LOCAL C^∞ CONJUGACY ON THE JULIA SET FOR SOME HOLOMORPHIC PERTURBATIONS OF $z \to z^2$

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Abstract. We consider holomorphic perturbations f of f_0, $f_0(z) = z^2$, which are small in a neighborhood of the unit circle (the Julia set of f_0). We show that if the C^1 conjugacy invariants of f and f_0 are identical, then f and f_0 are C^∞ conjugate on their part of the Julia set which remains near the unit circle.

I. INTRODUCTION

During the past few years, a lot of work has been dedicated to the question of stability of dynamical systems under small perturbations. One of the central questions is under what conditions do two dynamical systems have similar trajectories. In this paper we shall consider the case of small perturbations of the map f_0 of the complex plane given by

$$f_0(z) = z^2 .$$

The Julia set [1] of f_0 is the unit circle, and shall be denoted by J. J is a hyperbolic invariant set $\left(|f'_0|_J| = 2\right)$, and the unstable periodic points of f_0 are dense in J. Let r be a holomorphic function defined in a neighborhood of J. We now consider the function

$$f(z) = f_0(z) + r(z) .$$

If r is small enough, one would like to know how the Julia set of f is related to the Julia set of f_0. This question was investigated in [4], [6] and [7]. One of the results is the local J stability theorem for polynomial-like holomorphic mappings [4]. A consequence of this theorem is that if Ω denotes the set of points with an f orbit contained in some small neighborhood of J, there is a homeomorphism h between J and Ω such that on J we have

$$h \circ \tilde{f}_0 = f \circ h . \qquad \text{(with degree } \tilde{f}_0 = \text{degree } f) \qquad (1)$$

This homeomorphism h extends to a quasiconformal mapping which satisfies (1) [4], but we shall not use this result. In the appendix we give a C^2 version of this theorem on the Julia set J. This result is not new, however we shall need precise estimates on the amplitude of the perturbation which do not seem to be available in the current literature.

We shall be interested in the differentiability properties of the map h. We first observe that h gives a 1 - 1 correspondence between the unstable periodic points of f and f_0 near J. If h is C^1, we observe that if z is a periodic point of period p for f, the corresponding periodic point h(z) for f_0 must satisfy

$$(f^p)'(h(z)) = (f_0^p)'(z)$$

where $f^p = f \circ \ldots \circ f$ p times. It will be convenient to have a general notation for this result.

Let Γ_f be the set of periodic orbits of f. We define a map Sp(f) from Γ_f to $\mathbb{C}$ by

$$Sp(f)(\gamma) = f^{n'}(z)$$

where z is any point in the periodic orbit γ, and n is the minimal period of γ.

The local J structural stability theorem tells us that in a neighborhood of J, Γ_f and Γ_{f_0} are homeomorphic. In this case, if h is C^1, we have locally near J

$$Sp(f) \circ h = Sp(f_0) \ .$$

We shall now investigate the converse of this assumption. Let C_α, $(\alpha > 0)$ denote the corona

$$C_\alpha = \left\{ z \in \mathbb{C} \;\middle|\; e^\alpha \geqslant |z| \geqslant e^{-\alpha} \right\} \ .$$

Our perturbations r will be analytic and bounded in C_α. We shall denote by $\| \ \|_\alpha$ the sup norm in C_α, and by B_α the corresponding Banach space.

<u>THEOREM</u>. <u>Given</u> $\alpha > 0$, <u>there is a number</u> $\varepsilon(\alpha) > 0$ <u>such that if</u> $\|r\|_\alpha < \varepsilon(\alpha)$, <u>then there is a homeomorphism</u> h <u>defined on</u> J <u>such that</u> $\Gamma_{f|_{C_\alpha}} = h\left(\Gamma_{f_0|_{C_\alpha}}\right)$.

<u>If</u>, <u>moreover</u>

$$\mathrm{Sp}(f) \circ h\Big|_{\Gamma_{f_0} \cap C_\alpha} = \mathrm{Sp}(f_0)\Big|_{\Gamma_{f_0} \cap C_\alpha}$$

then h is a C^∞ diffeomorphism such that

$$h \circ f_0 = f \circ h$$

Moreover if $\|r\|_\alpha \to 0$, then $h \to \mathrm{Id}$ in C^p for any p.

This theorem implies that a part of the Julia set of f near J is a C^∞ curve.

The proof of the above theorem extends immediately to the case $f_0(z) = z^n$, $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$. We conjecture that this result is also true for hyperbolic invariant subsets of Julia sets such that the intersection of the Fatou set with some neighborhood of J has only finitely many connected components, and also for sufficiently differentiable perturbations. This problem is also related to a question of Carleson and Jones about maps on the interval.

A similar problem was investigated in [2] for the case of geodesic flow on surfaces of constant negative curvature. The second section contains various results about the linear version of equation (1). The theorem is proven in Section III using an adapted version of the usual Nash-Moser argument (see [8] for a related method). We shall denote by J the unit circle and by B_α the Banach space of analytic functions in C_α equipped with the norm $\| \ \|_\alpha$. Let ϕ be a function holomorphic in some domain D. If D_1 is a domain strictly contained in D, we can obtain estimates of the derivatives of ϕ in D_1 in terms of the sup norm of ϕ in D, and the distance between ∂D and ∂D_1. Such estimates will be called dimensional estimates.

II. PRELIMINARY RESULTS

We shall construct a solution of (1) with $\tilde{f}_0 = f_0$ of the form

$$h(z) = z + a(z)$$

with a small. If we replace this expression for h in (1) and keep only the lowest order terms, we obtain the equation

$$f_0'(z)\, a(z) - a\big(f_0(z)\big) = -r(z) \quad ,$$

namely

$$2z\, a(z) - a(z^2) = -r(z) \quad . \tag{2}$$

We shall now show that this equation is equivalent to the usual cocycle equation

$$b(z) - b(z^2) = \ell(z) \quad , \tag{3}$$

with $\ell(z) = -r'(z)/2z + r(z)/z^2$.

