ON THE SLOW MOTION OF THE INTERFACE OF LAYERED SOLUTIONS TO THE SCALAR GINZBURG–LANDAU EQUATION

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ON THE SLOW MOTION OF THE INTERFACE OF LAYERED SOLUTIONS TO THE SCALAR GINZBURG–LANDAU EQUATION

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1.Introduction. In this paper we consider the scalar Ginzburg–Landau equation,

$$u_t - \epsilon^2 u_{xx} - f(u) = 0 \quad , \quad x \in [0,1] \quad , \tag{1.1}$$

subject to the boundary-initial conditions

$$u_x(0,t) = u_x(1,t) = 0 \quad , \tag{1.2}$$

$$u(x,0) = u_0(x) \tag{1.3}$$

and where f is the derivative of a bistable potential; more precisely, we require that

$$\begin{cases} f \in C^2(\mathbb{R}) \, , \\ f \text{ is odd,} \\ f \text{ has exactly three zeros at } u = 0, \pm 1 \, , \\ f'(\pm 1) < 0 \quad , \quad f'(0) > 0. \end{cases} \tag{1.4}$$

Our objective is to analyze the behavior of solutions of (1.1)-(1.3) for small $\epsilon > 0$, when the initial condition u_0 has a *transition layer structure*, i.e. $u_0 \approx \pm 1$ except near a transition point. (For simplicity, we shall only consider the case where u_0 has only one transition point.)

Is was proved in [1],[14] that the number of zeros of the solution of (1.1)-(1.3) is nonincreasing with time, so that, at least for small time, u has a unique zero $z(t)$ at time t. A formal analysis due to J.Neu (see [5]) predicts that u will preserve its layer structure and that $z(t)$ will move according to the equation

$$\dot{z}(t) = -C_0 \epsilon \left\{ e^{-\mu 2z/\epsilon} - e^{-\mu 2(1-z)/\epsilon} \right\} \tag{1.5}$$

where C_0 , $\mu > 0$ are constants depending only on f. In particular, (1.5) implies that the motion is extremely slow and that the layered shape of u_0 will be preserved for periods of time which are exponentially large in ϵ.

Rigorous justifications of Neu's conclusions have been the object of several papers, in particular Fusco and Hale [12], Carr and Pego [6] and Bronsard and Kohn [3]; all these articles study solutions having finitely many transition points.

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The main idea in [12] is to construct a manifold $\mathcal{M} \subset H^1$ which approximates the "global attractor" of the semiflow defined by equation (1.1), i.e. the union of the unstable manifolds of the equilibria. This *slow motion manifold* $\mathcal{M}$ can be explicitly computed. However, it is not invariant under (1.1); further, no error estimates are presented in [12] between functions on $\mathcal{M}$ and actual solutions.

Carr and Pego [6] use another approach: they construct a *slow channel* Z within which solutions behave essentially according to the predictions of Neu. Based on energy estimates, they also prove that Z attracts nearby solutions. Further, if the initial data lie *in* Z, the transition points move approximately according to (1.5). In particular, if x_0 is the unique zero of u_0, then u_0 must satisfy

$$\|u_0 - W^0\|_{L^\infty} \le Ce^{-\mu 2 l_0/\epsilon} \quad , \; x_0 \ge M\epsilon\, ,\; C, M \text{ const.} \tag{1.6}$$

where W^0 is a given profile (depending on $u_0\,,\,\epsilon$) and $l_0 = \min\{x_0\,,\,(1-x_0)\}$. They also prove, in this case, that the solution u preserves its layered structure at least until $\min\{z(t)\,,\,(1-z(t))\}$ becomes $O(\epsilon)$.

The method in [3] is based on the derivation of a lower bound for the energy of a function with a finite number N of transition points. The authors show that the transition points do not move "appreciably" (i.e. by a positive constant) unless t is very large ($t = O(\frac{1}{\epsilon^{k+1}})$ if the energy of the initial condition does not exceed the minimum energy for functions having N transition points, by more than ϵ^k). Although the results in [3] are weaker than those in [6],[12], since they do not describe the motion of the interface, the approach in that paper is more elementary and it also applies to Dirichlet boundary conditions.

In this paper we present another approach to the problem. Our method consists of a careful construction of upper and lower solutions for (1.1)-(1.3) which will capture the expected behavior of the actual solution. These super and subsolutions have the form

$$v = U + \widetilde{V} \tag{1.7}$$

where U is a perturbation of a travelling wave solution of (1.1). The function $\widetilde{V}$ is a higher order term which is constructed as a perturbation of V, where V is the highest order term in the approximate solution described in [12].

Our method is far more elementary than [6],[12], since it only involves the construction of super and subsolutions. This allows us to handle a larger class of initial conditions; at the same time, it can be easily modified to obtain results like those in [3] (at least for Neumann boundary conditions). Finally, since we closely follow the constructions in [12], our method can be interpreted as providing "error estimates" for the work of Fusco and Hale.

In §2 we state some results about travelling wave solutions of equations of the type (1.1) (most of which are well-known). These results will be needed to construct the principal part U of the super and subsolutions. Section 3 is devoted to the construction of $\widetilde{V}$ and to the derivation of some estimates on it. The final section, §4, contains the complete construction of the upper and lower solutions, v_u , v_l , as well as the proof of the fact that

$$v_l \leq u \leq v_u \tag{1.8}$$

for exponentially large periods of time. Our final results are stated in two theorems under different assumptions on the initial data, where, for definiteness, we assume

$$0 < x_0 < \frac{1}{2} .$$

In Theorem 4.2 we assume that

$$\left| u_0(x) - U^0\left(\frac{x - x_0}{\epsilon}\right) \right| \leq \delta_0 \left\{ e^{-\mu(1-\gamma)x/\epsilon} + e^{-\mu(1-\gamma)x_0 \frac{(1-x)}{(1-x_0)}/\epsilon} \right\} ,$$
$$x \in [0,1] , \; x_0 \geq M_1 \epsilon , \tag{1.9}$$

while in Theorem 4.4 we assume the stronger condition

$$\left| u_0(x) - \left(U^0\left(\frac{x - x_0}{\epsilon}\right) + V(x_0, x) \right) \right| \leq \delta_1 e^{-\mu x_0/\epsilon} \left\{ e^{-\mu x/\epsilon} + e^{-\mu x_0 \frac{(1-x)}{(1-x_0)}/\epsilon} \right\} ,$$
$$x \in [0,1] , \; x_0 \geq K_0 \epsilon |\ln \epsilon| , \tag{1.10}$$

where γ is any constant, $0 < \gamma < \frac{1}{2}$, and $\delta_0 = \delta_0(\gamma)$, M_1 , δ_1 , K_0 are given constants. The function U^0 is the unique travelling wave solution of (1.1) satisfying

$$U^0(\pm\infty) = \pm 1 \quad , \quad U^0(0) = 0 \quad ,$$

and V is the function defined in [12], which is exponentially small throughout the interval $[0, 1]$. Notice that the right hand side of (1.10) is smaller than $e^{-\mu x_0/\epsilon}$ times the right hand side of (1.9) and that both of them majorize the right hand side of (1.6), for small $\epsilon > 0$.

The explicit construction of v_u , v_l satisfying (1.8) yields the following conclusions:

(i) Under assumption (1.9),

$$v_u - v_l = O(\epsilon) \tag{1.11}$$

as long as t is such that $z(t) > \frac{(1+\gamma)}{2} x_0$. Furthermore, the time it takes for $z(t)$ to become $O(\epsilon)$ can be estimated as in [6];

(ii) Under assumption (1.10), (1.11) holds as long as

$$z(t) \geq K_0 \epsilon |\ln \epsilon|.$$

Finally, we remark that the behavior of solutions of (1.1)-(1.3) changes dramatically in higher space dimensions. In fact, if we replace $\epsilon^2 u_{xx}$ by $\epsilon^2 \Delta u$ in (1.1), u_x in (1.2) by $\frac{\partial u}{\partial \nu}$ (normal derivative of u) and $[0,1]$ by $\Omega \subset \mathbb{R}^n$, then the transition hypersurface Γ moves according to the law

$$\mathrm{v} = \epsilon^2 \kappa$$

where v is the normal velocity and κ the mean curvature of Γ (see e.g. [4],[7],[8],[9]).

2.Properties of travelling wave solutions. A simple matched asymptotic analysis of equation (1.1) yields the following conclusion (see e.g. [10]):

if x_0 is the unique zero of the initial condition u_0 and $u_0 \approx 1$ for $x > x_0 + M\epsilon$, $u_0 \approx -1$ for $x < x_0 - M\epsilon$, then u preserves the layered structure of u_0; in fact, a first order approximation for u near the internal layer is given by

$$U\left(\frac{x - x(t)}{\epsilon}\right) \tag{2.1}$$

where

$$\dot{x} = \mathrm{v}_0 \quad , \quad x(0) = x_0 \tag{2.2}$$

and (U, v_0) is the unique solution of

$$U'' + \mathrm{v}_0 U' + f(U) = 0 \quad , \quad x \in \mathbb{R} \quad , \tag{2.3}$$

$$U(\pm\infty) = \pm 1 \quad , \quad U(0) = 0. \tag{2.4}$$

Furthermore, since f is odd, it is easy to verify that

$$\mathrm{v}_0 = 0 \tag{2.5}$$

and also (at least formally) that the approximation (2.1),(2.2),(2.5) is valid up to any order in ϵ.

Based on this fact, we expect to find upper and lower solutions of (1.1) whose principal part near an internal layer is given by

$$U\left(\beta, \frac{x - z(t)}{\epsilon}\right) \qquad (\beta = \beta(x,t)) \tag{2.6}$$

where $U(\beta, x)$ satisfies

$$U'' + \mathrm{v}(\beta) U' + f(U) - \beta = 0 \quad , \quad x \in \mathbb{R} \quad , \tag{2.7}$$

$$U(\beta, \pm\infty) = h_{\pm}(\beta) \quad , \tag{2.8}$$

$$U(\beta, 0) = 0. \tag{2.9}$$

Here, $' = \dfrac{\partial}{\partial x}$, $h_-(\beta) < h_0(\beta) < h_+(\beta)$ are the three zeros of $f(u) - \beta$ and $\mathrm{v}(\beta)$ is the unique constant that guarantees the existence of U satisfying (2.7)-(2.9) (see Lemma 2.1). Notice that, since $f'(\pm 1), f'(0) \neq 0$, the functions h_-, h_0, h_+ are well defined if β is sufficiently small, say

$$|\beta| < \rho_1 \quad , \quad \rho_1 > 0. \tag{2.10}$$

The following lemma states the properties of $U(\beta, x)$ that will be needed later to construct super and subsolutions of (1.1)-(1.3) (see §4).

