CARATHEODORY-TYPE SELECTIONS AND RANDOM FIXED POINT THEOREMS

by

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ABSTRACT

We provide some new Caratheodory-type selection theorems, i.e, selections for correspondences of two variables which are continuous with respect to one variable and measurable with respect to the other. These results generalize simultaneously Michael's [21] continuous selection theorem for lower-semicontinuous correspondences as well as a Caratheodory-type selection theorem of Fryszkowski [10]. Random fixed point theorems (which generalize ordinary fixed point theorems, e.g., Browder's [6]) follow as easy corollaries of our results.

1. INTRODUCTION

The two major types of selection theorems are continuous selection results of Michael-type [21], and measurable selection results of von Neumann-Aumann-type [1], [8], [19]. Both types of selection theorems have found important applications in general equilibrium theory as well as in other areas of analysis [11], [15], [17], [23], [29], [30], [31]. The present paper is concerned with so-called Caratheodory-type selections which combine both measurable and continuous selections via the setting of a product space. Thus Caratheodory-type selections are selections for correspondences of two variables which are continuous with respect to one of the variables and measurable with respect to the other. The main reference for this type of result appears to be Fryszkowski's paper [10], which in turn generalizes results of R. Cellina and C. Castaing[7]. We ourselves have encountered need for this type of result in [17], [31] while studying equilibria in economies with a measure space of agents.

Our main result (Theorem 3.1) is a simultaneous generalization of Michael's continuous selection theorem for lower-semicontinuous correspondences (Theorem 3.1''', [21, p. 368]) as well as of Fryszkowski's Theorem 1, [10, p. 44]. As an application of Theorem 3.1 we obtain a random fixed point theorem, Theorem 3.3, which generalizes the random fixed point theorem of Bohnenblust-Karlin [4] as well as a result of Itoh [16]. The random fixed point theorems, in turn, are generalizations of ordinary fixed point theorems, e.g., Browder's [6], but only when the underlying space is separable.

We would now like to comment on the relationship between our results and those of Fryszkowski. We consider the following setting. Let T be a measure space, and X and Y be topological spaces. Let $\phi : T \times X \to 2^Y$ be a (possibly empty valued) correspondence and

$$U = \{(t,x) \in T \times X : \phi(t,x) \neq \emptyset\}.$$

A function $f : U \to Y$ such that $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$, $f(\cdot,x)$ is measurable for each x and $f(t,\cdot)$ is continuous for each t is said to be a Caratheodory-type selection for the correspondence ϕ. In the actual results, ours as well as Fryszkowski's, it is assumed that T is a complete measure space, X is complete separable metric and Y is a Banach space. In addition, Fryszkowski assumes that X is locally compact and $U = T \times X$.

In our view, Fryszkowski's arguments are somewhat ad hoc. In effect, by an application of Michael's continuous selection theorem, he reduces the problem to an application of the measurable selection theorem of [19] to an auxiliary correspondence into a function space. This type of argument cannot be readily adapted to the setting when U is an arbitrary subset of $T \times X$. Likewise, we do not need the local compactness. By comparison, our arguments are more direct. To a large extent we simply just mimic Michael's proof. That is, we carry out a "parametrized" version of his proof, where the parameter t ranges over the measurable space T. We think that the details here are far from routine. We need a number of results scattered throughout the literature on measurable selections (Castaing-Valadier [8] is especially helpful) as well as some simple ideas of "descriptive set theoretic" character.

We finally suggest that allowing U to be an arbitrary subset of $T \times X$, rather than $T \times X$ itself, is a significant generalization. We needed this type of results in [17] and [31] to extend the theory of Nash equilibria, developed in [9], [11], [13], [14], [20], [22], [24], [26], [29], [30] to the setting of an arbitrary measure space of agents and an infinite dimensional strategy space. Moreover, random fixed point theorems also follow as easy corollaries of this more general version.

In [18] we have proved the Caratheodory-type selection theorem which we needed in [17]. This result did not require the full strength of Michael's methods but only a rather direct argument involving partitions of unity. The present paper deals primarily with the kinds of situations arising in applications of the theory of measurable selections in analysis (lower semicontinuous closed valued mappings), see [1], [2], [8]. By contrast, [18] is focused on a different and narrower situation which arose in [17].

The paper is organized as follows. Section 2 contains notation and definitions. Section 3 contains the statements of the main results of the paper. Several technical Lemmata needed for the proof of our main results are given in Section 4. Finally, Section 5 contains proofs of the main results.