LEMMA 1.

i) <u>If</u> $r \in B_\alpha$ <u>and</u> $\ell(z) = -r'(z)/2z + r(z)/z^2$, <u>then</u> $\oint_J \ell(\zeta)\, d\zeta/\zeta = 0$, $\ell \in B_{\alpha'}$ $\forall\, \alpha' \in \left]0, \alpha\right[$, <u>and</u> $\|\ell\|_{\alpha'} \leqslant e^{4\alpha}(\alpha - \alpha')^{-2} \|r\|_\alpha$.

ii) <u>If</u> $\ell \in B_\alpha$, <u>and</u> $\oint_J \ell(\zeta) d\zeta/\zeta = 0$, <u>then for any</u> $z_0 \in C_\alpha$, <u>and</u> $A \in \mathbb{C}$,

$$r(z) = -2z^2 \int_{z_0}^{z} \ell(\zeta) d\zeta/\zeta + Az^2$$

<u>belongs to</u> B_α <u>and satisfies</u>

$$-r'(z)/2z + r(z)/z^2 = \ell(z) \quad ,$$

and

$$\|r\|_\alpha \leqslant 2e^{3\alpha}\,[\pi + 2\alpha]\,\|\ell\|_\alpha + |A|\,e^{2\alpha} \quad .$$

Proof. It is easy to verify that the Laurent coefficients of ℓ are given by $\ell_n = -n\,r_{n+2}/2$. i) follows easily from the dimensional estimate

$$|r_n| \leqslant \|r\|_\alpha e^{-|n|\alpha} \quad .$$

To prove ii), we first observe that $\oint_J \ell(\zeta)d\zeta/\zeta = 0$ implies that $r(z)$ is a holomorphic function in C_α. The estimate on $\|r\|_\alpha$ is obvious.

QED

We can now give the relation between the solutions of (2) and (3).

LEMMA 2. Let $\ell(z) = -r'(z)/2z + r(z)/z^2$.

i) If (2) has a solution a in B_α, (3) has a solution b in $B_{\alpha'}$ for $\alpha' \in \,]0,\alpha[$. This solution is given by

$$b(z) = a'(z) - a(z)/z$$

and satisfies $\oint_J b(\zeta)d\zeta/\zeta = 0$.

ii) If (3) has a solution b in B_α such that $\oint_J b(\zeta)d\zeta/\zeta = 0$, (2) has a solution a in B_α given by

$$a(z) = z\int_{z_0}^{z} b(\zeta)d\zeta/\zeta - z\int_{z_0^2}^{z_0} b(\zeta)d\zeta/\zeta - z\,r(z_0)/z_0^2$$

where z_0 is any point in the interior of C_α.

Proof. i) is an easy computation. It is also easy to verify, using (3), that the expression of a does not depend on z_0, and is a solution of

$$b(z) = a'(z) - a(z)/z$$

holomorphic in C_α.

QED

From now on, we shall assume $\alpha < 1$, since this is no loss of generality.

It is well known [5] that a necessary condition for the existence of a C^0 solution of (3) is that for any periodic orbit γ of f_0 we must have

$$\sum_{z \in \gamma} \ell(z) = 0 .$$

This relation will be only approximately satisfied in our case. We shall first prove an estimate on the location of the periodic orbits of $f = f_0 + r$. From now on we shall tacitly assume that Theorem A of the appendix can be applied to f_0 and $f_0 + r$.

LEMMA 3. Let $\alpha \in]0, 1[$, then if $\|r\|_\alpha < e^{-\alpha}/3$ and Theorem A applies, any periodic orbit of $f = f_0 + r$ contained in C_α is in C_δ where

$$\delta = 2e^{\alpha}\|r\|_\alpha .$$

Proof. We first observe that if $z \in C_\alpha$, then

$$|z|\left[|z| + e^{\alpha}\|r\|_\alpha\right] > |f(z)| > |z|\left[|z| - e^{\alpha}\|r\|_\alpha\right] .$$

Therefore, if $z \in C_\alpha \setminus C_\delta$, there is an integer n such that $f^n(z) \notin C_\alpha$, hence any periodic orbit contained in C_α is inside C_δ.

QED

From now on we shall assume that

$$Sp(f)\Big|_{\Gamma_{f_0} \cap C_\alpha} = Sp(f_0)\Big|_{\Gamma_{f_0} \cap C_\alpha} \qquad (4) .$$

(We have identified $\Gamma_f \cap C_\alpha$ and $\Gamma_{f_0} \cap C_\alpha$ by the local J structural stability theorem.)

LEMMA 4. Assume condition (4) is satisfied for some $f = f_0 + r$, $r \in B_\alpha$, $\alpha < 1$. Let $\ell(z) = -r'(z)/2z + r(z)/z^2$. Let γ be a periodic orbit of f_0 of period $|\gamma|$. Then there is a number $\varepsilon_2 > 0$ such that if $\|r\|_\alpha \leqslant \varepsilon_2 \alpha^2$, and $2^{3|\gamma|} \leqslant \|r\|_\alpha^{-1}$, then

$$\left| \sum_{z \in \gamma} \ell(z) \right| \leqslant 11\, \alpha^{-4} \|r\|_\alpha^2 \, |\gamma|^4 (2e^{2\alpha})^{4|\gamma|} \quad .$$

Proof. Let $p = |\gamma|$, $\delta = 2e^\alpha \|r\|_\alpha$ and let $z \in \gamma$. Let $z + \delta z$ be the associated periodic point of f by the local stability theorem which applies if ε_2 is small enough, uniformly in $\alpha < 1$. The equation for δz is

$$f^p(z + \delta z) = z + \delta z \ ,$$

therefore,

$$\delta z = f^p(z) - z + f^{p'}(z)\delta z + f^{p''}(z_1)(\delta z)^2/2 \ ,$$

where $z_1 \in C_\delta$ by Lemma 3. Using again Lemma 3, we conclude that

$$\left| \delta z + \left(f^p(z) - z\right)/\left(f^{p'}(z) - 1\right) \right| \leqslant 3p\, e^{3\alpha}(\alpha - \delta)^{-3} \left(2e^\alpha + e^\alpha(\alpha - \delta)^{-2} \|r\|_\alpha\right)^p \ \|r\|_\alpha^2 \quad .$$

