# CONTROLLING THE FREE BOUNDARY OF ELLIPTIC VARIATIONAL INEQUALITIES ON A VARIABLE DOMAIN

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# Controlling the Free Boundary of Elliptic Variational Inequalities on a Variable Domain


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Abstract. A domain $\Omega_u$ in $\mathbb{R}^2$ depends upon a control variable $u$ and its boundary $\partial\Omega_u$ consists of two parts $\Gamma_\psi$, $\Gamma_u$ , $\Gamma_\psi \cap \Gamma_u = \varnothing$, where $\Gamma_\psi$ is prescribed independently of $u$. The problem is to choose $u$ so that $\Gamma_\psi$ becomes the free boundary and $\Omega_u$ the noncoincidence set of a given elliptic obstacle problem on $\Omega \supset \Omega_u$. One proves that this problem has a solution in a certain generalized sense. Numerical results are given.



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1. The formulation of the problem. Consider an annular domain $D_u \subset \mathbb{R}^2$

$$D_u = \{(x,y) \in \mathbb{R}^2;\ \alpha(\theta) < r < u(\theta),\ 0 \le \theta \le 2\pi\}$$

where $(r,\theta) \in \mathbb{R}^+ \times [0,2\pi]$ are usual polar coordinates, $u$ is a control function which belongs to the set

$$U = \{u \in H^3_\pi(0,2\pi);\ 0 < \psi(\theta) < u(\theta)\},$$

and $\alpha$, $\psi \in H^3_\pi(0,2\pi)$ are given functions such that

(1.1) $\qquad \psi(\theta) \ge \alpha(\theta) > 0 \qquad \forall\theta \in [0,2\pi].$

Here $H^3_\pi(0,2\pi)$ is the Sobolev space $\{u;\ u^{(k)} \in L^2(0,2\pi),\ k = 0,1,2,3;\ u(0) = u(2\pi)\}$. We set $\Gamma_u = \{(x,y);\ r = u(\theta),\ \theta \in [0,2\pi]\}$, $\Gamma_\alpha = \{(x,y);\ r = \alpha(\theta),\ \theta \in$

$[0,2\pi]\}$, and $\Gamma_\psi = \{(x,y);\ r = \psi(\theta),\ \theta \in [0,2\pi]\}$.

Consider in $D_u$ the variational inequality:

<u>Find</u> $z \in K_u$, <u>such</u> <u>that</u>

$$\int_{D_u} \nabla z \cdot \nabla(z-\zeta)dxdy + \int_{D_u} f(z-\zeta)dxdy \le 0 \qquad \forall \zeta \in K_u, \tag{1.2}$$

where

$$K_u = \{z \in H^1(D_u);\ z \ge 0 \text{ in } D_u,\ z = 1 \text{ on } \Gamma_u,\ z = 0 \text{ on } \Gamma_\alpha\}$$

Here $f$ is a given function in $\Omega_0 = \{(x,y);\ \theta \in [0,2\pi],\ \alpha(\theta) < r(\theta) < \infty\}$. Throughout this paper we shall use the same notation for $z(x,y)$ and $z(r,\theta)$. It is well known that for each $u$, problem (1.2) has a unique solution $z^u \in H^2(D_u)$ which is a generalized solution to the obstacle problem

$$\begin{aligned} &\Delta z = f \quad \text{in } \{(x,y);\ z(x,y) > 0\} \\ &\Delta z \le f,\quad z \ge 0 \text{ in } D_u \\ &z = 0 \quad \text{in } \Gamma_\alpha \\ &z = 1 \quad \text{in } \Gamma_u. \end{aligned} \tag{1.4}$$

The set $\{(x,y) \in D_u;\ z(x,y) = 0\}$ is the coincidence set of the problem and its boundary in $D_u$, i.e., the curve which separates the sets $\{(x,y) \in D_u;\ z(x,y) > 0\}$ and $\{(x,y) \in D_u;\ z(x,y) = 0\}$ is the free boundary of the obstacle problem (1.4), which will be denoted by $\Gamma_u^0$.

Here we will study the following problem

(1.5) <u>Find</u> $u \in U$ <u>such</u> <u>that</u> $\Gamma_\psi \equiv \Gamma_u^0$.

This is an optimal design controllability problem which is relevant to mathematical simulation of several physical processes and in particular that of electrochemical machining ([8]).

Following [1] (see also [2]) we shall reformulate problem (1.5) as

(1.6) <u>Find</u> $u \in U$ <u>such</u> <u>that</u> $\frac{\partial}{\partial \nu} z^u = 0$ a.e. on $\Gamma_\psi$ <u>and</u> $z^u > 0$ <u>on</u>

$$\Omega_u = \{(x,y) \in D_u;\ \psi(\theta) < r < u(\theta)\}.$$

Here $z^u$ is the solution to boundary value problem

(1.7) $\quad \Delta z = f$ in $\Omega_u$

$$z = 0 \text{ on } \Gamma_\psi;\ z = 1 \text{ on } \Gamma_u$$

and $\frac{\partial}{\partial\nu}$ is the outward normal derivative to $\partial\Omega_u$. It should be noted that under our hypotheses $z^u \in H^2(\Omega_u)$ and $\frac{\partial z^u}{\partial\nu}$ is well defined and belongs to $L^2(\partial\Omega_u)$.

If $(u^*, z^*)$ is a solution to (1.6) then it is easily seen that the function $\bar{z}^* \in H^1(D_u)$

$$\bar{z}^* = z^* \text{ in } \Omega_{u^*};\ \bar{z}^* = 0 \text{ in } D_{u^*}\backslash\Omega_{u^*}$$

is the solution to variational inequality (1.2) on $D_{u^*}$. Also, $\Omega_{u^*} = \{(x,y) \in D_{u^*};\ \bar{z}^*(x,y) > 0\}$ and so $\Gamma_\psi \equiv \Gamma^0_{u^*}$.

We need the following equivalent formulation of the problem (1.6):

(1.8) $\quad$ <u>Find</u> $u \in U$ <u>such that</u> $z^u > 0$ <u>on</u> $\Omega_u$ <u>and</u>

(1.9) $\quad \frac{\partial p}{\partial\nu}\frac{\partial z^u}{\partial r} = 0$ a.e. <u>in</u> $\Gamma_u$

<u>where</u> $p \in L^q(\Omega_u)$, $1 < q < 2$ <u>and</u> $\mu \in M(\overline{\Omega_u})$ <u>satisfy</u>

(1.10) $\quad \Delta p = \mu\cdot(r-\psi)^{-1} = \mu_\psi$ in $\Omega_u$

$$p = \frac{\partial z^u}{\partial\nu} \text{ in } \Gamma_\psi;\ p = 0 \text{ in } \Gamma_u$$

(1.11) $\quad supp\ \mu \subset \Gamma_\psi;\ \mu_\psi(z^u) = 0,\ \mu \le 0$

(1.12) $\quad p \in H^2(\hat{\Omega}^\delta_u)\quad \forall\delta > 0$ sufficiently small.

Here $\hat{\Omega}^\delta_u = \{(x,y) \in \Omega_u;\ \psi(\theta) + \delta < r < u(\theta)\}$ and $M(\overline{\Omega_u})$ is the space of all bounded Radon measures on $\overline{\Omega_u}$. The Radon measure $\mu_\psi = \mu\cdot(r-\psi)^{-1}$ is defined by $\mu_\psi(\varphi) = \mu(\varphi(x,y)(r-\psi(\theta))^{-1})$ for all $\varphi \in C_0(\Omega_u)$ ($C_0(\Omega_u)$ is the space of continuous functions on $\Omega_u$ with compact support in $\Omega_u$).