2. NOTATION AND DEFINITIONS

2.1 Notation

2^A denotes the set of all subsets of the set A

$c\ell A$ denotes the closure of the set A

$\setminus$ denotes the set theoretic subtraction

If $\phi : X \to 2^Y$ is a correspondence then $\phi|_U : U \to 2^Y$ denotes the restricion of ϕ to U

$B(0, \varepsilon)$ denotes the open ball centered at 0 of radius ε.

diam denotes diameter

dist denotes distance

proj denotes projection.

2.2 Definitions

Let X, Y be two topological spaces. The <u>graph</u> of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$. A correspondence $\phi : X \to 2^Y$ is said to be <u>lower semicontinuous</u> (l.s.c.) if the set $\{x \in X : \phi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y. An <u>open cover</u> of a topological space X is a collection $U = \{u_a : a \in A\}$ of open subsets of X whose union is X, i.e., $\bigcup_{a \in A} u_a = X$. An open cover $U = \{u_a : a \in A\}$ is <u>locally finite</u> if every $x \in X$ has a neighborhood intersecting only finitely many $u \in U$. Let (T, τ, μ) be a measure space, i.e., μ is a real-valued, non-negative, countably additive measure defined in a σ-field τ of subsets of T. The correspondence $\phi : T \to 2^X$ is said to have a <u>measurable graph</u> if $G_\phi \in \tau \otimes \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes Borel σ-algebra on X and $\otimes$ denotes σ-product field. The correspondence $\phi : T \to 2^Y$ is said to be <u>lower measurable</u> if the set $\{t \in T : \phi(t) \cap V \neq \emptyset\} \in \tau$ for every open subset V of Y.

3. MAIN RESULTS

Let X be a topological space and Y be a linear topological space. Let $\phi : X \to 2^Y$ be a nonempty valued correspondence. A function $f : X \to Y$ is said to be a continuous selection from ϕ if $f(x) \in \phi(x)$ for all $x \in X$, and f is continuous. Let T be an arbitrary measure space. Let $\psi : T \to 2^Y$ be a nonempty valued correspondence. A function $f : T \to Y$ is said to be a measurable selection from ψ if $f(t) \in \psi(t)$ for all $t \in T$, and f is measurable. The above notions have been extensively used in the literature, (see for instance Aumann [3] or Michael [21]). Let Z be a topological space and $\phi : T \times Z \to 2^Y$ be a nonempty valued correspondence. A function $f : T \times Z \to Y$ is said to be a Caratheodory-type selection from ϕ if $f(t, z) \in \phi(t, z)$ for all $(t, z) \in T \times Z$ and $f(\cdot, z)$ is measurable for all $z \in Z$ and $f(t, \cdot)$ is continuous for all $t \in T$.

Recently, several authors [10], [15], [18], have proved Caratheodory-type selection results. The theorem below generalizes significantly Fryszkowski's [10] result.

Theorem 3.1: Let (T,τ,μ) be a complete measure space, Y be a separable Banach space and Z be a complete, separable metric space. Let $\phi : T \times Z \to 2^Y$ be a convex, closed (possibly empty) valued correspondence such that

(i) $\phi(\cdot,\cdot)$ is lower measurable

(ii) for each $t \in T$, $\phi(t,\cdot)$ is l.s.c.

Let $U = \{(t, x) \in T \times Z : \phi(t, x) \neq \emptyset\}$ and for each $t \in T$, let $U^t = \{x \in Z : (t, x) \in U\}$ and for each $x \in Z$, let $U_x = \{t \in T : (t, x) \in U\}$. Then there exists a Caratheodory-type selection from $\phi|U$, i.e., there exists a function $f : U \to Y$ such that $f(t, x) \in \phi(t, x)$ for all $(t, x) \in U$ and for each $x \in Z$, $f(\cdot, x)$ is measurable on U_x and for each $t \in T$, $f(t,\cdot)$ is continuous in U^t. Moreover, $f(\cdot,\cdot)$ is jointly measurable.

Theorem 3.2: The statement of Theorem 3.1 remains true without closed valueness of $\phi : T \times Z \to 2^Y$ if either

(i) Y is finite dimensional, or

(ii) $\phi(t, x)$ has a nonempty interior for all $(t, x) \in U$.

Let T be any measure space and X be a nonempty subset of any linear topological space. Let ϕ be a correspondence from $T \times X$ into 2^X. The correspondence ϕ is said to have a random fixed point if there exists a measurable function $x : T \to X$ such that $x(t) \in \phi(t, x(t))$ for almost all t in T. Below we provide a random fixed point theorem. This result generalizes a theorem of Bohnenblust and Karlin [4]. For other random fixed point results see [16], [23].

