THE SHOCK GENERATION PROBLEM FOR A DISCRETE GAS WITH SHORT RANGE REPULSIVE FORCES

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IMA Preprint Series # 780

February 1991

The Shock Generation Problem for a Discrete Gas with Short Range Repulsive Forces

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1. Introduction

In this note we shall look at the shock generation problem for a discrete system of interacting particles. Greenberg and his coauthors have examined this and similar problems for discrete particle systems with billiard ball collisions [1-3]. Here we assume that the force between the k^{th} and $(k+1)^{\text{st}}$ particle is of the form $\hat{\sigma}(N(x_{k+1}-x_k))$ where N is an integer and $\frac{1}{N}$ represents both the initial particle spacing and the mass of each particle. A typical graph of $\hat{\sigma}$ versus γ is shown in Figure 1.

Prototypes of such force laws are

$$\hat{\sigma} = \begin{cases} \left(1-\left(\frac{\lambda}{\gamma}\right)^m\right), & 0<\gamma<\lambda<1 \\ 0, & \lambda \le \gamma \end{cases} \tag{1.1}$$

and

$$\hat{\sigma} = \begin{cases} (\lambda-\gamma)^p\left(1-\left(\frac{\lambda}{\gamma}\right)^m\right), & 0<\gamma<\lambda \\ 0, & \lambda \le \gamma. \end{cases} \tag{1.2}$$

What is really important is that $\hat{\sigma}(\gamma)\equiv 0$ for $\lambda\le\gamma$, $\hat{\sigma}(\gamma)<0$ for $\gamma<\lambda$, $\lim_{\gamma\to 0^+}\hat{\sigma}(\gamma)=-\infty$, and that the potential energy

$$P(\gamma) \stackrel{\text{def}}{=} \int_\lambda^\gamma \hat{\sigma}(s)ds \tag{1.3}$$

*This research was partially supported by the US National Science Foundation and the US Department of Energy.

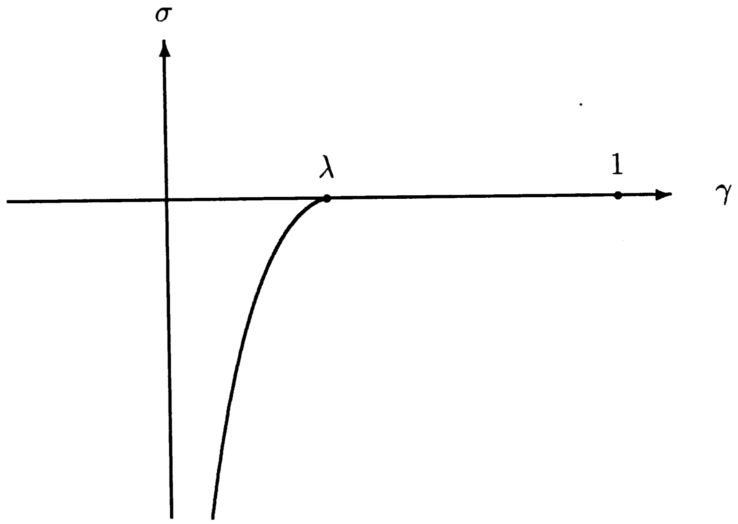

Figure 1.

satisfies $\lim_{\gamma \to 0^+} P(\gamma) = +\infty$. These guarantee for each $u_0 > 0$ there is a unique number $\gamma_1(u_0) \in (0, \lambda)$ such that

$$P(\gamma_1(u_0)) = u_0^2. \tag{1.4}$$

We let

$$\delta(u_0) = \int_{\gamma_1(u_0)}^{\lambda} \frac{ds}{\sqrt{u_0^2 - P(s)}} \tag{1.5}$$

and

$$\lambda_1(u_0) = \lambda - \delta(u_0)u_0 < \gamma_1(u_0) \tag{1.6}$$

and restrict our attention to $u_0 > 0$ small enough so that

$$\delta(u_0)u_0 < \min(\lambda, 1 - \lambda). \tag{1.7}$$

In the discrete shock generation problem we seek a solution to the the following infinite system of differential equations:

$$\dot{x}_1 = u_1 \ \text{ and } \ \frac{1}{N}\dot{u}_1 = \hat{\sigma}(N(x_2 - x_1)) - \hat{\sigma}(2Nx_1), \tag{1.8}$$

and for $k = 2, 3, \ldots$

$$\dot{x}_k = u_k \ \text{ and } \ \frac{1}{N}\dot{u}_k = \hat{\sigma}(N(x_{k+1} - x_k)) - \hat{\sigma}(N(x_k - x_{k-1})). \tag{1.9}$$

$x_k(t)$ and $u_k(t)$ represent the location and velocity of the k^{th} particle at time t. At time $t = 0$ we assume that

$$x_1(0) = \frac{\gamma_1(u_0)}{2N} \quad \text{and} \quad u_1(0) = 0, \tag{1.10}$$

and for $k = 2, 3, \ldots$ that

$$x_k(0) = \frac{(k-1)}{N} + \frac{\lambda_1(u_0)}{2N} \quad \text{and} \quad u_k(0) = -u_0 < 0. \tag{1.11}$$

The numbers $\gamma_1(u_0)$ and $\lambda_1(u_0)$ are defined in (1.4) and (1.6) and again we assume that $u_0 > 0$ is small enough so that (1.7) holds.