The boundary value problem (1.10) is interpreted in the following way

$$(1.13)\qquad \int_{\Omega_u} p\chi \, dxdy = \int_{\Gamma_\psi} \frac{\partial z^u}{\partial \nu} \frac{\partial \varphi}{\partial \nu} \, d\sigma + \mu_\psi(\varphi)$$

for all $\varphi$, $\chi \in C(\overline{\Omega_u})$ satisfying

$$\Delta\varphi = \chi \quad \text{in } \Omega_u$$

$$\varphi = 0 \quad \text{in } \partial\Omega_u.$$

Since $|\varphi(x,y)| \le C \, dist((x,y),\partial\Omega_u)$, $\mu_\psi(\varphi)$ is well defined.

It is readily seen, if one approximates $\mu$ by a sequence of $L^2(\Omega_u)$ functions, that $p$ also satisfies the equation

$$(1.14)\qquad \int_{\Omega_u} p\Delta\varphi \, dxdy = \int_{\Gamma_\psi} \frac{\partial z^u}{\partial \nu} \frac{\partial \varphi}{\partial \nu} \, d\sigma + \mu_\psi(\varphi) - \int_{\Gamma_u} \frac{\partial p}{\partial \nu} \varphi \, d\sigma$$

for all $\varphi \in H^2(\Omega_u)$ such that

$$(1.15)\qquad \Delta\varphi \in C(\overline{\Omega_u}), \ \varphi = 0 \text{ in } \Gamma_\psi.$$

PROPOSITION 1. <u>Problems</u> (1.6) <u>and</u> (1.8) <u>are</u> <u>equivalent.</u>

<u>Proof.</u> If $u$ is a solution to problem (1.8) then by (1.9) it follows that $\frac{\partial p}{\partial \nu} = 0$ in $\Gamma_u$ because by the Hopf maximum principle $\frac{\partial z^u}{\partial \nu} > 0$ on $\Gamma_u$ (if $z^u \not\equiv 0$) (see Lemma 2 below) and hence $\frac{\partial z^u}{\partial r} > 0$ on $\Gamma_u$. Since $p \in H^2(\hat{\Omega}_u^\delta)$ is the solution to homogeneous problem

$$\Delta p = 0 \quad \text{in } \hat{\Omega}_u^\delta; \ p = \frac{\partial p}{\partial \nu} = 0 \text{ in } \Gamma_u$$

we conclude that $p \equiv 0$ and so by (1.14) we have

$$\int_{\Gamma_\psi} \frac{\partial z^u}{\partial \nu} \frac{\partial \varphi}{\partial \nu} \, d\sigma + \mu_\psi(\varphi) = 0$$

for all $\varphi$ satisfying (1.15). In particular, we may take $\varphi = z^u$ to conclude that $\frac{\partial z^u}{\partial \nu} = 0$ in $\Gamma_\psi$. The implication (1.6) $\Rightarrow$ (1.8) is obvious.

The main result of this paper, Theorem 1 below amounts to saying that problem (1.8) is approximately solvable which as seen above implies that, in some sense, so are problems (1.6) and (1.5). The problem is first approximated (regularized) by one related shape optimization problem. This problem is studied in Section 3. The main difficulty there is that control set is not closed (and also constrained), so that some precise analysis is required to avoid that difficulty and to arrive at the optimality system of equations rather then inequalities. Theorem 1 is proved in Section 4, where the regularization is sent to zero. One could say that the treatment of the problem is constructive and it suggests a numerical algorithm which is discussed in Section 5.

For other results on optimum design problems associated with variational inequalities and partial differential equations we refer to [7] and the bibliography given there.

Here we shall use the standard notation $H^1(0,1)$, $H^2(\Omega)$, $C(\bar{\Omega})$, $L^p(\Omega)$ for Sobolev spaces, continuous functions space and $L^p$-spaces on $(0,1)$, $\Omega$ and $\bar{\Omega}$ respectively.

## 2. The main result.

Throughout in the following we shall assume that

*(i)*

$$f \in C^1(\Omega_0) \cap L^\infty(\Omega_0) \text{ and} \tag{2.1}$$

$$f \geq 0, \ (r^2 f)_r \leq 0 \text{ in } \Omega_0 = \{(x,y);\ 0 \leq \theta \leq 2\pi,\ \alpha(\theta) < r < \infty\}$$

*(ii)*

$$\psi \in H^3_\pi(0,2\pi),\ 0 < \psi(\theta) < \infty \quad \forall\theta \in [0,2\pi] \tag{2.2}$$

THEOREM 1. <u>There are sequences</u> $\{u_n\} \subset U$ <u>and</u> $\{\mu_n\} \subset M(\overline{\Omega_{u_n}})$ <u>satisfying</u> (1.11) <u>such that</u>

(2.3) $$0 < z^{u_n} < 1 \quad \text{in } \Omega_{u_n},$$

(2.4) $$\frac{\partial p^{u_n}}{\partial \nu}\frac{\partial z^{u_n}}{\partial r}\left((u_n(\theta))^2 + (u_n'(\theta))^2\right)^{1/2} \to 0 \quad \text{in } (H^3_\pi(0,2\pi))' \quad \text{as } n \to \infty,$$

<u>where</u> $z^u$ <u>and</u> $p^u$ <u>are</u> <u>the</u> <u>solutions</u> <u>to</u> (1.7) <u>and</u> (1.10), <u>respectively</u>. <u>Moreover,</u>

(2.5) $$\frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z^{u_n}}{\partial \nu}\right)^2 d\sigma \longrightarrow \inf\left\{\frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z^u}{\partial \nu}\right)^2 d\sigma;\ z^u \geq 0 \quad \text{in } \Omega_u,\ u \in U\right\}.$$

We would like to emphasize that system (1.9), (1.10), (1.12) is naturally associated with the original controllability problem since it is the dual optimality system of the shape optimization problem

(2.6) $$\inf\left\{\frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z^u}{\partial \nu}\right)^2 d\sigma;\ z^u \geq 0 \quad \text{in } \Omega_u,\ u \in U\right\}.$$

Indeed, by the same argument as that used for the proof of Theorem 1, it follows that if $u$ is optimal in problem (2.6) then there are $p$ and $\mu$ satisfying (1.9) to (1.12). Hence by Proposition 1 we may conclude that if the infimum in (2.6) is attained on some $u^* \in U$, then $\frac{\partial z^*}{\partial \nu} = 0$ in $\Gamma_\psi$ and so $u^*$ is a solution to problem (1.5). (In view of the well known regularity theorem for the free boundary ([3]) this is unlikely to happen however.)

Though it is not clear when the infimum in (2.6) is zero as one might expect under present conditions, we may view however $\{u_n\}$ as an approximating sequence for problem (2.5) and interpret Theorem 1 as an approximate controllability result for the free boundary $\Gamma_\psi$ through variable boundary $\Gamma_u$.