Theorem 3.3: Let (T,τ,μ) be a complete measure space, and X be a nonempty compact convex subset of a separable Banach space Y. Let $\phi : T \times X \to 2^X$ be a nonempty, convex, closed valued correspondence such that:

(i) $\phi(\cdot,\cdot)$ is lower measurable,

(ii) for each $t \in T$, $\phi(t,\cdot)$ is l.s.c.

Then ϕ has a random fixed point.

Theorem 3.4: The statement of Theorem 3.3 remains true without closed valueness of $\phi : T \times X \to 2^X$ if either

(i) Y is finite dimensional, or

(ii) $\phi(t,x)$ has a nonempty interior for all $(t,x) \in T \times X$.

Remark: Theorem 3.4 (i) can be seen as a generalization of a version of a theorem of Gale-Mas-Colell [11] given in [29, Theorem 3.4].

4. LEMMATA

Lemma 4.1: Let (T,τ) be a complete measure space, Z be an arbitrary topological space and W_n, $n = 1, 2,\ldots$ be correspondences from T into Z with measurable graphs. Then the correspondences $\bigcup_n W_n(\cdot)$, $\bigcap_n W_n(\cdot)$ and $Z \setminus W_n(\cdot)$ have measurable graphs.

Proof: See Theorem III.40 in Castaing-Valadier [8].

Lemma 4.2: Let (T, τ) be a complete measure space, Z be a separable metric space and $W : T \to 2^Z$ be a correspondence with measurable graph. Then for every $x \in Z$, $\mathrm{dist}(x,W(\cdot))$ is a measurable function.

Proof: See Theorem III.9 in Castaing-Valadier [8].

Lemma 4.3: Let (T,τ) be a measure space, Z be a complete, separable metric space, and $W : T \to 2^Z$ be a correspondence with measurable graph. Then the correspondence $V : T \to 2^Z$ defined by

$$V(t) = \{x \in Z : \mathrm{dist}(x, W(t)) < \lambda\}, \text{ (where } \lambda \text{ is any real number)}$$

is lower measurable.

Proof: See Himmelberg [12, Theorem 6.2].

Lemma 4.4: Let (T,τ) be a measure space and Z be a separable metric space. If $F : T \to 2^Z$ is a lower measurable correspondence, then the correspondence $\psi : T \to 2^Z$ defined by $\psi(t) = c\ell F(t)$ has a measurable graph.

Proof: See Himmelberg [12, Theorem 3.3].

Lemma 4.5: Let (T,τ) be a measure space, Z be a complete, separable metric space, and $W : T \to 2^Z$ be a correspondence with measurable graph. Then the correspondence $V : T \to 2^Z$ defined by

$$V(t) = \{x \in Z : \text{dist}(x, W(t)) < \lambda\}, \text{ (where } \lambda \text{ is any real number)}$$

has a measurable graph.

Proof: Notice that $V(t) = \{x \in Z : \text{dist}(x, W(t)) < \lambda\} = \bigcup_{\substack{r<\lambda \\ r:\text{rational}}} c\ell\, \{x \in Z : \text{dist}\,(x, W(t)) < r\}$. Therefore, by Lemmata 4.3 and 4.4 $V(t)$ has a measurable graph.

Lemma 4.6: Let $(S_i, \mathcal{Q}_i)$ for $i = 1, 2$ be measurable spaces, $h: S_1 \to S_2$ be a measurable function and $A \in \mathcal{Q}_1 \otimes \mathcal{Q}_2$. Then

$$\text{Proj}_{S_1}(G_h \cap A) \in \mathcal{Q}_1.$$

Proof: See [18, Lemma 4.4].

Lemma 4.7: Let (T_i, τ_i) for $i = 1,2,3$ be measurable spaces, $y : T_1 \to T_3$ be a measurable function and $\phi : T_1 \times T_2 \to 2^{T_3}$ be a correspondence with measurable graph, i.e., $G_\phi \in \tau_1 \otimes \tau_2 \otimes \tau_3$. Let $W : T_1 \to 2^{T_2}$ be defined by

$$W(t) = \{x \in T_2 : y(t) \in \phi(t,x)\}.$$

Then W has a measurable graph, i.e., $G_W \in \tau_1 \otimes \tau_2$.

Proof: See [18, Lemma 4.5].