Let $f_t(z) = f_0(z) + tr(z)$, we have

$$f^p(z) - z = f_1^p(z) - f_0^p(z) = \frac{d}{dt} f_t^p(z) \Big|_{t=0} + \int_0^1 (1 - t) \frac{d^2}{dt^2} f_t^p(z)\, dt$$

$$= \sum_{n=0}^{p-1} r\left(f_0^{p-n-1}(z)\right) f_0^{n'} \left(f_0^{p-n}(z)\right) + \int_0^1 (1-t) \frac{d^2}{dt^2} f_t^p(z)\, dz \quad .$$

Using Lemma 3 and a dimensional estimate, we have

$$\left| f^{p'}(z) - f_0^{p'}(z) \right| \leqslant 2^p \left[\left(1 + e^{2\alpha}(\alpha - \delta)^{-2} \|r\|_\alpha\right)^p - 1 \right]$$

and we finally obtain, after some easy computations

$$\left|\delta z + \left(f_0^{p'}(z) - 1\right) \sum_{n=0}^{p-1} r\left(f_0^{p-n-1}(z)\right) f_0^{n'}\left(f_0^{p-n}(z)\right)\right|$$
$$\leqslant 8p^4 \left(2e^{\alpha} + e^{2\alpha}(\alpha-\delta)^{-2}\|r\|_{\alpha}\right)^{3p} e^{2\alpha}(\alpha-\delta)^{-3}\|r\|_{\alpha}^2 .$$

We have $\gamma = \{z_1, \ldots, z_p\}$ and it will be useful to have an estimate for $\sum_{j=1}^{p} \delta z_j/z_j$. We observe that

$$z_j = \exp\left[2\pi i(2k+1)2^j/(2^p-1)\right]$$

for some $k \in \mathbb{N}$, $0 \leqslant k < 2^{p-1}-1$. Therefore

$$\left|(2^p-1)\sum_{j=1}^{p} \delta z_j/z_j + \sum_{j=1}^{p}\sum_{n=0}^{p-1} r\left(f_0^{p-n-1}(z_j)\right) 2^n z_j^{-1} \prod_{s=0}^{p-1} z_j^{2^{p-n+s}}\right|$$

$$= \left|(2^p-1)\sum_{j=1}^{p} \delta z_j/z_j + (2^p-1)\sum_{m=1}^{p} z_m^{-2}\, r(z_m)\right|$$

$$\leqslant 8p^4\left(2e^{\alpha}+e^{2\alpha}(\alpha-\delta)^{-2}\|r\|_{\alpha}\right)^{4p}(\alpha-\delta)^{-2}e^{3\alpha}\|r\|_{\alpha}^2 .$$

We shall now use condition (4). We obtain

$$2^p \prod_{j=1}^{p} z_j = \prod_{j=1}^{p}\left(2z_j + 2\delta z_j + r'(z_j+\delta z_j)\right)$$

which implies

$$\left|\sum_{j=1}^{p}\left(z_j^{-1}\delta z_j + r'(z_j)/2z_j\right)\right| \leqslant e^{4\alpha}2p^2(\alpha-\delta)^{-4}\|r\|_{\alpha}^2\left(1+(\alpha-\delta)^{-2}e^{3\alpha}\|r\|_{\alpha}\right)^p$$

Replacing $\sum_1^p z_j^{-1}\delta z_j$ by the above estimate, we obtain the result.

QED

If ℓ is a function holomorphic in C_α such that $\oint_J \ell(\zeta)d\zeta/\zeta = 0$, we have the Laurent expansion

$$\ell(z) = \sum_{n\in\mathbf{Z}\setminus\{0\}} \ell_n z^n \quad .$$

If $\ell(z) = -r'(z)/2z + r(z)/z^2$ with $r\in B_\alpha$ we have $\ell_n = -nr_{n+2}/2$, and

$$|\ell_n| \leqslant |n|e^{2\alpha}e^{-|n|\alpha}\|r\|_\alpha/2 \ .$$

It is easy to give a formal expression for the Laurent coefficients of a solution b of (3), namely

$$b_{2^s(2k+1)} = \sum_{j=0}^{s} \ell_{2^j(2k+1)} \quad .$$

We shall now estimate the Laurent coefficients of b.

<u>LEMMA 5.</u> <u>Assume</u> $\alpha\in\,]0,1[$, <u>and let</u> $r\in B_\alpha$, <u>and</u> $\ell(z) = -r'(z)/z + r(z)/z^2$. <u>There is a constant</u> $\varepsilon_3 > 0$ <u>such that if</u> $\|r\|_\alpha \leqslant \varepsilon_3\alpha^{200}$ <u>and condition</u> (4) <u>is satisfied, then for any</u> $k\in\mathbf{Z}$ <u>we have</u>

$$\left|\sum_{j=0}^{\infty} \ell_{2^j(2k+1)}\right| \leqslant \|r\|_\alpha^{7/4} \quad .$$

<u>Proof.</u> We shall first assume that $|2k+1| \leqslant 2^{-1/2}\|r\|_\alpha^{-1/66}$. This implies that we can find an integer p such that

$$2^{-1/2}\|r\|_\alpha^{-1/66} \leqslant 2^{p/2}, \quad \text{and} \quad 2^p \leqslant \|r\|_\alpha^{-1/33} \ .$$

In particular, we have $|2k+1| \leqslant 2^{p/2}$. We shall assume for the moment that