3. <u>The</u> <u>approximating</u> <u>shape</u> <u>optimization</u> <u>problem</u>. Consider the shape optimization problem:

<u>Minimize</u>

$$(3.1)\qquad \left\{ \frac{1}{2}\int_{\Gamma_\psi} \left(\frac{\partial z}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n} \|u\|^2_{H^3(0,2\pi)} \right\}$$

subject to $u \in U$ and to state constraints

$$(3.2)\qquad \Delta z = f \quad \text{in } \Omega_u$$

$$z = 0 \quad \text{on } \Gamma_\psi; \quad z = 1 \quad \text{on } \Gamma_u$$

$$(3.3)\qquad z \ge 0 \text{ on } \Omega_u.$$

A control $u \in U$, the shape of the domain, is said to be admissible for problem (3.1) if $z^u$ (the solution to (3.2) satisfies condition (3.3).

We shall prove first the following

LEMMA 1. Under assumptions (i), (ii) there is at least one admissible control $u$ for problem (3.1).

Proof. We need to show that there exists a function $u \in U$, for which (3.1) has a finite value, such that the corresponding $z^u$ satisfies (3.3). We will show that we can choose

$$(3.4)\qquad u_\varepsilon(\theta) = \psi(\theta) + \varepsilon,$$

where $\varepsilon > 0$ is small enough. To this end, consider

$$w = \exp\Big(\alpha(r - \psi(\theta))\Big) - 1 = w_0 - 1.$$

We have

$$\Delta w = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} \ge w_0 \left( \alpha^2 - \alpha \frac{1}{r^2} \psi_{\theta\theta} \right) \ge \|f\|_{L^\infty(\Omega_0)},$$

for $\alpha$ large enough. But then, if we take $\varepsilon = (\ln 2)/\alpha$ in (3.4), we have, by the maximum principle,

$$z^{u_\varepsilon} \ge w > 0 \quad \text{in } \Omega_{u_\varepsilon},$$

i.e., $u_\varepsilon$ is admissible.

LEMMA 2. Let $u \in U$ be admissible and let $z^u$ satisfy (3.2), (3.3). Then

(3.5) $$0 < z^u < 1 \quad \text{in } \Omega_u$$

(3.6) $$\frac{\partial z^u}{\partial r} > 0 \quad \text{in } \Omega_u \cup \Gamma_u.$$

Proof. By maximum principle it follows by (3.2) that $z^u < 1$ in $\Omega_u$ and $(z^u)_r > 0$ in $\Gamma_u$. We have

$$\frac{1}{r}(rz_r)_r + \frac{1}{r^2} z_{\theta\theta} = f.$$

Setting $w = rz_r$, multiplying with $r^2$, differentiating with respect to $r$, and multiplying with $r$, we obtain

$$r(rw_r)_r + w_{\theta\theta} = r(r^2 f)_r \le 0$$

$$w > 0 \quad \text{in } \{r = u(\theta)\}$$

$$w \ge 0 \quad \text{in } \{r = \psi(\theta)\}$$

$$w \text{ is } 2\pi\text{-periodic}$$

which implies, by the maximum principle, that $w > 0$ in $\Omega_u$. In particular, $z^u > 0$ in $\Omega_u$.

We associate with (3.1) the penalized shape optimization problem:

Minimize

(3.6) $$\left\{ \frac{1}{2}\int_{\Gamma_\psi} \left(\frac{\partial z}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n} \|u\|^2_{H^3(0,2\pi)} + \frac{1}{2\lambda} \int_{\Omega_u} (z^-)^2 dxdy \right\}$$

on all $u \in U$ and $z \in H^2(\Omega_u)$ subject to (3.2). Here $n \in \mathbb{N}$ and $\lambda$ is a positive parameter.

PROPOSITION 2. For every $\lambda > 0$ problem (3.6) has at least one solution $(u_\lambda, z_\lambda) \in U \times H^2(\Omega_{u_\lambda})$. Moreover, there is a sequence $\{\lambda_k\} \to 0$ such that

(3.7) $u_{\lambda_k} \to u_n$ weakly in $H^3_\pi(0,2\pi)$ and strongly in $C^2[0,2\pi]$

(3.8) $z_{\lambda_k} \to z_n$ strongly in $H^1(\Omega^\delta_{u_n})$ and weakly in $H^2(\Omega^\delta_{u_n})$

in every domain $\Omega^\delta_{u_n} = \{(x,y);\ 0 \le \theta \le 2\pi,\ \psi(\theta) < r < u_n(\theta) - \delta\},\ 0 < \delta < \delta_0$

where $(z_n, u_n) \in H^2(\Omega_{u_n}) \times U$ is a solution to problem (3.1).

Proof. The main difficulty with existence in problem (3.6) as well as for (3.1) is that the control set $U$ is not closed in $H^3_\pi(0,2\pi)$. Denote by $d$ the infimum of functional (3.7) on $U$, and consider a sequence $\{u_m\} \subset U$ such that

$$(3.9)\qquad d \le \phi(u_m) = \frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z^m}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n}\|u_m\|^2_{H^3(0,2\pi)} + \frac{1}{2\lambda}\int_{\Omega_{u_m}}((z^m)^-)^2 dxdy$$

$$\le d + \frac{1}{m}$$

where $z^m = z^{u_m}$. Then on a subsequence which for simplicity we denote again $\{u_m\}$, we have

$$(3.10)\qquad u_m \to u_\lambda \text{ strongly in } C^2[0,2\pi] \text{ and weakly in } H^3_\pi(0,2\pi).$$

Now if we multiply (3.2) where $u = u_m$ and $z = z^m$ by 1 and $z^m$ respectively, we get

$$\int_{\Gamma_{u_m}} \frac{\partial z^m}{\partial \nu}\, d\sigma + \int_{\Gamma_\psi} \frac{\partial z^m}{\partial \nu}\, d\sigma = \int_{\Omega_{u_m}} f\, dx$$

and

$$(3.11)\qquad \int_{\Omega_{u_m}} |\nabla z^m|^2 dxdy - \int_{\Gamma_{u_m}} \frac{\partial z^m}{\partial \nu}\, d\sigma = -\int_{\Omega_{u_m}} f z^m\, dxdy.$$

respectively. Since by the maximum principle $z^m \le 1$ whilst by (3.10) $\left\{\frac{\partial z^m}{\partial \nu}\right\}$ is bounded in $L^2(\Gamma_\psi)$ and $\{(z^m)^-\}$ is bounded in $L^2(\Omega_{u_m})$ we conclude that

$$(3.12) \qquad \int_{\Omega_{u_m}} |\nabla z^m|^2 dxdy + \int_0^{2\pi} |\frac{\partial z^m}{\partial \nu}(u_m(\theta),\theta)|d\theta \le C$$

where $C$ is independent of $m$.

Next from the obvious equality

$$-1 = \int_{u_m(\theta)}^{\psi(\theta)} \frac{\partial z^m}{\partial r}(r,\theta)\,dr \qquad \forall\theta \in [0,2\pi]$$

and by (3.12) we see that

$$|E| \le \left(\int_E |u_m-\psi|d\theta\right)^{1/2} \left(\int_{\Omega_{u_m}} \left(\frac{\partial z^m}{\partial r}\right)^2 drd\theta\right)^{1/2} \le$$

$$\le C\left(\int_E |u_m-\psi|d\theta\right)^{1/2}$$

for any measurable subset E of $[0,2\pi]$. In particular the latter implies that the Lebesgue measure of the set $F_\lambda = \{\theta;\ u_\lambda(\theta) = \psi(\theta)\}$ is zero.