Variants of this problem have been considered by many authors, most notably von Neumann [4]. Von Neumann thought that solutions of (1.8) and (1.9) should approximate solutions of the nonlinear wave equation

$$\frac{\partial^2 \chi}{\partial t^2} - \frac{\partial \hat{\sigma}\left(\frac{\partial \chi}{\partial X}\right)}{\partial X} = 0; \tag{WE}$$

specifically that solutions to (1.8) and (1.9) should converge weakly to solutions of (WE). Lax and his coauthors [5-7] have amassed a body of evidence, both theoretical and for this particular problem computational, which tends to refute this conjecture; the critical issue is not the weak convergence of the fields but the rather whether the limiting fields satisfy the force law $\sigma^\infty = \hat{\sigma}\left(\frac{\partial \chi^\infty}{\partial X}\right)$ or not. Our main result, which we shall state presently, shows that for force laws of the type considered here Lax was correct; that is, behind the limiting shock $\sigma^\infty \neq \hat{\sigma}\left(\frac{\partial \chi^\infty}{\partial X}\right)$.

Our main result is a closed form solution to (1.8)-(1.11). This solution is characterized by translates of a single periodic function. More interestingly, we are able to evaluate the continuum limit of this solution. In particular, we obtain the limit motion $\chi^\infty(X, t)$ as the pointwise limit of the individual trajectories $x_k(t)$ as N tends to infinity along with $X = \frac{k}{N}$ fixed; the result is

$$\chi^\infty(X,t) = \begin{cases} X - u_0 t, & 0 < t < \frac{(1-\lambda_1(u_0))X}{2u_0} \\ \frac{(1+\lambda_1(u_0))X}{2}, & t > \frac{(1-\lambda_1(u_0))X}{2u_0}. \end{cases} \tag{1.12}$$

The strains $\gamma_{(k+1,k)} = N(x_{k+1} - x_k)$, velocities u_k, and stresses $\sigma_{(k+1,k)} = \hat{\sigma}(\gamma_{(k+1,k)})$ also converge weakly to the functions $\gamma^\infty(X,t)$, $u^\infty(X,t)$, and $\sigma^\infty(X,t)$ given below:

$$\gamma^\infty(X,t) = \chi_X^\infty(X,t) = \begin{cases} 1, & 0 < t < \frac{(1-\lambda_1(u_0))X}{2u_0} \\ \frac{(1+\lambda_1(u_0))}{2}, & t > \frac{(1-\lambda_1(u_0))X}{2u_0}, \end{cases} \tag{1.13}$$

$$u^\infty(X,t) = \chi_t^\infty(X,t) = \begin{cases} -u_0, & 0 < t < \frac{(1-\lambda_1(u_0))X}{2u_0} \\ 0, & t > \frac{(1-\lambda_1(u_0))X}{2u_0}, \end{cases} \tag{1.14}$$

and

$$\sigma^\infty(x,t) = \begin{cases} 0, & 0 \le t < \frac{(1-\lambda_1(u_0))x}{2u_0} \\ \frac{-2u_0^2}{1-\lambda_1(u_0)}, & t > \frac{(1-\lambda_1(u_0))x}{2u_0}. \end{cases} \tag{1.15}$$

These functions are weak solutions of

$$\frac{\partial \gamma^\infty}{\partial t} - \frac{\partial u^\infty}{\partial x} = 0 \text{ and } \frac{\partial u^\infty}{\partial t} - \frac{\partial \sigma^\infty}{\partial x} = 0, x > 0 \text{ and } t > 0, \tag{1.16}$$

$$(\gamma^\infty, u^\infty)(x,0) = (1, -u_0), x > 0, \tag{1.17}$$

and

$$u^\infty(0^+, t) = 0, t > 0 \tag{1.18}$$

with a single shock wave

$$x = \frac{2u_0}{1-\lambda_1(u_0)} t, \quad t > 0. \tag{1.19}$$

This validity of the Lax conjecture for this limit flow follows from examining the stress behind the shock wave $x = \frac{2u_0 t}{1-\lambda_1(u_0)}, t > 0$. The limit stress is $\sigma^\infty = \frac{-2u_0^2}{1-\lambda_1(u_0)}, t > 0$ whereas $\hat{\sigma}(\gamma^\infty) = \hat{\sigma}\left(\frac{1+\lambda_1(u_0)}{2}\right) = 0$. The last identity follows from the observation that (1.7) implies that $\lambda < \frac{1+\lambda_1(u_0)}{2}$ and the identity $\hat{\sigma} \equiv 0, \lambda \le \gamma$.

2. The Solution

In order to solve the shock generation problem (1.8)–(1.110) we introduce the functions $(\xi, u)(t)$ defined as the unique solution of

$$\dot{\xi} = u \text{ and } \frac{1}{N}\dot{u} = -\hat{\sigma}(2N\xi) + \hat{\sigma}\left(2N\left(\frac{1+\lambda_1(u_0)}{2N} - \xi\right)\right) \tag{2.1}$$

and

$$\xi(0) = \frac{\gamma_1(u_0)}{2N} \text{ and } u(0) = 0. \tag{2.2}$$

Once again $\gamma_1(u_0)$ and $\lambda_1(u_0)$ are defined in (1.4) and (1.6) and $u_0 > 0$ is small enough so that (1.7) holds. Since solutions of (2.1) satisfy

$$\frac{d}{dt}\left(u^2 + \int_\lambda^{2N\xi} \hat{\sigma}(s)ds + \int_\lambda^{(1+\lambda_1(u_0)-2N\xi)} \hat{\sigma}(s)ds\right) = 0, \tag{2.3}$$