$2k+1>0$, and we set $R = 2^p - 2k - 2$. We observe that

$$\sum_{t=0}^{2^p-2} \exp\left[4\pi i(2^r n+R)t/(2^p-1)\right]$$

is different from zero (and equal to 2^p-1) if and only if

$$2(2^r n+R) \equiv 0 \qquad \mathrm{mod}(2^p-1) \ .$$

This is equivalent to

$$2^r n+R \equiv 0 \qquad \mathrm{mod}(2^p-1) \tag{5}$$

since 2^p-1 is odd. We shall only consider the case $0<r\leqslant p$. If $n\in\mathbb{Z}$ is a solution of (5) then $n+q(2^p-1)$ is also a solution for any $q\in\mathbb{Z}$. From now on, we shall look for a solution n with $0\leqslant n\leqslant 2^p-2$. Assume n and n' are two solutions for the same r. We can assume $n>n'$. We must have

$$2^r(n-n') \equiv 0 \qquad \mathrm{mod}(2^p-1)$$

which implies

$$n-n' \equiv 0 \qquad \mathrm{mod}(2^p-1).$$

However, since $n-n'\leqslant 2^p-2$, this implies $n=n'$. We have proven that for a given r, equation (5) has at most one solution n_r such that $0\leqslant n_r\leqslant 2^p-2$. We can now write all the solutions of (5):

$$n_r \equiv 2^{p-r}(2k+1) \qquad \mathrm{mod}(2^p-1).$$

We shall now derive some properties of the numbers n_r. If r satisfies $2^r>(2k+1)$, we have $2^{p-r}(2k+1)<2^p-1$, and therefore $n_r = 2^{p-r}(2k+1)$. Assume now $1<2^r<(2k+1)$. We have

$$(2k+1) = 2^r u+v, \qquad \text{with } 0<v\leqslant 2^r-1 \text{ and } 2^r u\leqslant(2k+1)\leqslant 2^{p/2}.$$

This implies

$$n_r \equiv 2^{p-r}(2k+1) \equiv u+2^{p-r}v \quad \mod(2^p-1) \;,$$

and if $r \neq 0$ and $p > 1$,

$$u+2^{p-r}v \leqslant u+2^p-2^{p-r} \leqslant 2^p-2^{p/2}+\frac{2^{p/2}}{2} \leqslant 2^p-1 \;.$$

Therefore, we have $n_r = u+2^{p-r}v$ and also

$$n_r \geqslant 2^{p-r}v \geqslant 2^{p-r} \geqslant 2^{p/2}$$

$$n_r - 2^p + 1 \leqslant 2^{p/2}2^{-r} + 1 - 2^{p-r} \leqslant 2 - 2^{p/2} \;.$$

This implies that the numbers n_r are of the form $2^j(2k+1)$, for some $j \in \mathbb{N}$, or their residues modulo $2^p - 1$ have a modulus greater than $2^{p/2} - 2$ (which is greater than 1 by our hypothesis). Let S_k be defined by

$$S_k = \sum_{t=0}^{2^p-2} \sum_{r=0}^{p-1} \sum_{n=-\infty}^{+\infty} \ell_n \exp\left[2\pi i\left[2^r(2t+1)n+2Rt\right]/(2^p-1)\right] \;.$$

From the above computation we derive

$$S_k = (2^p-1)\exp\left[2\pi iR/(2^p-1)\right] \sum_{r=0}^{p-1} \sum_{q=-\infty}^{+\infty} \ell_{n_r+q(2^p-1)} \;.$$

Using Lemma 4 and the estimate $|\ell_n| \leqslant |n|e^{2\alpha}e^{-|n|\alpha}\|r\|_\alpha/2$, we have

$$|S_k| \leqslant 11\alpha^{-4}\|r\|_\alpha^2 p^4(2^p-1)(2e^{2\alpha})^{4p} + p2^p e^{3\alpha} e^{-\alpha 2^p} \alpha^{-2}\|r\|_\alpha \;.$$

Combining these two estimates, and the estimates on the coefficients ℓ_n, we derive

$$\left|\sum_{s=0}^{+\infty} \ell_{2^s(2k+1)}\right| \leqslant 11\alpha^{-4}\|r\|_\alpha^2 p^4 2^p(2e^{2\alpha})^{4p} + 2p2^p e^{3\alpha}\alpha^{-2}e^{-\alpha 2^{p/2}}\|r\|_\alpha < \|r\|_\alpha^{7/4} \;.$$

If $(2k+1) < 0$, we take $R = -(2k+1)$. A similar argument implies that the unique solution $n_r \in \mathbb{Z}$ of

$$2^r n_r + R \equiv 0 \qquad \mod (2^p - 1)$$

with $0 \geqslant n_r \geqslant 2 - 2^p$ is either of the form $2^j(2k+1)$ for some $j \in \mathbb{N}$, or its residue modulo $2^p - 1$ has a modulus greater than $2^{p/2} - 2$. The estimation is then completed as before.

We now consider the case $|2k+1| > 2^{-1/2} \|r\|_\alpha^{-1/66}$. In this case we have

$$\left| \sum_{s=0}^{\infty} \ell_{2^s(2k+1)} \right| \leqslant \left| \sum_{|n| \geqslant |2k+1|} \ell_n \right| \leqslant e^{3\alpha} \alpha^{-2} \exp\left[-2^{-1/2} \alpha \|r\|_\alpha^{-1/66} \right] \|r\|_\alpha \leqslant \|r\|_\alpha^{7/4} .$$

<u>QED</u>

<u>Corollary</u> 6. <u>Under the hypothesis of Lemma</u> 4 <u>we have</u>

$$\left| \sum_{j=0}^{s} \ell_{2^j(2k+1)} \right| \leqslant \|r\|_\alpha^{7/4} + e^{3\alpha} \alpha^{-2} e^{-2^{s+1} |2k+1| \alpha} \|r\|_\alpha$$

<u>Proof</u>. The proof is obvious from $\ell_n = -n r_{n+2}/2$, and the equality

$$\sum_{j=0}^{s} \ell_{2^j(2k+1)} = \sum_{j=0}^{+\infty} \ell_{2^j(2k+1)} - \sum_{j=s+1}^{\infty} \ell_{2^j(2k+1)} .$$

<u>QED</u>

Given an integer m, we shall denote by $b^{(m)}$ the function

$$b^{(m)}(z) = \sum_{\substack{s \in \mathbb{N},\, k \in \mathbb{Z} \\ |2^s(2k+1)| \leqslant m}} z^{2^s(2k+1)} \sum_{j=0}^{s} \ell_{2^j(2k+1)} .$$

We shall now show that $b^{(m)}$ is a good approximation to a solution of (3).

