemes (hence all corresponding configurations) are virtual complete intersections.

Theorem 4.2.1. *For any zero-dimensional subscheme X of $\mathbb{P}^1 \times \mathbb{P}^1$, there is an ideal J so that $\sqrt{J} = \sqrt{I_X}$ and S/J has a virtual resolution that is a length two Koszul complex.*

Proof. Let $\text{Supp}(X)$ be the underlying set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We will show that there is an ideal generated by two bihomogeneous forms f and g so that $\text{Supp}(\mathbb{V}(f,g)) = \text{Supp}(X)$.

Let k be the number of distinct vertical rulings containing points of $\text{Supp}(X)$, and let ℓ_i be the $(1,0)$-form defining the i^{th} of these vertical rulings. Set $f = \ell_1 \cdots \ell_k$, so $\mathbb{V}(f)$ is the smallest union of vertical rulings containing the set of points.

Next, let n be the maximum number of points of $\text{Supp}(X)$ that lie on a single vertical ruling. For each vertical ruling i, let g_i be a $(0,n)$ form that is the the product of n $(0,1)$-forms such that $\text{Supp}(X \cap \mathbb{V}(\ell_i)) = \text{Supp}(\mathbb{V}(g_i) \cap \mathbb{V}(\ell_i))$. That is, we multiply the forms defining horizontal rulings containing points of the set on $\mathbb{V}(\ell_i)$, and repeat some of them so that the degree is $(0,n)$. Set $g = \sum_{i=1}^{k} \frac{fg_i}{l_i}$.

Now we check that $\text{Supp}(X \cap \mathbb{V}(\ell_i)) = \text{Supp}(\mathbb{V}(g) \cap \mathbb{V}(\ell_i))$. Let p be contained in the left hand side, which is equal to $\text{Supp}(\mathbb{V}(g_i) \cap \mathbb{V}(\ell_i))$. We have $g_i(p) = 0$ and $\frac{f}{\ell_j}(p) = 0$ for $j \neq i$, so $g(p) = 0$. This shows $\text{Supp}(X \cap \mathbb{V}(\ell_i)) \subseteq \text{Supp}(\mathbb{V}(g) \cap \mathbb{V}(\ell_i))$.

To show $\text{Supp}(X \cap \mathbb{V}(\ell_i)) \supseteq \text{Supp}(\mathbb{V}(g) \cap \mathbb{V}(\ell_i))$, suppose q is contained in the right hand side. Then $g(q) = 0$ forces $g_i(q) = 0$ since $\frac{f}{\ell_j}$ vanishes if and only if $i \neq j$. Therefore $q \in \text{Supp}(\mathbb{V}(g_i) \cap \mathbb{V}(\ell_i)) = \text{Supp}(X \cap \mathbb{V}(\ell_i))$, and thus $\text{Supp}(X \cap \mathbb{V}(\ell_i)) = \text{Supp}(\mathbb{V}(g) \cap \mathbb{V}(\ell_i))$.

Taking the union over all i yields $\text{Supp}(X) = \text{Supp}(\mathbb{V}(f) \cap \mathbb{V}(g)) = \text{Supp}(\mathbb{V}(f,g))$ as desired. □

Example 4.2.2. We demonstrate the procedure described in the proof of Theorem 4.2.1 to show that configuration of six points in Figure 4.5 is set theoretic virtual complete intersection.

Set

$$\ell_1 = x_0, \qquad \ell_2 = x_0 - x_1, \qquad \ell_3 = x_0 - 2x_1.$$

We can then choose the g_i to be

$$\begin{aligned} g_1 &= (x_1 - x_0)(x_0 - 2x_1)y_0^3 \\ g_2 &= x_0(x_0 - 2x_1)y_0(y_0 - y_1)^2 \\ g_3 &= x_0(x_0 - x_1)y_0(y_0 - y_1)(y - 2y_1). \end{aligned}$$

Letting $f = \ell_1\ell_2\ell_3$ and $g = g_1 + g_2 + g_3$, then $\mathbb{V}(f, g)$ is exactly the 6 points in the figure, however, $([0 : 1], [0 : 1])$ is counted with multiplicity 2 and $([1 : 1], [1 : 1])$ is counted with multiplicity 3. ◇

Remark 4.2.3. Notice that this procedure makes the sum of the multiplicities of points on each horizontal ruling equal. Perhaps an easier way to visualize that this configuration is a set theoretic VCI is visualizing the points with the above multiplicities as the scheme-theoretic intersection of the dotted curves indicated in Figure 4.5,

$$\mathbb{V}(x_0(x_0 - x_1)(x_0 - 2x_1))$$

which is the vanishing set of a $(3, 0)$ form, and the dashed curves,

$$\mathbb{V}((y_0)(2x_1^2y_0 + x_0^2y_1 - 3x_0x_1y_1)(x_0y_1 - x_1y_0))$$

which is the vanishing set of a $(3, 3)$ form. By [Sha13, §4.2.1] (see Theorem 4.3.3), the intersection should have order 9, but the curves intersect at $([0 : 1], [0 : 1])$ with multiplicity 3 and at $([1 : 1], [1 : 1])$ with multiplicity 2, so the intersection set is indeed 6 points.

4.3 Determination of VCIs

We now consider reduced zero-dimensional subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$, which we refer to as "sets of points." This requires that the homogeneous ideal generated by the two forms equal I_X after saturation by B, instead of first taking the radical and then saturating

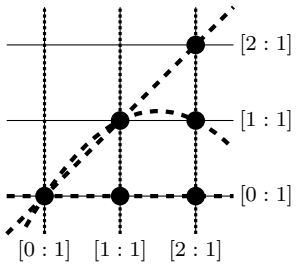


Figure 4.5: Configuration from Example 4.2.2, forms from Remark 4.2.3

by B, which leads to a richer classification of configurations into VCIs, non-VCIs, and coordinate dependent cases.

In the previous section, we proved that set-theoretically all configurations are VCIs by assigning multiplicities so that along each ruling, there are the same total multiplicity of points. When this condition is satisfied without having to artificially "boost up" the multiplicity of any point, we have a natural environment for VCIs.

Theorem 4.3.1. *If X has the same number of points in each vertical (or each horizontal) ruling, it is a VCI.*

Proof. By symmetry, it is enough to prove the vertical case. The proof is nearly identical to the proof of Theorem 4.2.1. In the notation of that proof, we construct f and g as before, noting that each $\mathbb{V}(l_i)$ contains n points of $\mathrm{Supp}(X)$, so $\mathbb{V}(g_i) \cap \mathbb{V}(l_i)$ is supported on n distinct points each having multiplicity 1. Therefore, the computations in the set-theoretic case of the support are exactly the same as in the scheme-theoretic case, showing that $X = \mathbb{V}(f) \cap \mathbb{V}(g) = \mathbb{V}(f, g)$, and thus that X is a VCI. □

Notice that Example 4.1.7 illustrates this theorem. On the other hand, Theorem 4.3.2 below names conditions that guarantee when a set of points can never be a VCI.

Theorem 4.3.2. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let m be the maximum number of points of X on a single horizontal ruling, and let n be the maximum number of points on a single vertical ruling. If $|X| < mn$, and there is at least one point in X that is on a horizontal ruling with m points and a vertical ruling with n points, then X is not a VCI.*

Before proving Theorem 4.3.2, we introduce the generalized Bézout's theorem and two technical lemmas to serve as tools for providing bounds on multidegrees.

Theorem 4.3.3 (Generalized Bézout's Theorem, [Sha13, §4.2.1])**.** *Let $f, g \in S$ be two bihomogeneous forms in $\mathbb{P}^1 \times \mathbb{P}^1$. If f and g of multidegree (a, b) and (c, d) respectively, are in general position, i.e., f, g have no common factor, then $|\mathbb{V}(f) \cap \mathbb{V}(g)| = ad + bc$, counting multiplicities.*

This theorem will be used extensively to help combinatorially determine virtual complete intersections.

Lemma 4.3.4. *Given a configuration of finitely many points X in $\mathbb{P}^1 \times \mathbb{P}^1$, let m be the maximum number of points on a single horizontal ruling and n be the maximum number of points on a single vertical ruling. If $K(f, g)$ is a virtual resolution of S/I_X, where polynomials f and g are of degrees (a, b) and (c, d), respectively, then* $\max(a, c) \geq m$ *and* $\max(b, d) \geq n$.

Proof. Assume, for the sake of contradiction, that both a and c are less than m. Without loss of generality, we can change coordinates to assume that the m points are on the horizontal ruling with coordinates $[1 : 0]$ and assume none of the m points lie on the vertical ruling $[0 : 1]$. We can restrict f to the horizontal ruling $[1 : 0]$ by substituting $y_0 = 1, y_1 = 0$, and $x_0 = 1$ yielding a single variable polynomial of degree a with m roots. By the assumption that $a < m$, this restriction of m must be identically 0, and so $\mathbb{V}(f)$ contains the entire ruling $[1 : 0]$. By an identical argument on g using $c < m$, we have $\mathbb{V}(g)$ also containing the entire ruling $[1 : 0]$. Therefore, $\mathbb{V}(f) \cap \mathbb{V}(g)$ contains that entire ruling, and so cannot be the original finite set of points. Thus our assumption that both a and c are less than m was false, and so $\max(a, c) \geq m$. The proof that $\max(b, d) \geq n$ is analogous. □

Lemma 4.3.5. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let m and n be as in Lemma 4.3.4. If $K(f, g)$ is a virtual resolution for S/I_X, where f and g have multidegrees (a, b) and (c, d) respectively, and $|X| < mn$, then,*

1. *Either (i) $a \geq m$ and $b \geq n$, or (ii) $c \geq m$ and $d \geq n$.*
2. *In case (i), $\mathbb{V}(g)$ contains the horizontal ruling containing the m points, and the vertical ruling on the containing the n points. In case (ii), the same is true of f.*

Proof. By Lemma 4.3.4, we have

$$\max(a, c) \geq m \quad \text{and} \quad \max(b, d) \geq n.$$

Without loss of generality, suppose $a \geq c$ and $d \geq b$. Then, $a \geq m$ and $d \geq n$. However, in this case $ad \geq mn$, so $ad + bc \geq mn$, which contradicts $|X| < mn$. Therefore, we must have $a \geq c$ and $b \geq d$, so $a \geq m$ and $b \geq n$. This proves (1).