(2.2) and (2.3) imply that

$$u^2 + \int_\lambda^{2N\xi} \hat{\sigma}(s)ds + \int_\lambda^{(1+\lambda_1(u_0)-2N\xi)} \hat{\sigma}(s)ds \equiv \int_\lambda^{\gamma_1(u_0)} \hat{\sigma}(s)ds + \int_\lambda^{(1+\lambda_1(u_0)-\gamma_1(u_0))} \hat{\sigma}(s)ds. \tag{2.4}$$

Equation (1.4) implies that

$$\int_{\lambda}^{\gamma_1(u_0)} \hat{\sigma}(s)ds = u_0^2, \tag{2.5}$$

while (1.6) and (1.7) imply that

$$(1+\lambda_1(u_0)-\gamma_1(u_0)) = (1+\lambda-\delta(u_0)u_0-\gamma_1(u_0)) > 1-\delta(u_0)u_0 > \lambda, \tag{2.6}$$

and

$$\int_{\lambda}^{(1+\lambda_1(u_0)-\gamma_1(u_0))} \hat{\sigma}(s)ds = 0. \tag{2.7}$$

Combining (2.4), (2.5), and (2.7) then gives

$$u^2 + \int_{\lambda}^{2N\xi} \hat{\sigma}(s)ds + \int_{\lambda}^{(1+\lambda_1(u_0)-2N\xi)} \hat{\sigma}(s)ds \equiv u_0^2. \tag{2.8}$$

A sketch of the energy surface is shown in Figure 2,

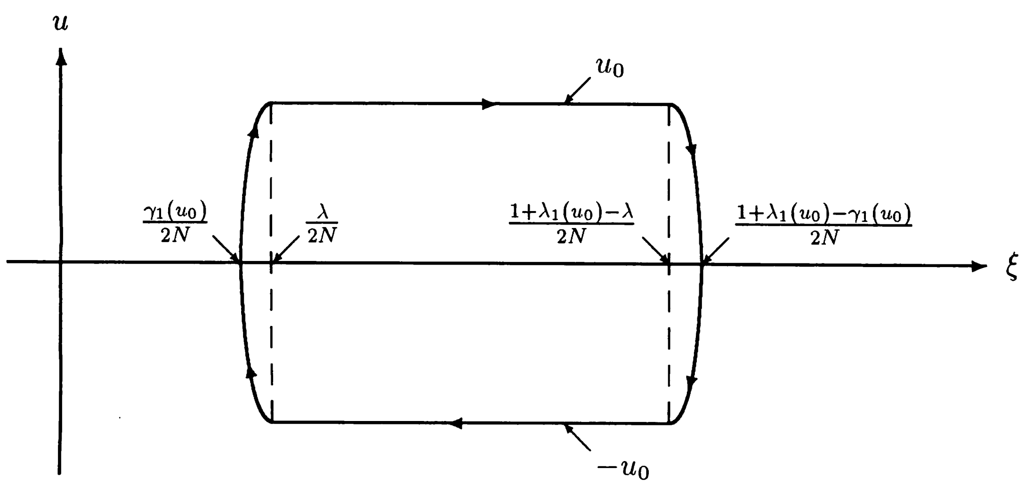

Figure 2.

and a graph of $t \mapsto \xi(t)$ is shown in Figure 3.

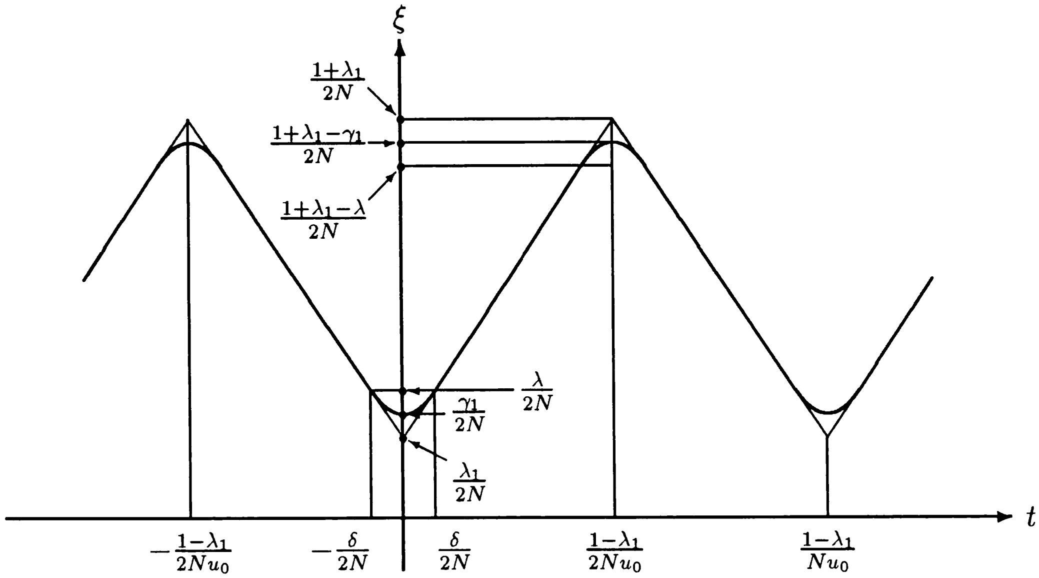

Figure 3.