LEMMA 7. *Let* $\alpha \in]0,1[$ *be given. There is a number* $\varepsilon_4 > 0$ *such that if* $\|r\|_\alpha \leq \varepsilon_4 \alpha^{200}$, *and if* $m \in \mathbb{N}$ *satisfies* $m^4 < \|r\|_\alpha^{-1}$, *and* $m^2 > 1 + (\log_2 m)^4 + 2(\log_2 m)^2$, *then for any* $\alpha' \in]0,\alpha[$, *and any* $z \in C_{\alpha'}$, *we have*

$$\left|\ell(z) - b^{(m)}(z) + b^{(m)}(z^2)\right|$$

$$\leq 4e^{3\alpha} m^2 (\alpha - \alpha')^{-2} \left[\|r\|_\alpha^{7/4} e^{2m\alpha'} + e^{-m\alpha} \|r\|_\alpha + e^{-2m(\alpha-\alpha')} \|r\|_\alpha\right] .$$

Proof. It is easy to verify that

$$\ell(z) - b^{(m)}(z) + b^{(m)}(z^2) = \sigma_1(z) + \sigma_2(z)$$

where

$$\sigma_1(z) = \sum_{\substack{s \in \mathbb{N},\ k \in \mathbb{Z} \\ m < |2^s(2k+1)| \leq 2m}} z^{2^s(2k+1)} \sum_{j=0}^{s} \ell_{2^j(2k+1)}$$

and

$$\sigma_2(z) = \sum_{\substack{n \in \mathbb{Z} \\ |n| \geq 2m}} \ell_n z^n \quad .$$

We shall now estimate $\|\sigma_1\|_{\alpha'}$ and $\|\sigma_2\|_{\alpha'}$. From the bounds on the coefficients ℓ_n we obtain

$$\|\sigma_2\|_{\alpha'} \leq e^{2\alpha} e^{-2m(\alpha-\alpha')} (\alpha - \alpha')^{-2} \|r\|_\alpha .$$

We now estimate $\|\sigma_1\|_{\alpha'}$. Using Corollary 6, we have

$$\|\sigma_1\|_{\alpha'} < 2m \log_2 m \; e^{2m\alpha'} \|r\|_\alpha^{7/4}$$

$$+ \sum_{\substack{s,k \\ m<|2^s(2k+1)|<2m}} e^{3\alpha} \alpha^{-2} e^{-2^{(s+1)}|2k+1|\alpha} e^{2^s|2k+1|\alpha'} \|r\|_\alpha$$

$$\leqslant 2m^2(1+e^{3\alpha}\alpha^{-2})\left(\|r\|_\alpha^{7/4} e^{2m\alpha'} + e^{-m\alpha}\|r\|_\alpha\right) .$$

Collecting the estimate we obtain the result.

QED

Corollary 8. Assume the same hypothesis as in Lemma 7, and let $a^{(m)}$ be such that

$$a^{(m)'}(z) - a^{(m)}(z)/z = b^{(m)}(z), \quad \text{and} \quad \oint_J a^{(m)}(\xi)d\xi/\xi^2 = \oint_J r(\xi)d\xi/\xi^3 ,$$

then for any $\alpha' \in \left]0,\alpha\right[$, and any $z \in C_{\alpha'}$, we have

$$\left|r(z) + 2za^{(m)}(z) - a^{(m)}(z^2)\right|$$

$$\leqslant 30\pi e^{7\alpha} m^2(\alpha-\alpha')^{-2}\left[\|r\|_\alpha^{7/4} e^{2m\alpha'} + e^{-m\alpha}\|r\|_\alpha + e^{2m(\alpha-\alpha')}\|r\|_\alpha\right] .$$

Proof. The proof follows at once from the formula

$$r(z)+2za^{(m)}(z) - a^{(m)}(z^2) = -2z^2\int_{z_0}^{z} \left[\ell(\xi) - b^{(m)}(\xi) + b^{(m)}(\xi^2)\; d\xi/\xi\right]$$

$$- z^2\left[r(z_0) - r^{(m)}(z_o)\right]/z_0^2$$

where $r^{(m)}$ satisfies

$$b^{(m)}(z) - b^{(m)}(z^2) = -r^{(m)'}(z)/2z + r^{(m)}(z)/z^2, \quad \text{and} \quad \oint_J \left[r^{(m)}(\xi) - r(\xi)\right]d\xi/\xi^3 = 0$$

QED

III. PROOF OF THE THEOREM

The proof is an adaptation of the usual Arnold's argument. For a related method see [8].

We assume that $\alpha > 0$ is given and denote by α_n the sequence

$$\alpha_n = \alpha 3^{-n} .$$

Let $\mathcal{E}_n = \mathcal{E}_0^{(10/9)^n}$, and $m_{n+1} = [-7/32\, \alpha_n^{-1} \log \mathcal{E}_n]$, $m_0 = 1$, where $\mathcal{E}_0$ is small enough. We shall define a sequence of maps h_n by $h_0(z) = z$, and

$$h_n(z) = z + a_n(z)$$

where a_n is in $B_{2\alpha_{n-1}}$, $h_n(C_{2\alpha_n}) \subset C_{2\alpha_{n-1}}$, $\|a_n\|_{2\alpha_{n-1}} \leqslant \mathcal{E}_{n-1}^{7/8}$, and $a_0 = 0$.

We shall also define a sequence g_n by

$$g_n = f_0 + r_n$$

where $r_n \in B_{\alpha_n}$, $\|r_n\|_{\alpha_n} \leqslant \mathcal{E}_n$, and $\mathrm{Sp}(g_n)|_{C_{\alpha_n}}$ is isomorphic to $\mathrm{Sp}(f_0)|_{C_\alpha}$. Moreover,

$$h_{n+1} \circ g_{n+1} = g_n \circ h_{n+1} .$$

The construction is recursive, and we shall assume that h_n and g_n have already been constructed with all the above properties.

We first observe that we can apply Theorem A3 of the appendix for any value of n.