If g does not contain the entire line of the m collinear points, then g restricted to that line is a nonzero polynomial with m roots, and so has degree at least m. This means that $c \geq m$, which gives the contradiction $|X| = ad + bc \geq bc \geq mn$. Similarly, if g does not contain the ruling with n points, then its restriction to that line must have degree at least n giving the contradiction $|X| = ad + bc \geq ad \geq mn$. This completes the proof. □

Proof of Theorem 4.3.2. Assume that $|X| < mn$, and there is at least one point in X that lies both on a horizontal ruling with m points and a vertical ruling with n points. If $K(f, g)$ is a virtual resolution for S/I_X, where f has multidegree (a, b) and g has multidegree (c, d), then by the first part of Lemma 4.3.5, we may assume $a \geq m$ and $b \geq n$. Suppose $\mathbb{V}(g)$ includes s horizontal lines and t vertical lines. Now using the second part of Lemma 4.3.4, s and t are at least one, and by assumption, the intersection of these $s + t$ lines contains at least one point of X. Factoring g, and changing coordinates if necessary so that no points have coordinate $[0 : 1]$, yields

$$\begin{aligned} g &= \lambda h_0 g_0, \text{ where} \\ h_0 &= (x_1 - \alpha_1 x_0)(x_1 - \alpha_2 x_0) \cdots (x_1 - \alpha_s x_0)(y_1 - \beta_1 y_0) \cdots (y_1 - \beta_t y_0) \end{aligned}$$

is the product of the $s + t$ components, and g_0 is a bihomogeneous polynomial of multidegree $(p, q) = (c - s, d - t)$. Let $Y = \mathbb{V}(h_0, f) \subseteq X$ be the points covered by the $s + t$ components of g. We have $|Y| \leq ms + nt - 1$, because we are certainly double counting the point lying on the intersection of the vertical and horizontal rulings. The remaining set of points, $X \setminus Y$, must be precisely $\mathbb{V}(g_0, f)$, whose cardinality is $aq + bp$ according to Theorem 4.3.3.

Applying Theorem 4.3.3 again to f and g, it follows that

$$a(s+q)+b(t+p) = |X| \leq ms+nt-1+aq+bp.$$

Simplifying the inequality above yields

$$as+bt \leq ms+nt-1.$$

Since $a \geq m$, $b \geq n$, and both s and t are at least one, we have a contradiction. Thus, X cannot be a VCI. □

Using Lemma 4.3.4, we can now flesh out Remark 4.1.6 into a theorem.

Theorem 4.3.6. *There exist sets of points $X_1, X_2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that X_1 and X_2 are equivalent up to configuration, but X_1 is a VCI and X_2 is not.*

Proof. Consider the configuration of six points in Figure 4.3, and suppose it is the VCI of f and g with multidegree (a,b) and (c,d) respectively. Through the six points of Figure 4.3 there is a form of degree $(0,5)$ and a form of degree $(5,0)$. If any of the degrees were 0, say a, then $\mathbb{V}(f)$ would be parallel lines, and since there are five distinct coordinates, f would have degree $(0,5)$. There is no choice of c and d that satisfies $ad+bc=6$, so none of the degrees are 0.

By Lemma 4.3.4, since $m=2$ and $n=2$, we can assume that $a \geq 2$ and b or d is at least 2. But we also have $ad+bc=6$ by Theorem 4.3.3, and since none of the degrees are 0, we find that the only possibility is that the degrees of f and g are $(2,1)$ and $(2,2)$ (or vice versa).

Without loss of generality, let f have degree $(2,1)$. By Theorem 4.3.3, f will intersect each vertical ruling in exactly one point unless it contains the whole ruling. Since there are two points in our configuration sharing a vertical ruling (and symmetrically a horizontal ruling), f must have a degree $(1,0)$ component passing through that ruling, and therefore must have a degree $(1,1)$ component passing through the remaining four points. Conics in the projective plane (i.e. forms of degree $(1,1)$) are determined by three points, though, so this is impossible in most cases. Thus, in these cases the set of points could not be a VCI.

However, in the cases where the remaining four points do lie on a conic, the points may be a VCI. For instance, letting $\Bbbk = \mathbb{C}$ for a moment, if the points have coordinates:

$$X = \begin{Bmatrix} ([1:1],[1:1]),([2:1],[1:2]),([3:1],[1:3]), \\ ([4:1],[1:4]),([1:0],[1:1]),([1:0],[1:0]) \end{Bmatrix},$$

then $K(f,g)$ is a virtual resolution of S/I_X, where

$$\begin{aligned} f &= x_0x_1y_0 - x_1^2y_1, \\ g &= 24x_1^2y_0^2 - x_0^2y_0y_1 - 50x_1^2y_0y_1 + x_0^2y_1^2 - 9x_0x_1y_1^2 + 35x_1^2y_1^2. \end{aligned}$$ □

Theorem 4.3.2 enables us to give a complete classification of VCI points in $\mathbb{P}^1 \times \mathbb{P}^1$ that lie in a configuration that forms a Ferrers diagram. A Ferrers diagram of points in an m by n grid will be called a ***rectangle***. As mentioned in Section 4.1, [GVT15] show a set of points is arithmetically Cohen–Macaulay exactly when it forms a Ferrers diagram. The corollary below states that if X is arithemetically Cohen–Macaulay, then it is a VCI if and only if it is a complete intersection.

Corollary 4.3.7. *If X is a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ forming a Ferrers diagram, then X is a virtual complete intersection if and only if it is a rectangle.*

Proof. Defining m and n as before, if X forms a Ferrers diagram that is not a rectangle, then the number of points is strictly lower than mn. Further, the corner of the diagram is one of m points on its horizontal ruling and n points on its vertical ruling, so applying Theorem 4.3.2 proves that X is not a VCI.

The converse was proven in [GVT15], but we reproduce it here for completeness. If the configuration is a rectangle, then let f denote the $(m,0)$-form whose vanishing set consists of the m vertical rulings the points lie on, and let g denote the $(0,n)$-form whose vanishing set consists of the n horizontal rulings the points lie on. Then (f,g) is a regular sequence, indicating X is a complete intersection. □

4.4 Bounds on Multidegrees and Size of A Configuration

In some cases, the property of being a VCI cannot be directly determined based on the maximum number of points on a single vertical or horizontal ruling. In this section, we

provide characterizations for VCIs by more closely examining the relationship between the multidegrees of f and g and the total number of points in the configuration.

Theorem 4.4.1. *If $|X| < mn$ and $\gcd(m,n)$ does not divide $|X|$, then X is not a VCI.*

Before proving Thereom 4.4.1, we first prove the following lemma.

Lemma 4.4.2. *Let f be a bihomogeneous polynomial of multidegree (a,b) and g of multidegree (c,d). Let $K(f,g)$ be a virtual resolution for S/I_X with $|X| < mn$. By Lemma 4.3.5, without loss of generality, we can assume that $a \geq m$ and $b \geq n$. Then,*

1. *$a = m$, $b = n$, and*
2. *$\mathbb{V}(g)$ has vertical components exactly on rulings with n points of X and has horizontal components exactly on rulings with m points of X.*

Proof. As in the proof of Theorem 4.3.2, let s be the number of factors of g of the form $x_1 - \alpha x_0$ (i.e. the number of horizontal lines in $\mathbb{V}(g)$) and t be the number of factors of the form $y_1 - \beta y_0$) (i.e. the number of vertical lines in $\mathbb{V}(g)$), where α, β are constants. Set Y to be the points of g covered by those lines, and let (p,q) be the multidegree of the remaining components of g. By Lemma 4.3.5, we have $s,t \geq 1$ and

$$as + bt = |Y| \leq ms + nt.$$

By hypothesis, $a \geq m$ and $b \geq n$. Either of these being strict would contradict the above, so $a = m$ and $b = n$. This implies

$$ms + nt = |Y|.$$

Thus, each vertical component of $\mathbb{V}(g)$ must contain n points of X and each horizontal component must contain m points of X, because Theorem 4.3.2 guarantees no point can lie on both a horizontal and vertical component. □

Note that when $|X| < mn$, the values of s and t are intrinsically determined by the configuration of X: in this case, we must have $s = m$ and $t = n$.