It is easily checked that ξ and u are endowed with the following symmetries:

$$(\xi, u)(-t) = (\xi, -u)(t), \tag{2.9}$$

$$(\xi, u)\left(\frac{1-\lambda_1(u_0)}{2Nu_0} + t\right) = (\xi, -u)\left(\frac{1-\lambda_1(u_0)}{2Nu_0} - t\right), \tag{2.10}$$

$$(\xi, u)\left(\frac{1-\lambda_1(u_0)}{Nu_0} + t\right) = (\xi, u)(t), \tag{2.11}$$

$$\xi(t) + \xi\left(t - \frac{1-\lambda_1(u_0)}{2Nu_0}\right) = \frac{1+\lambda_1(u_0)}{2N}, \tag{2.12}$$

and

$$u(t) = -u\left(t - \frac{1-\lambda_1(u_0)}{2Nu_0}\right). \tag{2.13}$$

The reason for constraining u_0 by (1.7) is apparent from Figure 3; we want $\lambda_1(u_0) > 0$ and $\frac{1-\lambda_1(u_0)}{2Nu_0} - \frac{\delta(u_0)}{N} > 0$. The first inequality is guaranteed by $\delta(u_0)u_0 < \lambda$ and the second by $\delta(u_0)u_0 < (1-\lambda)$. It is also worth noting that $u(t) \equiv u_0$ for $t \in \left[\frac{\delta(u_0)}{2N}, \frac{1-\lambda_1(u_0)}{2N} - \frac{\delta(u_0)}{2N}\right]$.

We are now in a position to write down the solution to (1.8)–(1.11); the result is

$$x_1(t) = \xi(t) \text{ and } u_1(t) = u(t), \tag{2.14}$$

and for $k = 2, 3, \ldots$

$$\left.\begin{array}{ll} x_k(t) = & \frac{(k-1)}{N} + \frac{\lambda_1(u_0)}{2N} - u_0 t \\ u_k(t) = & -u_0 \end{array}\right\} 0 \leq t \leq \frac{(k-1)(1-\lambda_1(u_0))}{2Nu_0} - \frac{\delta(u_0)}{2N}, \tag{2.15}$$

and

$$\left.\begin{array}{ll} x_k(t) = & \frac{(k-1)(1+\lambda_1(u_0))}{2N} + \xi\left(t - \frac{(k-1)(1-\lambda_1(u_0))}{2Nu_0}\right) \\ u_k(t) = & u\left(t - \frac{(k-1)(1-\lambda_1(u_0))}{2Nu_0}\right) \end{array}\right\} \frac{(k-1)(1-\lambda_1(u_0))}{2Nu_0} - \frac{\delta(u_0)}{2N} \leq t. \tag{2.16}$$

To verify that the above functions satisfy (1.8)–(1.11) is a simple exercise provided one exploits the identities (2.9)–(2.13) and the fact that $\hat{\sigma}(\gamma) \equiv 0$ for $\lambda \leq \gamma$. For each $k = 1, 2, \ldots$ particles are free streaming for times $t \notin \left[\frac{(k-1)(1-\lambda_1(u_0))}{2Nu_0} - \frac{\delta(u_0)}{2N}, \frac{(k-1)(1-\lambda_1(u_0))}{2Nu_0} + \frac{\delta(u_0)}{2N}\right]$ and for times $t \in \left[\frac{(k-1)(1-\lambda_1(u_0))}{2Nu_0} - \frac{\delta(u_0)}{2N}, \frac{(k-1)(1-\lambda_1(u_0))}{2Nu_0} + \frac{\delta(u_0)}{2N}\right]$ when the particles interact only one of the two terms $\hat{\sigma}(N(x_{k+1} - x_k))$ and $\hat{\sigma}(N(x_k - x_{k-1}))$ is nonzero; thus all interactions are binary.

Our final task is to verify that the limit relations (1.12)–(1.19) hold. For each integer N we introduce the functions

$$\chi^N(x,t) = \begin{cases} 2Nx_1(t)x, & 0 \leq x \leq \frac{1}{2N} \\ \text{and for } k = 1, 2, \ldots & \\ x_k(t) + N(x_{k+1} - x_k)(t)\left(x - \frac{2k-1}{2N}\right), & \frac{2k-1}{2N} \leq x \leq \frac{2k+1}{2N}, \end{cases} \tag{2.17}$$

$$\gamma^N(x,t) = \chi^N_x(x,t) = \begin{cases} 2Nx_1(t), & 0 \leq x \leq \frac{1}{2N} \\ \text{and for } k = 1, 2, \ldots & \\ N(x_{k+1} - x_k)(t), & \frac{(2k-1)}{2N} \leq x \leq \frac{2k+1}{2N}, \end{cases} \tag{2.18}$$

$$u^N(x,t) = \chi^N_t(x,t) = \begin{cases} 2Nu_1(t)x, & 0 \leq x \leq \frac{1}{2N} \\ \text{and for } k = 1, 2, \ldots & \\ u_k(t) + N(u_{k+1} - u_k)(t)\left(x - \frac{(2k-1)}{2N}\right), & \frac{(2k-1)}{2N} \leq x \leq \frac{2k+1}{2N}, \end{cases} \tag{2.19}$$

and

$$\sigma^N(x,t) = \hat{\sigma}(\gamma^N(x,t)) = \begin{cases} \hat{\sigma}\left(2Nx_1(t)\right), & 0 \leq x \leq \frac{1}{2N} \\ \text{and for } k = 1, 2, \ldots & \\ \hat{\sigma}\left(N(x_{k+1} - x_k)(t)\right), & \frac{(2k-1)}{2N} \leq x \leq \frac{2k+1}{2N}. \end{cases} \tag{2.20}$$

The numbers $x_k(t)$ and $u_k(t)$ are the particle trajectories and velocity defined in (2.14)–(2.16).