In order to construct a_{n+1}, we first form the approximate solution $b^{(m_{n+1})}$ of the equation

$$b(z) - b(z^2) = -r_n'(z)/2z + r_n(z)/z^2 ,$$

and we define a_{n+1} by

$$a_{n+1}(z) = +z\int_{z_0}^{z} b^{(m_{n+1})}(\xi)d\xi/\xi - z\int_{z_0^2}^{z_0} b^{(m_{n+1})}(\xi)d\xi/\xi - z\rho_n(z_0)/z_0^2$$

where ρ_n satisfies

$$b^{(m_{n+1})}(z) - b^{(m_{n+1})}(z^2) = -\rho_n'(z)/2z + \rho_n(z)/z^2 ,$$

and z_0 belongs to $C_{2\alpha_n}$. We first observe that a_{n+1} does not depend on z_0. We also notice that if $(a_{n+1}(z), \rho_n(z))$ is a solution of the above equations, $(a_{n+1}(z) + Az, \rho_n(z) - Az^2)$ is also a solution. We eliminate this indeterminacy by imposing that

$$\oint_J [\rho_n(\xi) - r_n(\xi)]d\xi/\xi^3 = 0,$$

i.e., the terms of degree two in the Laurent expansion of r_n and ρ_n coincide.

$b^{(m_{n+1})}$ is a polynomial, therefore this function belongs to $B_{2\alpha_n}$. Using Corollary 6, we have

$$\|b^{(m_{n+1})}\|_{2\alpha_n} \leqslant 2\alpha_n^{-1} e^{(2m_{n+1}+1)\alpha_n} \|r\|_{\alpha_n}^{7/4} + 2e^{3\alpha_n}\alpha_n^{-2} m_{n+1}\|r\|_{\alpha_n}$$

$$\leqslant 2e^{3\alpha_n}\alpha_n^{-2}\left(e^{2m_{n+1}\alpha_n}\|r\|_{\alpha_n}^{7/4} + m_{n+1}\|r\|_{\alpha_n}\right)$$

Using Lemma 1 and a dimensional estimate, we derive

$$\|a_{n+1}\|_{2\alpha_n} \leqslant 17e^{5\alpha_n}\alpha_n^{-2}\left(e^{2m_{n+1}\alpha_n}\|r\|_{\alpha_n}^{7/4} + m_{n+1}\|r\|_{\alpha_n}\right) \leqslant \varepsilon_n^{7/8} .$$

Since $2\alpha_{n+1} + \varepsilon_n^{1/2} \leqslant 2\alpha_n$, $h_{n+1} = \mathrm{Id} + a_{n+1}$ maps $C_{2\alpha_{n+1}}$ into $C_{2\alpha_n}$. Moreover, using the usual implicit function theorem, one can deduce that h_{n+1} has a well defined inverse on C_{α_n}. This inverse h^{-1}_{n+1} is holomorphic in C_{α_n} and satisfies

$$h^{-1}_{n+1}(z) = z - a_{n+1}(z) + d_{n+1}(z) = z + D_{n+1}(z) \; ,$$

with

$$\|d_{n+1}\|_{\alpha_n} \leqslant 2\|a_{n+1}\|^2_{2\alpha_n}(\alpha_n - 2\varepsilon_n^{7/8})^{-2} \leqslant \varepsilon_n^{5/4} \; .$$

We now observe that for ε_0 small enough,

$$g_n \circ h_{n+1}(C_{\alpha_{n+1}}) \subset C_{\alpha_n} \; .$$

Therefore, for $z \in C_{\alpha_{n+1}}$, we can define g_{n+1} by

$$g_{n+1}(z) = h^{-1}_{n+1} \circ g_n \circ h_{n+1}(z) \; .$$

We shall now estimate $r_{n+1}(z) = g_{n+1}(z) - f_0(z)$. We have

$$\begin{aligned} r_{n+1}(z) &= 2z\,a_{n+1}(z) + a_{n+1}(z)^2 + r_n\big(z + a_{n+1}(z)\big) + D_{n+1}\Big(\big(z + a_{n+1}(z)\big)^2 + r_n\big(z + z_{n+1}(z)\big)\Big) \\ &= R_1(z) + R_2(z) + R_3(z) + R_4(z) \end{aligned}$$

where

$$R_1(z) = 2z\,a_{n+1}(z) - a_{n+1}(z^2) + r_n(z)$$

$$R_2(z) = a_{n+1}(z^2) + D_{n+1}(z^2) = d_{n+1}(z^2)$$

$$R_3(z) = D_{n+1}\big(z + a_{n+1}(z)\big)^2 + r_n\big(z + a_{n+1}(z)\big) - D_{n+1}(z^2) + \big(a_{n+1}(z)\big)^2$$

$$R_4(z) = r_n\big(z + a_{n+1}(z)\big) - r_n(z) \; .$$

We shall now estimate the α_{n+1} norm of each R_i. Using Corollary 8, we have

$$\|R_1\|_{\alpha_{n+1}} \leqslant 100\, e^{7\alpha_n}(\alpha_n - \alpha_{n+1})^{-2}\, m_{n+1}^2 \Big[\varepsilon_n^{7/4} e^{2m_{n+1}\alpha_n} + e^{-m_{n+1}\alpha_n}\,\varepsilon_n + e^{-2m_{n+1}(\alpha_n - \alpha_{n+1})}\,\varepsilon_n\Big]$$

$$\leqslant \varepsilon_n^{9/8} .$$

Since $\alpha_n > 2\alpha_{n+1}$, we have already estimated $\|R_2\|_{\alpha_{n+1}}$, i.e.,

$$\|R_2\|_{\alpha_{n+1}} \leqslant \|d_{n+1}\|_{\alpha_n} \leqslant \varepsilon_n^{5/4} .$$

We now estimate R_3. We have for $z \in C_{\alpha_{n+1}}$

$$R_3(z) = \frac{dD_{n+1}}{dz}(z_1)\Big[2z\,a_{n+1}(z) + a_{n+1}(z)^2 + r_n\big(z + a_{n+1}(z)\big)\Big] + a_{n+1}(z)^2$$

for some $z_1 \in C_{\beta_n}$ where $\beta_n = 2\alpha_{n+1} + \varepsilon_n^{5/4}$. From the dimensional estimate we obtain

$$\|R_3\|_{\alpha_{n+1}} \leqslant (\alpha_n - 2\alpha_{n+1} - \varepsilon_n^{5/4})^{-1} \times 2\,\varepsilon_n^{7/8} \times 3\,e^{\alpha_{n+1}}\,\varepsilon_n^{7/8} + \varepsilon_n^{7/4} \leqslant \varepsilon_n^{5/4}$$

if ε_0 is small enough.