We may represent the set $(0,2\pi) \setminus F_\lambda = E_\lambda$ as

$$E_\lambda = \bigcup_{k=1}^{\infty} (\alpha_k,\beta_k)$$

where $\alpha_k,\ \beta_k \in F_\lambda$ for all k. Denote by $D_k$ the domain $\{(x,y);\ \alpha_k < \theta < \beta_k;\ \psi(\theta) < r < u_\lambda(\theta)\}$ and observe that, without loss of generality we can assume, $u_\lambda(\alpha_k) = \psi(\alpha_k)$, $u_\lambda(\beta_k) = \psi(\beta_k)$. By interior and boundary estimates it follows that

$$(3.13) \qquad \|z^m\|_{C^{2,\alpha}(D_k^\varepsilon)} \le C_\varepsilon \qquad \forall m,$$

where $0 < \alpha < 1$, $D_k^\varepsilon = \{(x,y) \in D_k;\ \alpha_k + \varepsilon < \theta < \beta_k - \varepsilon,\ \psi(\theta) < r < u_\lambda(\theta)\}$.

On the other hand, we have

(3.14) $\quad z^m \to z_\lambda^k$ weakly in $L^2(D_k)$

where $z_\lambda^k$ is the weak solution to equation (3.2) in $D_k$ i.e.,

(3.15) $\quad \Delta z_\lambda^k = f$ in $D_k$

$$z_\lambda^k = 0 \text{ on } \Gamma_\psi;\ z_\lambda^k = 1 \text{ on } \Gamma_{u_\lambda}.$$

By (3.13) it follows that $\left\{\frac{\partial z^m}{\partial \nu}\right\}$ is compact in $L^2(\Gamma_\psi \cap \partial D_k^\varepsilon)$ and since $\frac{\partial z_\lambda^k}{\partial \nu}$ is well defined on $\Gamma_\psi \cap \partial D_k^\varepsilon$ it follows by (3.14) that on a subsequence

$$\frac{\partial z^m}{\partial \nu} \to \frac{\partial z_\lambda^k}{\partial \nu} \text{ in } L^2 \text{ and a.e. on } \Gamma_\psi \cap \partial D_k^\varepsilon \text{ as } m \to 0.$$

Thus letting m tend to $+\infty$ in inequality (3.10) it follows via Fatou's lemma that

$$(3.16) \qquad \frac{1}{2} \sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} \left(\frac{\partial z_\lambda^k}{\partial \nu}\right)^2 \left(\psi^2 + (\psi')^2\right)^{1/2} d\theta + \frac{1}{2n} \|u_\lambda\|^2_{H^3(0,2\pi)} \le d.$$

Let us prove now that $u_\lambda(\theta) > \psi(\theta)$ for all $\theta \in [0,2\pi]$. Otherwise, we can assume, there is a point $\alpha$ such that $u_\lambda(\alpha) = \psi(\alpha)$ and $u_\lambda(\theta) > \psi(\theta)$ for $\theta \in (\alpha,\beta)$. Then as seen above the solution $z$ to equation (3.15) on $D = \{(x,y);\ \alpha < \theta < \beta,\ \psi(\theta) < r < u_\lambda(\theta)\}$, i.e.,

(3.17) $\quad \Delta z = f$ in D

$$z = 0 \text{ in } \Gamma_\psi;\ z = 1 \text{ in } \Gamma_{u_\lambda}$$

has the property that $\frac{\partial z}{\partial \nu}$ exists on $(\alpha,\beta)$ and

$$(3.18) \qquad \int_\alpha^\beta \left(\frac{\partial z}{\partial \nu}\right)^2 \left(\psi^2 + (\psi')^2\right)^{1/2} d\theta < \infty.$$

Without loss of generality we may assume that $\alpha = 0$ and that the part of the domain D has the form (in Cartesian coordinates) $\{0 < x < \beta;\ \psi(x) < y < u_\lambda(x)\}$ where $\psi(0) = \psi'(0) = 0$. Moreover, since $u_\lambda(0) = u_\lambda'(0) = 0$ we have

(3.19) $\quad u_\lambda(x) \le ax^r,$

(3.20) $\quad \psi(x) \ge -bx^r,$

in a neighborhood of 0 for some $r > 1$ and $a > 0$, $b > 0$.

Consider the following function

$$z_0(x,y) = \exp\left(\frac{y - \psi(x)}{u_\lambda(x) - \psi(x)} \ln 2\right) - 1 = \hat{z} - 1 \qquad \forall(x,y) \in D.$$

After some elementary calculation we get

$$\Delta z_0 = \hat{z}\,(\ln 2)^2\,(u_\lambda(x) - \psi(x))^{-2} + A(x,y),$$

where $|A(x,y)| \le |u_\lambda(x) - \psi(x)|^{-2} O(|x|)$ $\forall(x,y) \in D$. Hence, by (3.19) and (3.20)

$$\Delta z_0 \ge (u_\lambda(x) - \psi(x))^{-2}(C + O(|x|)) \longrightarrow +\infty, \qquad \text{as } |x| \to 0.$$

Let $B_R = \{(x,y);\ x^2 + y^2 < R^2\}$. The function $w = z_0 - z$ satisfies (for R small enough)

$$\Delta w \ge 0 \quad \text{in } B_R \cap D$$

$$w = 0 \quad \text{in } \partial D \cap B_R.$$

Consider now the solution $v$ to the boundary value problem

$$\Delta v = 0 \quad \text{in } B_R \cap D$$

$$v = 0 \quad \text{in } \partial D \cap B_R;\ v = w \text{ in } \partial B_R \cap D.$$

Clearly $v \in C^{0,1}(B_R \cap D)$ and by the maximum principle we have

$$w \le v \quad \text{in } B_R \cap D$$

and therefore

$$\frac{\partial w}{\partial \nu} \ge \frac{\partial v}{\partial \nu} \ge -C(R) = -C \quad \text{in } \Gamma_\psi \cap B_R.$$

This yields

$$(3.21) \qquad \frac{\partial z_0}{\partial \nu} \geq \frac{\partial z}{\partial \nu} - C \quad \text{in } \Gamma_\psi \cap B_R$$

for some constant $C > 0$. On the other hand, by an elementary calculation it follows that

$$\frac{\partial z_0}{\partial \nu} = -\hat{z}\,(1 + (\psi')^2)^{1/2}\,(u_\lambda(x) - \psi(x))^{-1} \ln 2 \quad \text{in } \Gamma_\psi \cap B_R.$$

By (3.19) and (3.20), this implies that

$$\int_{\Gamma_\psi \cap B_R} \frac{\partial z_0}{\partial \nu}\, d\sigma = -\infty$$

and by (3.21), since $\frac{\partial z}{\partial \nu} < 0$ in $\Gamma_\psi \cap B_R$, we conclude that

$$\int_{\Gamma_\psi \cap B_R} \left(\frac{\partial z}{\partial \nu}\right)^2 d\sigma = +\infty,$$

which clearly contradicts (3.18). The contradiction we arrived at shows that $u_\lambda > \psi$ everywhere on $[0,2\pi]$.