We first observe that $\gamma^N = \chi^N_x$ and $u^N = \chi^N_t$ are weak solutions of $\frac{\partial \gamma^N}{\partial t} - \frac{\partial u^N}{\partial x} = 0$ in $x > 0$ and $t > 0$; that is they satisfy

$$\int_a^b (\gamma^N(x,t_2) - \gamma^N(x,t_1))dx - \int_{t_1}^{t_2} (u^N(b,t) - u^N(a,t))dt = 0 \tag{2.21}$$

for all $0 < a < b$ and $0 < t_1 < t_2$. The defining relations (2.14)-(2.16) imply that as N tends to infinity χ^N converges pointwise to the function χ^∞ defined in (1.12) and that the derivatives $\gamma^N = \chi^N_x$ and $u^N = \chi^N_t$ converge weakly to the functions $\gamma^\infty = \chi^\infty_x$ and $u^\infty = \chi^\infty_t$ defined in (1.13) and (1.14). The proof that γ^N goes weakly to γ^∞ is straightforward, but the proof that u^N goes weakly to u^∞ is a little tricky; the argument hinges on the fact that (2.13) implies that for $1 \le j \le k$ and $t \ge \frac{k(1-\lambda_1(u_0))}{2Nu_0} - \frac{\delta(u_0)}{2N}$, $(u_{j+1} - u_j)(t) = 0$. That γ^∞ and u^∞ are weak solutions of $\frac{\partial \gamma^\infty}{\partial t} - \frac{\partial u^\infty}{\partial x} = 0$ follows from (2.21); moreover, they satisfy the Rankine-Hugoniot relation $\frac{2u_0}{1-\lambda_1(u_0)}(\gamma^\infty_- - \gamma^\infty_+) + (u^\infty_- - u^\infty_+) = 0$ across the shock wave $x = \frac{2u_0}{1-\lambda_1(u_0)}t$.

To show that σ^N has a weak limit is a bit subtle. Equations (2.14)-(2.16) and (2.20) imply that $\sigma^N \equiv 0$ in $x \ge \frac{2u_0 t}{1-\lambda_1(u_0)} + \frac{1}{N}$ and this guarantees that they converge pointwise to $\sigma^\infty \equiv 0$ in $x > \frac{2u_0 t}{1-\lambda_1(u_0)}$. In the region $0 < x < \frac{2u_0 t}{1-\lambda_1(u_0)} + \frac{1}{N}$ we know that $\hat{\sigma}(\gamma_1(u_0)) \le \sigma^N \le 0$ independently of N. A rather tedious calculation, which exploits (2.19) and (2.20) and the fact that the $x_k(t)$'s and $u_k(t)$'s satisfy (1.8) and (1.9), implies that for all $0 < a < b$ and $0 < t_1 < t_2$

$$\left| \int_a^b (u^N(x,t_2) - u^N(x,t_1))dx - \int_{t_1}^{t_2} (\sigma^N(b,t) - \sigma^N(a,t))dt \right| \le \frac{C}{N} \tag{2.22}$$

where C depends only on u_0 and is independent of N. If we now let $t_2 = \frac{(1-\lambda_1(u_0))b}{2u_0}$ and $t_1 = \frac{(1-\lambda_1(u_0))a}{2u_0}$ and exploit the limit relations:

$$\lim_{N\to\infty} \int_a^b u^N\left(x, \frac{(1-\lambda_1(u_0))a}{2u_0}\right) dx = -(b-a)u_0, \tag{2.23}$$

$$\lim_{N\to\infty} \int_a^b u^N\left(x, \frac{(1-\lambda_1(u_0))b}{2u_0}\right) dx = 0, \tag{2.24}$$

and

$$\lim_{N\to\infty} \int_{\frac{(1-\lambda_1(u_0))a}{2u_0}}^{\frac{(1-\lambda_1(u_0))b}{2u_0}} \sigma^N(b,t)dt = 0, \tag{2.25}$$

we find that (2.22) implies that for all $0 < a < b$

$$\lim_{N\to\infty} \int_{\frac{(1-\lambda_1(u_0))a}{2u_0}}^{\frac{(1-\lambda_1(u_0))b}{2u_0}} \sigma^N(a,t)dt = -(b-a)u_0. \tag{2.26}$$

The last relation guarantees that σ^N converges weakly to the constant function $\sigma^\infty = -\frac{2u_0^2}{1-\lambda_1(u_0)}$ in $0 < x < \frac{2u_0 t}{1-\lambda_1(u_0)}$. The above construction also implies that the pair

$$u^\infty = \begin{cases} 0, & 0 < x < \frac{2u_0}{1-\lambda_1(u_0)}t \\ -u_0, & \frac{2u_0}{1-\lambda_1(u_0)}t < x \end{cases} \tag{2.27}$$

and

$$\sigma^\infty = \begin{cases} \frac{-2u_0^2}{1-\lambda_1(u_0)}, & 0 < x < \frac{2u_0}{1-\lambda_1(u_0)}t \\ 0, & \frac{2u_0}{1-\lambda_1(u_0)}t < x \end{cases} \tag{2.28}$$

is a weak solution of

$$\frac{\partial \gamma^\infty}{\partial t} - \frac{\partial \sigma^\infty}{\partial x} = 0$$

in $x > 0$ and $t > 0$ and satisfies the Rankine-Hugoniot equation

$$\frac{2u_0}{(1-\lambda_1(u_0))}(u_-^\infty - u_+^\infty) + (\sigma_-^\infty - \sigma_+^\infty) = 0 \tag{2.29}$$

across the shock $x = \frac{2u_0}{(1-\lambda_1(u_0))}t$.