Similarly we have for $z \in C_{\alpha_{n+1}}$

$$R_4(z) = \frac{dr_n}{dz}(z_2)\,a_{n+1}(z)$$

for some $z_2 \in C_{\gamma_n}$ where $\gamma_n = \alpha_{n+1} + \varepsilon_n^{5/4}$. Therefore

$$\|R_4\|_{\alpha_{n+1}} \leqslant (\alpha_n - \alpha_{n+1} - \varepsilon_n^{5/4})^{-1}\,\varepsilon_n\,\varepsilon_n^{7/8} \leqslant \varepsilon_n^{5/4} ,$$

provided ε_0 is small enough.

Summing up the above estimates we obtain

$$\|r_{n+1}\|_{\alpha_{n+1}} \leqslant 4\,\varepsilon_n^{9/8} \leqslant \varepsilon_n^{10/9} = \varepsilon_{n+1} .$$

We define a sequence of functions H_n, holomorphic in C_{α_n} by

$$H_n(z) = h_1 \circ \dots \circ h_n(z) \;;$$

notice that $H_{n+1} = H_n \circ h_{n+1}$.

For $z \in J$, let $\Delta_n(z) = H_{n+1}(z) - H_n(z)$. We have

$$\Delta_n(z) = \int_z^{h_{n+1}(z)} \frac{dH_n}{dt}(t)\, dt$$

where the integral is along J. Let $\| \;\|_J$ denote the sup norm on J. We have, using the chain rule,

$$\|\Delta_n\|_{\alpha_{n+1}} \leqslant \pi \prod_{j=1}^{n} \left\|\frac{dh_j}{dz}\right\|_{2\alpha_j} \|a_{n+1}\|_{2\alpha_{n+1}}$$

$$\leqslant \pi \prod_{j=1}^{n} (1 + \alpha_j^{-2} e^{\alpha_j} \varepsilon_j^{7/8})\, \varepsilon_n^{7/8}$$

$$\leqslant 4\varepsilon_n^{7/8} \text{ if } \varepsilon_0 \text{ is small enough (independently of n).}$$

Tnerefore H_n converges uniformly on J to a continuous function h. Let $\Phi_n = H_n \circ g_n - H_n \circ f_0$, we have as before for $z \in J$

$$\Phi_n(z) = \int_{f_0(z)}^{g_n(z)} \frac{dH_n}{dt}(t)\, dt ,$$

therefore

$$\|\Phi_n\|_J \leqslant \pi \prod_{j=1}^{n} \left\|\frac{dh_j}{dz}\right\|_{2\alpha_j} \cdot \|a_n\|_{2\alpha_n}$$

$$\leqslant \Theta(1)\, \varepsilon_n^{7/8} .$$

This implies

$$\|H_n \circ f_0 - g_0 \circ H_n\|_J = \|H_n \circ f_0 - H_n \circ g_n\|_J \to 0 \quad \text{if } n \to \infty,$$

namely

$$h \circ f_0 = g_0 \circ h .$$

We now show that h is a C^{∞} diffeomorphism. Let p be any integer, then from $\|\Delta_n\|_{\alpha_{n+1}} \leqslant 4\,\varepsilon_n^{7/8}$, we derive

$$\|\Delta_n^{(p)}\|_J \leqslant 4\,\alpha_{n+1}^{-p-1}\, e^{p\alpha_{n+1}}\, \varepsilon_n^{7/8} .$$

Therefore the sequence $H_n^{(p)}$ is uniformly convergent on J and hence h is in $C^p(J)$ for any integer p.

We have also the following three inequalities,

$$\|h - Id\|_J \leqslant 5\,\varepsilon_0^{7/8} , \qquad \|h' - 1\|_J \leqslant 5\,\alpha^{-2} e^{\alpha}\, \varepsilon_0^{7/8} , \qquad \|h''\|_J \leqslant 5\,\alpha^{-3} e^{2\alpha}\, \varepsilon_0^{7/8} ,$$

which imply that h is a C^{∞} diffeomorphism (see [3] for example). This ends the proof of the theorem.

APPENDIX

In this appendix we give a proof of the local Ω stability theorem for C^2 expanding endomorphisms. This is a well know result, however explicit estimates do not seem to be available in the current literature. Let Ω be a closed compact set in $\mathbb{R}^p$, and let Φ be a C^2 map from a closed neighborhood V of Ω into $\mathbb{R}^p$ such that $V \subset \Phi(V)$, and

$$\Omega = \left\{ x \in V \,\middle|\, \Phi^n(x) \in V \;\forall\; n \in \mathbb{N} \right\} .$$

We also assume that $D\Phi_x$ is invertible for any x in V, and that there is an integer m and a number $K > 1$ such that

$$\sup_{x \in V} \left\| (D\Phi_x)^{-m} \right\| \leqslant K^{-1} ,$$

i.e., Ω is a repeller.

Let δ be a positive number, we define W_δ by

$$W_\delta = \left\{ x \,\middle|\, d(x, \Omega) \leqslant \delta \right\} .$$

We now define the inner radius $\beta(V)$ of V by

$$\beta(V) = \sup\left\{ \delta \,\middle|\, W_\delta \subset V \right\} .$$

Notice that $W_{\beta(V)} \subset V$ is a neighborhood of Ω.