LEMMA 2.1. *Let F be defined by*

$$F'(u) = -2f(u) \quad , \quad F(\pm 1) = 0 \tag{2.11}$$

and write

$$f'(\pm 1) \equiv -\mu^2 \quad , \quad \int_0^1 F(u)^{1/2} du = \int_{-1}^0 F(u)^{1/2} du \equiv K$$
$$K_1 \equiv \exp\left\{ \mu \int_0^1 F(u)^{-1/2} \left(1 - \frac{F(u)^{1/2}}{\mu(1-u)} \right) du \right\}. \tag{2.12}$$

Then,

(i) *there exists a unique solution $(U(\beta, x), \mathrm{v}(\beta))$ of (2.7)-(2.9);*

(ii) *the solution (U, v) satisfies*

$$U'(\beta, x) > 0 \quad , \quad x \in \mathbb{R} \quad , \tag{2.13}$$

$$\mathrm{v}(\beta) = \beta \left(h_+(\beta) - h_-(\beta) \right) \left(\int_{-\infty}^{\infty} (U'(\beta, x))^2 dx \right)^{-1} ; \tag{2.14}$$

(iii) *there exists ρ_2, $0 < \rho_2 \leq \rho_1$, such that, for $|\beta| < \rho_2$,*

$$\mathrm{v}, U(., x) \in C^2 \quad , \quad U(\beta, .), \frac{\partial U}{\partial \beta}(\beta, .), \frac{\partial^2 U}{\partial \beta^2}(\beta, .) \in C^2 \tag{2.15}$$

with derivatives bounded uniformly in (β, x), $x \in \mathbb{R}$;

(iv) *given $r > 0$, there exist constants M_0, ρ_0, $0 < \rho_0 \leq \rho_2$, such that, if $|\beta| < \rho_0$, we have*

$$\frac{\partial U}{\partial \beta} - h'_+(\beta), \frac{\partial^2 U}{\partial x \partial \beta}, \frac{\partial^3 U}{\partial^2 x \partial \beta} = O\left(e^{-\mu(1-r)x} \right) \quad , \quad x > M_0 \quad , \tag{2.16}$$

$$\frac{\partial U}{\partial \beta} - h'_-(\beta), \frac{\partial^2 U}{\partial x \partial \beta}, \frac{\partial^3 U}{\partial^2 x \partial \beta} = O\left(e^{\mu(1-r)x} \right) \quad , \quad x < -M_0 ; \tag{2.17}$$

(v) *if* $\beta = 0$, *then*

$$1 - U\,,\ \frac{U'}{\mu}\,,\ -\frac{U''}{\mu^2} = K_1 e^{-\mu x}\left(1 + O\left(e^{-\mu x}\right)\right) \quad , \quad x > 0 \quad , \tag{2.18}$$

$$1 + U\,,\ \frac{U'}{\mu}\,,\ \frac{U''}{\mu^2} = K_1 e^{\mu x}\left(1 + O\left(e^{\mu x}\right)\right) \quad , \quad x < 0 \quad , \tag{2.19}$$

$$\frac{\partial U}{\partial \beta} + \frac{1}{\mu^2}\,,\ \frac{\partial U'}{\partial \beta} = O\left((|x| + 1)e^{-\mu|x|}\right) \quad , \quad x \in \mathbb{R} \tag{2.20}$$

and

$$\mathrm{v}'(0) = \frac{1}{K}. \tag{2.21}$$

Proof. Assertion (i) was proved in [13] and [2;Thm.4.1], while (2.13) follows from [11]. The representation (2.14) for $\mathrm{v}(\beta)$ can be obtained by multiplying (2.7) by U' and integrating on the whole real line.

The statements (iii) and (iv) follow from the analysis in [10;Appendix].

When $\beta = 0$, (2.7)-(2.9) become

$$U'' + f(U) = 0 \tag{2.22}$$

$$U(\pm\infty) = \pm 1 \quad , \quad U(0) = 0. \tag{2.23}$$

Upon integrating (2.22), (2.18) and (2.19) easily follow.

Finally, differentiating (2.7)-(2.9) with respect to β and setting $\beta = 0$, $\frac{\partial U}{\partial \beta}(0, x) = \phi(x)$ we get

$$\phi'' + f'(U(0, x))\phi = 1 - \mathrm{v}'(0)U'(0, x) \quad , \tag{2.24}$$

$$\phi(\pm\infty) = -\frac{1}{\mu^2} \quad , \quad \phi(0) = 0. \tag{2.25}$$

To compute ϕ, make the change of variables

$$v = U(0, x)$$

and use (2.11) and the fact that

$$U'(0, x) = F(U(0, x))^{1/2}\,.$$

The equation for $\phi = \phi(v)$ becomes

$$(F\phi' + f\phi)' = 1 - \mathrm{v}'(0)F^{1/2} \tag{2.26}$$

with

$$\phi(\pm 1) = -\frac{1}{\mu^2} \quad , \quad \phi(0) = 0. \tag{2.27}$$

Since ϕ is bounded (from (iii)), the unique solution of (2.26),(2.27) is given by

$$\phi(v) = \frac{F(v)^{1/2}}{K} \int_0^v F(s)^{-3/2} \left(\int_0^s (K - F(r)^{1/2}) dr \right) ds \tag{2.28}$$

and $v'(0) = \frac{1}{K}$ as asserted in (2.21). Now, (2.20) is an easy consequence of (2.28) and the properties of $v = U(0, x)$ in (2.18),(2.19). □

3. Approximation of layered solutions. As discussed at the beginning of §2, we expect (2.1),(2.2),(2.5) to be a good approximation of the solution $u(x,t)$ of (1.1)-(1.3) near $x = x_0$ as long as

$$t = O(\frac{1}{\epsilon^k}) \quad \text{for some } k > 0. \tag{3.1}$$

Furthermore, since

$$\begin{aligned} U'(0, \frac{x - x_0}{\epsilon})\Big|_{x=0} &= O\left(e^{-\mu x_0/\epsilon}\right) \quad , \\ U'(0, \frac{x - x_0}{\epsilon})\Big|_{x=1} &= O\left(e^{-\mu(1-x_0)/\epsilon}\right) \quad , \end{aligned} \tag{3.2}$$

$U(0, \frac{x-x_0}{\epsilon})$ will be very close to $u(x,t)$ throughout the interval $[0,1]$, provided t satisfies (3.1).

However, (3.2) also shows that, in general, we cannot expect $U(0, \frac{x-x_0}{\epsilon})$ to continue to approximate u for "larger" time than in (3.1), e.g. for $t \geq e^{\rho/\epsilon}$, $\rho > 0$.

A formal calculation (see [5]) shows that for exponentially large times, an approximation of u is given by

$$U(0, \frac{x - \bar{z}(t)}{\epsilon})$$

where

$$\dot{\bar{z}} = p(\bar{z}) \equiv -\frac{\epsilon \mu^2}{K} K_1^2 \left\{ e^{-\mu 2 \bar{z}/\epsilon} - e^{-\mu 2(1-\bar{z})/\epsilon} \right\}. \tag{3.3}$$

Furthermore, a higher order approximation was computed in [12]: it has the form

$$U(0, \frac{x - z(t)}{\epsilon}) + V(z, x) \tag{3.4}$$

where $V = V^\epsilon$ and (V, z) is the solution of

$$\dot{z} = c(z) \quad , \tag{3.5}$$

$$\epsilon^2 V_{xx} + f'(U(0,\frac{x-z}{\epsilon}))V = -\frac{c(z)}{\epsilon}U'(0,\frac{x-z}{\epsilon}) \quad , \quad x \in [0,1] \quad , \tag{3.6a}$$

$$V_x = -\frac{1}{\epsilon}U'(0,\frac{x-z}{\epsilon}) \quad , \quad x = 0,1 \quad , \tag{3.7}$$

$$\int_0^1 V(z,x)U'(0,\frac{x-z}{\epsilon})dx = 0. \tag{3.6b}$$

Notice that $c(z)$ should be treated as an unknown for the problem (3.6a,b),(3.7).

This section is devoted to the study of the function V and some perturbations of it which will be used as the highest order term of the upper and lower solutions to be constructed in the next section.

We begin by recalling the main result of [12]. We state it in a slightly different form although the proof is exactly the same. For simplicity, we write

$$U^0(x) \equiv U(0,x) \tag{3.8}$$

and, for definiteness, assume

$$0 < z < \frac{1}{2}. \tag{3.9}$$

THEOREM 3.1 ([12;p.86]). *Let $a, b \in \mathbb{R}$. There exists a constant M_1 such that, if*

$$z > M_1\epsilon \quad , \tag{3.10}$$

then there is a unique solution (V,c) of (3.6a,b) satisfying

$$V_x(z,0) = -\frac{a}{\epsilon}U_x^0(\frac{-z}{\epsilon}) \quad , \quad V_x(z,1) = -\frac{b}{\epsilon}U_x^0(\frac{1-z}{\epsilon}). \tag{3.6c}$$

The functions $V, c \in C^1$ and $V(z,.) \in C^2$. Moreover,

$$\begin{aligned} V(z,x) &= [F^{1/2}(\alpha + \beta\int_0^v F^{-3/2} - \frac{c}{\epsilon}\int_0^v (F^{-3/2}\int_0^s F^{1/2}))]\Big|_{v=U^0((x-z)/\epsilon)} \\ &\equiv V(z,v)\big|_{v=U^0((x-z)/\epsilon)} \quad , \end{aligned} \tag{3.11}$$

$$\begin{aligned} \frac{c(z)}{\epsilon} &= -a(1+d(z/\epsilon))\left\{p_1(z/\epsilon) + \frac{\mu^2 K_1^2}{K}e^{-\mu 2z/\epsilon}(1+d_3(z/\epsilon))(1+d_4((1-z)/\epsilon))\right\} \\ &\quad + b(1+d(z/\epsilon))\left\{p_2(z/\epsilon) + \frac{\mu^2 K_1^2}{K}e^{-\mu 2(1-z)/\epsilon}(1+d_3(z/\epsilon))(1+d_1(z/\epsilon))\right\} \end{aligned} \tag{3.12}$$

where

$$\alpha = -ae^{-\mu 2z/\epsilon}(\frac{\mu^2 K_1^2}{2K}\frac{z}{\epsilon} + O(1)) + be^{-\mu 2(1-z)/\epsilon}(\frac{\mu^2 K_1^2}{2K}\frac{(1-z)}{\epsilon} + O(1)) \quad , \tag{3.13}$$

$$
\begin{aligned}
\beta = a\big(1+d(z/\epsilon)\big)\Big\{p_3(z/\epsilon) - \mu^2 K_1^2 e^{-\mu 2z/\epsilon}\big(1+d_3(z/\epsilon)\big)\big(1+d_5((1-z)/\epsilon)\big)\Big\} \\
+ b\big(1+d(z/\epsilon)\big)\Big\{p_4(z/\epsilon) - \mu^2 K_1^2 e^{-\mu 2(1-z)/\epsilon}\big(1+d_3(z/\epsilon)\big)\big(1+d_2(z/\epsilon)\big)\Big\}
\end{aligned}
\tag{3.14}
$$

and

$$
p_i(s) = O\Big(\frac{e^{-\mu 2/\epsilon}}{\epsilon}\Big) \quad , \quad d(s)\,,\, d_j(s) = O\big(e^{-\mu s}\big) \tag{3.15}
$$

with p_i, d_j independent of a, b.

REMARK 3.2. Notice that the principal part of $c(z)$ in (3.12) coincides with $p(z)$ (defined in (3.3)) when $a = b = 1$.