Proof of Theorem 4.4.1. Suppose $|X| < mn$ with $\gcd(m,n)$ not dividing $|X|$. Assume $K(f,g)$ is a virtual resolution for S/I_X. From Lemma 4.4.2, it can be assumed f has

multidegree (m, n). Letting g have multidegree (c, d), Theorem 4.3.3 implies $|X| = dm + cn$. This is divisible by $\gcd(m, n)$, which is a contradiction. □

Proposition 4.4.3. *If $K(f, g)$ is a virtual resolution for S/I_X with $|X| < mn$, $\gcd(m, n) = 1$, and (m, n) the multidegree of f, then the multidegree of g is (c, d), where $0 \leq c < m$ and $0 \leq d < n$ are unique integers satisfying:*

$$c = n^{-1}|X| \mod m, \qquad \text{and} \qquad d = m^{-1}|X| \mod n.$$

Proof. By Theorem 4.3.3,

$$dm + cn = |X|.$$

Considering modulo m and n, we have:

$$c \equiv n^{-1}|X| \mod m \qquad \text{and} \qquad d \equiv m^{-1}|X| \mod n.$$

Since both cn and dm are less than mn we must have $c < m$ and $d < n$. Thus c and d must have the desired values. □

Proposition 4.4.4. *Assume X is a finite set of points with $|X| < mn$, $\gcd(m, n) = 1$, and let*

$$c = n^{-1}|X| \mod m \qquad \text{and} \qquad d = m^{-1}|X| \mod n.$$

Let s and t be defined as in the proof of Theorem 4.3.2, and set $p := d - s$ and $q := c - t$. If any of the following are true, X will not be a VCI.

1. *$dm + cn \neq |X|$*
2. *$d < s$ or $c < t$*
3. *There is a horizontal ruling with strictly between q and m points of X, or a vertical ruling with strictly between p and n points of X.*

Proof. We will prove the contrapositive of (1). Assume that $K(f, g)$ is a virtual resolution for S/I_X. Then by Lemma 4.4.2, $\deg(f) = (m, n)$ and by Proposition 4.4.3, $\deg(g) = (c, d)$. Thus Theorem 4.3.3 guarantees $|X| = dm + cn$.

If X is a VCI of f and g, then g has s horizontal line components, so $d \geq s$. Similarly since it has t vertical line components, then $c \geq t$. Thus (2) is proved.

By Lemma 4.4.2, any horizontal ruling with fewer than m points of X cannot be contained in $\mathbb{V}(g)$. However, a polynomial of multidegree (p, q) cannot vanish on more than p points of a horizontal ruling without containing the entire ruling. Analogously, g cannot vanish on between q and n points of a vertical ruling, completing (3). □

Note that when X is a VCI of f and g with $|X| < mn$ and $\deg(f) = (m, n)$, we have determined not only the multidegree of g but also the multidegree of components that are not degree 1 lines, that is, p and q are intrinsically determined.

Using these restrictions on VCIs, it is possible to classify all possible configurations of VCIs when the parameters are small and eliminate a sizable number of cases in general. Nevertheless, it is important to keep in mind Remark 4.1.6, which illustrates the limit of the applicability of combinatorics of configurations to determine VCIs when the size of the configuration gets sufficiently large (with respect to m and n).

4.5 Complete Classifications of Points on at Most Three Rulings

In this section, we give a complete classification of VCIs when all points lie on at most three rulings. The case where all points lie on a single ruling is automatically a VCI (in fact, a complete intersection), and we state the conditions for 2-ruling VCIs in Theorem 4.5.1, 3-ruling VCIs in Theorem 4.5.2.

Theorem 4.5.1. *If all points lie on two horizontal rulings, they form a VCI if and only if either:*

1. *no two of them lie on the same vertical ruling, or*
2. *both horizontal rulings contain the same number points.*

Proof. If no two points lie on the same vertical ruling, then each vertical ruling contains exactly one point, so the configuration is a VCI by Theorem 4.3.1. If both horizontal rulings contain the same number of points, we can match point p_i in one ruling with

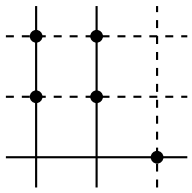


Figure 4.6: f is the product of solid lines; g of dashed lines.

point p_i' in the other. The configuration is the VCI of two horizontal lines through the two rulings and the product of $(1,1)$ forms each passing through one pair of p_i and p_i'.

Inversely, suppose two points lie on the same vertical ruling, and the two horizontal rulings have different numbers of points. Then the maximum number of points on the same horizontal ruling times 2 (the maximum number of points on the same vertical ruling) is greater than the total number of points. So by Theorem 4.3.2, this configuration is not a VCI. □

Theorem 4.5.2. *If all points lie on three horizontal rulings, the configuration is a VCI if and only if it satisfies one of the following conditions:*

1. *All horizontal rulings contain the same number of points.*
2. *All vertical rulings contain the same number of points.*
3. *On two of the horizontal rulings all points are in pairs on the same vertical ruling, and no vertical rulings contain* 3 *points.*
4. *Two of the horizontal rulings contain the same number of points, k, and all points lie on a $(k,1)$-curve.*

Proof. All four conditions are sufficient. Conditions (1) and (2) follow from Theorem 4.3.1. Configurations satisfying condition (3) can always be decomposed into two rectangular blocks and be seen as an intersection of two forms that resemble the structure in Figure 4.6. That is, $\mathbb{V}(f)$ consists of the two paired horizontal rulings and vertical rulings through each remaining point, and $\mathbb{V}(g)$ consists of vertical rulings through each pair of points and a horizontal ruling through the remaining points. Condition (4) can always be decomposed into the intersection of a $(k,1)$-curve, and the union of two horizontal lines and one vertical line through each point of X on the remaining ruling, as

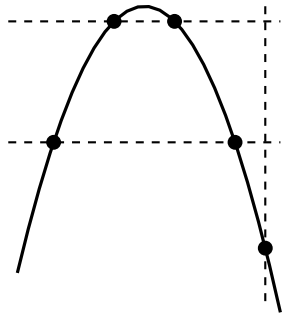


Figure 4.7: f is the solid parabola; g is the product of dashed lines.

demonstrated in Figure 4.7. By Theorem 4.3.3, the $(k, 1)$-curve will intersect the two horizontal rulings in exactly the k points each, and will intersect each vertical ruling in exactly one point, as desired. The vertical rulings need to be carefully chosen so that their intersections with the curve lie on the same horizontal ruling.

We now show that these are the only cases. Assume X is a VCI that satisfies none of the conditions above. Let n be the maximum number of points on a single vertical ruling so n may be $1, 2$ or 3. We will show that in each case X must satisfy one of the conditions above.

Notice that if $n = 1$, then condition (2) is satisfied. If $n = 3$, one of the 3 points on such a ruling will also be on the horizontal ruling with the maximal number m of points. Theorem 4.3.2 implies X will not be a VCI unless all three horizontal rulings contain m points, in which case (1) holds.

We now consider the the most difficult case of $n = 2$. Assume the three horizontal rulings contain $\alpha \geq \beta \geq \gamma$ points, respectively. Notice that α, β, γ are not all equal since otherwise X satisfies condition (1). Since X is a VCI, $K(f, g)$ is a virtual resolution of S/I_X for some f, g of multidegrees (a, b) and (c, d). According to Lemma 4.3.4, $\max\{a, c\} \geq \alpha$ and $\max\{b, d\} \geq 2$. Without loss of generality, suppose $a \geq c$. Then $a \geq \alpha$. This means $d < 3$ since $3\alpha > |X| = ad + bc \geq d\alpha$.

There are two cases:

Case 1: $b \geq 2$. If $d = 0$, then $\mathbb{V}(g)$ is the union of vertical rulings. By Theorem 4.3.3, the number of intersection points of $\mathbb{V}(f)$ and each vertical ruling in $\mathbb{V}(g)$ is b, so the configuration must satisfy (2).

Therefore, $d > 0$. Notice that g having a high degree in y-dimension implies a low

degree in x-dimension. In particular, degree $c \leq \beta$, because

$$\alpha + 2\beta \geq \alpha + \beta + \gamma = |X| = ad + bc \geq \alpha + 2c \tag{6.1}$$

(using $a \geq \alpha$ and $b \geq 2$ in the last step).

If $c < \beta$, then $\mathbb{V}(g)$ must contain the entirety of the horizontal rulings of α points and β points. This is because when restricting to those rulings g would be a polynomial of degree c, and so could vanish at β points only if it vanished on the whole ruling. Thus, we have $d \geq 2$, and so $\alpha + \beta + \gamma = ad + bc \geq 2\alpha + 2c$. From this we find $c < \gamma$, and again using the same argument, $\mathbb{V}(g)$ vanishes on all 3 horizontal rulings, which violates the fact that $d < 3$.

On the other hand, if $c = \beta$, then using the inequality (6.1) and recalling $b \geq 2, d > 0$, we have

$$c = \beta = \gamma, \qquad a = \alpha, \qquad b = 2, \qquad \text{and} \qquad d = 1.$$

In this case, $\alpha > \beta = c$, so $\mathbb{V}(g)$ must contain the entire horizontal ruling of α points. Since $d = 1$, the rest of $\mathbb{V}(g)$ can only consist of vertical rulings, and there must be β of them. Each vertical ruling intersects $\mathbb{V}(f)$ at 2 points by Theorem 4.3.3, and so the configuration satisfies (3).