An alternate proof that σ^N converges weakly to $\frac{-2u_0^2}{1-\lambda_1(u_0)}$ follows from the observation that for $0 < \frac{(1-\lambda_1(u_0))X}{2u_0} < t_1 < t_2$ and $t_2 - t_1 \gg \frac{(1-\lambda_1(u_0))}{Nu_0}$

$$\int_{t_1}^{t_2} \sigma^N(x,t)dt = \int_{t_1}^{t_2} \hat{\sigma}(2N\xi(t))dt + O(\frac{1}{N}), \tag{2.30}$$

$$\int_{t_1}^{t_2} \hat{\sigma}(2N\xi(t))dt = \frac{4(t_2-t_1)u_0 N}{(1-\lambda_1(u_0))}\int_0^{\delta(u_0)/2N} \hat{\sigma}(2N\xi(t))dt + O(\frac{1}{N}), \tag{2.31}$$

and

$$2N\int_0^{\delta(u_0)/2N} \hat{\sigma}(2N\xi(t))dt = 2\int_{\gamma_1(u_0)}^{\lambda} \frac{\hat{\sigma}(\gamma)d\gamma}{\sqrt{u_0^2 - P(\gamma)}} = -u_0 < 0. \tag{2.32}$$

These identities imply that for $0 < \frac{(1-\lambda_1(u_0))X}{2u_0} < t_1 < t_2$

$$\lim_{N\to\infty}\int_{t_1}^{t_2} \sigma^N(x,t)dt = \frac{-2u_0^2}{(1-\lambda_1(u_0))}(t_1 - t_2) \tag{2.33}$$

which is the desired result.

We note that we could have avoided introducing the interpolated motion χ^N (see (2.17)) as well as the functions γ^N, u^N, and σ^N (see (2.18)–(2.20)) and worked directly with the particle trajectories and velocities (see (2.14)–(2.16)) if we were interested in the Eulerian rather than the Lagrangian description of the limit flow.

The identities (2.14)–(2.16) imply that for $j = 1, 2, \dots, k_{\#}(t)$ there is exactly one particle in the interval $\left[\frac{(j-1)(1+\lambda_1(u_0))}{2N}, \frac{j(1+\lambda_1(u_0))}{2N}\right]$ (see Figure 4). Here $k_{\#}(t)$ is the largest integer k such that

$$\frac{(1+\lambda_1(u_0))}{2}\frac{k}{N} \le s_N(t) = \frac{\lambda_1(u_0)}{2N} + \frac{\delta(u_0)u_0}{N(1-\lambda_1(u_0))} + \left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right)u_0 t. \tag{2.34}$$

Additionally, for $j = k_{\#}(t)+1, k_{\#}(t)+2, \dots$ there is exactly one particle in the interval $\left[s_N(t) + \frac{j-k_{\#}(t)-1}{N}, s_N(t) + \frac{j-k_{\#}(t)}{N}\right)$.

These observations imply that if we introduce the emperical mass density

$$\rho^{N,\delta}(x,t) \stackrel{\text{def}}{=} \frac{1}{\delta} \sum_{\{k | x_k(t) \in [(j-1)\delta, j\delta)\}} \frac{1}{N} \tag{2.35}$$

where $(j-1)\delta \le x < j\delta$ and $j = 1, 2, \dots$, then

$$\rho^{\infty,\delta}(x,t) \stackrel{\text{def}}{=} \lim_{N\to\infty} \rho^{\infty,\delta}(x,t) = \begin{cases} \frac{2}{1+\lambda_1(u_0)}, & 0 < x \le j_{\max}(t)\delta \\ 1, & (j_{\max}(t)+1)\delta < x \end{cases} \tag{2.36}$$

where $j_{\max}(t)$ is the largest integer such that

$$j_{\max}(t)\delta \le \left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right)u_0 t, \tag{2.37}$$

and thus that

$$\rho^{\infty,0}(x,t) \stackrel{\text{def}}{=} \lim_{\delta\to 0^+} \rho^{\infty,\delta}(x,t) = \begin{cases} \frac{2}{1+\lambda_1(u_0)}, & 0 \le x < \left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right)u_0 t, \\ 1, & \left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right)u_0 t < x. \end{cases} \tag{2.38}$$

Similarly, if we introduce the emperical momentum density

$$m^{N,\delta}(x,t) \stackrel{\text{def}}{=} \frac{1}{\delta} \sum_{\{k | x_k(t) \in [(j-1)\delta, j\delta)\}} \frac{u_k(t)}{N} \tag{2.39}$$

where $(j-1)\delta \le x < j\delta$ and $j = 1, 2, \dots$, we obtain

$$m^{\infty,\delta}(x,t) \stackrel{\text{def}}{=} \lim_{N\to\infty} m^{N,\delta}(x,t) = \begin{cases} 0, & 0 \le x \le j_{\max}(t)\delta \\ -u_0, & j_{\max}(t)\delta < x, \end{cases} \tag{2.40}$$

and

$$m^{\infty,0}(x,t) \stackrel{\text{def}}{=} \lim_{\delta\to 0^+} m^{\infty,\delta}(x,t) = \begin{cases} 0, & 0 \le x < \left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right)u_0 t \\ -u_0, & \left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right)u_0 t < x. \end{cases} \tag{2.41}$$

Again, $j_{\max}(t)$ is defined in (2.37). These indentities rely an the fact that behind the discrete shock $x = s_N(t)$ the particle velocities satisfy $u_{j+1}(t) = -u_j(t)$. The limits satisfy the Rankine-Hugoniot relation

$$\left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right) u_0 \left(\rho_-^{\infty,0} - \rho_+^{\infty,0}\right) = \left(m_-^{\infty,0} - m_+^{\infty,0}\right) \tag{2.42}$$

across the limit shock $x = \left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right) u_0 t$.