Now coming back to estimate (3.12) we infer that $\{z^m\}$ is bounded in every $H^1(\Omega^\delta_{u_\lambda})$ as $m \to \infty$, and so by boundary and interior estimates it follows that $\{z^m\}$ is bounded in every $H^2(\Omega^\delta_{u_\lambda})$. Hence there is $z_\lambda \in \bigcap_{0<\delta<\delta_0} H^2(\Omega^\delta_{u_\lambda})$ such that

$$z^m \to z_\lambda \qquad \text{weakly in } H^2(\Omega^\delta_{u_\lambda}) \text{ and strongly in } H^1(\Omega^\delta_{u_\lambda})$$

for all $0 < \delta < \delta_0$. Obviously $z_\lambda$ satisfies equation (3.2) and the boundary conditions $z_\lambda = 0$ in $\Gamma_\psi$. It is also clear that

$$(3.22) \qquad \frac{\partial z^m}{\partial \nu} \to \frac{\partial z_\lambda}{\partial \nu} \qquad \text{weakly in } L^2(\Gamma_\psi) \text{ as } m \to \infty.$$

We claim that $z_\lambda$ takes boundary value equal to 1 on $\Gamma_{u_\lambda}$. For any $\varepsilon > 0$

and any $(x,y) \in \Omega_{u_\lambda}$, there exists $m_0$, such that for any $m \ge m_0$

(i) $\qquad (x,y) \in \Omega_{u_m}$

(ii) $\qquad |z^m(x,y) - z_\lambda(x,y)| \le \varepsilon_1$

(iii) $\qquad dist(\Gamma_{u_\lambda}, \Gamma_{u_m}) \le \varepsilon_2$

where $\varepsilon_1$ and $\varepsilon_2$ are to be specified later. We observe that, by elliptic estimates,

$$\|\nabla z^m\|_{L^\infty(\Omega_{u_m})} \le C,$$

where $C$ does not depend on $m$. Hence,

$$\begin{aligned}|z_\lambda(x,y) - 1| &\le |z_\lambda(x,y) - z^m(x,y)| + |z^m(x,y) - 1| \\ &\le \varepsilon_1 + C\, dist(y, \Gamma_{u_m}) \\ &\le \varepsilon_1 + C\left[dist(y,\Gamma_{u_\lambda}) + dist(\Gamma_{u_\lambda}, \Gamma_{u_m})\right] \\ &\le \varepsilon\end{aligned}$$

if $dist(y,\Gamma_{u_\lambda})$ is small enough (here, $\varepsilon_1$ and $\varepsilon_2$ are chosen appropriately).

Next, letting m tend to $+\infty$ in (3.10) it follows

by (3.22) that

$$\frac{1}{2}\int_{\Gamma_\psi} \left(\frac{\partial z_\lambda}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n}\|u_\lambda\|^2_{H^3(0,2\pi)} + \frac{1}{2\lambda}\int_{\Omega_{u_\lambda}} (z_\lambda^-)^2 dxdy = d$$

i.e., $(z_\lambda, u_\lambda)$ is optimal in problem (3.7). Here, $d = d(\lambda, n)$.

Now let $u_0 \in U$ be an admissible control for problem 3.1 (Lemma 1), and $z_0$ the corresponding solution. We have

$$\frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z_\lambda}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n}\|u_\lambda\|^2_{H^3(0,2\pi)} + \frac{1}{2\lambda}\int_{\Omega_{u_\lambda}}(z_\lambda^-)^2 dxdy \le$$

$$\le \frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z_0}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n}\|u_0\|^2_{H^3(0,2\pi)} \qquad \forall\lambda > 0.$$

Hence $\{u_\lambda\}$ is bounded in $H^3(0,2\pi)$ and

$$\frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z_\lambda}{\partial \nu}\right)^2 d\sigma + \frac{1}{2\lambda}\int_{\Omega_{u_\lambda}}(z_\lambda^-)^2 dxdy \le C \qquad \forall\lambda > 0. \tag{3.23}$$

Then, on a subsequence $\{\lambda_k\} \longrightarrow 0$ we have

$$u_{\lambda_k} \to u_n \qquad \text{weakly in } H^3(0,2\pi) \text{ and strongly in } C^2[0,2\pi].$$

On the other hand, by standard estimates for elliptic boundary value problems it follows that

$$\|z_\lambda\|_{H^2(\Omega_{u_\lambda})} \le C \qquad \forall\lambda > 0$$

and arguing as above it follows that there is a subsequence of $\{\lambda_k\}$ again denoted $\{\lambda_k\}$ such that

$$z_{\lambda_k} \to z_n = z^{u_n} \qquad \text{weakly in } H^2(\Omega^\delta_{u_n}) \text{ and strongly in } H^1(\Omega^\delta_{u_n}),$$

for $0 < \delta < \delta_0$ and

$$\frac{\partial z_{\lambda_k}}{\partial \nu} \to \frac{\partial z_n}{\partial \nu} \qquad \text{weakly in } L^2(\Gamma_\psi).$$

Moreover, by estimate (3.23) we see that $z_n \ge 0$ in $\Omega_{u_n}$. Then letting $\lambda$ tend to zero in the inequality

$$\frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z_\lambda}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n}\|u_\lambda\|^2_{H^3(0,1)} + \frac{1}{2\lambda}\int_{\Omega_{u_\lambda}}(z_\lambda^-)^2 dxdy \le$$

$$\leq \frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n}\|u\|^2_{H^3(0,2\pi)}$$

for all $(z,u)$ satisfying (3.2), (3.3), $u \in U$, we get

$$\frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z_n}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n}\|u_n\|^2_{H^3(0,2\pi)} = \inf\ (3.1)$$

On the other hand, by the same argument as that used for $u_\lambda$ it follows that $u_n(\theta) > \psi(\theta)$ $\forall\theta \in [0,2\pi]$. Hence $u_n \in U$ and the pair $(z_n,u_n)$ is optimal in problem (3.1). This completes the proof of Proposition 2.

LEMMA 3. Let $(z_\lambda,u_\lambda)$ be optimal in problem (3.7) then there are $p_\lambda \in L^2(\Omega_{u_\lambda})$ such that

$$\Delta p_\lambda = -\frac{1}{\lambda}\, z_\lambda^- \text{ in } \Omega_{u_\lambda} \tag{3.24}$$

$$p_\lambda = \frac{\partial z_\lambda}{\partial \nu} \text{ in } \Gamma_\psi;\ p_\lambda = 0 \text{ in } \Gamma_{u_\lambda}$$

$$\frac{\partial p_\lambda}{\partial \nu}(\theta,u_\lambda(\theta))\ \frac{\partial z_\lambda}{\partial r}(\theta,u_\lambda(\theta))\left(u_\lambda^2(\theta) + (u_\lambda')^2(\theta)\right)^{1/2} = n^{-1}Au_\lambda \text{ in } [0,2\pi] \tag{3.25}$$

where $A$: $H^3_\pi(0,2\pi) \to (H^3_\pi(0,2\pi))'$ is defined by

$$(Au,v) = \sum_{k=0}^{3}\int_0^{2\pi} u^{(k)}v^{(k)}d\theta \quad \forall v \in H^3_\pi(0,2\pi) \tag{3.26}$$

where $(\cdot,\cdot)$ is the pairing between $H^3_\pi(0,2\pi)$ and its dual $(H^3_\pi(0,2\pi))'$.