COROLLARY 3.3. *Under the assumption (3.10), the function V in (3.11) satisfies*

$$
\begin{aligned}
|V(z,x)| \le \Bigg\{|a|\Bigg[e^{-\mu(2z+(1-z))/\epsilon}\frac{1}{1-v} + e^{-\mu 2z/\epsilon}\frac{1}{1+v} \\
+ e^{-\mu 2z/\epsilon}\frac{z}{\epsilon}(1-v^2)\Bigg] + |b|\Bigg[\Big(e^{-\mu 2(1-z)/\epsilon} + \frac{e^{-\mu 2/\epsilon}}{\epsilon}\Big)\frac{1}{1-v} \\
+ \Big(e^{-\mu(2(1-z)+z)/\epsilon} + \frac{e^{-\mu 2/\epsilon}}{\epsilon}\Big)\frac{1}{1+v} + \frac{e^{-\mu 2(1-z)/\epsilon}}{\epsilon}(1-v^2)\Bigg]\Bigg\}\Bigg|_{v=U^0((x-z)/\epsilon)}
\end{aligned}
\tag{3.16}
$$

Proof. Let

$$
L = L(v) \equiv F(v)^{1/2}\int_0^v F^{-3/2} \quad , \quad M = M(v) \equiv F(v)^{1/2}\int_0^v \Big(F^{-3/2}\int_0^s F^{1/2}\Big)\,. \tag{3.17}
$$

Then, (see [12])

$$
L \text{ is odd} \quad , \quad M \text{ is even} \tag{3.18a}
$$

$$
L\,,\ \frac{M}{K} = \frac{1}{2\mu^2}(1-v)^{-1}\big(1+O(1-v)\big) \quad , \quad \text{for } v \text{ near } 1, \tag{3.18b}
$$

and

$$
V = \Big(F^{1/2}\alpha + \beta L - \frac{c}{\epsilon}M\Big)\Big|_{v=U^0((x-z)/\epsilon)}. \tag{3.19}
$$

On the other hand, using (3.12),(3.14), we get

$$
\beta - \frac{cK}{\epsilon} = aO\big(e^{-\mu(2z+(1-z))/\epsilon}\big) + bO\big(e^{-\mu 2(1-z)/\epsilon} + \frac{e^{-\mu 2/\epsilon}}{\epsilon}\big) \tag{3.20a}
$$

and

$$
-\beta - \frac{cK}{\epsilon} = aO\big(e^{-\mu 2z/\epsilon}\big) + bO\big(e^{-\mu(2(1-z)+z)/\epsilon} + \frac{e^{-\mu 2/\epsilon}}{\epsilon}\big). \tag{3.20b}
$$

Now, (3.16) follows from (3.13) and (3.18)-(3.20). □

Next, we prove existence, uniqueness and certain properties of a particular perturbation $\widetilde{V}$ of V. The main feature that we look for in constructing $\widetilde{V}$ is that $U+\widetilde{V}$ should approximate a solution of (1.1),(1.2) even better than $U+V$. For this, let ρ be a small number, say $0<\rho<\frac{1}{4}$, and let $h:\mathbb{R}\to\mathbb{R}$ be a smooth function satisfying

$$h(v)=\begin{cases}1, & \text{if } v\le-\rho,\\ 0, & \text{if } v\ge 0.\end{cases} \tag{3.21}$$

Let $\chi:\mathbb{R}\to\mathbb{R}$ be a smooth function such that

$$\chi(z)=\begin{cases}0, & \text{if } z\ge\frac{2}{5}-\frac{1}{3}(\frac{2}{5}-\frac{1}{3})=\frac{17}{45},\\ 1, & \text{if } z\le\frac{1}{3}+\frac{1}{3}(\frac{2}{5}-\frac{1}{3})=\frac{16}{45},\end{cases} \tag{3.22}$$

and set

$$H(v,z)=1+\chi(z)(h(v)-1). \tag{3.23}$$

Notice that

$$H(v,z)=\begin{cases}1-\chi(z), & \text{if } v\ge 0,\\ 1, & \text{if } v\le-\rho.\end{cases} \tag{3.24a}$$

Also note that

$$z\le\frac{17}{45}\quad\text{implies}\quad 2(1-l)(1-z)>3(1+l)z \tag{3.24b}$$

and that

$$1-\chi(z)\equiv 0\quad\text{if}\quad 2(1-l)z<(1+l)(1-z), \tag{3.24c}$$

where l is (fixed) small number, e.g. $0<l<\frac{1}{40}$.

Consider the problem

$$\begin{aligned}\epsilon^2\widetilde{V}_{xx}+f'(U^0(\frac{x-z}{\epsilon}))\widetilde{V}&=-\frac{\widetilde{c}(z)}{\epsilon}U^0_x(\frac{x-z}{\epsilon})\\ &-\frac{1}{2}f''(U^0(\frac{x-z}{\epsilon}))H(U^0(\frac{x-z}{\epsilon}),z)\big(V(z,x)\big)^2\quad,\quad x\in[0,1]\quad,\end{aligned} \tag{3.25a}$$

$$\widetilde{V}_x=-\frac{a}{\epsilon}U^0_x(\frac{-z}{\epsilon})\quad,\quad\widetilde{V}_x=-\frac{b}{\epsilon}U^0_x(\frac{1-z}{\epsilon})\quad, \tag{3.25b}$$

$$\int_0^1\widetilde{V}(z,x)U^0_x(\frac{x-z}{\epsilon})dx=0\quad, \tag{3.25c}$$

where $a,b\in\mathbb{R}$, V is the solution of (3.6) and H is defined in (3.23).

THEOREM 3.4. *Fix a small number r*

$$0 < r < \frac{l}{10} \quad (l \text{ appearing in (3.24)}) \tag{3.26}$$

and let

$$a = \lambda_1^0 + \lambda_1 e^{\mu r z/\epsilon} \tag{3.27a}$$
$$b = \lambda_2^0 + \lambda_2 e^{\mu r(1-z)/\epsilon} \tag{3.27b}$$

where λ_1^0 , λ_2^0 *are constants bounded independently of* ϵ *and*

$$|\lambda_1| , |\lambda_2| \le \lambda \le 1 \quad (\lambda \text{ const.}). \tag{3.28}$$

There exists a constant $M_2 > 0$ *such that, if*

$$z > M_2 \epsilon \quad , \tag{3.29}$$

then there is a unique solution $(\widetilde{V}, \widetilde{c})$ *of (3.25).*

The functions $\widetilde{V}, \widetilde{c} \in C^1$ *and* $\widetilde{V}(z,.) \in C^2$ *and they satisfy:*

$$\begin{aligned}\widetilde{V}(z,x) = \Bigg[F(v)^{1/2} \Bigg(\widetilde{\alpha}(z) + \widetilde{\beta}(z) \int_0^v F(s)^{-3/2} ds - \frac{\widetilde{c}(z)}{\epsilon} \int_0^v \big(F(s)^{-3/2} \int_0^s F(\omega)^{1/2} d\omega\big) ds \\ - \int_0^v \big(F(s)^{-3/2} \int_0^s \frac{f''(\omega)}{2} H(\omega,z) V(z,\omega)^2 d\omega\big) ds \Bigg) \Bigg] \Bigg|_{v=U^0((x-z)/\epsilon)} \\ \equiv \widetilde{V}(z,v)\Big|_{v=U^0((x-z)/\epsilon)} \quad ,\end{aligned} \tag{3.30}$$

$$\widetilde{\alpha} = \overline{\alpha} + \alpha_0 \quad , \quad \widetilde{\beta} = \overline{\beta} + \beta_0 \quad , \quad \widetilde{c} = \overline{c} + c_0 \quad , \tag{3.31}$$

where

$$\alpha_0 = r\Big(\frac{3z}{\epsilon}\Big)\big(1 + O(e^{-\mu z/\epsilon})\big) \quad , \tag{3.32a}$$

$$\begin{aligned}\beta_0 = r\Big(\frac{3z}{\epsilon}\Big)\Big\{ - \frac{K}{2\mu K_1^2} e^{-\mu 2z/\epsilon} \big(1 + d_6(z/\epsilon)\big)\big(1 + d_5((1-z)/\epsilon)\big) \\ + \frac{K}{2\mu K_1^2} e^{-\mu 2(1-z)/\epsilon} \big(1 + d_2(z/\epsilon)\big)\big(1 + d_7((1-z)/\epsilon)\big) \Big\} \quad ,\end{aligned} \tag{3.32b}$$

$$\begin{aligned}\frac{c_0}{\epsilon} = r\Big(\frac{3z}{\epsilon}\Big)\Big\{ - \frac{e^{-\mu 2z/\epsilon}}{2\mu K_1^2} \big(1 + d_6(z/\epsilon)\big)\big(1 + d_4((1-z)/\epsilon)\big) \\ + \frac{e^{-\mu 2(1-z)/\epsilon}}{2\mu K_1^2} \big(1 + d_1(z/\epsilon)\big)\big(1 + d_7((1-z)/\epsilon)\big) \Big\} \quad ,\end{aligned} \tag{3.32c}$$

and $\overline{\alpha}$, $\overline{\beta}$, $\overline{c}$ are defined as α, β, c in (3.12)-(3.14) but with

$$
\begin{aligned}
&a \text{ replaced by } a + q_1(z/\epsilon) \\
&b \text{ replaced by } b + (1-\chi(z))\left\{ q_2((1-z)/\epsilon) + \left(\frac{z}{\epsilon}\right)^2 e^{-\mu(3z-2(1-z))/\epsilon} q_3(z/\epsilon) \right\}.
\end{aligned}
\tag{3.33}
$$

Here,

$$
r(s)\,,\, q_i(s) = O\big(e^{-\mu s} + \lambda e^{-\mu(1-2r)s}\big) \quad , \tag{3.34a}
$$

$$
d_6(s)\,,\, d_7(s) = O\big(e^{-\mu s}\big) \quad (\textit{independent of } a, b) \tag{3.34b}
$$

and $d_1, \ldots, d_5$ are the functions appearing in Theorem 3.1.

Proof. The proof uses the same ideas as the proof of Theorem 3.1. As in [12] we make the change of variables

$$
v = U^0\left(\frac{x-z}{\epsilon}\right) \tag{3.35}
$$

to transform (3.25) into

$$
(F\widetilde{V}' + f\widetilde{V})' = -\frac{\widetilde{c}}{\epsilon}F^{1/2} - gHV^2 \quad , \tag{3.36a}
$$

$$
\widetilde{V}'(v^0) = -a \quad , \quad \widetilde{V}'(v^*) = -b \quad , \tag{3.36b}
$$

$$
\int_{v^0}^{v^*} \widetilde{V} dv = 0 \quad , \tag{3.36c}
$$

where $' = \frac{d}{dv}$, $g(v) = \frac{1}{2}\left(\frac{d^2 f}{du^2}\right)(v)$,

$$
v^0 = U^0\left(\frac{-z}{\epsilon}\right) \quad \text{and} \quad v^* = U^0\left(\frac{1-z}{\epsilon}\right). \tag{3.37}
$$

From (3.36a), it is clear that $\widetilde{V}$ is given by (3.30) for some constants $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{c}$.

Let

$$
N = N(v) \equiv F(v)^{1/2} \int_0^v F(s)^{-3/2} \left(\int_0^s g(\omega) H(\omega, z) V(z,\omega)^2 d\omega \right) ds \tag{3.38}
$$

and notice that

$$
F(v)^{1/2} = \mu(1-v)(1 + O(1-v))\,, \quad v \text{ near } 1; \quad F^{1/2} \text{ is even.} \tag{3.39}
$$

If a, b satisfy (3.27), then (3.16) implies the existence of a constant $C > 0$ such that

$$|V(z,x)| \leq C\Big\{ \big(1+\lambda e^{\mu r z/\epsilon}\big)\Big[e^{-\mu(2z+(1-z))/\epsilon}\frac{1}{1-v} + e^{-\mu 2z/\epsilon}\frac{1}{1+v} + e^{-\mu 2z/\epsilon}\frac{z}{\epsilon}\Big] + \big(1+\lambda e^{\mu r(1-z)/\epsilon}\big)e^{-\mu 2(1-z)/\epsilon}\frac{1}{1-v}\Big\}$$

and therefore

$$V^2 \leq C\Big\{ \big(1+\lambda e^{\mu 2rz/\epsilon}\big)e^{-\mu 4z/\epsilon}\Big(\frac{z}{\epsilon}\Big)^2 + \big(1+\lambda e^{\mu 2r(1-z)/\epsilon}\big)e^{-\mu 4(1-z)/\epsilon}\frac{1}{(1-v)^2}\Big\} \quad , \quad v \geq 0 \quad , \tag{3.40}$$
$$V^2 \leq C\big(1+\lambda e^{\mu 2rz/\epsilon}\big)e^{-\mu 4z/\epsilon}\Big(\Big(\frac{z}{\epsilon}\Big)^2 + \frac{1}{(1+v)^2}\Big) \quad , \quad v \leq 0.$$