Case 2: $b < 2$. Since $\max\{b, d\} \geq 2$, d must equal 2. If $b = 0$, then $\mathbb{V}(f)$ is the union of multiple vertical rulings. Since the number of intersections of $\mathbb{V}(g)$ and each vertical ruling is always c, the configuration must satisfy (2). Thus, $b = 1$. As before, notice that

$$\alpha + \beta + \gamma = |X| = ad + bc \geq 2\alpha + c.$$

Then, $c \leq \gamma$. If $c < \gamma$, then $\mathbb{V}(g)$ has to contain all three horizontal rulings, which contradicts $d = 2$. Therefore, $c = \gamma$, and $\alpha = \beta > \gamma$. In this case, $\mathbb{V}(g)$ must contain the entire horizontal rulings of α points and β points. Since $d = 2$, the rest of $\mathbb{V}(g)$ can only be vertical rulings, and there must be γ of them, and recalling $b = 1$, the configuration satisfies (4). □

Chapter 5

Virtual Resolutions M2 Package

5.1 Chapter Overview

This chapter is an adaptation of the article [ABLS19] which introduces the *Macaulay2* package `VirtualResolutions`. The package provides tools for constructing and studying virtual resolutions over products of projective spaces.

The `VirtualResolutions` project began in 2018 at the *Macaulay2 Workshop at the University of Wisconsin - Madison*, building on previous work of Christine Berkesch, David Eisenbud, Daniel Erman, and Gregory G. Smith. Along with them, the authors of [ABLS19] developed the `VirtualResolutions` package to generate examples of virtual resolutions in products of projective spaces. In particular, the package provides tools to generate examples of virtual resolutions and to check whether chain complexes are virtual resolutions. These methods are introduced and demonstrated in Section 2.

In Section 3, we concentrate on virtual resolutions arising from curves in $\mathbb{P}^1 \times \mathbb{P}^2$. This is the simplest case after points in a product of projective spaces. We demonstrate several methods for finding the defining ideals of curves in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2$, including monomial curves, rational curves, and curves from $\mathbb{P}^3$.

5.2 Constructing Virtual Resolutions

If X is a smooth projective toric variety, we denote its $\mathrm{Pic}(X)$-graded Cox ring, as defined in [Cox95], by $\mathrm{Cox}(X)$ (see also Section 5.2 of [CLS11]). Denote the associated

irrelevant ideal of X by $B \subset \mathrm{Cox}(X)$. In addition to Definition 2.4.1, a virtual resolution of a $\mathrm{Pic}(X)$-graded module over $\mathrm{Cox}(X)$ can be equivalently defined in terms of chain complexes of $\mathcal{O}_X$-modules.

Definition 5.2.1 ([BES17, Definition 1.1])**.** *A* ***virtual resolution*** *of a* $\mathrm{Pic}(X)$*-graded* $\mathrm{Cox}(X)$*-module* M *is a chain complex of* $\mathrm{Pic}(X)$*-graded free* $\mathrm{Cox}(X)$*-modules*

$$F_\bullet := \left[F_0 \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots\right]$$

such that the corresponding complex of $\mathcal{O}_X$*-modules is a locally free resolution of the sheaf* $\widetilde{M}$*.*

One can rephrase the definition of a virtual resolution in a way that is more practical for computations: a complex $F_\bullet$ of $\mathrm{Pic}(X)$-graded free $\mathrm{Cox}(X)$-modules is a virtual resolution of M if and only if the following two conditions are satisfied:

1. The sheaf $\widetilde{H_0(F_\bullet)}$ on X is isomorphic to the sheaf $\widetilde{M}$.
2. The homology module $H_i(F_\bullet)$ is supported only on B for all $i > 0$.

For the remainder of this chapter, we will focus on the case when X is a product of projective spaces, since [BES17] demonstrates several ways to construct virtual resolutions in this case. We let $\mathbf{n} = (n_1, n_2, ..., n_r) \in (\mathbb{Z}_{\geq 0})^r$ and $\mathbb{P}^{\mathbf{n}} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$. Further, we write S for $\mathrm{Cox}(\mathbb{P}^{\mathbf{n}})$, which is graded by $\mathrm{Pic}(\mathbb{P}^{\mathbf{n}}) \cong \mathbb{Z}^r$.

Given this setup, Berkesch, Erman, and Smith show that it is possible to construct a virtual resolution of a $\mathbb{Z}^r$-graded S-module M from the graded minimal free resolution of M and an element of the multigraded Castelnuovo-Mumford regularity $\mathbf{d} \in \mathbb{Z}^r$ as introduced in [MS04]. They call a virtual resolution constructed in this manner the **virtual resolution of the pair** $(M, \mathbf{d})$.

Theorem 5.2.2 ([BES17, Theorem 1.3])**.** *Let* M *be a finitely-generated* $\mathbb{Z}^r$*-graded* B*-saturated* S*-module that is* $\mathbf{d}$*-regular. If* G *is the free subcomplex of a minimal free resolution of* M *consisting of all summands generated in degrees at most* $\mathbf{d} + \mathbf{n}$*, then* G *is a virtual resolution of* M*.*

Notice that the virtual resolution of the pair $(M, \mathbf{d})$ is a subcomplex of the minimal free resolution, so its length is at most the projective dimension of M. Also note that we

are justified in saying "the" virtual resolution of the pair $(M, \mathbf{d})$ because this complex is unique up to isomorphism.

Example 5.2.3. Consider again three points $([1:1],[1:4])$, $([1:2],[1:5])$, and $([1:3],[1:6])$ in $\mathbb{P}^1 \times \mathbb{P}^1$ from Example 2.4.3. Using *Macaulay2*, we may compute the B-saturated ideal defining these three points.

```
i1 : needsPackage "VirtualResolutions";
i2 : X = toricProjectiveSpace(1) ** toricProjectiveSpace(1);
i3 : S = ring X; B = ideal X;
o4 : Ideal of S
i5 : J = saturate(intersect(
         ideal(x_1 - x_0, x_3 - 4*x_2),
         ideal(x_1 - 2*x_0, x_3 - 5*x_2),
        ideal(x_1 - 3*x_0, x_3 - 6*x_2)), B);
o5 : Ideal of S
```

One can show the $\mathbb{Z}^2$-graded minimal free resolution of S/J is the following complex.

$$S \longleftarrow \begin{matrix} S(0,-3) \\ \oplus \\ S(-1,-2) \\ \oplus \\ S(-2,-1) \\ \oplus \\ S(-3,0) \\ \oplus \\ S(-1,-1) \end{matrix} \longleftarrow \begin{matrix} S(-1,-3)^2 \\ \oplus \\ S(-2,-2)^2 \\ \oplus \\ S(-3,-1)^2 \end{matrix} \longleftarrow \begin{matrix} S(-2,-3) \\ \oplus \\ S(-3,-2) \end{matrix} \longleftarrow 0$$

We may view the multigraded Betti table for the above graded minimal free resolution using the code below.

```
i6 : minres = res J;
i7 : multigraded betti minres
        0             1               2         3
o7 = 0: 1             .               .         .
     2: .            ab               .         .
     3: . a3+a2b+ab2+b3               .         .
     4: .             . 2a3b+2a2b2+2ab3         .
     5: .             .               . a3b2+a2b3
o7 : MultigradedBettiTally
```

In order to compute a virtual resolution of S/J, we find an element of the multigraded Castelnuovo-Mumford regularity of the module [MS04, Definition 4.1]. The minimal elements in the multigraded regularity of S/J can be found by making use of the `multigradedRegularity` command.

```
i8 : multigradedRegularity(X, S^1/J)
o8 = {{0, 2}, {1, 1}, {2, 0}}
o8 : List
```

This computation relies on the fact that a B-saturated S-module M over a product of projective spaces $\mathbb{P}^{\mathbf{n}}$ is $\mathbf{d}$-regular provided that the following conditions are satisfied:

1. the Hilbert function $H(M, \mathbf{d})$ agrees with the Hilbert polynomial $P_M(\mathbf{d})$;
2. $H^i\left(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{a})\right) = 0$ for all $i > 0$ and twists $\mathbf{a}$ such that $d_j - i \leq a_j$.

In particular, the function `multigradedRegularity` utilizes the *Macaulay2* package `TateOnProducts` [EES19] to compute the sheaf cohomology of twists of $\widetilde{M}$, which allows one to find elements of the multigraded regularity of the sheaf [EES15, Proposition 3.11].

The function `virtualOfPair` implements Theorem 5.2.2 to compute the virtual resolution of a pair. We call the function below in order to compute the virtual resolution of the pair $(S/J, (2, 0))$. Note that since we must remove all twists generated in degrees not less than or equal to $\mathbf{n} + \mathbf{d}$ from the graded minimal free resolution, we input the element $(3, 1) = (1, 1) + (2, 0)$ in `virtualOfPair`.

```
i9 : vres = virtualOfPair(J, {{3, 1}})

      1      3      2
o9 = S  <-- S  <-- S  <-- 0

     0      1      2      3

i10 : multigraded betti vres

          0      1    2
o10 = 0: 1      .    .
      2: .     ab    .
      3: . a3+a2b    .
      4: .      . 2a3b
o10 : MultigradedBettiTally
```

The above virtual resolution of the pair $(S/J, (2, 0))$ is shorter and thinner than the graded minimal free resolution of S/J.

By default, `virtualOfPair` computes a virtual resolution by using the induced Schreyer orders to iteratively compute syzygies in the desired degrees. When a minimal free resolution is already known or provided as input, `virtualOfPair` takes the appropriate subcomplex of that resolution.