*Proof*. Let $(u_\lambda,z_\lambda)$ be optimal pair in (3.7). Let $v \in H^3_\pi(0,2\pi)$ be arbitrary but fixed. For all $\varepsilon > 0$ sufficiently small, $u_\lambda + \varepsilon v \in U$. Thus we have

$$\frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z^{u_\lambda+\varepsilon v}}{\partial \nu}\right)^2 d\sigma + \frac{1}{2\lambda}\int_{\Omega_{u_\lambda+\varepsilon v}}\left(\left(z^{u_\lambda+\varepsilon v}\right)^-\right)^2 dxdy + \frac{1}{2}\|u_\lambda+\varepsilon v\|^2_{H^3_\pi(0,2\pi)} \geq$$

$$\frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z_\lambda}{\partial \nu}\right)^2 d\sigma + \frac{1}{2\lambda}\int_{\Omega_{u_\lambda}} (z^-)^2 dxdy + \frac{1}{2}\|u_\lambda\|^2_{H^3(0,2\pi)}.$$

We set

$$\zeta_\lambda = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left(z^{u_\lambda+\varepsilon v} - z_\lambda\right)$$

where the limit exists in every $H^1(\Omega^\delta_{u_\lambda})$ for $0 < \delta < \delta_0$. Arguing as in the proof of Proposition 2 it follows that $\zeta_\lambda \in L^2(\Omega_{u_\lambda}) \cap \left(\bigcap_{0<\delta<\delta_0} H^2(\Omega^\delta_{u_\lambda})\right)$ is the solution to boundary value problem

$$(3.27)\qquad \Delta\zeta_\lambda = 0 \quad \text{in } \Omega_{u_\lambda},$$

$$\zeta_\lambda = 0 \quad \text{in } \Gamma_\psi; \quad \zeta_\lambda = -\frac{\partial z_\lambda}{\partial r}\, v \quad \text{in } \Gamma_u,$$

This yields

$$(3.28)\qquad \int_{\Gamma_\psi}\frac{\partial z_\lambda}{\partial\nu}\frac{\partial\zeta_\lambda}{\partial\nu}\, d\sigma - \frac{1}{\lambda}\int_{\Omega_{u_\lambda}} z^-_\lambda\,\zeta_\lambda\, dxdy + (Au_\lambda, v) = 0$$

for all $v \in H^3_\pi(0,2\pi)$.

Now let $p_\lambda \in L^2(\Omega_{u_\lambda})$ be the solution to system (3.24). By boundary regularity for elliptic equations it follows that $p_\lambda \in H^2(\Omega')$ in every domain $\Omega'$ of the form $\{(x,y) \in \Omega_{u_\lambda};\ \psi(\theta) + \eta < r < u_\lambda(\theta)\}$, $\eta > 0$, and so $\frac{\partial p_\lambda}{\partial\nu}$ is well defined on $\Gamma_{u_\lambda}$ and belongs to $H^{1/2}(\Gamma_{u_\lambda}) \subset \bigcap_{q\geq 2} L^q(\Gamma_{u_\lambda})$.

Now by some calculation involving equations (3.24) and (3.27) we get

$$\frac{1}{\lambda}\int_{\Omega_{u_\lambda}} z^-_\lambda\,\zeta_\lambda\, dxdy = \int_{\Gamma_\psi}\frac{\partial z_\lambda}{\partial\nu}\frac{\partial\zeta_\lambda}{\partial\nu}\, d\sigma + \int_{\Gamma_{u_\lambda}}\frac{\partial p_\lambda}{\partial\nu}\frac{\partial z_\lambda}{\partial r}\, v\, d\sigma$$

and substituting in (3.28) yields

$$- \int_0^{2\pi} \frac{\partial p_\lambda}{\partial \nu} (\theta, u_\lambda(\theta)) \frac{\partial z_\lambda}{\partial r}(\theta, u_\lambda(\theta)) \left(u_\lambda^2(\theta) + (u_\lambda')^2(\theta)\right)^{1/2} v(\theta)\, d\theta +$$

$$+ n^{-1}(Au_\lambda, v) = 0 \quad \forall v \in H^3_\pi(0,2\pi)$$

and (3.25) follows thereby completing the proof.

4. Proof of Theorem 1. Let $(z_n, u_n)$ be the solution to problem (3.1) obtained by Proposition 2. We will prove that the sequence $\{(z_n, u_n)\}$ satisfies (2.3) and (2.4). Let $w_\lambda$ be the solution to boundary value problem

$$(4.1) \qquad \Delta w_\lambda = 0 \quad \text{in } \Omega_{u_\lambda}$$

$$w_\lambda = 0 \quad \text{in } \Gamma_\psi; \ w_\lambda = \frac{\partial z_\lambda}{\partial r} \quad \text{in } \Gamma_{u_\lambda}$$

As $\lambda \to 0$ we have (see the proof of Proposition 2) $w_\lambda \to w_n$ weakly in every $H^2(\Omega^\delta_{u_n})$, where $w_n \in H^2(\Omega^\delta_{u_n})$, $\forall \delta > 0$ is the solution to

$$(4.2) \qquad \Delta w_n = 0 \quad \text{in } \Omega_{u_n}$$

$$w_n = 0 \quad \text{in } \Gamma_\psi, \ w_n = \frac{\partial z_n}{\partial r} \quad \text{in } \Gamma_{u_n}.$$

Now coming back to system (3.24), we multiply Eq. (4.1) by $p_\lambda$ and integrate on $\Omega_{u_\lambda}$. We get

$$(4.4) \qquad \frac{1}{\lambda} \int_{\Omega_{u_\lambda}} \bar{z_\lambda}\, w_\lambda \, dxdy = \int_{\Gamma_\psi} \frac{\partial w_\lambda}{\partial \nu} \frac{\partial z_\lambda}{\partial \nu}\, d\sigma - \int_{\Gamma_{u_\lambda}} \frac{\partial p_\lambda}{\partial \nu} \frac{\partial z_\lambda}{\partial r}\, d\sigma.$$

Since $\left\{\frac{\partial w_\lambda}{\partial \nu}\right\}$ and $\left\{\frac{\partial z_\lambda}{\partial \nu}\right\}$ are bounded in $L^2(\Gamma_\psi)$, while by (3.25)

$$\int_{\Gamma_{u_\lambda}} \frac{\partial p_\lambda}{\partial \nu} \frac{\partial z_\lambda}{\partial r}\, d\sigma = n^{-1}(Au_\lambda, 1) \le C \quad \forall \lambda > 0$$

we conclude that

$$(4.5) \qquad \frac{1}{\lambda}\int_{\Omega_{u_\lambda}} z_\lambda^- \, w_\lambda \, dxdy \le C \qquad \forall\lambda > 0.$$

On the other hand by Eq. (4.2) it follows via the maximum principle that $w_n > 0$ in $\Omega_{u_n}$ and $\frac{\partial w_n}{\partial r} > 0$ in $\Gamma_\psi$ and since $w_n$ is of class $C^2$ in every $\overline{\Omega^\delta_{u_n}}$ we have $\frac{\partial w_n}{\partial r} > 0$ in a neighborhood $V$ of $\Gamma_\psi$. Since $w_\lambda \to w_n$ and $\frac{\partial w_\lambda}{\partial r} \to \frac{\partial w_n}{\partial r}$ uniformly in $V$, we conclude that

$$w_\lambda(r,\theta) \ge C(r - \psi(\theta)) \qquad \text{in } V,$$

where $C$ is independent of $\lambda$. Then by estimate (4.5) we infer that

$$(4.6) \qquad \frac{1}{\lambda}\int_{\Omega_{u_\lambda}} z_\lambda^- \, (r - \psi(\theta)) \, dxdy \le C.$$

(We have denoted by $C$ several positive constants independent of $\lambda$.)