Lemma 4.8: Let (T,τ) be a complete measure space, Z be a complete separable metric space, and $W : T \to 2^Z$ be a correspondence with measurable graph. Then the correspondence $V : T \to 2^Z$ defined by

$V(t) = \{x \in Z : \text{dist}(x, W(t)) \geq \lambda\}$, (where λ is any real number)

has a measurable graph.

Proof: See Lemma 4.2 and Himmelberg [12, theorem 6.4].

Lemma 4.9: Let (T,τ) be a measurable space, Z be a separable metric space, Y be a metric space and $f : T \times Z \to Y$ be a function which is measurable in $t \in T$ and continuous in $x \in Z$. Then f is jointly measurable.

Proof: See Lemma III.14 in Castaing-Valadier [8].

Lemma 4.10: Let (T,τ) be a measurable space and Y be a complete, separable metric space. Let $\phi : T \to 2^Y$ be a lower measurable correspondence and $f : T \to Y$ be a measurable function. Then the correspondence $\psi : T \to 2^Y$ defined by

$$\psi(t) = \phi(t) \cap (f(t) + B(0,\varepsilon))$$

is lower measurable.

Proof: We must show that $\{t \in T : \psi(t) \cap U \neq \emptyset\} \in \tau$ for every open subset U of Y. Observe that

$$\{t \in T : \psi(t) \cap U \neq \emptyset\} = \{t \in T : (\psi(t) \cap U) \cap (f(t) + B(0,\varepsilon)) \neq \emptyset\}$$

$$= \{t \in T : f(t) \in (\phi(t) \cap U) + B(0,\varepsilon)\}.$$

Since U is open, $\phi(t) \cap U$ is lower measurable, and since $\phi(t) \cap U) + B(0,\varepsilon) = \{y \in Y : \text{dist}(y, \phi(t) \cap U) < \varepsilon\}$, $(\phi(t) \cap U) + B(0,\varepsilon)$ has a measurable graph by Lemma 4.5. By Lemma 4.1, $\{t \in T : \psi(t) \cap U \neq \emptyset\} \in \tau$. This completes the proof of the Lemma.

Lemma 4.11: Let X, Y be topological spaces and $\phi : X \to 2^Y$, $\psi : X \to 2^Y$ be nonempty valued l.s.c. correspondences. Let V be an open entourage for some uniform structure on Y. Suppose that $\phi(x) \cap V(\psi(x)) \neq \phi$ for all $x \in X$. Then the correspondence $\theta : X \to 2^Y$ defined by $\theta(x) = \phi(x) \cap V(\psi(x))$ is l.s.c.

Proof: See Michael [21, Proposition 2.5].

Lemma 4.12: (Aumann). If (T, τ, μ) is a complete finite measure space, Y is a complete, separable metric space, and $F : T \to 2^Y$ is a correspondence with measurable graph, then there is a measurable function $f : T \to Y$ such that $f(t) \in F(t)$ for all $t \in T$.

Proof: See Aumann [3], or Castaing-Valadier [8].

5. PROOF OF THEOREMS

We begin with an approximate Caratheodory-type Selection result which is needed for the proof of Theorem 3.1.

Main Lemma 5.1: (Approximate Caratheodory-Type Selection): Let $(T,\mathcal{T},\mu)$ be a complete measure space, Y be a separable Banach space and Z be a complete, separable, metric space.
Let $\phi : T \times Z \to 2^Y$ be a convex valued correspondence (possibly empty) such that

(i) $\phi(\cdot,\cdot)$ is lower measurable

(ii) for each $t \in T$, $\phi(t,\cdot)$ is l.s.c.

Let $U = \{(t,x) \in T \times Z : \phi(t, x) \neq \emptyset\}$, and for each $x \in X$, let $U_x = \{t \in T : (t, x) \in U\}$ and for each $t \in T$, let $U^t = \{x \in Z : (t,x) \in U\}$. Then there exists an approximate or ε- Caratheodory-type Selection from $\phi|U$ i.e., given $\varepsilon > 0$, there exists a function $f^\varepsilon : U \to Y$ such that $f^\varepsilon(t, x) \in \phi(t, x) + B(0,\varepsilon)$, and for each $x \in Z$, $f^\varepsilon(\cdot, x)$ is measurable on U_x and for each $t \in T$, $f^\varepsilon(t,\cdot)$ is continuous on U^t.