We may check that `vres` is indeed a virtual resolution by using the `isVirtual` function.

```
i11 : isVirtual(B, vres)
o11 = true
```

By default, `isVirtual` checks whether the homology of the given chain complex is supported only on the irrelevant ideal. More specifically, `isVirtual` checks whether the annihilator of the homology of the given chain complex saturates to the entire ring.

Remark 5.2.4. Computing the saturation of an ideal is a critical but computationally costly step in many aspects of studying virtual resolutions. As such, our package utilizes new saturation methods which are under development by Michael Stillman for future release in *Macaulay2*. These methods are currently stored in the auxiliary file `Colon.m2`, but will be removed once they have been released separately.

As an alternative approach, we also implement the method presented in Theorem 1.3 of [Lop19] for checking whether a complex is virtual. This is done by setting the option `Strategy => Determinantal`. In this case, two conditions are checked: one involving the ranks of the maps in the chain complex, and the other involving the depths of the B-saturated ideals of minors of the maps. Typically the default strategy is faster than the determinantal strategy, as the determinantal strategy must compute the ideal of minors for each map in the chain complex, which is generally computationally expensive. ◇

Another way to generate virtual resolutions is by using the following theorem of [BES17], which provides a method for producing virtual resolutions of ideals defining zero-dimensional subschemes of $\mathbb{P}^{\mathbf{n}}$.

Theorem 5.2.5. *[BES17, Theorem 4.1] If $Z \subset \mathbb{P}^{\mathbf{n}}$ is a zero-dimensional scheme and I is the corresponding B-saturated S-ideal, then there exists $\mathbf{a} \in \mathbb{N}^r$ with $a_r = 0$ such that the minimal free resolution of $S/(I \cap B^{\mathbf{a}})$ has length equal to $|\mathbf{n}| = \dim \mathbb{P}^{\mathbf{n}}$. Moreover, any $\mathbf{a} \in \mathbb{N}^r$ with $a_r = 0$ and other entries sufficiently positive yields such a virtual resolution of S/I.*

The function `resolveViaFatPoint` computes $B^{\mathbf{a}} = \bigcap_{i=1}^{r} (x_{i,0}, x_{i,1}, \ldots, x_{i,n_i})^{a_i}$, intersects it with the input ideal J, and computes the minimal free resolution of $S/(J \cap B^{\mathbf{a}})$.

Example 5.2.6. Again consider the ideal of three points in $\mathbb{P}^1 \times \mathbb{P}^1$ as in Example 5.2.3. We use the function `resolveViaFatPoint` to compute a virtual resolution of J.

```
i12 : C = resolveViaFatPoint(J, B, {2, 0})

       1      4      3
o12 = S  <-- S  <-- S  <-- 0

      0      1      2      3

o12 : ChainComplex
i13 : isVirtual(B, C)
o13 = true
```

Note that the virtual resolution for J produced by `resolveViaFatPoint` and the one produced by `virtualOfPair` are different. In fact, virtual resolutions obtained by

`resolveViaFatPoint` may not necessarily be obtained using `virtualOfPair`. Thus, `resolveViaFatPoint` is a useful method for producing new and interesting examples of virtual resolutions. ◇

5.3 Constructing Curves in $\mathbb{P}^1 \times \mathbb{P}^2$

One source of potentially interesting virtual resolutions comes from studying curves in $\mathbb{P}^1 \times \mathbb{P}^2$. With this in mind, the `VirtualResolutions` package contains functions for constructing a limited class of curves in $\mathbb{P}^1 \times \mathbb{P}^2$, which we believe may be of interest to other researchers. In particular, there are functions for generating monomial curves, rational curves, and curves arising from curves in $\mathbb{P}^3$.

The main function in this direction is `curveFromP3toP1P2`. Given the defining ideal of a curve in $\mathbb{P}^3$, this function returns the defining ideal of a curve in $\mathbb{P}^1 \times \mathbb{P}^2$ constructed in the following way: given projections $\pi_1 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ and $\pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, there is an induced rational map $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^2$, and this is the map under which we are computing the image of our curve C.

The projections that `curveFromP3toP1P2` uses are fixed. In particular, it uses the coordinate projections

$$\pi_1([z_0 : z_1 : z_2 : z_3]) = [z_0 : z_1] \quad \text{and} \quad \pi_2([z_0 : z_1 : z_2 : z_3]) = [z_1 : z_2 : z_3].$$

As one might wish to preserve the degree of the curve $C \subset \mathbb{P}^1 \times \mathbb{P}^2$, the function `curveFromP3toP1P2` has an option called `PreserveDegree`. When this option is set to `true`, the function will return an error if the given curve intersects the base locus of these projections.

Example 5.3.1. The code below uses `curveFromP3toP1P2` to construct a curve in $\mathbb{P}^1 \times \mathbb{P}^2$ from the defining ideal of the twisted cubic $C \subset \mathbb{P}^3$.

```
i14 : R = ZZ/101[z_0, z_1, z_2, z_3];
i15 : I = ideal(z_0*z_2-z_1^2, z_1*z_3-z_2^2, z_0*z_3-z_1*z_2);
o15 : Ideal of R
```

```
i16 : J = curveFromP3toP1P2 I
              2
o16 = ideal (x    - x   x   , - x   x    + x   x   , - x   x    + x   x   )
              1,1    1,0 1,2     0,1 1,1    0,0 1,2     0,1 1,0    0,0 1,1

               ZZ
o16 : Ideal of ---[x   , x   , x   , x   , x   ]
               101  0,0   0,1   1,0   1,1   1,2
```

We can check that the ideal J defines a curve in $\mathbb{P}^1 \times \mathbb{P}^2$ by computing its dimension. Note that since J is an ideal in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2$, we expect J to be three dimensional, since in general the dimension of a subscheme defined by a $\mathrm{Pic}(X)$-graded ideal $J \subset \mathrm{Cox}(X)$ is given by $\dim J - \operatorname{rank} \mathrm{Pic}(X)$.

```
i17 : dim J
o17 = 3
```

◇

The function `randomCurveP1P2` constructs a random curve in $\mathbb{P}^1 \times \mathbb{P}^2$ by calling the `curve` function from the package `SpaceCurves` [Zha18] to randomly generate a curve $C \subset \mathbb{P}^3$ of specified degree and genus; see [Zha18, Section 2] for a detailed discussion of the inner workings of `curve`. The function `randomCurveP1P2` then constructs a curve $C' \subset \mathbb{P}^1 \times \mathbb{P}^2$ from C using `curveFromP3toP1P2`.

Example 5.3.2. Using the `randomCurveP1P2` function we produce a random curve in $\mathbb{P}^1 \times \mathbb{P}^2$ coming from a curve of degree 7 and genus 3 in $\mathbb{P}^3$. Additionally, we check that the resulting ideal defines a curve.

```
i18 : I = randomCurveP1P2(7, 3);
               ZZ
o18 : Ideal of ---[x   , x   , x   , x   , x   ]
               101  0,0   0,1   1,0   1,1   1,2
i19 : S = ring I;
i20 : dim I
o20 = 3
```

Using the `multigradedRegularity` function, one can see that $(2, 3)$ is a minimal element in the multigraded regularity of this curve. Note that we first saturate the ideal defining our curve; this is necessary to ensure that the output of `multigradedRegularity` is correct.

```
i21 : B = intersect(ideal(x_(0,0), x_(0,1)),
                    ideal(x_(1,0), x_(1,1), x_(1,2)));
o21 : Ideal of S
i22 : J = saturate(I, B);
o22 : Ideal of S
i23 : multigradedRegularity(S, S^1/J)
o23 = {{1, 4}, {2, 3}}
o23 : List
```

Finally, we can use this element of the multigraded regularity to compute a virtual resolution of S/J.

```
i24 : minres = res J;
i25 : vres = virtualOfPair(J, {{3, 5}})

       1      11      18      8
o25 = S  <-- S   <-- S   <-- S  <-- 0

      0      1       2       3      4

o25 : ChainComplex
```

Comparing the multigraded Betti tables of these two resolutions we see that once again the virtual resolution is shorter and less complicated than the graded minimal free resolution.

```
i26 : multigraded betti minres
         0              1                       2                3          4
o26 = 0: 1              .                       .                .          .
      5: .      a3b2+3a2b3                     .                .          .
      6: . a4b2+2a2b4+5ab5                     .                .          .
      7: .              b7 3a4b3+6a3b4+12a2b5+3ab6                .          .
      8: .               .                    2ab7 3a4b4+8a3b5+6a2b6          .
      9: .               .                       .             a2b7 a4b5+3a3b6
o26 : MultigradedBettiTally
```

```
i27 : multigraded betti vres
         0          1            2     3
o27 = 0: 1          .            .     .
      5: . a3b2+3a2b3            .     .
      6: . 2a2b4+5ab5            .     .
      7: .          . 6a3b4+12a2b5     .
      8: .          .            . 8a3b5
o27 : MultigradedBettiTally                                              ◇
```

Remark 5.3.3. Since the curve generated by `randomCurveP1P2` is, in some sense, random, one will often get different virtual resolutions when running the above example. As an example of this phenomena, working over a slightly larger finite field than in Example 5.3.1, we compute the multigraded regularity for 500 curves arising from curves of genus 3 and degree 7 in $\mathbb{P}^3$. The resulting distribution of multigraded regularities is shown below.

```
i28 : tally apply(500,i->(
          I := randomCurveP1P2(7, 3, ZZ/32003);
          S := ring I;
          B := intersect(ideal(x_(0,0), x_(0,1)),
          ideal(x_(1,0), x_(1,1), x_(1,2)));
          J := saturate(I, B);
          multigradedRegularity(S, S^1/J)
       ))
o28 = Tally{{{1, 4}, {2, 3}} => 167}
             {{1, 5}} => 170
             {{1, 6}} => 163
o28 : Tally
```