Thus, using the definition of H, (3.38)-(3.40) imply the following estimates:

$$N(v,z) = (1-\chi(z))\Big\{O\Big(e^{-\mu 4z/\epsilon}\big(\frac{z}{\epsilon}\big)^2\big(1+\lambda e^{\mu 2rz/\epsilon}\big)\Big)\frac{1}{(1-v)} + O\Big(e^{-\mu 4(1-z)/\epsilon}\big(1+\lambda e^{\mu 2r(1-z)/\epsilon}\big)\Big)\frac{1}{(1-v)^2}\Big\} + O\Big(e^{-\mu 4z/\epsilon}\big(1+\lambda e^{\mu 2rz/\epsilon}\big)\Big)\Big(\big(\frac{z}{\epsilon}\big)^2\frac{1}{(1+v)} + \frac{1}{(1+v)^2}\Big)I_{v\leq 0} \quad , \tag{3.41a}$$

$$N'(v,z) = (1-\chi(z))\Big\{O\Big(e^{-\mu 4z/\epsilon}\big(\frac{z}{\epsilon}\big)^2\big(1+\lambda e^{\mu 2rz/\epsilon}\big)\Big)\frac{1}{(1-v)^2} + O\Big(e^{-\mu 4(1-z)/\epsilon}\big(1+\lambda e^{\mu 2r(1-z)/\epsilon}\big)\Big)\frac{1}{(1-v)^3}\Big\} + O\Big(e^{-\mu 4z/\epsilon}\big(1+\lambda e^{\mu 2rz/\epsilon}\big)\Big)\Big(\big(\frac{z}{\epsilon}\big)^2\frac{1}{(1+v)^2} + \frac{1}{(1+v)^3}\Big)I_{v\leq 0} \quad , \tag{3.41b}$$

and

$$\int_{v^0}^{v^*} N(v,z)dv = (1-\chi(z))\Big\{O\Big(e^{-\mu 4z/\epsilon}\big(\frac{z}{\epsilon}\big)^2\big(1+\lambda e^{\mu 2rz/\epsilon}\big)\Big)|\ln(1-v^*)| + O\Big(e^{-\mu 4(1-z)/\epsilon}\big(1+\lambda e^{\mu 2r(1-z)/\epsilon}\big)\Big)\frac{1}{(1-v^*)}\Big\} + O\Big(e^{-\mu 4z/\epsilon}\big(1+\lambda e^{\mu 2rz/\epsilon}\big)\Big)\Big(\big(\frac{z}{\epsilon}\big)^2|\ln(1+v^0)| + \frac{1}{(1+v^0)}\Big) \quad , \tag{3.41c}$$

where $I_{v\leq 0}$ =characteristic function of $\{v \leq 0\}$.

Now, using (3.30),(3.41) and the asymptotic formulas for the terms in $\widetilde{V}$ (see [12]), we may write the conditions (3.25b,c) in the form

$$A\begin{pmatrix}\widetilde{\alpha}\\ \widetilde{\beta}\\ \widetilde{c}/\epsilon\end{pmatrix}=\begin{pmatrix}-\widetilde{a}\\ -\widetilde{b}\\ \widetilde{\gamma}\end{pmatrix}\quad, \tag{3.42}$$

where

$$A=\begin{pmatrix}\mu(1+d_6(z/\epsilon)) & \frac{e^{\mu 2z/\epsilon}}{(2\mu^2K_1^2)}(1+d_1(z/\epsilon)) & \frac{Ke^{\mu 2z/\epsilon}}{(2\mu^2K_1^2)}(1+d_2(z/\epsilon))\\ -\mu(1+d_7((1-z)/\epsilon)) & \frac{e^{\mu 2(1-z)/\epsilon}}{(2\mu^2K_1^2)}(1+d_4((1-z)/\epsilon)) & -\frac{Ke^{\mu 2(1-z)/\epsilon}}{(2\mu^2K_1^2)}(1+d_5((1-z)/\epsilon))\\ 2K(1+d_3(z/\epsilon)) & \frac{(1-2z)}{(2\mu\epsilon)}+O(1) & -\frac{K}{(2\mu\epsilon)}+O(1)\end{pmatrix}, \tag{3.43}$$

and

$$\widetilde{a}=a-N'(v^0,z)\equiv a+q_1(z/\epsilon)\quad, \tag{3.44a}$$

$$\begin{aligned}\widetilde{b}&=b-N'(v^*,z)\\ &\equiv b+(1-\chi(z))\left\{q_2((1-z)/\epsilon)+\left(\frac{z}{\epsilon}\right)^2e^{-\mu(3z-2(1-z))/\epsilon}q_3(z/\epsilon)\right\}\quad,\end{aligned} \tag{3.44b}$$

$$\widetilde{\gamma}=\int_{v^0}^{v^*}N\,dv=O\left(e^{-\mu 3z/\epsilon}(1+\lambda e^{\mu 2rz/\epsilon})\right). \tag{3.44c}$$

Notice that

$$(\det(A))^{-1}=-\frac{(\mu^2K_1^2)^2}{K^2}e^{-\mu 2/\epsilon}(1+d(z/\epsilon))\quad, \tag{3.45}$$

where $d(s)$ is the function in Theorem 3.1.

Let α_0 , β_0 , c_0 be defined by the equation

$$A\begin{pmatrix}\alpha_0\\ \beta_0\\ c_0/\epsilon\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \widetilde{\gamma}\end{pmatrix}. \tag{3.46}$$

Then, by (3.31),

$$A\begin{pmatrix}\overline{\alpha}\\ \overline{\beta}\\ \overline{c}/\epsilon\end{pmatrix}=\begin{pmatrix}-\widetilde{a}\\ -\widetilde{b}\\ 0\end{pmatrix}\quad,$$

that is, $\overline{\alpha}$, $\overline{\beta}$, $\overline{c}$ are defined like α, β, c in (3.12)-(3.14) but with a, b replaced by $\widetilde{a}$, $\widetilde{b}$, respectively. Finally, (3.43),(3.44c) and (3.46) immediately yield (3.32) if we choose

$$r(3z/\epsilon)=\widetilde{\gamma}\left(-\frac{(\mu^2K_1^2)^2}{K^2}\right)(1+d(z/\epsilon)).\quad\Box \tag{3.47}$$

We now derive the estimates on (V,c), $(\widetilde{V},\widetilde{c})$ and their difference, which will be needed in §4.

LEMMA 3.5. *Let (V, c) be the solution of (3.6), let $(\widetilde{V}, \widetilde{c})$ be the solution of (3.25) and assume a , b satisfy (3.27),(3.28). Then:*

$$|V| + |\widetilde{V}| + \epsilon|V_z| + \epsilon|\widetilde{V}_z| = \Big\{ \big(1 + \lambda e^{\mu r z/\epsilon}\big) \Big[O\big(e^{-\mu(2z+(1-z))/\epsilon}\big) \frac{1}{1-v} + O\big(e^{-\mu 2z/\epsilon}\big) \frac{1}{1+v} + O\big(e^{-\mu 2z/\epsilon} \frac{z}{\epsilon}\big) \Big] + \big(1 + \lambda e^{\mu r(1-z)/\epsilon}\big) O\big(e^{-\mu 2(1-z)/\epsilon}\big) \frac{1}{1-v} \Big\} \Big|_{v=U^0((x-z)/\epsilon)} ; \tag{3.48a}$$

$$|\frac{\partial V}{\partial a}| + |\frac{\partial \widetilde{V}}{\partial a}| = \Big\{ O\big(e^{-\mu(2z+(1-z))/\epsilon}\big) \frac{1}{1-v} + O\big(e^{-\mu 2z/\epsilon}\big)\big(\frac{z}{\epsilon} + \frac{1}{1+v}\big) \Big\} \Big|_{v=U^0((x-z)/\epsilon)} ; \tag{3.48b}$$

$$|\frac{\partial V}{\partial b}| + |\frac{\partial \widetilde{V}}{\partial b}| = \Big\{ O\big(e^{-\mu 2(1-z)/\epsilon} + \frac{e^{-\mu 2/\epsilon}}{\epsilon}\big) \frac{1}{1-v} + O\big(e^{-\mu(2(1-z)+z)/\epsilon} + \frac{e^{-\mu 2/\epsilon}}{\epsilon}\big) \frac{1}{1+v} + O\big(\frac{e^{-\mu 2(1-z)/\epsilon}}{\epsilon}\big) \Big\} \Big|_{v=U^0((x-z)/\epsilon)} ; \tag{3.48c}$$

$$|V - \widetilde{V}| = \Big\{ \Big[\big(1 + \lambda e^{\mu 2rz/\epsilon}\big) O\big(e^{-\mu(3z+(1-z))/\epsilon} + (1-\chi(z))\big(\frac{z}{\epsilon}\big)^2 e^{-\mu 4z/\epsilon}\big) + \big(1 + \lambda e^{\mu 2r(1-z)/\epsilon}\big) O\big(e^{-\mu 3(1-z)/\epsilon}\big) \Big] \frac{1}{1-v} + \big(1 + \lambda e^{\mu 2rz/\epsilon}\big) O\big(e^{-\mu 3z/\epsilon}\big)\big(\frac{z}{\epsilon} + \frac{1}{1+v}\big) \Big\} \Big|_{v=U^0((x-z)/\epsilon)} ; \tag{3.49a}$$

$$|H(U^0, z)V^2 - \widetilde{V}^2| = O\Big(\big(1 + \lambda e^{\mu 3rz/\epsilon}\big) \Big[e^{-\mu(3z+2x)/\epsilon} + \big(\frac{z}{\epsilon}\big)^2 e^{-\mu 4z/\epsilon} \Big] \Big) + O\big(e^{-\mu(3z+(1-x))/\epsilon}\big) ; \tag{3.49b}$$

$$|c(z)| + |\widetilde{c}(z)| = O\Big(\epsilon\big(1 + \lambda e^{\mu 2rz/\epsilon}\big) e^{-\mu 2z/\epsilon} \Big) ; \tag{3.50a}$$

$$|c(z) - \widetilde{c}(z)| = O\Big(\epsilon\big(1 + \lambda e^{\mu 2rz/\epsilon}\big) e^{-\mu 3z/\epsilon} \Big) ; \tag{3.50b}$$

and

$$|(c(z) - p(z))/\epsilon| = |a-1| O\Big(e^{-\mu 2z/\epsilon}\Big) + |b-1| O\Big(e^{-\mu 2(1-z)/\epsilon}\Big) + O\Big(\big(1 + \lambda e^{\mu rz/\epsilon}\big) e^{-\mu 3z/\epsilon} \Big) . \tag{3.50c}$$

Before proving the Lemma, we state an immediate consequence of (3.48),(3.49), which can be obtained using the asymptotic formulas (2.18),(2.19).

COROLLARY 3.6. *Under the assumptions of Lemma 3.5, there holds:*

$$|V| + |\widetilde{V}| + \epsilon|V_z| + \epsilon|\widetilde{V}_z| = O\left((1 + \lambda e^{\mu r z/\epsilon})e^{-\mu z/\epsilon}\right) \quad , \tag{3.51a}$$

$$|\frac{\partial V}{\partial a}| + |\frac{\partial \widetilde{V}}{\partial a}| = O\left(e^{-\mu z/\epsilon}\right) \quad , \tag{3.51b}$$

$$|\frac{\partial V}{\partial b}| + |\frac{\partial \widetilde{V}}{\partial b}| = O\left(e^{-\mu(1-z)/\epsilon} + \frac{e^{-\mu((1-z)+2z)/\epsilon}}{\epsilon}\right) = O\left(e^{-\mu(1-r)(1-z)/\epsilon}\right) \tag{3.51c}$$

and

$$|V - \widetilde{V}| = O\left((1 + \lambda e^{\mu 2rz/\epsilon})e^{-\mu 2z/\epsilon}\right) . \tag{3.52}$$

Proof of Lemma 3.5. Notice first that the estimates for V, $\frac{\partial V}{\partial a}$, $\frac{\partial V}{\partial b}$ follow from (3.16) and the fact that V and c are linear in a, b.