Proof: Since Y is separable we may choose $\{y_n : n = 1,2,\ldots\}$ to be a countable dense subset of Y. For each $t \in T$ and $\varepsilon > 0$, let $W_n^\varepsilon(t) = \{x \in Z : y_n \in [\phi(t, x) + B(0,\varepsilon)]\}$. It follows from l.s.c. of ϕ that for each $t \in T$ and $n = 1,2,\ldots$, $W_n^\varepsilon(t)$ is open in Z. Since for each $(t,x) \in U$, $\phi(t, x) \neq \emptyset$, the set $\{W_n^\varepsilon(t) : n = 1,2,\ldots\}$ is an open cover of U^t. Notice that $\phi(t,x) + B(0,\varepsilon) = \{y \in Y : \text{dist}(y, \phi(t, x)) < \varepsilon\}$. By Lemma 4.5, $\phi(\cdot,\cdot) + B(0,\varepsilon)$ has a measurable graph. By Lemma 4.7, $W_n^\varepsilon(\cdot)$ has a measurable graph. For each $m = 1, 2,\ldots$, define the operator $(\ )_m$ by

$$(W)_m = \{w \in W : \text{dist}(w, Z \setminus W) \geq \frac{1}{2^m}\}.$$

For n = 1,2,..., let $V_n^\varepsilon(t) = W_n^\varepsilon(t) \setminus \bigcup_{k=1}^{n-1} (W_k^\varepsilon(t))_n$. It can be easily checked that $\{V_n^\varepsilon(t) : n=1,2,\ldots\}$ is a locally finite open cover of the set $U^t = \{x \in Z : (t, x) \in U\}$. Since $W_n^\varepsilon(\cdot)$ has a measurable graph, by Lemmata 4.1 and 4.8, $V_n^\varepsilon(\cdot)$ has a measurable graph. Let $\{g_n^\varepsilon(t, x) : n=1,2,\ldots\}$ be the partition of unity subordinated to the open cover $\{V_n^\varepsilon(t) : n = 1,2,\ldots\}$, for instance, for each n = 1,2,..., let

$$g_n^\varepsilon(t,x) = \frac{\text{dist}(x, Z \setminus V_n^\varepsilon(t))}{\sum_{k=1}^{\infty} \text{dist}(x, Z \setminus V_k^\varepsilon(t))}.$$

Then $\{g_n^\varepsilon(t,\cdot) : n=1,2,\ldots\}$ is a family of continuous functions $g_n^\varepsilon : U \to [0,1]$ such that $g_n^\varepsilon(t,x) = 0$ for $x \notin V_n^\varepsilon(t)$ and $\sum_{n=1}^{\infty} g_n^\varepsilon(t,x) = 1$ for all $(t,x) \in U$. Define $f^\varepsilon : U \to Y$ by $f^\varepsilon(t,x) = \sum_{n=1}^{\infty} g_n^\varepsilon(t,x) y_n$. Since $\{V_n^\varepsilon(t) : n=1,2,\ldots\}$ is locally finite, each x has a neighborhood N_x which intersects only finitely many $V_n^\varepsilon(t)$. Hence, $f^\varepsilon(t,\cdot)$ is a finite sum of continuous functions on N_x and it is therefore continuous on N_x. Consequently, $f^\varepsilon(t,\cdot)$ is a continuous function. Moreover, for any n such that $g_n^\varepsilon(t,x) > 0$, $x \in V_n^\varepsilon(t) \subset W_n^\varepsilon(t) = \{z \in Z : y_n \in [\phi(t,z) + B(0,\varepsilon)]\}$, i.e., $y_n \in \phi(t,x) + B(0,\varepsilon)$. So $f^\varepsilon(t,x)$ is a convex combination of elements, y_n from the convex set $\phi(t,x) + B(0,\varepsilon)$. Therefore, $f^\varepsilon(t,x) \in \phi(t,x) + B(0,\varepsilon)$ for all $(t,x) \in U$. Since $V_n^\varepsilon(\cdot)$ has a measurable graph by Lemmata 4.1 and 4.2, $\text{dist}(x, Z \setminus V_n^\varepsilon(\cdot))$ is a measurable function. Hence, for each n, x, $g_n^\varepsilon(\cdot,x)$ is a measurable function. Consequently, $f^\varepsilon(\cdot,x)$ is measurable for each x. Therefore f^ε is an approximate or ε-Caratheodory-type selection from $\phi|U$. This completes the proof of the Main Lemma.