Remark 5.3.4. It would be interesting to have an understanding of how the geometry of the curves in $\mathbb{P}^3$ affects multigraded regularity of the resulting curves in $\mathbb{P}^1 \times \mathbb{P}^2$. For example, in the computation in Remark 5.3.3 every smooth curve of genus 3 and degree 7 lies on either a smooth cubic surface or a rational quartic surface with a double line. In fact, using the notation from [Zha18] each curve of genus 3 and degree 7 lies in one of three possible divisor classes: $(4,1,1,1,1,1,0)$, $(5,3,1,1,1,1,1)$, and $(6,1,2,2,2,2,2,2,1)$, the first two being divisors on a smooth cubic surface and the last being a divisor on a rational quartic surface with a double line. An experiment

similar to the one in Remark 5.3.3 suggests the following relationship between divisors and multigraded regularity:

1. The minimal elements in the multigraded regularity of any curve in $\mathbb{P}^1 \times \mathbb{P}^2$ arising from a smooth curve in $\mathbb{P}^3$ lying on smooth cubic surface in the divisor class $(4, 1, 1, 1, 1, 1, 0)$ are $\{(1, 6)\}$.

2. The minimal elements in the multigraded regularity of any curve in $\mathbb{P}^1 \times \mathbb{P}^2$ arising from a smooth curve in $\mathbb{P}^3$ lying on smooth cubic surface in the divisor class $(5, 3, 1, 1, 1, 1, 1)$ are $\{(1, 5)\}$.

3. The minimal elements in the multigraded regularity of any curve in $\mathbb{P}^1 \times \mathbb{P}^2$ arising from a smooth curve in $\mathbb{P}^3$ lying on a rational quartic surfaces with a double line in the divisor class $(6, 1, 2, 2, 2, 2, 2, 2, 2, 1)$ are $\{(1, 4), (2, 3)\}$.

How this generalizes to all curves remains unclear, but we hope these functions will stimulate others to consider this and other similar problems.

The functions `randomRationalCurve` and `randomMonomialCurve` are alternative methods for generating curves in $\mathbb{P}^1 \times \mathbb{P}^2$. Given two positive integers d and e, these functions construct the defining ideal of the curve arising as the image of the map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \text{ given by } [t_0 : t_1] \longmapsto ([f_0 : f_1], [g_0 : g_1 : g_2])$$

where the f_i and g_i are forms in $\Bbbk[t_0, t_1]$ of degrees d and e, respectively. For the command `randomMonomialCurve`, these forms are chosen to be monomials, while for `randomRationalCurve` they can be any possible forms of the correct degree.

Example 5.3.5. Here we use the `randomRationalCurve` function to construct a rational curve of bidegree $(5, 7)$. Once again we verify that it is, in fact, a curve by computing its dimension.

```
i29 : I = randomRationalCurve(5, 7);
                ZZ
o29 : Ideal of ---[x , x , y , y , y ]
               101  0   1   0   1   2
i30 : dim I
o30 = 3
```