We now define a number $\alpha(V)$ by

$$\alpha(V) = \left(\sup_{x \in W_{\beta(V)}} \left\| (D\Phi_x)^{-1} \right\| \cdot \sup_{x \in W_{\beta(V)}} \left\| D^2\Phi_x \right\| \right)^{-1}$$

Notice that $\alpha(V) \neq 0$.

<u>LEMMA A1.</u> <u>Let</u> x <u>and</u> x' <u>in</u> $W_{\beta(V)/2}$ <u>be such that</u> $x \neq x'$, <u>and</u> $\Phi(x) = \Phi(x')$. <u>Then</u> $\|x - x'\| \geqslant \mathrm{Inf}\big(\beta(V)/2,\, 2\alpha(V)\big)$.

Proof. We can assume $\|x - x'\| < \beta(V)/2$, otherwise there is nothing to prove. We have

$$\Phi(x') - \Phi(x) = D\Phi_x(x' - x) + \int_0^1 (1-t)\, D^2\Phi_{x+t(x'-x)}(x'-x,\, x'-x)\, dt \ .$$

Therefore

$$(x' - x) = (D\Phi_x)^{-1} \int_0^1 (1-t)\, D^2\Phi_{x+t(x'-x)}(x'-x,\, x'-x)\, dt \ ,$$

which implies

$$\|x' - x\| \leqslant \alpha(V)^{-1} \|x' - x\|^2 \quad ,$$

Hence the result.

QED

Let $L = \sup_{x \in V} \|D\Phi_k\|$, and define $\mathcal{E}(V)$ by

$$\mathcal{E}(V) = (L-1)L^{-3(m+1)}(K-1)\,\mathrm{Inf}\left[\beta(V)/4,\ (K-1)\,\alpha(V)/4,\ \alpha(V)/8\right].$$

We now have the following version of the shadowing lemma:

LEMMA A2. For $\varepsilon \in [0, \mathcal{E}(V)[$, let $(x_n)_{n \in \mathbb{N}}$ be an ε pseudo orbit contained in $W_{\beta(V)/2}$, i.e.,

$$\|\Phi(x_n) - x_{n+1}\| < \varepsilon \ , \quad \text{and} \quad x_n \in W_{\beta(V)} \ \forall\ n \in \mathbb{N}$$

then there is a unique x in Ω such that

$$\|\Phi^n(x) - x_n\| < 2\varepsilon L^{2m+1}/(L-1)(K-1) \quad \forall \quad n \in \mathbb{N}.$$

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$y_n = x_{nm} \qquad n\in\mathbb{N} .$$

Let $\varepsilon_1 = \varepsilon L^m/(L-1)$, it is easy to verify that $(y_n)_{n\in\mathbb{N}}$ is an ε_1 pseudo orbit for Φ^m which is contained in $W_{\beta(V)/2}$. Let $\gamma = 2\varepsilon_1/(K-1)$, and denote by $\bar{B}(y_n,\rho)$ the closed ball of radius ρ centered at x. Let $B_n = \bar{B}(y_n,\gamma)$, it is easy to verify that

$$f^m(\bar{B}_n) \supset \bar{B}\left(\Phi^m(y_n), K\gamma - L^{2m}\gamma^2/\alpha(V)(L-1)\right) \supset \bar{B}_{n+1} .$$

Moreover, by Lemma A1, there is a well defined inverse of Φ^m from $\bar{B}_{n+1}$ to $\bar{B}_n$. Let $K_n \subset B_0$ be defined by $\Phi^{mn}(K_n) = \bar{B}_n$, and $\Phi^{mj}(K_n) \subset \bar{B}_j$ for $j = 0, 1, \ldots, n-1$. We have

$$\mathrm{diam}(K_n) \leqslant 2\gamma K^{-n} , \qquad \text{and} \qquad K_{n+1} \subset K_n .$$

Therefore $\bigcap_n K_n$ is a point x. This point γ shadows the sequence $(y_n)_{n\in\mathbb{N}}$, and γL^m shadows the sequence $(x_n)_{n\in\mathbb{N}}$. Using Lemma A1 and hyperbolicity, it is easy to verify that x is unique and belongs to Ω.

QED

We can now prove the local Ω stability theorem.

THEOREM A3. Let V be as in Lemma A1, and let ε be such that

$$0 < \varepsilon < (L-1)(K-1)(2L)^{-(3m+3)} \mathrm{Inf}\left[\alpha(V), \beta(V), (K-1)\alpha(V), (K-1), L^{-1}, (L-1)/2, \left(1+\alpha(V)L\right)^{-1}\right] .$$

Let r be C^2 in V and such that

$$\sup_{x\in W_{2\beta(V)/3}} \left[\|r\| + \|D_x r\| + \|D_x^2 r\|\right] < \varepsilon .$$

Let $\Psi = \Phi + r$, and let Ω' be defined by

$$\Omega' = \left\{ x \in V \mid \Psi^n(x) \in W_{\beta(V)/2} \ \forall\ n \in \mathbb{N} \right\} .$$

There is a homeomorphism h from Ω to Ω' such that

$$\sup_{x \in \Omega} \| h(x) - x \| \leqslant 64 \varepsilon (2L)^{2m+1} / (L-1)(K-1) ,$$

and on Ω we have

$$h \circ \Phi = \Psi \circ h .$$

Proof. Let x be a point in Ω, and define $x_n = \Phi^n(x)$. $(x_n)_{n \in \mathbb{N}}$ is an ε pseudo orbit for Ψ. We can now apply Lemma A2 to Ψ, and denote by $h(x)$ the unique point in Ω' which shadows $(x_n)_{n \in \mathbb{N}}$. It is easy to verify that h is continuous. We can now apply the same construction starting with a point x' in Ω'. Let h' denote the associated map. From the uniqueness property in Lemma A2 we have $h' \circ h = \mathrm{Id}|_{\Omega}$. Moreover, again from the uniqueness property we have

$$h \circ \Phi = \Psi \circ h \quad \text{on} \quad \Omega .$$

QED

ACKNOWLEDGMENTS. It is a great pleasure to thank P. Blanchard, A. Douady and G. Wayne for helpful discussions, and H. Weinberger and G. Sell for their warm hospitality at IMA.

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