<u>Proof of Theorem 3.1</u>: We now construct inductively, functions $f_\ell : U \to Y$, $\ell = 1,2,\ldots$, such that

(a) $f_\ell(t,\cdot)$ is continuous on U^t and $f_\ell(\cdot,x)$ is measurable on U_x,

(b) $f_\ell(t,x) \in \phi(t,x) + B(0,\frac{1}{2^\ell})$, $\ell = 1,2,\ldots,$

(c) $f_\ell(t,x) \in f_{\ell-1}(t,x) + 2B(0, \frac{1}{2^{\ell-1}})$, $\ell = 2,3,\ldots.$

The existence of f_1 satisfying (a) and (b) for $\ell = 1$, is guaranteed by the Main Lemma 5.1. Suppose that we have $f_1,\ldots,f_k$ satisfying (a), (b) and (c) for $\ell = 1,2,\ldots,k$. We must find $f_{k+1} : U \to Y$ which satisfies (a), (b) and (c) for $\ell = k+1$. Now define $\phi_{k+1}(t,x) = \phi(t,x) \cap (f_k(t,x) + B(0,\frac{1}{2^k}))$. Then $\phi_{k+1}(t,x)$ is nonempty, by the induction hypothesis, and $\phi_{k+1}(t,\cdot)$ is l.s.c. by Lemma 4.11. It follows from Lemma 4.10 that $\phi_{k+1}(\cdot,\cdot)$ is lower measurable. By the Main Lemma 5.1 there exists $f_{k+1} : U \to Y$ such that $f_{k+1}(t,x) \in \phi_{k+1}(t,x) + B(0, \frac{1}{2^{k+1}})$. But then $f_{k+1}(t,x) \in (f_k(t,x) + B(0,\frac{1}{2^k})) + B(0,\frac{1}{2^{k+1}})) \subset (f_k(t,x) + 2B(0,\frac{1}{2^k}))$ which is (c) and $f_{k+1}(t,x) \in \phi(t,x) + B(0,\frac{1}{2^{k+1}})$ which is (b). By (c), $\{f_\ell : \ell = 1,2,\ldots\}$ is uniformly Cauchy, and therefore converges uniformly to $f : U \to Y$. Since ϕ is closed valued $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$. Furthermore, $f(t,\cdot)$ is continuous in U^t and $f(\cdot,x)$ is measurable on U_x. By Lemma 4.9, $f(\cdot,\cdot)$ is jointly measurable This completes the proof of the Theorem.

<u>Main Lemma 5.2</u>: Under the conditions of Theorem 3.1 there exists a countable collection $\mathfrak{F}$ of Caratheodory-type selections from $\phi|U$ such that for every $(t,x) \in U$, $\{f(t,x) : f \in \mathfrak{F}\}$ is dense in $\phi(t,x)$.

Proof: Let $\{E^n : n=1,2,\ldots\}$ be a convex open basis of Y. For each $n = 1,2,\ldots$, $U^n = \{(t,x) \in T \times Z : \phi(t,x) \cap E^n \neq \phi\}$. For each $t \in T$, define $U^n(t) = \{x \in Z : (t,x) \in U^n\}$. Notice that for each $t \in T$, $U^n(t)$ is open in Z. Moreover, $U^n(\cdot)$ has a measurable graph. For each $k = 1,2,\ldots$, and $t \in T$, let $A_k^n(t) = \{x \in Z : \text{dist}(x, \; Z \setminus U_n(t)) \geqslant \frac{1}{2^k}\}$. By Lemma 4.8, $A_k^n(\cdot)$ has a measurable graph. Notice that $\bigcup_{k=1}^{\infty} A_k^n(t) = U^n(t)$, and for each $t \in T$, A_k^n is closed in Z. Define $\phi_k^n : T \times Z \to 2^Y$ by

$$\phi_k^n(t,x) = \begin{cases} c\ell(\phi(t,x) \cap E^n) & \text{if } x \in A_k^n(t) \\ \phi(t,x) & \text{if } x \notin A_k^n(t). \end{cases}$$

Since for each $t \in T$, $A_k^n(t)$ is closed in Z, $\phi_k^n(t,\cdot)$ is l.s.c. Moreover, since for every open subset V of Y, $\{(t,x) : \phi_k^n(t,x) \cap V \neq \phi\} = \{(t,x) : c\ell(\phi(t,x) \cap E^n) \cap V \neq \phi, \; x \in A_k^n(t)\} \cup \{(t,x) : \phi(t,x) \cap V \neq \phi, \; x \notin A_k^n(t)\} \in \tau \otimes \mathcal{B}(Z)$, $\phi_k^n(\cdot,\cdot)$ is lower measurable. By Theorem 3.1 there exist Caratheodory-type selection $f_k^n(\cdot,\cdot)$ from $\phi_k^n(\cdot,\cdot)$. Let $\mathcal{F}$ be the collection of all f_k^n, $n,k=1,2,\ldots$. Then $\mathcal{F}$ is a countable collection of Caratheodory-type selections from $\phi|U$, and it can be easily seen that $\{f(t,x) : f \in \mathcal{F}\}$ is dense in $\phi(t,x)$ for all $(t,x) \in U$. This completes the proof of the Lemma.