If we define the Eulerian stress fields by

$$\tilde{\sigma}^N(x,t) = \hat{\sigma}(N(x_{k+1} - x_k)(t)), \quad x_k(t) < x < x_{k+1}(t), \tag{2.43}$$

then the reasoning used to establish (2.30)–(2.33) implies the existence of a weak limiting stress

$$\tilde{\sigma}^\infty(x,t) \stackrel{\text{def}}{=} \begin{cases} \frac{-2u_0^2}{1-\lambda_1(u_0)}, & 0 < x < \left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right) u_0 t \\ 0, & \left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right) u_0 t < x \end{cases} \tag{2.44}$$

with the property that for $x > 0$

$$\lim_{N\to\infty} \int_{t_1}^{t_2} \tilde{\sigma}^N(x,t)dt = \int_{t_1}^{t_2} \tilde{\sigma}^\infty(x,t)dt. \tag{2.45}$$

Moreover, the triple $(\rho^{\infty,0}, m^{\infty,0}, \tilde{\sigma}^\infty)$ is a weak solution of the Euler form of the momentum equation

$$\frac{\partial m^{\infty,0}}{\partial t} + \frac{\partial}{\partial x}\left(\frac{(m^{\infty,0})^2}{\rho^{\infty,0}} - \tilde{\sigma}^\infty\right) = 0 \tag{2.46}$$

in $x > 0$ and $t > 0$. This last assertion follows from the fact that across the shock $x = \left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right) u_0 t$ the Rankine-Hugoniot equation

$$\left(\frac{1+\lambda_1(u_0)}{1-\lambda_1(u_0)}\right) u_0(m_-^\infty - m_+^\infty) - \left(\frac{(m_-^{\infty,0})^2}{\rho_-^{\infty,0}} - \frac{(m_+^{\infty,0})^2}{\rho_+^{\infty,0}}\right) + \left(\tilde{\sigma}_-^\infty - \tilde{\sigma}_+^\infty\right) = 0 \tag{2.47}$$

is satisfied.

3. Concluding Remarks

Again we note that behind the Lagrangian shock, that is for $0 < X < \frac{2u_0 t}{1-\lambda_1(u_0)}$, the inequality (1.7) implies that the limit strain γ^∞ satisfies $\gamma^\infty = \frac{1+\lambda_1(u_0)}{2} > \lambda$ and thus the limit stress $\sigma^\infty = \frac{-2u_0^2}{1-\lambda_1(u_0)}$ is not given by the interparticle force law since $\hat{\sigma}\left(\frac{1+\lambda_1}{2}\right) = 0$. This observation establishes Lax's conjecture for the model considered here.

We contrast the limit flow with the more conventional shock solution to the system

$$\frac{\partial\gamma}{\partial t}-\frac{\partial u}{\partial x}=0 \quad \frac{\partial u}{\partial t}-\frac{\partial\sigma}{\partial x}=0, \text{ and } \sigma=\hat{\sigma}(\gamma), \quad x>0 \text{ and } t>0, \tag{3.1}$$

$$(\gamma,u)(x,0)=(1,-u_0), \quad x>0, \tag{3.2}$$

and

$$u(0^+,t)=0, \tag{3.3}$$

namely the pair

$$(\gamma,u)=\begin{cases}(\gamma_-(u_0),0), & 0<x<c(u_0)t\\ (1,-u_0), & c(u_0)t<x\end{cases} \tag{3.4}$$

where $\gamma_-(u_0)<\lambda$ is the unique solution to

$$\sqrt{(1-\gamma_-(u_0))(-\hat{\sigma}(\gamma_-(u_0)))}=u_0>0 \tag{3.5}$$

and

$$c(u_0)=\sqrt{\frac{-\hat{\sigma}(\gamma_-(u_0))}{1-\gamma_-(u_0)}}. \text{ [1]} \tag{3.6}$$

This particular shock is obtainable as the $\mu=0^+$ limit of travelling wave solutions to the Navier Stokes equation

$$\frac{\partial\gamma}{\partial t}-\frac{\partial u}{\partial x}=0 \text{ and } \frac{\partial u}{\partial t}-\frac{\partial\hat{\sigma}(\gamma)}{\partial x}=\mu\frac{\partial}{\partial x}\left(\frac{1}{\gamma}\frac{\partial u}{\partial x}\right) \tag{3.7}$$

and is normally regarded as the acceptable solution to the problem; specifically

$$(\gamma,u)=(\Gamma,U)\left(\frac{x-c(u_0)t}{\mu}\right), \tag{3.8}$$

where

$$U(\xi)=-u_0+c(u_0)(1-\Gamma(\xi)), \tag{3.9}$$

$$\frac{c(u_0)}{\Gamma}\frac{d\Gamma}{d\xi}=\hat{\sigma}(\Gamma)+c^2(u_0)(1-\Gamma), \tag{3.10}$$

$$\lim_{\xi\to-\infty}\Gamma(\xi)=\gamma_-(u_0) \quad \text{and} \quad \lim_{\xi\to\infty}\Gamma(\xi)=1, \tag{3.11}$$

and

$$\xi=x-c(u_0)t. \tag{3.12}$$

[1] Recall, von Neumann's proposal was that the shock solution to (3.1)–(3.3) should be the weak limit of the solution to the discrete problem (1.8)–(1.11).