Using (3.16) and (3.33), it is clear that to prove the estimate for $\widetilde{V}$, it suffices to prove it for W_1 where

$$W_1(z,x) = \left(F^{1/2}\alpha_0 + \beta_0 L - \frac{c_0}{\epsilon}M - N\right)|_{v=U^0((x-z)/\epsilon)} \tag{3.53}$$

with L, M defined in (3.17) and N in (3.38).

Using (3.32) we obtain

$$|\alpha_0| = O((1 + \lambda e^{\mu 2rz/\epsilon})e^{-\mu 3z/\epsilon}) \quad , \tag{3.54a}$$

$$\beta_0 - K\frac{c_0}{\epsilon} = r(3z/\epsilon)O(e^{-\mu(2z+(1-z))/\epsilon} + e^{-\mu 2(1-z)/\epsilon}) \quad , \tag{3.54b}$$

$$-\beta_0 - K\frac{c_0}{\epsilon} = r(3z/\epsilon)O(e^{-\mu 2z/\epsilon} + e^{-\mu(2(1-z)+z)/\epsilon}). \tag{3.54c}$$

Now the estimate (3.48a) for W_1 follows from (3.53),(3.54), upon using (3.18) and (3.41a). In fact, for W_1 we obtain

$$\begin{aligned}|W_1| = O((1 + \lambda e^{\mu 2rz/\epsilon})e^{-\mu 3z/\epsilon})\Big\{&(e^{-\mu(2z+(1-z))/\epsilon} + e^{-\mu 2(1-z)/\epsilon})\frac{1}{1-v} \\ &+ e^{-\mu 2z/\epsilon}\frac{1}{1+v} + 1\Big\} + (1-\chi(z))\Big\{O(e^{-\mu 4z/\epsilon}(\frac{z}{\epsilon})^2(1 + \lambda e^{\mu 2rz/\epsilon}))\frac{1}{1-v} \\ &+ (1 + \lambda e^{\mu 2r(1-z)/\epsilon})O(e^{-\mu 4(1-z)/\epsilon})\frac{1}{(1-v)^2}\Big\} \\ &+ O(e^{-\mu 4z/\epsilon}(1 + \lambda e^{\mu 2rz/\epsilon}))((\frac{z}{\epsilon})^2\frac{1}{1+v} + \frac{1}{(1+v)^2})I_{v\le 0}.\end{aligned} \tag{3.55}$$

Notice that all the terms in the right hand side of (3.55) are dominated by the right hand side of (3.49a), so that

$$|W_1| \text{ is bounded by the right hand side of (3.49a).} \tag{3.56}$$

Now, we recall that we can write

$$V = \left(\alpha F^{1/2} + \beta L - \frac{c}{\epsilon} M\right) |_{v=U^0((x-z)/\epsilon)} \tag{3.57}$$

and

$$\widetilde{V} = \left(\overline{\alpha} F^{1/2} + \overline{\beta} L - \frac{\overline{c}}{\epsilon} M + W_1\right) |_{v=U^0((x-z)/\epsilon)} \tag{3.58}$$

and also that

$$U^0_x = F(U^0)^{1/2}. \tag{3.59}$$

Thus,

$$\begin{aligned} \epsilon V_z &= \left(\epsilon \alpha_z F^{1/2} + \epsilon \beta_z L - \frac{(\epsilon c_z)}{\epsilon} M\right) |_{v=U^0((x-z)/\epsilon)} \\ &+ \left(\alpha G + \beta D - \frac{c}{\epsilon} E\right) |_{v=U^0((x-z)/\epsilon)} \equiv P_1 + P_2 \end{aligned} \tag{3.60}$$

and

$$\begin{aligned} \epsilon \widetilde{V}_z &= \left(\epsilon \overline{\alpha}_z F^{1/2} + \epsilon \overline{\beta}_z L - \frac{(\epsilon \overline{c}_z)}{\epsilon} M\right) |_{v=U^0((x-z)/\epsilon)} \\ &+ \left(\overline{\alpha} G + \overline{\beta} D - \frac{\overline{c}}{\epsilon} E\right) |_{v=U^0((x-z)/\epsilon)} + \left(\epsilon \alpha_{0z} F^{1/2} + \epsilon \beta_{0z} L - \frac{(\epsilon c_{0z})}{\epsilon} M\right) |_{v=U^0((x-z)/\epsilon)} \\ &+ \left(\alpha_0 G + \beta_0 D - \frac{c_0}{\epsilon} E\right) |_{v=U^0((x-z)/\epsilon)} + \left(\frac{dN}{dv} F^{1/2}\right) |_{v=U^0((x-z)/\epsilon)} \\ &\equiv P_3 + P_4 + P_5 + P_6 + P_7 , \end{aligned} \tag{3.61}$$

where

$$G \equiv -\frac{dF^{1/2}}{dv} F^{1/2} = -f \quad , \quad G = -\mu(1-v)\big(1 + O(1-v)\big) \text{ for } v \text{ near } 1 \quad , \quad G \text{ is odd} \quad , \tag{3.62a}$$

$$D \equiv -\frac{dL}{dv} F^{1/2} \quad , \quad D = -\frac{1}{2\mu}(1-v)^{-1}\big(1 + O(1-v)\big) \text{ for } v \text{ near } 1 \quad , \quad D \text{ is even} \quad , \tag{3.62b}$$

$$E \equiv -\frac{dM}{dv} F^{1/2} \quad , \quad E = -\mu(1-v)\big(1 + O(1-v)\big) \text{ for } v \text{ near } 1 \quad , \quad E \text{ is odd.} \tag{3.62c}$$

Due to (3.62), we can proceed as in the proof of Corollary 3.3 to prove (3.48a) for P_1 , P_2 , and, using Theorem 3.4, also for P_3 , P_4 ; the bound for P_5 , P_6 follows arguing as in the proof of the estimate for W_1 given above. Finally, since

$$\frac{dN}{dv} F^{1/2} = \frac{f}{F^{1/2}} N + \frac{1}{F^{1/2}} \int_0^v gHV^2 \quad , \tag{3.63}$$

we have that $\frac{dN}{dv}F^{1/2}$ satisfies exactly the same estimate as N in (3.41a), and therefore P_7 can be estimated by the right hand side of (3.48a). This completes the proof of (3.48a).

To prove (3.48b) we set

$$W = \frac{\partial \widetilde{V}}{\partial a} \quad , \quad \widehat{c} = \frac{\partial \widetilde{c}}{\partial a} \tag{3.64}$$

so that

$$\begin{cases} \epsilon^2 W_{xx} + f'(U^0)W = -\frac{\widehat{c}}{\epsilon}U_x^0 - f''(U^0)HV\frac{\partial V}{\partial a} \quad , \\ W_x(z,0) = -\frac{1}{\epsilon}U_x^0(\frac{-z}{\epsilon}) \quad , \quad W_x(z,1) = 0 \quad , \\ \int_0^1 WU_x^0 dx = 0 \quad , \end{cases} \tag{3.65}$$

and proceed as in the proof of Theorem 3.4. In the present case, we define

$$\widehat{N} \equiv F^{1/2}\int_0^v F(s)^{-3/2}\left(\int_0^s f''HVV_a\right) \tag{3.66}$$

so

$$W = \left(F^{1/2}\widehat{\alpha} + \widehat{\beta}L - \frac{\widehat{c}}{\epsilon}M - \widehat{N}\right)|_{v=U^0((x-z)/\epsilon)} \tag{3.67}$$

for some constants $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{c}$.

From Corollary 3.3 we get

$$\begin{aligned} |\frac{\partial V}{\partial a}| \le C(1+\lambda e^{\mu rz/\epsilon})\,\{\, & e^{-\mu(2z+(1-z))/\epsilon}\frac{1}{1-v} \\ & + e^{-\mu 2z/\epsilon}\frac{1}{1+v} + e^{-\mu 2z/\epsilon}\frac{z}{\epsilon}(1-v^2)\,\} \end{aligned} \tag{3.68}$$

and therefore

$$\begin{aligned} |V\frac{\partial V}{\partial a}| &\le C\Big\{(1+\lambda e^{\mu 2rz/\epsilon})e^{-\mu 4z/\epsilon}(\frac{z}{\epsilon})^2 \\ &\quad + (1+\lambda e^{\mu r/\epsilon})e^{-\mu 2/\epsilon}\left((\frac{z}{\epsilon}) + \frac{e^{-\mu(1-z)/\epsilon}}{(1-v)^2}\right)\Big\} \quad , \quad v \ge 0 \quad , \\ |V\frac{\partial V}{\partial a}| &\le C(1+\lambda e^{\mu 2rz/\epsilon})e^{-\mu 4z/\epsilon}\left((\frac{z}{\epsilon})^2 + \frac{1}{(1+v)^2}\right) \quad , \quad v \le 0 . \end{aligned} \tag{3.69}$$

Hence,

$$\begin{aligned} \widehat{N}(v,z) = (1-\chi(z))\Big\{ & O\big(e^{-\mu 4z/\epsilon}(\frac{z}{\epsilon})^2(1+\lambda e^{\mu 2rz/\epsilon})\big)\frac{1}{(1-v)} \\ & + O\big(e^{-\mu(3(1-z)+2z)/\epsilon}(1+\lambda e^{\mu r/\epsilon})\big)\frac{1}{(1-v)^2}\Big\} \\ & + O\big(e^{-\mu 4z/\epsilon}(1+\lambda e^{\mu 2rz/\epsilon})\big)\left((\frac{z}{\epsilon})^2\frac{1}{(1+v)} + \frac{1}{(1+v)^2}\right)I_{v\le 0} \quad , \end{aligned}$$

or, equivalently,

$$\widehat{N}(v,z) = (1+\lambda e^{\mu 2rz/\epsilon})O(e^{-\mu 4z/\epsilon})\Big\{(1-\chi(z))(\frac{z}{\epsilon})^2\frac{1}{(1-v)} + I_{v\le 0}\Big((\frac{z}{\epsilon})^2\frac{1}{(1+v)}+\frac{1}{(1+v)^2}\Big)\Big\} \quad , \tag{3.70a}$$

and also

$$\widehat{N}_v(v^0,z) = (1+\lambda e^{\mu 2rz/\epsilon})O(e^{-\mu z/\epsilon}) \equiv -\widehat{a}_1 \quad , \tag{3.70b}$$

$$\widehat{N}_v(v^*,z) = (1+\lambda e^{\mu 2rz/\epsilon})\Big\{(1-\chi(z))(\frac{z}{\epsilon})^2 O(e^{-\mu(4z-2(1-z))/\epsilon})\Big\} \equiv -\widehat{b}_1 \tag{3.70c}$$

and

$$\int_{v_0}^{v^*}\widehat{N}dv = (1+\lambda e^{\mu 2rz/\epsilon})O(e^{-\mu 4z/\epsilon})\Big\{(1-\chi(z))(\frac{z}{\epsilon})^2\frac{1}{\epsilon} + (\frac{z}{\epsilon})^3 + e^{\mu z/\epsilon}\Big\} \equiv \widehat{\gamma}_3 \, . \tag{3.70d}$$