◇

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Appendix A

Virtual Resolutions M2 Package

Throughout this thesis, the computer algebra software *Macaulay2* has been instrumental in constructing examples and testing conjectures. The author, along with collaborators has developed the following software package that is available in current installations of *Macaulay2*. The code for this package can be found here.

```
-----------------------------------------------------------------------------
-- PURPOSE : Construct, display, and study virtual resolutions for
↪  products
--           of projective spaces.
--
-- PROGRAMMERS : Ayah Almousa, Christine Berkesch, Juliette Bruce,
-- David Eisenbud, Daniel Erman, Michael Loper, Mahrud Sayrafi,
-- Greg Smith.
--
-- UPDATE HISTORY : created 14 April 2018 at M2@UW;
--                  updated 15 April 2019 at IMA Coding Sprint.
-----------------------------------------------------------------------------
newPackage ("VirtualResolutions",
    Version => "1.2",
    Date => "February 2, 2020",
```

```
    Headline => "Methods for virtual resolutions on products of
    ↪  projective spaces",
    Authors =>{
        {Name => "Ayah Almousa",       Email => "aka66@cornell.edu",
        ↪  HomePage => "http://pi.math.cornell.edu/~aalmousa "},
        {Name => "Christine Berkesch", Email => "cberkesc@umn.edu",
        ↪  HomePage => "http://math.umn.edu/~cberkesc/"},
        {Name => "Juliette Bruce",     Email => "jebruce2@wisc.edu",
        ↪  HomePage => "https://juliettebruce.github.io"},
        {Name => "David Eisenbud",     Email => "de@msri.org",
        ↪  HomePage => "http://www.msri.org/~de/"},
        {Name => "Michael Loper",      Email => "loper012@umn.edu",
        ↪  HomePage => "http://math.umn.edu/~loper012/"},
        {Name => "Mahrud Sayrafi",     Email => "mahrud@umn.edu",
        ↪  HomePage => "http://math.umn.edu/~mahrud/"}
        },
    PackageExports => {
        "SpaceCurves",
        "TateOnProducts",
        "NormalToricVarieties",
        "Elimination",
        "Depth"
        },
    AuxiliaryFiles => true,
    DebuggingMode => false
    )

export{
    "curveFromP3toP1P2",
    "idealSheafGens",
    "isVirtual",
    "virtualOfPair",
```

```
    "resolveViaFatPoint",
    "randomRationalCurve",
    "randomMonomialCurve",
    "randomCurveP1P2",
    "multigradedRegularity",
    -- Options
    "Attempt",
    "PreserveDegree",
    "GeneralElements"
    }

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- CODE
-----------------------------------------------------------------------
-----------------------------------------------------------------------

debug Core;

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (I,irr) = (ideal, ideal)
----- Output: saturation of I with respect to irr.
----- Description: This is the fast saturation from Colon.m2. Since
----- Colon.m2 might change at some point we have created this wrap
----- function to easily implement other saturations that might be
----- created. We hope this is eventually removed.
-----------------------------------------------------------------------
-----------------------------------------------------------------------
load("./VirtualResolutions/Colon.m2")
ourSaturation = (I,irr) -> saturationByElimination(I, decompose irr);
```

```
-----------------------------------------------------------------------
-----------------------------------------------------------------------
--Input: F a free chain complex on Cox(X), alphas a list of degrees
--Output: A subcomplex of summands generated only in degrees in the list
↪  alphas.
--Given a ring and its free resolution, keeps only the summands in
↪  resolution of specified degrees
--If the list alphas contains only one element, the output will be
↪  summands generated in degree less than or equal to alpha.
--See Algorithm 3.4 of [BES]
-----------------------------------------------------------------------
-----------------------------------------------------------------------
virtualOfPair = method(Options => {LengthLimit => infinity})
virtualOfPair (Ideal,  List) := Boolean => opts -> (I, alphas) ->
↪  virtualOfPair((ring I)^1/I, alphas, opts) -- TODO: add opts
↪  everywhere
virtualOfPair (Module, List) := Boolean => opts -> (M, alphas) -> (
    R := ring M;
    if M.cache.?resolution then return virtualOfPair(M.cache.resolution,
    ↪  alphas, opts);
    if any(alphas, alpha -> #alpha =!= degreeLength ring M) then error
    ↪  "degree has wrong length";
    m := schreyerOrder gens gb presentation M;
    apply(alphas, alpha -> m = submatrixByDegrees(m, (,alpha),
    ↪  (,alpha)));
    i := 2;
    L := {m} | while m != 0 and i <= opts.LengthLimit list (
        i = i + 1;
        m = map(R, rawKernelOfGB raw m); apply(alphas, alpha -> m =
        ↪  submatrixByDegrees(m, (,alpha), (,alpha))); m);
    chainComplex L
```

```
    )
virtualOfPair (ChainComplex, List) := Boolean => opts -> (F, alphas) ->
↪  (
    if any(alphas, alpha -> #alpha =!= degreeLength ring F) then error
    ↪  "degree has wrong length";
    L := apply(length F, i -> (
            m := F.dd_(i+1); apply(alphas, alpha -> m =
            ↪  submatrixByDegrees(m, (,alpha), (,alpha))); m));
    chainComplex L
    );


-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (J,irr,A)=(Ideal,Ideal,List) where J defines a 0-dim
----- subscheme and irr is the irrelevant ideal
----- Output: A virtual resolution of S/J, which is potentially short.
----- Description: This function implements Theorem 4.1 of [BES].
----- In particular, it computes a virutal resolution of S/J by
----- computing a graded minimal free resolution of S/(J\cap B^A).
----- By the theorem this might be a short virtual resolution
-----------------------------------------------------------------------
-----------------------------------------------------------------------
resolveViaFatPoint = method()
resolveViaFatPoint(Ideal, Ideal, List) := ChainComplex => (J, irr, A) ->
↪  (
    L := decompose irr;
    if #A != #L then error "intersectionRes: expected exponent vector of
    ↪  the right length.";
    -- note: decompose doesn't necessarily return in the right order
    Q := intersect for X in L list (
        D := degree X_0;
```

```
        d := (select((0..#D-1), i -> D#i == 1))_0;
        X ^ (A#d)
        );
    res intersect (Q, J)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
-- This method checks if a given complex is a virtual resolution by
↪  computing
-- homology and checking whether its annihilator saturates to the whole
↪  ring.
-- Input: Ideal irr - the irrelevant ideal of the ring
--       Chain Complex C - proposed virtual resolution
-- Output: Boolean - true if complex is virtual resolution, false
↪  otherwise
-- Note: the Determinatal strategy is based on Theorem 1.3 of
↪  [Loper2019].
-----------------------------------------------------------------------
-----------------------------------------------------------------------
isVirtual = method(Options => {Strategy => null})
isVirtual (Ideal, ChainComplex) := Boolean => opts -> (irr, C) -> (
-- if strategy "determinantal is selected, the method checks virtuality
-- via the depth criterion on the saturated ideals of minors
    if opts.Strategy === "Determinantal" then (
        for i from 1 to length(C) do (
            if rank(source(C.dd_i)) != (rank(C.dd_i) + rank(C.dd_(i+1)))
            ↪  then (
                if debugLevel >= 1 then print "isVirtual failed at
                ↪  homological degree " | toString i;
                return false;
                );
```

```
            );
        for i from 1 to length(C) do (
            minor := minors(rank(C.dd_i),C.dd_i);
            minorSat := ourSaturation(minor,irr);
            if depth(minorSat,ring(minorSat)) < i then (
                if debugLevel >= 1 then print "isVirtual failed at
                ↪  homological degree " | toString i;
                return false;
            );
        );
    true
    );
-- default strategy is calculating homology and checking homology is
-- supported on irrelevant ideal
    for i from 1 to length(C) do (
        annHHi := ann HH_i(C);
        if annHHi != ideal(sub(1,ring C)) then (
            if annHHi == 0 or ourSaturation(annHHi,irr) !=
            ↪  ideal(sub(1,ring C)) then (
                if debugLevel >= 1 then print "isVirtual failed at
                ↪  homological degree " | toString i;
                return false;
                );
            );
        );
    true
    )


isVirtual (NormalToricVariety, ChainComplex) := Boolean => opts -> (X,
↪  C) -> (
```

```
    if ring C != ring X then error "chain complex is not in Cox ring of
    ↪  normal toric variety";
    isVirtual(ideal X, C)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
-- Input: ZZ n - size of subset of generators to check
--       Ideal J - ideal of ring
--       Ideal irr - irrelevant ideal
-- Output: a list of ideals generated by subsets of size n of the
↪  generators of J
--         that give the same ideal as J up to saturation by the
↪  irrelevant ideal
-----------------------------------------------------------------------
-----------------------------------------------------------------------
idealSheafGens = method(Options => {GeneralElements => false})
idealSheafGens(ZZ, Ideal, Ideal) := List => opts -> (n, J, irr) -> (
    R := ring(J);
    k := coefficientRing(R);
    Jsat := ourSaturation(J,irr);
    if opts.GeneralElements == true then (
        degs := degrees(J);
        -- place of all unique degrees
        allmatches := unique(apply(degs, i -> positions(degs, j -> j ==
        ↪  i)));
        -- creates an ideal where if degrees of generators match
        -- those generators are replaced by one generator that
        -- is a random combination of all generators of that degree
        J = ideal(apply(allmatches, i -> sum(apply(i, j -> random(k) *
        ↪  J_j))));
        );
```

```
    lists := subsets(numgens(J), n);
    output := {};
    apply(lists, l -> (
            I := ideal(J_*_l);
            if ourSaturation(I, irr) == Jsat then (
                output = append(output, I);
                );
            )
        );
    output
    )
idealSheafGens(ZZ, Ideal, NormalToricVariety) := List => opts -> (n, J,
↪  X) -> (
    idealSheafGens(n, J, ideal X)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (d,e,F)=(degree,degree,base ring)
----- Output: The ideal of a random rational curve in P1xP2 of
----- degree (d,e) defined over F.
----- Description: This randomly generates 2 forms of degree
----- d and 3 forms of degree e in the ring S (locally defined),
----- and computes the ideal defining the image of the map of the
----- associated map P^1---->P^1xP^2.
-----------------------------------------------------------------------
-----------------------------------------------------------------------
randomRationalCurve = method()
randomRationalCurve (ZZ,ZZ,Ring) := (d,e,F)->(
    -- Defines P1
    s := getSymbol "s";
    t := getSymbol "t";
```

```
    R := F(monoid[s,t]);
    --- Defines P1xP2
    x := getSymbol "x";
    y := getSymbol "y";
    S1 := F(monoid[x_0, x_1]);
    S2 := F(monoid[y_0,y_1,y_2]);
    S := tensor(S1,S2);
    --- Defines P1x(P1xP2)
    U := tensor(R,S);
    uVars := flatten entries vars U;
    --- Defines graph of morphisms in P1x(P1xP2)
    --M1 := matrix {apply(2,i->random({d,0,0},U)),{x_0,x_1}};
    M1 := matrix {apply(2,i->random({d,0,0},U)),{uVars#2,uVars#3}};
    --M2 := matrix {apply(3,i->random({e,0,0},U)),{y_0,y_1,y_2}};
    M2 := matrix
    ↪  {apply(3,i->random({e,0,0},U)),{uVars#4,uVars#5,uVars#6}};
    J := minors(2,M1)+minors(2,M2);
    --- Computes saturation and then eliminates producing curve in P1xP2
    J' := ourSaturation(J,ideal(uVars#0,uVars#1));
    --J' := saturate(J,ideal(uVars#0,uVars#1),MinimalGenerators=>false);
    I := sub(eliminate({uVars#0,uVars#1},J'),S);
    (T, E) := productOfProjectiveSpaces({1, 2},CoefficientField=>F);
    G := map(T,S,(flatten entries vars T));
    G(I)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (d,e)=(degree,degree)
----- Output: The ideal of a random rational curve in P1xP2 of
----- degree (d,e) defined over ZZ/101
-----------------------------------------------------------------------
```

```
-----------------------------------------------------------------------