We will need the following notions. If K is a closed, convex subset of a normed linear space, then a <u>supporting set</u> of K is a closed convex subset S of K, $S \neq K$, such that if an interior point of a segment in K is in S, then the whole segment is in S. The set of all elements of K which are not in any supporting set of K will be denoted by I(K). The following facts below are due to Michael [21, p. 372].

Fact 5.1: If any convex subset K of Y is either closed or has an interior point or is finite dimensional, then $I(c\ell K) \subset K$.

Fact 5.2: Let K be a nonempty, closed, convex separable subset of a Banach space Y, and $\{y_i : i = 1,2,\ldots\}$ be a dense subset of K. If

$$z_i = y_i + \frac{(y_i - y_1)}{\max(1, \|y_i - y_1\|)} \quad \text{for all } i \text{ and } z = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i z_i ,$$

then $z \in I(K)$.

Proof: See Michael [17, Lemma 5.1, p. 372].

Proof of Theorem 3.2: Define $\psi : T \times Z \to 2^Y$ by $\psi(t,x) = c\ell\phi(t,x)$. Since $\phi(t,\cdot)$ is l.s.c. so is $\psi(t,\cdot)$. Moreover, ψ is lower measurable. By the Main Lemma 5.2 there exist Caratheodory-type selections $\{g_k(t,x) : k=1,2,\ldots\}$ dense in $\psi(t,x)$ for all $(t,x) \in U$. Let for each $k=1,2,\ldots$

$$f_k(t,x) = g_1(t,x) + \frac{g_k(t,x) - g_1(t,x)}{\max(1, \|g_k(t,x) - g_1(t,x)\|)}$$

$$f(t,x) = \sum_{k=1}^{\infty} \frac{1}{2^k} f_k(t,x).$$

By Fact 5.2, $f(t,x) \in I(\psi(t,x))$ for all $(t,x) \in U$. Since the series defining f converges uniformly, it follows that for each $t \in T$, $f(t,\cdot)$ is continuous and for each $x \in X$, $f(\cdot,x)$ is measurable. By Fact 5.1, $f(t,x) \in I(\psi(t,x)) \subset \phi(t,x)$ if either (i) or (ii) of Theorem 3.2 are satisfied. This completes the proof of the Theorem.

Proof of Theorem 3.3: It follows from Theorem 3.1 that there exists a function $f : T \times X \to X$ such that $f(t,x) \in \phi(t,x)$ for all $(t,x) \in T \times X$, and for each $x \in X$, $f(\cdot,t)$ is measurable and for each $t \in T$, $f(t,\cdot)$ is continuous. Moreover, $f(\cdot,\cdot)$ is jointly measurable.

For each $t \in T$, let $F(t) = \{x \in X : g(t,x) = 0\}$, where $g(t,x) = f(t,x) - x$. It follows from the Tychonoff fixed point theorem that the function $f(t,\cdot) : X \to X$ has a fixed point. Therefore, for each $t \in T$, $F(t) \neq \phi$. Since g is jointly measurable, F has a measurable graph. Hence by Aumann's measurable selection theorem (Lemma 4.12), there exists a measurable function $x^* : T \to X$ such that for almost all t in T, $x^*(t) \in F(t)$, i.e., $x^*(t) = f(t,x^*(t)) \in \phi(t,x^*(t))$. This completes the proof of the theorem.

Proof of Theorem 3.4: The argument is similar to that adopted in the proof of Theorem 3.3 except that one must use now Theorem 3.2 instead of Theorem 3.1.

REFERENCES

[1] Aubin, P. J. and A. Cellina, Differential Inclusions, (1984), Springer-Verlag: New York.

[2] Aubin, P. J. and J. Ekeland, Applied Nonlinear Analysis, (1983), Wiley-Interscience.

[3] Aumann, R. J., "Measurable Utility and the Measurable Choice Theorem," La Decision, C.N.R.S., Aix-en-Provence (1967), 15-26.

[4] Bohnenblust, H. and S. Karlin, "On a Theorem of Ville," in Contributions to the Theory of Games I, eds., H. Kuhn and A. Tucker, (1950), Princeton University Press: Princeton, NJ.