Next we write

$$\widehat{\alpha} = \widehat{\alpha}_1 + \widehat{\alpha}_2 + \widehat{\alpha}_3 \quad , \quad \widehat{\beta} = \widehat{\beta}_1 + \widehat{\beta}_2 + \widehat{\beta}_3 \quad , \quad \widehat{c} = \widehat{c}_1 + \widehat{c}_2 + \widehat{c}_3 \quad ,$$

where

$$A\begin{pmatrix}\widehat{\alpha}_1\\ \widehat{\beta}_1\\ \widehat{c}_1/\epsilon\end{pmatrix} = \begin{pmatrix}-\widehat{a}_1\\ -\widehat{b}_1\\ 0\end{pmatrix} \quad , \quad A\begin{pmatrix}\widehat{\alpha}_2\\ \widehat{\beta}_2\\ \widehat{c}_2/\epsilon\end{pmatrix} = \begin{pmatrix}-1\\ 0\\ 0\end{pmatrix} \quad \text{and} \quad A\begin{pmatrix}\widehat{\alpha}_3\\ \widehat{\beta}_3\\ \widehat{c}_3/\epsilon\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \widehat{\gamma}_3\end{pmatrix} .$$

and we also notice that

$$\widehat{a}_1 = O(1) \quad , \quad \widehat{b}_1 = O\Big((1+\lambda e^{\mu 2rz/\epsilon})(1-\chi(z))(\frac{z}{\epsilon})^2 e^{-\mu(4z-2(1-z))/\epsilon}\Big) \quad \widehat{\gamma}_3 = O\Big((1+\lambda e^{\mu 2rz/\epsilon})e^{-\mu 3z/\epsilon}\Big) \tag{3.71}$$

and that

$$W = \sum_{i=1}^{3}\widehat{W}_i - \widehat{N} \tag{3.72a}$$

where

$$\widehat{W}_i = F^{1/2}\widehat{\alpha}_i + \widehat{\beta}_i L - \frac{\widehat{c}_i}{\epsilon}M \, . \tag{3.72b}$$

Using (3.16) and (3.71) we immediately obtain (3.48b) for $\widehat{W}_1$ and $\widehat{W}_2$. To prove the estimate $\widehat{W}_3$ we first get a representation for $(\widehat{\alpha}_3 \, , \, \widehat{\beta}_3 \, , \, \widehat{c}_3/\epsilon)$ as was done in Theorem 3.4 for

$(\alpha_0\,,\,\beta_0\,,\,c_0/\epsilon)$. Then, (3.54) holds for $(\widehat{\alpha}_3\,,\,\widehat{\beta}_3\,,\,\widehat{c}_3/\epsilon)$ and this implies the desired bound for $\widehat{W}_3$. Finally, the bound (3.48b) for $\widehat{N}$ follows from (3.70a).

The proof of (3.48c) for $\dfrac{\partial \widetilde{V}}{\partial b}$ is similar to the proof of (3.48b) for $\dfrac{\partial \widetilde{V}}{\partial a}$ and is therefore omitted.

Next, we prove (3.49a). By (3.56)-(3.58), it suffices to prove the estimate for W_2 where

$$W_2 = \big((\alpha-\overline{\alpha})F^{1/2} + (\beta-\overline{\beta})L - (\frac{c}{\epsilon}-\frac{\overline{c}}{\epsilon})M\big)\,|_{v=U^0((x-z)/\epsilon)} \tag{3.73}$$

However, for W_2, (3.49a) follows easily from (3.16),(3.33) and (3.34a).

To establish (3.49b), first notice that (3.48a) and (3.49a) imply

$$\begin{aligned}|\widetilde{V}^2 - V^2| = &\frac{m_1}{(1-v)^2} + \frac{m_2}{(1+v)^2} + \frac{m_3}{(1-v^2)} + \frac{m_4}{(1-v)}\\ &+\frac{m_5}{(1+v)} + m_6\end{aligned} \tag{3.74}$$

and that

$$\widetilde{V}^2\,,\,V^2 = \frac{n_1}{(1-v)^2} + \frac{n_2}{(1+v)^2} + n_3 \quad , \tag{3.75}$$

where

$$\begin{aligned}
m_1 &= \big(1+\lambda e^{\mu 3rz/\epsilon}\big)O\big(e^{-\mu(5z+2(1-z))/\epsilon} + e^{-\mu(3z+3(1-z))/\epsilon}\\
&\quad + (1-\chi(z))\big(\frac{z}{\epsilon}\big)^2 e^{-\mu(4z+2(1-z))/\epsilon}\big)\\
&\quad + \big(1+\lambda e^{\mu 3r(1-z)/\epsilon}\big)O\big(e^{-\mu 5(1-z)/\epsilon}\big) \quad ,\\
m_2 &= \big(1+\lambda e^{\mu 3rz/\epsilon}\big)O\big(e^{-\mu 5z/\epsilon}\big) \quad ,\\
m_3 &= \big(1+\lambda e^{\mu 3rz/\epsilon}\big)O\big(e^{-\mu(5z+(1-z))/\epsilon} + (1-\chi(z))\big(\frac{z}{\epsilon}\big)^2 e^{-\mu 6z/\epsilon}\big)\\
&\quad + \big(1+\lambda e^{\mu 3r(1-z)/\epsilon}\big)O\big(e^{-\mu(2(1-z)+3z)/\epsilon}\big) \quad ,\\
m_4 &= \big(1+\lambda e^{\mu 3rz/\epsilon}\big)O\big(e^{-\mu(5z+(1-z))/\epsilon}\big(\frac{z}{\epsilon}\big) + (1-\chi(z))\big(\frac{z}{\epsilon}\big)^3 e^{-\mu 6z/\epsilon}\big)\\
&\quad + \big(1+\lambda e^{\mu 3r(1-z)/\epsilon}\big)O\big(e^{-\mu(2(1-z)+3z)/\epsilon}\big)\big(\frac{z}{\epsilon}\big) \quad ,\\
m_5 &= \big(1+\lambda e^{\mu 3rz/\epsilon}\big)O\big(e^{-\mu 5z/\epsilon}\big)\big(\frac{z}{\epsilon}\big) \quad ,\\
m_6 &= \big(1+\lambda e^{\mu 3rz/\epsilon}\big)O\big(e^{-\mu 5z/\epsilon}\big)\big(\frac{z}{\epsilon}\big)^2 \quad ,\\
n_1 &= \big(1+\lambda e^{\mu 2rz/\epsilon}\big)O\big(e^{-\mu(4z+2(1-z))/\epsilon}\big)\\
&\quad + \big(1+\lambda e^{\mu 2r(1-z)/\epsilon}\big)O\big(e^{-\mu 4(1-z)/\epsilon}\big) \quad ,\\
n_2 &= \big(1+\lambda e^{\mu 2rz/\epsilon}\big)O\big(e^{-\mu 4z/\epsilon}\big) \quad ,\\
n_3 &= \big(1+\lambda e^{\mu 2rz/\epsilon}\big)O\big(e^{-\mu 4z/\epsilon}\big)\big(\frac{z}{\epsilon}\big)^2 .
\end{aligned}$$

Thus, for $x \in [0,1]$,

$$\begin{aligned}|\widetilde{V}^2 - V^2| &= \big(1+\lambda e^{\mu 3rz/\epsilon}\big)O\big(e^{-\mu(3z+2x)/\epsilon} + \big(\frac{z}{\epsilon}\big)^2 e^{-\mu 5z/\epsilon}\big) \\ &\quad + \big(1+\lambda e^{\mu 3r(1-z)/\epsilon}\big)O\bigg(e^{-\mu(3z+3(1-z)-2(x-z))/\epsilon} + e^{-\mu(5z+(1-z)-(x-z))/\epsilon} \\ &\quad + (1-\chi(z))\big(\frac{z}{\epsilon}\big)^2\big(e^{-\mu(4z+2(1-z)-2(x-z))/\epsilon} + \big(\frac{z}{\epsilon}\big)e^{-\mu(6z-(x-z))/\epsilon}\big)\bigg)\end{aligned} \tag{3.76}$$

$$\begin{aligned}\widetilde{V}^2\,,\, V^2 &= \big(1+\lambda e^{\mu 2rz/\epsilon}\big)O\bigg(e^{-\mu(4z+2(1-z)-2(x-z))/\epsilon} + \big(\frac{z}{\epsilon}\big)^2 e^{-\mu 4z/\epsilon} \\ &\quad + e^{-\mu(2z+2x)/\epsilon}\bigg) + \big(1+\lambda e^{\mu 2r(1-z)/\epsilon}\big)O\big(e^{-\mu(4(1-z)-2(x-z))/\epsilon}\big)\,.\end{aligned} \tag{3.77}$$

It is easy to check that the right hand side of (3.76) is always bounded by the right hand side of (3.49b). Hence, by (3.22)-(3.24), (3.49b) holds if either

$$x \le z - M_\rho \epsilon \quad \text{or} \quad x \ge z \text{ and } z \ge \frac{17}{45} \quad ,$$

where M_ρ is such that

$$U^0(y) \le -\rho \quad \text{if} \quad y < -M_\rho.$$

On the other hand, if either

$$x \ge z \text{ and } z \le \frac{17}{45} \quad (\text{so that } 4(1-z) - 2(x-z) > 3z + (1-x)) \quad , \text{ or}$$

$$z - M_\rho \epsilon \le x \le z \quad ,$$

then the right hand side of (3.77) is dominated by the right hand side of (3.49b), thereby completing the proof of the equality.

Finally, the estimates (3.50) are an immediate consequence of Theorems 3.1,3.4 and the definition of $p(z)$ (see(3.3)). □

4.Construction of upper and lower solutions; bounds on the velocity of the internal layer. In this section we construct upper and lower solutions of (1.1)-(1.3) when the initial condition u_0 is close to a certain profile. The main feature of these functions is that they stay close together for periods of time that are exponentially large in ϵ.

Fix a small number γ, say

$$0 < \gamma < \frac{1}{2} \quad , \tag{4.1}$$

and let

$$\beta(x,t)=\beta(x,t,z)\equiv \delta e^{-t\nu}\left\{e^{-\mu(1-\gamma)x/\epsilon}+e^{-\mu(1-\gamma)\frac{z}{1-z}(1-x)/\epsilon}\right\}+C_1e^{-\mu 3z/\epsilon} \tag{4.2}$$

and

$$\frac{\alpha(z)}{\epsilon}\equiv C_2\left\{e^{-t\nu}e^{-\mu(1-\gamma)z/\epsilon}+e^{-\mu 3z/\epsilon}\right\} \tag{4.3}$$

for some constants δ, ν, C_1, C_2 to be specified later.

Let $(V(z,x,a,b),c(z,a,b))$ denote the solution of (3.6), $(\widetilde{V}(z,x,a,b),\widetilde{c}(z,a,b))$ the solution of (3.25), and let $z_l=z_l(t,a,b)$, $z_u=z_u(t,a,b)$ be defined by

$$\dot{z}_l=\widetilde{c}(z_l,a,b)+\alpha(z_l)\quad,\quad z(0)=z_0^l<\frac{1}{2}\quad, \tag{4.4}$$

$$\dot{z}_u=\widetilde{c}(z_u,a,b)-\alpha(z_u)\quad,\quad z(0)=z_0^u<\frac{1}{2}\quad, \tag{4.5}$$

where $\dot{}=\dfrac{\partial}{\partial t}$.