randomRationalCurve (ZZ,ZZ) := (d,e)->(
    randomRationalCurve(d,e,ZZ/101)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (d,e,F)=(degree,degree,base ring)
----- Output: The ideal of a random rational curve in P1xP2 of degree
↪  (d,e).
----- Description: This randomly generates 2 monomials of degree
----- d and 3 monomials of degree e in the ring S (locally defined),
----- and computes the ideal defining the image of the map of the
----- associated map P^1---->P^1xP^2.
-----------------------------------------------------------------------
-----------------------------------------------------------------------
randomMonomialCurve = method()
randomMonomialCurve (ZZ,ZZ,Ring) := (d,e,F)->(
    --- Defines P1
    s := getSymbol "s";
    t := getSymbol "t";
    R := F[s,t];
    --- Defines P1xP2
    x := getSymbol "x";
    y := getSymbol "y";
    S1 := F(monoid[x_0, x_1]);
    S2 := F(monoid[y_0,y_1,y_2]);
    S := tensor(S1,S2);
    --- Defines P1x(P1xP2)
    U := tensor(R,S);
    uVars := flatten entries vars U;
    --- Choose random monomial to define map to P2.
```

```
    B := drop(drop(flatten entries basis({e,0,0},U),1),-1);
    f := (random(B))#0;
    --- Defines graph of morphisms in P1x(P1xP2)
    M1 := matrix {{(uVars#0)^d,(uVars#1)^d},{uVars#2,uVars#3}};
    M2 := matrix
    ↪  {{(uVars#0)^e,(uVars#1)^e,f},{uVars#4,uVars#5,uVars#6}};
    J := minors(2,M1)+minors(2,M2);
    --- Computes saturation and then eliminates producing curve in P1xP2
    J' := saturate(J,ideal(uVars#0,uVars#1),MinimalGenerators=>false);
    I := sub(eliminate({uVars#0,uVars#1},J'),S);
    (T, E) := productOfProjectiveSpaces({1, 2},CoefficientField=>F);
    G := map(T,S,(flatten entries vars T));
    G(I)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (d,e)=(degree,degree)
----- Output: The ideal of a random rational curve in P1xP2 of
----- of degree (d,e) defined over ZZ/101.
-----------------------------------------------------------------------
-----------------------------------------------------------------------
randomMonomialCurve (ZZ,ZZ) := (d,e)->(
    randomMonomialCurve(d,e,ZZ/101)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (J)=(ideal of curve in P3)
----- Output: The ideal of a corresponding curve in P1xP2.
----- Description: Given a curve defined by the ideal J in P3,
----- this outputs the ideal I of the curve in P1xP2 given by
```

```
----- considering the projection P3---->P1 on the first two variables,
----- and the projection P3----->P2 on the last three variables.
-----------------------------------------------------------------------
-----------------------------------------------------------------------
curveFromP3toP1P2 = method(Options => {PreserveDegree => true})
curveFromP3toP1P2 (Ideal) := opts -> (J) ->(
    --- Defines P3
    w := getSymbol "w";
    R := (coefficientRing ring J) monoid([w_0,w_1,w_2,w_3]);
    rVars := flatten entries vars R;
    J = sub(J,matrix{{R_0,R_1,R_2,R_3}});
    --- Base locus of projection
    BL1 := ideal(rVars#0,rVars#1);
    BL2 := ideal(rVars#1,rVars#2,rVars#3);
    BL := intersect(BL1,BL2);
    --- If PreserveDegree => true checks whether curve intersects base
    ↪  locus;
    --- this ensures the curve has the correct degree and genus.
    if opts.PreserveDegree == true then (
        if (saturate((J+BL1))==ideal(rVars)) or
        ↪  (saturate((J+BL2))==ideal(rVars)) then error "Given curve
        ↪  intersects places of projection.";
        );
    --- Defines P1xP2
    x := getSymbol "x";
    y := getSymbol "y";
    S1 := (coefficientRing ring J) monoid([x_0, x_1]);
    S2 := (coefficientRing ring J) monoid([y_0,y_1,y_2]);
    S := tensor(S1,S2);
    --- Defines P3x(P1xP2)
    U := tensor(R,S);
    uVars := flatten entries vars U;
```

```
    --- Place curve in P3x(P1xP2)
    C' := sub(J,U);
    --- Defines graph of projection
    M1 := matrix {{uVars#0,uVars#1},{uVars#4,uVars#5}};
    M2 := matrix {{uVars#1,uVars#2,uVars#3},{uVars#6,uVars#7,uVars#8}};
    D := minors(2,M1)+minors(2,M2);
    --- Intersects irrelevant ideal with base locus
    B1 := ideal(apply(4,i->uVars#i));
    B2 := ideal(apply(2,i->uVars#(4+i)));
    B3 := ideal(apply(3,i->uVars#(6+i)));
    B := intersect(B1,B2,B3,sub(BL,U));
    --- Computes saturation and then eliminates producing curve in P1xP2
    K := ourSaturation(C'+D,B);
    I := sub(eliminate({uVars#0,uVars#1,uVars#2,uVars#3},K),S);
    (T, E) := productOfProjectiveSpaces({1,
    ↪  2},CoefficientField=>(coefficientRing ring J));
    G := map(T,S,(flatten entries vars T));
    G(I)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (d,e,F)=(degree,genus,base ring)
----- Output: The ideal of a random curve in P1xP2 defined over F.
----- Description: This randomly generates a curve of degree d
----- and genus g in P3, and then computes the ideal of the
↪  corresponding
----- curve in P1xP2 given by considering the projection
----- P3---->P1 on the first two variables.
----- and the projection P3----->P2 on the last three variables.
-----------------------------------------------------------------------
-----------------------------------------------------------------------
```

```
randomCurveP1P2 = method(Options => {Attempt => 1000})
randomCurveP1P2 (ZZ,ZZ,Ring) := opts -> (d,g,F)->(
    --- Defines P3
    z := getSymbol "z";
    R := F(monoid[z_0,z_1,z_2,z_3]);
    rVars := flatten entries vars R;
    --- Base locus of porjection
    BL1 := ideal(rVars#0,rVars#1);
    BL2 := ideal(rVars#1,rVars#2,rVars#3);
    BL := intersect(BL1,BL2);
    --- Randomly generates curve in P3 until finds one not intersecting
    --- base locus of projection or until Bound is reached.
    C := ideal(0);
    apply(opts.Attempt,i->(
            C = curve(d,g,R);
            if class(C) === Curve then C = ideal(C);
            if (saturate(C+BL1)!=ideal(rVars)) and
            ↪  (saturate(C+BL2)!=ideal(rVars)) then break C;
            )
        );
    --- Checks whether curve in P3 intersects base locus of projection;
    --- this ensures the curve has the correct degree and genus.
    if (saturate(C+BL1)==ideal(rVars)) or
    ↪  (saturate(C+BL2)==ideal(rVars)) then error "Unable to find curve
    ↪  not intersecting places of projection.";
    --- Defines P1xP2
    curveFromP3toP1P2(C)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (d,e)=(degree,genus)
```

```
----- Output: The ideal of a random curve in P1xP2 over ZZ/101
-----------------------------------------------------------------------
-----------------------------------------------------------------------
randomCurveP1P2 (ZZ,ZZ) := randomCurveP1P2 => opts -> (d,g)->(
    randomCurveP1P2(d,g,ZZ/101)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: S = Cox ring of a product of projective spaces.
-- OR
----- Input: X = normalToricVariety of a product of projective spaces.
----- Output: The dimension vector for the product of projective spaces.
----- Note the dimension is ordered assuming the degree {1,0,...} is
↪  first.
-----------------------------------------------------------------------
-----------------------------------------------------------------------
dimVector = method()
dimVector(Ring) := (S) -> (
    deg := degrees S;
    degTally := tally deg;
    apply(rsort unique deg, i->(degTally_i - 1))
    )

dimVector(Thing) := (X) -> (
    S := ring X;
    deg := degrees S;
    degTally := tally deg;
    apply(rsort unique deg, i->(degTally_i - 1))
    )
```

```
-- Helper function for multigradedRegularity
-- borrowed from LinearTruncations:
multigradedPolynomialRing = n -> (
    x := local x;
    xx := flatten apply(#n, i -> apply(n_i+1, j -> x_(i,j)));
    degs := flatten apply(#n, i -> apply(n_i+1, k ->
            apply(#n, j -> if i == j then 1 else 0)));
    ZZ/32003[xx, Degrees=>degs]
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (S,M) = (Ring, Module)
-- OR
----- Input: (X,M) = (NormalToricVariety,Module)
----- Output: A list consisting of the minimal elements of the
----- multigraded regularity of M.
----- Description: This computes the multigraded regularity of a
----- module as defined in Definition 1.1 of [Maclagan, Smith 2004].
----- It returns a list of the minimal elements.
----- Caveat: This assumes M is B-saturated already i.e. H^0_B(M)=0
-----------------------------------------------------------------------
-----------------------------------------------------------------------
multigradedRegularity = method()
multigradedRegularity(Thing,              Ideal)  := List => (T, I)  ->
↪  multigradedRegularity(T, comodule I)
multigradedRegularity(Ring,               Module) := List => (S, M') ->
↪  multigradedRegularity(null, S, M')
multigradedRegularity(NormalToricVariety, Module) := List => (X, M)  ->
↪  multigradedRegularity(X, null, M)
-- Note: some hacking is involved to deal with the differences between
↪  productOfProjectiveSpaces and toricProjectiveSpaces
```

```
multigradedRegularity(Thing, Thing, Module) := List => (X, S, M) -> (
    if class X === NormalToricVariety then (
        -- go from module over NormalToricVariety to module over
        ↪  productOfProjectiveSpaces
        -- assuming that the NormalToricVariety is a tensor product of
        ↪  toricProjectiveSpaces
        S = ring X;
        (S', E') := productOfProjectiveSpaces(dimVector X,
        ↪  CoefficientField => coefficientRing S);
        M' := coker (map(S', S, gens S'))(presentation M);
        ) else (
        -- go from module over productOfProjectiveSpaces to module over
        ↪  tensor product of toricProjectiveSpaces
        (S', E') = productOfProjectiveSpaces(dimVector S,
        ↪  CoefficientField => coefficientRing S);
        M' = coker (map(S', S, gens S'))(presentation M);
        X = fold((A,B) -> A**B, dimVector(S)/(i->toricProjectiveSpace(i,
        ↪  CoefficientRing => coefficientRing S)));
        M = coker (map(ring X, S, gens ring X))(presentation M);
        S = ring X;
        );
    n := #(degrees S)_0;
    r := regularity M;
    H := hilbertPolynomial(X, M);
    -- We only search in the positive cone and up to the regularity of M
    L := pairs cohomologyHashTable(M', toList(n:0), toList(n:r));
    -- Based on findHashTableCorner from TateOnProducts
    P := multigradedPolynomialRing toList(n:0);
    gt := new MutableHashTable;
    apply(L, ell -> (
            -- Check that Hilbert function and Hilbert polynomial match
```

```
            -- (this imposes a condition on the alternating sum of local
            ↪  cohomology dimensions)
            if hilbertFunction(ell_0_0, M) != (map(QQ, ring H,
            ↪  ell_0_0))(H) then (
                gt#(ell_0_0) = true;
                );
            -- Check that higher local cohomology vanishes (i.e.,
            ↪  H^i_B(M) = 0 for i > 1)
            if ell_1 != 0 and ell_0_1 > 0 then (
                gt#(ell_0_0) = true;
                apply(n, j -> gt#(ell_0_0 + degree P_j) = true);
                );
            )
        );
    low := apply(n, i -> min (L / (ell -> ell_0_0_i - 1)));
    I := ideal apply(L, ell -> if not gt#?(ell_0_0) then product(n, j ->
    ↪  P_j^(ell_0_0_j - low_j)) else 0);
    sort apply(flatten entries mingens I, g -> (flatten exponents g) +
    ↪  low)
    )

-----------------------------------------------------------------------
-----------------------------------------------------------------------
----- Input: (C)=(ChainComplex)
----- Output: A resolution of the tail end of the complex appended
----- to the given complex.
----- Description: This function is not currently being exported,
----- but we hope it will eventually be useful in generating new
----- virtual resolutions. The key is we need a way, like for example
----- module primary decomposition to add irrlevence to a chain
----- complex before we apply resolveTail. (See comment.)
-----------------------------------------------------------------------
```

```
-----------------------------------------------------------------------

--TODO: Finish test
--      Add length limit
resolveTail = method()
resolveTail(ChainComplex) := C ->(
    N := max support C;
    M := coker syz C.dd_N;
    -- TODO: add some component of the irrelevant ideal to M here.
    T := res M;
    L1 := for i from min C to max support C - 1 list matrix C.dd_(i+1);
    L2 := for i from min T to max support T - 1 list matrix T.dd_(i+1);
    chainComplex(L1 | L2)
    );
```