The particular profile satisfying $\Gamma(0) = \lambda$ is given by

$$\int_{\Gamma(\xi)}^{\lambda} \frac{d\gamma}{\gamma(\hat{\sigma}(\gamma) - \hat{\sigma}(\gamma_-(u_0)) - c^2(u_0)(\gamma - \gamma_-(u_0)))} = \frac{-\xi}{c(u_0)}, \quad \xi < 0, \tag{3.13}$$

and

$$\Gamma(\xi) = \frac{\lambda e^{\xi/c(u_0)}}{1 + \lambda(e^{\xi/c(u_0)} - 1)}, \quad \xi \geq 0. \tag{3.14}$$

These results raise a number of issues which will not be resolved here. They tell us that plausible formal approximation methods to solve conservation laws must be taken with a grain of salt since they may produce unexpected results. They also raise an issue about the relation between constitutive equations (here $\sigma = \hat{\sigma}(\gamma)$) and the force laws between particles; clearly these laws do not scale as hypothesized, namely as $\hat{\sigma}(N(x_{k+1} - x_k))$. Also open is the proper evolution equation for the weak limits of the discrete system (1.8) and (1.9); at this point all we know for sure is the equations for the limit flow are not (3.1).

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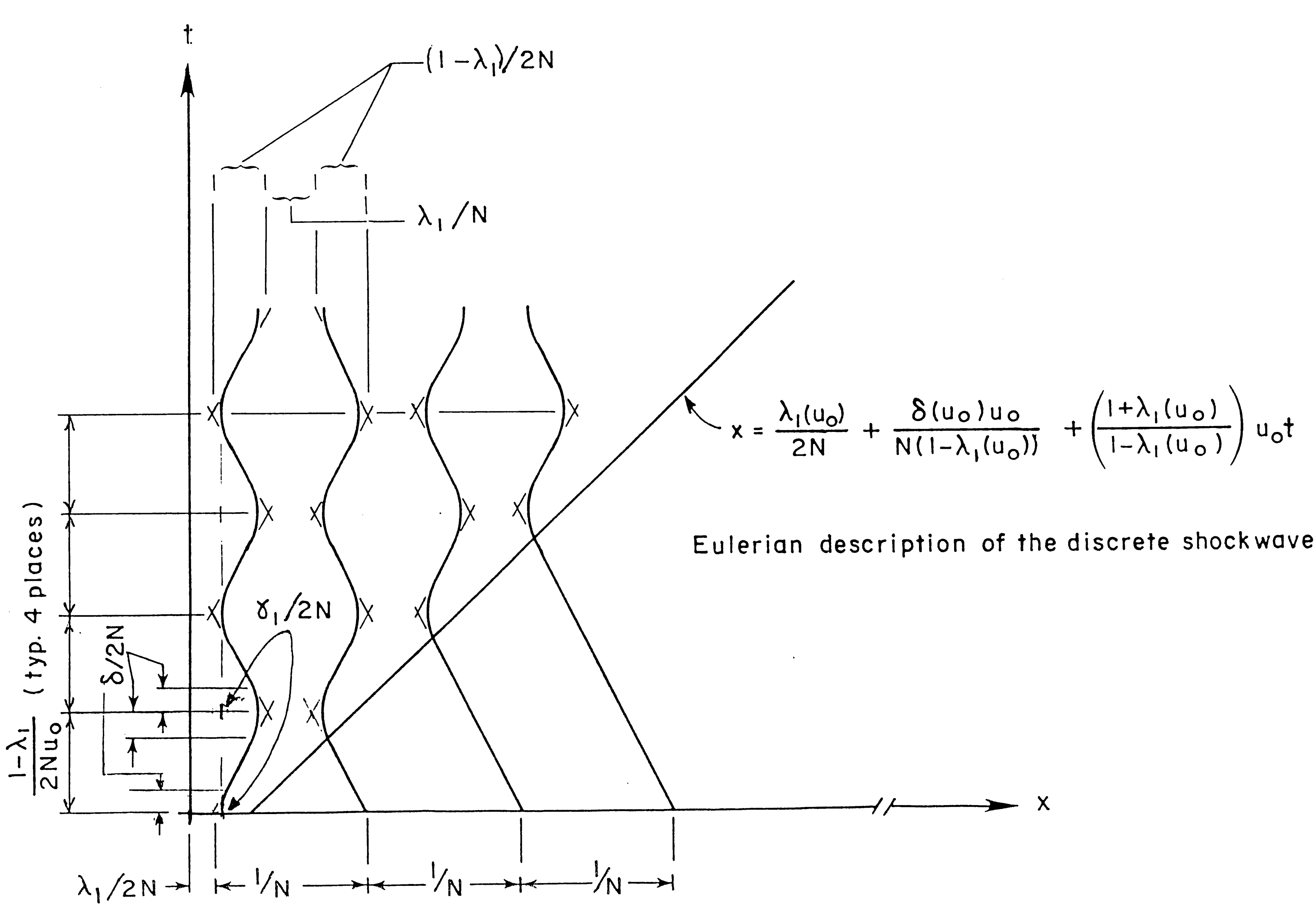
t
(1 − λ₁)/2N
λ₁ /N
x = λ₁(u₀)/2N + δ(u₀)u₀/N(1−λ₁(u₀)) + ((1+λ₁(u₀))/(1−λ₁(u₀))) u₀t
Eulerian description of the discrete shockwave
γ₁ /2N
δ/2N
(typ. 4 places)
1−λ₁/2Nu₀
x
λ₁/2N
1/N
1/N
1/N

Figure 4

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