This implies that the family of measures defined by

$$(4.7) \qquad \mu_\lambda(\varphi) = -\frac{1}{\lambda}\int_{\Omega_{u_\lambda}} z_\lambda^- \, (r - \psi(\theta)) \, \varphi \, dxdy \qquad \forall\varphi \in C(\overline{\Omega_{u_\lambda}})$$

is bounded and so on a subsequence (eventually generalized) we have

$$\mu_\lambda \longrightarrow \mu_n \qquad \text{weak star in every } M(\overline{\Omega^\delta_{u_n}}),$$

where $\mu_n \in M(\overline{\Omega^\delta_{u_n}})$.

Then, letting $\lambda$ tend to zero in the obvious equality

$$(4.8) \qquad \int_{\Omega_{u_\lambda}} p_\lambda \chi \, dxdy = \int_{\Gamma_\psi} \frac{\partial z^{u_\lambda}}{\partial \nu} \frac{\partial \varphi}{\partial \nu} \, d\sigma + \mu_\lambda(\hat{\varphi})$$

where $\hat{\varphi} = \varphi\cdot(r - \psi(\theta))^{-1}$, $\Delta\varphi = \chi$ in $\Omega_{u_\lambda}$ and $\varphi = 0$ in $\Gamma_\psi \cup \Gamma_{u_\lambda}$, $\chi \in C(\overline{\Omega_{u_\lambda}})$, we conclude that (observe that $\hat{\varphi} \in C(\overline{\Omega_{u_\lambda}})$)

$$p_\lambda \longrightarrow p_n \quad \text{weak star in every } M(\overline{\Omega^\delta_{u_n}})$$

where $p_n \in M(\overline{\Omega^\delta_{u_n}})$ is the solution to

$$(4.9) \qquad \Delta p_n = \mu_n \cdot (r - \psi)^{-1} \quad \text{in } \Omega_{u_n}$$

$$p_n = \frac{\partial z^{u_n}}{\partial \nu} \quad \text{in } \Gamma_\psi, \ p_n = 0 \quad \text{in } \Gamma_{u_n}.$$

Now we proceed to prove (1.12). Let $\kappa \in L^{q'}(\Omega_{u_\lambda})$ and $\varphi \in W^{2,q'}(\Omega_{u_\lambda})$ be such that

$$\Delta\varphi = \kappa \quad \text{in } \Omega_{u_\lambda}, \ \varphi = 0 \quad \text{in } \partial\Omega_{u_\lambda}.$$

For $q' > 2$ we know that $\varphi \in C^1(\Omega_{u_\lambda})$ and so

$$|\varphi(x,y)| \le C\|\kappa\|_{L^{q'}(\Omega_{u_\lambda})} dist((x,y),\Gamma_\psi).$$

Then by (4.6) and (4.8) it follows that

$$|\int_{\Omega_{u_\lambda}} p_\lambda \kappa \; dxdy| \le \int_{\Gamma_\psi} |\frac{\partial z^{u_\lambda}}{\partial \nu} \frac{\partial \varphi}{\partial \nu}| \; d\sigma + \frac{1}{\lambda} \int_{\Omega^\delta_{u_n}} z^-_\lambda \; |\varphi| \; dxdy \le$$

$$\le C\|\kappa\|_{L^{q'}(\Omega_{u_\lambda})} + C\|\kappa\|_{L^{q'}(\Omega_{u_\lambda})} \frac{1}{\lambda} \int_{\Omega_{u_\lambda}} z^-_\lambda \; (r - \psi(\theta)) \; dxdy \le$$

$$\le C\|\kappa\|_{L^{q'}(\Omega_{u_\lambda})}.$$

This implies that $p_\lambda \in L^q(\Omega_{u_\lambda})$ where $1 < q < 2$. Now consider the solution $\varphi$ to the problem

$$\Delta\varphi = f_0 + \sum_{i=1}^{2} \frac{\partial f_i}{\partial x_i} \quad \text{in } \Omega_{u_\lambda}$$

$$\varphi = 0 \quad \text{in } \partial\Omega_{u_\lambda}$$

where $f_0$, $f_1$, $f_2 \in L^{q'}(\Omega_{u_\lambda})$, $q' > 2$. We know that (see e.g. [4] p.462) $\varphi \in W^{1,q'}(\Omega_{u_\lambda}) \cap L^\infty(\Omega_{u_\lambda})$ and

$$\|\varphi\|_{1,q'} \le C \sum_{i=0}^{2} \|f_i\|_{q'} .$$

Let $\alpha \in C^\infty(\Omega_{u_\lambda})$ such that $\alpha = 0$ in a neighborhood of $\Gamma_\psi$. We have

$$\Delta(\alpha\varphi) = \alpha \left( f_0 + \sum_{i=1}^{2} \frac{\partial f_i}{\partial x_i} \right) + 2\, \nabla\varphi \cdot \nabla\alpha + \varphi\Delta\alpha \quad \text{in } \Omega_{u_\lambda}$$

$$\alpha\varphi = 0 \quad \text{in } \partial\Omega_{u_\lambda} .$$

Then using (3.24) we get, for $\lambda$ sufficiently small ($z_\lambda$ converges uniformly to $z_n$ in $\Omega^\delta_{u_n}$, and $z_n > 0$; this implies that $\alpha\Delta p_\lambda = 0$)

$$\int_{\Omega_{u_\lambda}} \alpha p_\lambda \left( f_0 + \sum_{i=1}^{2} \frac{\partial f_i}{\partial x_i} \right) dxdy = - \int_{\Omega_{u_\lambda}} \alpha p_\lambda (2\nabla\varphi \cdot \nabla\alpha + \varphi\Delta\alpha)\, dxdy.$$

Since $\nabla\varphi \in L^{q'}(\Omega_{u_\lambda})$ and $\{p_\lambda\}$ is bounded in $L^q(\Omega_{u_\lambda})$, $1/q + 1/q' = 1$, we conclude that

$$\|\alpha p_\lambda\|_{W^{1,q}(\Omega_{u_\lambda})} \le C \quad \forall\lambda > 0.$$

Hence $p_n \in W^{1,q}(\hat{\Omega}^\delta_{u_n})$ for any domain $\hat{\Omega}^\delta_{u_n} = \{(x,y);\ \psi(\theta) + \delta < r < u_n(\theta)\}$. However since the boundary $\Gamma_{u_n}$ is smooth we infer that $p_n \in H^2(\hat{\Omega}^\delta_{u_n})$ as claimed.