Finally, let

$$\widetilde{a}=\widetilde{a}(t)=U_x(\beta(0,t),-z_l/\epsilon)\left[U_x^0(-z_l/\epsilon)\right]^{-1}\quad, \tag{4.6a}$$
$$\widetilde{b}=\widetilde{b}(t)=U_x(\beta(1,t),(1-z_l)/\epsilon)\left[U_x^0((1-z_l)/\epsilon)\right]^{-1}\quad, \tag{4.6b}$$
$$\overline{a}=\overline{a}(t)=U_x(-\beta(0,t),-z_u/\epsilon)\left[U_x^0(-z_u/\epsilon)\right]^{-1}\quad, \tag{4.7a}$$
$$\overline{b}=\overline{b}(t)=U_x(-\beta(1,t),(1-z_u)/\epsilon)\left[U_x^0((1-z_u)/\epsilon)\right]^{-1}\quad, \tag{4.7b}$$

and define

$$v_l(x,t)=U\big(\beta(x,t),(x-z_l)/\epsilon\big)+\widetilde{V}\big(z_l,x,\widetilde{a}(t),\widetilde{b}(t)\big)\quad, \tag{4.8}$$
$$v_u(x,t)=U\big(-\beta(x,t),(x-z_u)/\epsilon\big)+\widetilde{V}\big(z_u,x,\overline{a}(t),\overline{b}(t)\big). \tag{4.9}$$

By Lemma 2.1, the quantities $\widetilde{a}$, $\overline{a}$, $\widetilde{b}$, $\overline{b}$ satisfy (3.27) if β is sufficiently small (independently of ϵ). In fact,

$$\widetilde{a}=1+\beta(0,t)k_1e^{\mu r z_l/\epsilon}\quad,\quad \widetilde{b}=1+\beta(1,t)k_2e^{\mu r(1-z_l)/\epsilon} \tag{4.10}$$

where

$$k_1=\big(U_x\big(\beta(0,t),-z_l/\epsilon\big)-U_x^0\big(-z_l/\epsilon\big)\big)\left[\beta(0,t)U_x^0\big(-z_l/\epsilon\big)\right]^{-1}e^{-\mu r z_l/\epsilon},$$
$$k_2=\big(U_x\big(\beta(1,t),(1-z_l)/\epsilon\big)-U_x^0\big((1-z_l)/\epsilon\big)\big)\left[\beta(1,t)U_x^0\big((1-z_l)/\epsilon\big)\right]^{-1}e^{-\mu r(1-z_l)/\epsilon},$$

so that

$$|k_1|\,,\ |k_2|\leq 1 \tag{4.11}$$

if $|\beta| < \overline{\rho} = \overline{\rho}(r)$ and $0 < \epsilon < \epsilon_0$ for some constants $\overline{\rho}$, ϵ_0. Notice that, in this case, we can take

$$\lambda = \beta(0,t) = k(t)\beta(1,t) \tag{4.12}$$

in (3.28), and that

$$0 < k_0 < k(t) < k_* < \infty \tag{4.13}$$

for constants k_0, k_* independent of ϵ.

Also, again from Lemma 2.1,

$$\begin{aligned}\dot{\tilde{a}} &= \left(U_{x\beta}\beta_t - U_{xx}\left(\dot{z}_l/\epsilon\right)\right)\left(U_x^0\right)^{-1} + \left(U_x^0\right)^{-2} U_{xx}^0 U_x \left(\dot{z}_l/\epsilon\right) \\ &= O\left(e^{\mu r z_l/\epsilon}\right)\beta_t(0,t) + O(1)\left(\dot{z}_l/\epsilon\right) + O\left(e^{\mu r z_l/\epsilon}\right)\beta(0,t) \quad ,\end{aligned} \tag{4.14}$$

$$\dot{\tilde{b}} = O\left(e^{\mu r(1-z_l)/\epsilon}\right)\beta_t(1,t) + O(1)\left(\dot{z}_l/\epsilon\right) + O\left(e^{\mu r(1-z_l)/\epsilon}\right)\beta(1,t). \tag{4.15}$$

Similar estimates can be obtained for $\overline{a}$, $\overline{b}$, $\dot{\overline{a}}$, $\dot{\overline{b}}$.

LEMMA 4.1. *There exist constants ϵ_0, M_0, δ, ν, C_1, C_2 such that, if*

$$0 < \epsilon < \epsilon_0 \quad , \tag{4.16}$$

then

$$\mathcal{L}v_l \equiv \partial_t v_l - \epsilon^2\partial_x^2 v_l - f(v_l) < 0 \tag{4.17}$$

and

$$\mathcal{L}v_u > 0 \quad , \tag{4.18}$$

as long as z_u, $z_l \geq M_0\epsilon$.

Proof. We shall only prove (4.17), since (4.18) is proved in a similar way.

First of all, by choosing δ, ϵ_0 sufficiently small, we may assume that r in (4.10) satisfies (3.26) and

$$0 < r < \frac{\gamma}{10}. \tag{4.19}$$

Computing explicitly, we have

$$\begin{aligned}\mathcal{L}v_l &= U_\beta\beta_t - U_x\dot{z}_l\epsilon^{-1} + \widetilde{V}_z\dot{z}_l + \widetilde{V}_a\dot{\tilde{a}} + \widetilde{V}_b\dot{\tilde{b}} \\ &\quad - \epsilon^2\left\{U_\beta\beta_{xx} + U_{\beta\beta}\beta_x^2 + 2U_{\beta x}\beta_x\epsilon^{-1} + U_{xx}\epsilon^{-2} + \widetilde{V}_{xx}\right\} \\ &\quad - f(U) - f'(U)\widetilde{V} - \frac{1}{2}f''(U)\widetilde{V}^2 - R(U,\widetilde{V})\widetilde{V}^3\end{aligned}$$

where

$$R(U,\widetilde{V}) = \frac{1}{2}\int_0^1 (1-s)^2 f'''(U + s\widetilde{V})\,ds. \tag{4.20}$$

Thus, by (2.7) and (3.25a),

$$\mathcal{L}v_l = T_1 - \epsilon^2 T_2 - T_3 + T_4 - U_x^0 \left(\alpha/\epsilon - \mathrm{v}(\beta)\right) - \beta \tag{4.21}$$

where

$$T_1 = U_\beta \beta_t + \widetilde{V}_z \dot{z}_l + \widetilde{V}_a \dot{\widetilde{a}} + \widetilde{V}_b \dot{\widetilde{b}} \quad , \tag{4.22a}$$

$$T_2 = U_\beta \beta_{xx} + U_{\beta\beta}\beta_x^2 + 2U_{\beta x}\beta_x \epsilon^{-1} \quad , \tag{4.22b}$$

$$\begin{aligned} T_3 = {} & (U_x - U_x^0)\frac{\widetilde{c}}{\epsilon} + (f'(U) - f'(U^0))\widetilde{V} \\ & + \frac{1}{2}(f''(U) - f''(U^0))\widetilde{V}^2 + (U_x - U_x^0)\left(\frac{\alpha}{\epsilon} - \mathrm{v}(\beta)\right) \quad , \end{aligned} \tag{4.22c}$$

$$T_4 = \frac{1}{2} f''(U^0)\left(H(U^0, z_l)V^2 - \widetilde{V}^2\right) - R(U, \widetilde{V})\widetilde{V}^3 \, . \tag{4.22d}$$

In what follows K_0 will denote a (fixed, large) known constant, not always the same, and we shall write $z_l \equiv z$.

In order to estimate T_i , we first notice that, from (3.50a),(3.51a),

$$\begin{aligned} |\widetilde{V}_z||\dot{z}| & \leq K_0\left(1 + \lambda e^{\mu r z/\epsilon}\right)e^{-\mu z/\epsilon}\left\{\left|\frac{\widetilde{c}}{\epsilon}\right| + \left|\frac{\alpha}{\epsilon}\right|\right\} \\ & \leq K_0\left(1 + \lambda e^{\mu r z/\epsilon}\right)e^{-\mu z/\epsilon}\left\{\left(1 + \lambda e^{\mu 2 r z/\epsilon}\right)e^{-\mu 2 z/\epsilon} + C_2(\delta^{-1} + C_1^{-1})\beta(z,t)\right\} \end{aligned}$$

so that, by (4.12),

$$|\widetilde{V}_z||\dot{z}| \leq K_0\left\{\left(1 + \lambda e^{\mu 3 r z/\epsilon}\right)e^{-\mu 3 z/\epsilon} + C_2(\delta^{-1} + C_1^{-1})e^{-\mu(1-r)z/\epsilon}\beta(z,t)\right\} . \tag{4.23}$$

Also, from (4.2),(4.3),

$$z \leq z_0^l + \epsilon\nu^{-1}C_2 e^{-\mu(1-\gamma)z_0^l/\epsilon} < \frac{1}{2} \quad , \tag{4.24}$$

$$|\beta_t| \leq \left(\nu + K_0\left|\frac{\dot{z}}{\epsilon}\right|\right)\beta \leq 2\nu\beta \quad , \tag{4.25a}$$

$$|\beta_x| \leq (1-\gamma)\frac{\mu}{\epsilon}\beta \quad , \tag{4.25b}$$

$$|\beta_{xx}| \leq (1-\gamma)\frac{\mu^2}{\epsilon^2}\beta \, . \tag{4.25c}$$

On the other hand, by Lemma 2.1,

$$\left|U_x - U_x^0\right| + \left|f'(U) - f'(U^0)\right| + \left|f''(U) - f''(U^0)\right| \leq K_0\beta \, . \tag{4.26}$$

Finally, we also have

$$|\frac{\alpha}{\epsilon} - \mathrm{v}(\beta)| \leq K_0 \left\{\beta(x,t) + C_2(\delta^{-1} + C_1^{-1})\beta(z,t)\right\} \quad , \tag{4.27}$$

$$\begin{aligned} |\widetilde{V}_a\dot{\tilde{a}}| &\leq K_0 e^{-\mu z/\epsilon} \left\{ e^{\mu r z/\epsilon}\beta_t(0,t) + |\frac{\dot{z}}{\epsilon}| + e^{\mu r z/\epsilon}\beta(0,t)\right\} \\ &\leq K_0 e^{-\mu z/\epsilon} \{ \, e^{\mu r z/\epsilon}\beta(0,t) + \left(1 + \lambda e^{\mu 2 r z/\epsilon}\right) e^{-\mu 2 z/\epsilon} \\ &+ C_2(\delta^{-1} + C_1^{-1})\beta(z,t) \, \} \quad , \end{aligned} \tag{4.28a}$$

$$\begin{aligned} |\widetilde{V}_b\dot{\tilde{b}}| &\leq K_0 e^{-\mu(1-r)(1-z)/\epsilon} \{ \, e^{\mu r(1-z)/\epsilon}\beta_t(1,t) + \left(1 + \lambda e^{\mu 2 r z/\epsilon}\right) e^{-\mu 2 z/\epsilon} \\ &+ C_2(\delta^{-1} + C_1^{-1})\beta(z,t) \, \} \quad , \end{aligned} \tag{4.28b}$$

and

$$\begin{aligned} \lambda = \beta(0,t) &\leq K_0 e^{\mu(1-\gamma)z/\epsilon}\beta(z,t) \quad , \\ \lambda \leq K_0\beta(1,t) &\leq K_0 e^{\mu(1-\gamma)z/\epsilon}\beta(z,t) \quad , \\ \beta(z,t) &\leq K_0\beta(x,t) \quad , \quad x \in [0,1] \, . \end{aligned} \tag{4.29}$$

Hence,

$$\begin{aligned} |T_1| &\leq |U_\beta \beta_t| + |\widetilde{V}_z||\dot{z}| + |\widetilde{V}_a||\dot{\tilde{a}}| + |\widetilde{V}_b||\dot{\tilde{b}}| \\ &\leq K_0 \{ \, \nu\beta + e^{-\mu 3 z/\epsilon} + \beta(0,t)e^{-\mu 3(1-r)z/\epsilon} + C_2(\delta^{-1} + C_1^{-1})e^{-\mu(1-r)z/\epsilon}\beta(z,t) \\ &+ e^{-\mu(1-r)z/\epsilon}\beta(0,t) \, \} \\ &\leq K_0 \left\{ \nu\beta + e^{-\mu 3 z/\epsilon} + \beta \left[e^{-\mu(\gamma - r)z/\epsilon} + C_2(\delta^{-1} + C_1^{-1})e^{-\mu(1-r)z/\epsilon} \right] \right\} \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} \epsilon^2|T_2| &\leq \epsilon^2 \{ \, |U_\beta - U_\beta^0||\beta_{xx}| + |U_{\beta\beta}|\beta_x^2 + 2|U_{\beta x} - U_{\beta x}^0||\beta_x|\epsilon^{-1} \\ &+ |U_\beta^0||\beta_{xx}| + 2|U_{\beta x}^0||\beta_x|\epsilon^{-1} \, \} \\ &\leq K_0\beta^2 + (1-\gamma)\beta + K_0\beta\chi_{\{|x-z|<K_1\epsilon\}} + \frac{\gamma}{10}\beta \quad , \end{aligned} \tag{4.31}$$

where $K_1 > 0$ is such that

$$|U_\beta^0 + \frac{1}{\mu^2}|(1-\gamma)\mu^2 + 2|U_{\beta x}^0|(1-\gamma)\mu < \frac{\gamma}{10} \quad \text{if } |\frac{x-z}{\epsilon}| \geq K_1 \tag{4.32}$$

and $\chi_{\{|x-z|<K_1\epsilon\}}$ =characteristic function of $\{|x-z| < K_1\epsilon\}$.