[5] Borglin, P. A. and H. Keiding, "Existence of Equilibrium Actions and Equilibrium: A Note on the 'New' Existence Theorems," Journal of Mathematical Economics 3, (1976), 313-316.

[6] Browder, F., "The Fixed Point Theory of Multivalued Mappings in Topological Vector Spaces," Mathematische Annalen 177, (1968), 283-301.

[7] Castaing, C., "Sur l'existence des sections separement measurable et separement continues d'une multi-application," Travaux du Seminaire d'analyse Convexe, Univ. des. Sci. et Techniques du Languedoc, 5, (1975), 14.

[8] Castaing, C. and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics No. 480, (1977), Springer-Verlag: New York.

[9] Debreu, G., "A Social Equilibrium Existence Theorem," Proceedings of the National Academy of Sciences of the U.S.A. 38, (1952), 886-893.

[10] Fryszkowski,A., "Caratheodory Type Selectors of Set-Valued Maps of Two Variables," Bulletin De L'Academie Polonaise Des Sciences 25, (1977), 41-46.

[11] Gale, D. and A. Mas-Colell, "An Equilibrium Existence Theorem for a General Model without Ordered Preferences," Journal of Mathematical Economics 2, (1975), 9-15.

[12] Himmelberg, J. C., "Measurable Relations," Fundamenta Mathematicae, LXXXVII, (1975), 53-72.

[13] Khan, M. Ali, "Equilibrium Points of Nonatomic Games over a Non-Reflexive Banach Space," Journal of Approximation Theory, (to appear).

[14] Khan, M. Ali, "On Extensions of the Cournot-Nash Theorem," in Advances in Equilibrium Theory, eds., C. D. Aliprantis, et al., (1985), Springer-Verlag.

[15] Khan, M. Ali and N. S. Papageorgiou, "On Cournot-Nash Equilibria in Generalized Quantitative Games with a Continuum of Players," University of Illinois, (1985).

[16] Itoh, S., "Random Fixed Point Theorems with Applications to Random Differential Equations in Banach Spaces," Journal of Mathematical Analysis and Applications 67, (1979), 261-273.

[17] Kim, T., K. Prikry, and N. C. Yannelis, "Equilibria in Abstract Economies with a Measure Space of Agents and with an Infinite Dimensional Strategy Space," University of Minnesota, (1985).

[18] Kim, T., K. Prikry and N. C. Yannelis, "On a Caratheodory-Type Selection Theorem," University of Minnesota, (1985).

[19] Kuratowski, K. and C. Ryll-Nardzewski, "A General Theorem on Selectors," Bull. Acad. Polom. Sci. Ser. Sci. Marsh. Astronom. Phys. 13, (1965), 397-403.

[20] Mas-Colell, A., "On a Theorem of Schmeidler," Journal of Mathematical Economics 13, (1984), 201-206.

[21] Michael, E., "Continuous Selections I.," Annals of Mathematics 63, (1956), 363-382.

[22] Nash, J. F., "Non-Cooperative Games," Annals of Mathematics 54, (1951), 286-295.

[23] Papageorgiou, N. S., "Random Fixed Point Theorems for Measurable Multifunctions in Banach Spaces," Proceedings of the American Mathematical Society, (to appear).

[24] Schmeidler, D., "Equilibrium Points of Nonatomic Games," Journal of Statistical Physics 7, (1973), 295-300.

[25] Shafer, W. and H. Sonnenschein, "Equilibrium in Abstract Economies without Ordered Preferences," Journal of Mathematical Economics 2, (1975), 345-348.

[26] Toussaint, S., "On the Existence of Equilibrium with Infinitely Many Commodities and Without Ordered Preferences," Journal of Economic Theory 33, (1984), 98-115.

[27] Wesley, E., "Borel Preference Orders in Markets with a Continuum of Traders," Journal of Mathematical Economics 3, (1976), 155-165.

[28] Wieczorek, A., "On the Measurable Utility Theorem," Journal of Mathematical Economics 7, (1980), 165-173.

[29] Yannelis, N. C. and N. D. Prabhakar, "Existence of Maximal Elements and Equilibria in Linear Topological Spaces," Journal of Mathematical Economics 12, (1983), 233-245.

[30] Yannelis, N. C. and N. D. Prabhakar, "Equilibrium in Abstract Economies with an Infinite Number of Agents, an Infinite Number of Commodities and without Ordered Preferences," University of Minnesota, (1983).

[31] Yannelis, N. C., "Equilibria in Non-Cooperative Models of Competition," University of Minnesota, (1985).