We shall prove now (1.11). By (4.7) it is also clear that $\mu_n(\varphi(r - \psi)^{-1}) = 0$ for every $\varphi \in C(\overline{\Omega_{u_n}})$ with $\overline{supp\ \varphi} \subset \{(x,y) \in \Omega_{u_n};\ \psi(\theta) < r < u_n(\theta)\}$. Let us show now that $\mu_n(z_n(r - \psi)^{-1}) = 0$. Indeed, since

$$\frac{1}{2} \int_{\Gamma_\psi} \left(\frac{\partial z_\lambda}{\partial \nu}\right)^2 d\sigma + \frac{1}{2\lambda} \int_{\Omega_{u_\lambda}} (z_\lambda^-)^2\, dxdy + \frac{1}{2n}\, \|u_\lambda\|^2_{H^3(0,2\pi)} \le$$

$$\le \frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z_n}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n}\,\|u_n\|^2_{H^3(0,2\pi)},$$

where $z_\lambda = z^{u_\lambda}$ and as seen earlier,

$$\liminf_{\lambda\to 0}\left\{\frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z_\lambda}{\partial \nu}\right)^2 + \frac{1}{2n}\,\|u_\lambda\|^2_{H^3(0,2\pi)}\right\} \ge$$

$$\ge \frac{1}{2}\int_{\Gamma_\psi}\left(\frac{\partial z_n}{\partial \nu}\right)^2 d\sigma + \frac{1}{2n}\,\|u_n\|^2_{H^3(0,2\pi)},$$

we conclude that

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\int_{\Omega_{u_\lambda}}(z_\lambda^-)^2\,dxdy = 0$$

and so $\lim_{\lambda\to 0}\mu_\lambda(z_\lambda(r-\psi)^{-1}) = 0$. Finally, since $z_\lambda \to z_n$ uniformly on every $\Omega^\delta_{u_n}$ and $\mu_\lambda \to \mu_n$ weakly, we infer that $\mu_n(z_n(r-\psi)^{-1}) = 0$ as claimed.

We conclude the proof by noticing that letting $\lambda$ tend to zero in (3.25) we obtain

$$\frac{\partial p_n}{\partial \nu}(\theta,u_n(\theta))\,\frac{\partial z_n}{\partial r}(\theta,u_n(\theta))\,\left(u_n^2(\theta) + (u_n')^2(\theta)\right)^{1/2} = n^{-1}Au_n.$$

(The family of functions $\{p_\lambda\}$ is bounded in $H^2(\Omega')$ in a neighborhood $\Omega'$ of $\Gamma_{u_n}$ and so we may pass to the limit by standard device.)

Now since $n^{-1}Au_n \longrightarrow 0$ in $(H^3_\pi(0,2\pi))'$ as $n\to\infty$ $\left(\text{because } \left\{n^{-1}\|u_n\|_{H^3(0,2\pi)}\right\}\text{ is bounded}\right)$ we see that the sequence $\{(u_n,\mu_n,p_n,z_n)\}$ satisfies the requirements of Theorem 1. (Condition (2.5) is satisfied by construction of $u_n$ as solution to problem (3.1).)

REMARK. Theorem 1 may be viewed as a maximum principle type result for the optimization problem (2.6) which as seen earlier is not well posed. This fact becomes clear if one uses Ekeland's variational principle to obtain equations (1.9) - (1.12). More precisely, let $\phi\colon H^3_\pi(0,2\pi) \longrightarrow \overline{\mathbb{R}} = (-\infty,+\infty]$ be the function

$$(4.10) \qquad \phi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Gamma_\psi} \left(\dfrac{\partial z^u}{\partial \nu}\right)^2 d\sigma & \text{if } z^u \ge 0,\ u \in U \\ +\infty & \text{otherwise.} \end{cases}$$

The function $\phi$ is lower semicontinuous, i.e.,

$$\liminf_{u \to u_0} \phi(u) \ge \phi(u_0) \qquad \forall u_0 \in H^3_\pi(0,2\pi)$$

because either $u_0 > \psi$ in $[0,2\pi]$ or $\liminf\limits_{u \to u_0} \phi(u) = +\infty$. Then by the Ekeland Variational principle there is a sequence $\{u_\varepsilon\} \subset H^3_\pi(0,2\pi)$ such that

$$\phi(u_\varepsilon) \le \inf\left\{ \phi(u);\ u \in U \right\} + \varepsilon$$

and

$$(4.11) \qquad u_\varepsilon = \arg\min_{u \in U} \left\{ \phi(u) + \varepsilon^{1/2}\|u - u_\varepsilon\|_{H^3(0,2\pi)} \right\}.$$

This implies as in the previous proof that there are $p_\varepsilon$ and $\mu_\varepsilon \in M(\overline{\Omega_{u_\varepsilon}})$ satisfying (1.10) - (1.12) and such that

$$\frac{\partial p_\varepsilon}{\partial \nu}\,\frac{\partial z^{u_\varepsilon}}{\partial r} = \varepsilon^{1/2}\eta_\varepsilon \quad \text{in } \mathcal{D}'(0,2\pi)$$

where $\|\eta_\varepsilon\|_{(H^3_\pi(0,2\pi))'} \le 1$. This is exactly the conclusion of Theorem 1.

5. Numerical example. Consider the problem (1.5) in the case when

$$(5.1) \qquad \psi(\theta) = 1.$$

$$(5.2) \qquad f(r,\theta) = \frac{1}{r^2},$$

The explicit solution is easy to compute. It is given by

(5.3) $$u(\theta) = e^{\sqrt{2}},$$

(5.4) $$z^{u}(r,\theta) = \frac{(\ln r)^2}{2}.$$

On the other hand, following the approach of this paper, we consider the following approximate problem:

(5.5) $$\inf_{u} \left\{ \frac{1}{2} \int_{\Gamma_\psi} \left(\frac{\partial z}{\partial \nu}\right)^2 d\sigma + \frac{1}{2\lambda} \int_{\Omega_u} (z^-)^2 dxdy \right\} = \inf_{u} \phi_\lambda(u).$$

We apply the steepest descent method. The gradient of the functional $\phi_\lambda$ in (5.5) is computed (as it is not difficult to see) by

(5.6) $$\nabla\phi_\lambda(u) = -\frac{\partial p}{\partial \nu}\frac{\partial z}{\partial r}\Big|_{\Gamma_u} (u^2 + u_\theta^2)^{1/2},$$

where $z$ and $p$ are computed by

(5.7) $$\Delta z = \frac{1}{r^2} \quad \text{in } \Omega_u$$

$$z = 0 \quad \text{in } \Gamma_\psi, \quad z = 1 \quad \text{in } \Gamma_u$$

(5.8) $$\Delta p = -\frac{1}{\lambda} z^- \quad \text{in } \Omega_u$$

$$p = -z_r \quad \text{in } \Gamma_\psi, \quad p = 0 \quad \text{in } \Gamma_u.$$

Then, the steepest descent iteration is given by

(5.9) $$u_{n+1} = u_n + \rho\, (p_r z_r)\big|_{r=u(\theta)}$$

$$\cdot\left[\left(\cos\theta + u_\theta \frac{\sin\theta}{u}\right)\left(u_\theta \sin\theta + u\cos\theta\right)\right.$$

$$\left. + \left(\sin\theta - u_\theta \frac{\cos\theta}{u}\right)\left(u\sin\theta - u_\theta\cos\theta\right)\right], \qquad \rho > 0.$$

So, for $\lambda = 0.00001$, $u_0 = 4.5$, and $\rho = 1$, we compute using (5.7) - (5.9), $u_{10}$

$= 4.26$, $u_{20} = 4.24$. Similarly, if one takes $\lambda = 0.000001$, $u_0 = 4.24$, and $\rho = 2$, then $u_{10} = 4.21$. We recall that the exact value is $e^{\sqrt{2}} \approx 4.11$. Computations were done on 32×32 polar grid with piecewise linear finite elements.

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