Also, using (4.26),(4.27),

$$\begin{aligned} |T_3| &\leq K_0\beta \{ \, \left(1 + \lambda e^{\mu 2 r z/\epsilon}\right)e^{-\mu 2 z/\epsilon} + \left(1 + \lambda e^{\mu 2 r z/\epsilon}\right)e^{-\mu z/\epsilon} \\ &+ \left(1 + \lambda e^{\mu 2 r z/\epsilon}\right)^2 e^{-\mu 2 z/\epsilon} + \beta + C_2(\delta^{-1} + C_1^{-1})\beta(z,t) \, \} \\ &\leq K_0\beta \left\{ (1 + \lambda e^{\mu 2 r z/\epsilon})e^{-\mu z/\epsilon} + \beta + C_2(\delta^{-1} + C_1^{-1})\beta(z,t) \right\} \end{aligned} \tag{4.33}$$

and (from (3.49b),(3.51a))

$$|T_4| \le K_0 (1 + \lambda e^{\mu 3 r z/\epsilon}) e^{-\mu 3 z/\epsilon} . \tag{4.34}$$

Thus, using (4.29)-(4.34),

$$\begin{aligned} \mathcal{L} v_l \le K_0 \{ \, \beta [\, \nu + e^{-\mu(\gamma - r) z/\epsilon} + C_2(\delta^{-1} + C_1^{-1}) e^{-\mu(1-r)z/\epsilon} \\ + \chi_{\{|x-z|<K_1\epsilon\}} + C_2(\delta^{-1} + C_1^{-1}) \beta(z,t) \,] + \beta^2 \\ + e^{-\mu 3 z/\epsilon} + \beta(z,t) e^{-\mu(2+\gamma-3r)z/\epsilon} \, \} \\ - U_x^0 \chi_{\{|x-z|<K_2\epsilon\}} \big(\frac{\alpha}{\epsilon} - \mathrm{v}(\beta) \big) + |U_x^0| \chi_{\{|x-z|\ge K_2\epsilon\}} \mathrm{v}(\beta) \\ - \frac{9}{10} \gamma \beta . \end{aligned} \tag{4.35}$$

If we now choose ν , C_1 , K_2 such that

$$\begin{aligned} K_0 \nu < \frac{\gamma}{10} \quad , \\ \frac{10 K_0}{\gamma} < C_1 \quad , \\ \mathrm{v}(\beta) U_x^0(K_2) < \frac{\gamma}{10} \beta \quad , \quad K_2 > K_1 \end{aligned}$$

and

$$\begin{aligned} &\delta \text{ small enough so that} \quad K_0 \beta < \frac{\gamma}{10} \quad , \\ &C_2 \text{ large enough so that} \quad U_x^0 \big(\frac{\alpha}{\epsilon} - \mathrm{v}(\beta) \big) > K_0 \beta \quad \text{if } \big| \frac{x-z}{\epsilon} \big| < K_2 , \end{aligned}$$

then (4.35) implies that, if $\frac{z}{\epsilon}$ is sufficiently large,

$$\mathcal{L} v_l < 0$$

thereby completing the proof of the Lemma. □

THEOREM 4.2. *Let x_0 be the unique zero of the initial condition u_0 in (1.3) and set*

$$\begin{aligned} z_0^l &= x_0 + \Gamma \epsilon e^{-\mu(1-\gamma) x_0/\epsilon} \quad , \\ z_0^u &= x_0 - \Gamma \epsilon e^{-\mu(1-\gamma) x_0/\epsilon} \quad (\Gamma \text{ const.}). \end{aligned} \tag{4.36}$$

There exist positive constants Λ , Γ such that, if

$$|u_0(x) - U((x - x_0)/\epsilon)| \le \Lambda \beta(x, 0) \quad , \tag{4.37}$$

then the solution u of (1.1)-(1.3) satisfies

$$v_l(x,t) < u(x,t) < v_u(x,t) \tag{4.38}$$

as long as z_l , $z_u \geq M_0 \epsilon$.

Proof. First notice that

$$\partial_x v_l = U_\beta \beta_x \quad \text{at } x = 0, 1 \quad , \tag{4.39a}$$

$$\partial_x v_u = -U_\beta \beta_x \quad \text{at } x = 0, 1 \quad , \tag{4.39b}$$

and that

$$\beta_x(0,t) < 0 \quad , \quad \beta_x(1,t) > 0 \, . \tag{4.40}$$

Thus, using Lemma 2.1 (in particular (2.20)), we get

$$\begin{aligned} &\partial_x v_l(0,t) = \left\{ \left[U_\beta(\beta(0,t), -z_l/\epsilon) - U_\beta^0(-z_l/\epsilon) \right] + U_\beta^0(-z_l/\epsilon) \right\} \beta_x(0,t) > 0 \, , \\ &\partial_x v_l(1,t) < 0 \, , \end{aligned} \tag{4.41a}$$

and

$$\partial_x v_u(0,t) < 0 \quad , \quad \partial_x v_u(1,t) > 0 \, . \tag{4.41b}$$

On the other hand, it is easily checked that

$$U^0\left((x - x_0)/\epsilon\right) - U\left(\beta(x,0), (x - z_0^l)/\epsilon\right) \geq \Lambda_1 \beta(x,0) \geq \delta \Lambda_1 e^{-\mu(1-\gamma)z/\epsilon} \quad , \tag{4.42a}$$

$$U\left(-\beta(x,0), (x - z_0^u)/\epsilon\right) - U^0\left((x - x_0)/\epsilon\right) \geq \Lambda_1 \beta(x,0) \geq \delta \Lambda_1 e^{-\mu(1-\gamma)z/\epsilon} \quad , \tag{4.42b}$$

for some small constant Λ_1 (independent of ϵ), provided Γ in (4.36) is chosen large enough (again, independently of ϵ).

Hence, if Λ is sufficiently small, (3.51a) together with (4.42) implies

$$v_l(x,0) \leq u(x,0) \leq v_u(x,0) \, . \tag{4.43}$$

Now, (4.38) follows from (4.17),(4.18),(4.41),(4.43) and the maximum principle. □

Now, using the estimates (3.50) together with (3.3), it is not difficult to obtain the bound

$$0 < z_l - z_u \leq \frac{A_1 \epsilon e^{\mu(1+\gamma)x_0/\epsilon}}{\left(e^{\mu 2x_0/\epsilon} - A_2 e^{\mu x_0/\epsilon} - 2\mu^3 K_1^2 t/K\right)} \quad , \tag{4.44}$$

for some constants A_1 , $A_2 > 0$; also, from (4.8),(4.9),

$$0 < v_l - v_u \leq A_3 \left(\beta(x,t) + \left(\frac{z_l - z_u}{\epsilon}\right) + e^{-\mu(1-\gamma)x_0/\epsilon} \right) \, . \tag{4.45}$$

From (4.44),(4.45) we see that the profile of the solution u resembles that of v_l and v_u only as long as

$$z_l - z_u = o(\epsilon)\,;$$

this last equality can be shown to hold provided

$$z_u > \frac{(1+\gamma)}{2} x_0 \quad (\text{recall that } z_l > z > z_u\,).$$

However, due to (3.50) we can still obtain the same bounds as in [6;§6] for the time it takes for the zero $z(t)$ of u to become $O(\epsilon)$.

On the other hand, for a smaller class of initial conditions u_0, it is possible to construct upper and lower solutions which will stay "close together" for even longer periods of time. In particular, their difference will be $O(\epsilon)$ as long as the zero $z(t)$ of u satisfies

$$z(t) \geq \widehat{K}\epsilon |\ln \epsilon| \quad \text{for some constant } \widehat{K} > 0 \text{ (independent of } \epsilon). \tag{4.46}$$

LEMMA 4.3. *Let*

$$\beta(x,t) \equiv \left\{ \widehat{\delta} e^{-t\widehat{\nu}\epsilon} + \frac{\widehat{C_1}}{\epsilon} \left(\frac{z}{\epsilon}\right)^2 e^{-\mu 3z/\epsilon} \right\} \left\{ e^{-\mu(1-\epsilon)x/\epsilon} + e^{-\mu(1-\epsilon)\frac{z}{1-z}(1-x)/\epsilon} \right\}, \tag{4.47}$$

$$\frac{\alpha(z)}{\epsilon} \equiv \frac{\widehat{C_2}}{\epsilon^2} \left\{ e^{-t\widehat{\nu}\epsilon} e^{-\mu 2z/\epsilon} + \frac{1}{\epsilon}\left(\frac{z}{\epsilon}\right)^2 e^{-\mu 4z/\epsilon} \right\}, \tag{4.48}$$

and define V, c, $\widetilde{V}$, $\widetilde{c}$, $\widetilde{a}$, $\widetilde{b}$, $\overline{a}$, $\overline{b}$, v_l, v_u *as in (4.4)-(4.9). Then, there exist constants* $\widehat{\delta}$, $\widehat{\nu}$, $\widehat{C_1}$, $\widehat{C_2}$, $\widehat{K}$ *such that*

$$\mathcal{L}v_l < 0 \quad , \tag{4.49}$$

$$\mathcal{L}v_u > 0 \quad , \tag{4.50}$$

provided z_u, $z_l \geq \widehat{K}\epsilon|\ln\epsilon|$.

The proof of Lemma 4.3 is similar to the proof of Lemma 4.1; the only difference is that one should now use the full force of Lemma 3.5 instead of just using Corollary 3.6.

Using Lemma 4.3 and arguing as in Theorem 4.2 we obtain

THEOREM 4.4. *Let* x_0 *be the unique zero of the initial condition* u_0 *in (1.3) and set*

$$\begin{aligned} z_0^l &= x_0 + \widehat{\Gamma}\epsilon e^{-\mu 2x_0/\epsilon} \quad , \\ z_0^u &= x_0 - \widehat{\Gamma}\epsilon e^{-\mu 2x_0/\epsilon} \quad (\widehat{\Gamma} \text{ const.}). \end{aligned} \tag{4.51}$$

There exist constants $\widehat{\Lambda}$, $\widehat{\Gamma}$ *such that, if*

$$\big|u_0(x) - \big(U\big((x-x_0)/\epsilon\big) + V(x_0, x, 1, 1)\big)\big| \le \widehat{\Lambda}\beta(x,0) \quad (\beta \textit{ defined in (4.47)}), \tag{4.52}$$

then the solution u *of (1.1)-(1.3) satisfies*

$$v_l(x,t) < u(x,t) < v_u(x,t) \tag{4.53}$$

as long as z_l , $z_u \ge \widehat{K}\epsilon|\ln\epsilon|$.

Under the hypotheses of Theorem 4.4, the difference between z_l and z_u satisfies

$$0 < z_l - z_u \le \frac{\widehat{A_1}}{\epsilon^5} e^{-\mu 2 z_u/\epsilon} \le \frac{\widehat{A_2}}{\epsilon^5} e^{-\mu 2 z_l/\epsilon} . \tag{4.54}$$

for some constants $\widehat{A_1}$, $\widehat{A_2}$ > 0 (independent of ϵ). Therefore, if $\widehat{K}$ is large enough, $v_u - v_l = O(\epsilon)$ as long as the zero $z(t)$ of u satisfies (4.46).

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