Symplectic Flexibility and the Hard Lefschetz Property

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Heidi M. Andersen

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

TIAN-JUN LI

August, 2017

Acknowledgements

I would like to express thanks to my advisor and to the University of Minnesota, both of which helped provide me an incredible environment in which to study and grow. I deeply appreciate my advisor's mathematical suggestions, as well as his patience, understanding, and career advice. I also owe thanks to the Director of Graduate Studies in Mathematics for his authenticity and unfailing support, as well as to my initial advisor, for his excellent advice in doing well on qualifying exams as well as in doing well in life. No thanks must be spared for all of the professors who employed me as a teaching assistant and taught me the craft of teaching math, as well as for the math department's administrative assistants, who tirelessly helped me navigate the labyrinthine path of paperwork along the way to a doctoral degree. Last of all, but certainly not least, I must thank my colleagues - inside and outside the math department - for their companionship and their diverse points of view, and I must thank my family for their solicitousness and their emotional support as well as job hunting advice. "It takes a village," as they say, and I don't think I could have asked for a better one.

Abstract

In this thesis, we begin our study with the classical notion of symplectic flexibility - the existence on symplectic (X^{2n}, ω) of a continous family ω_t of symplectic forms along which at least one symplectic harmonic number $h_{k,hr}(\omega_t)$ varies. We use this theory, which was put forth by Ibañez, Rudyak, Tralle, and Ugarte, to analyze the flexibility of the new symplectic Bott-Chern and Aeppli cohomologies invented by Tsai, Tseng, and Yau. We next explore Guillemin and Sternberg's notion of symplectic birational cobordism (especially the symplectic blow-up), where we improve the existing theory on its capacity for shrinking the kernels of the Lefschetz maps $\cup[\omega]^k : H^{n-k}_{dR}(X) \to H^{n+k}_{dR}(X)$, $k \leq n$. Examples are given of Hard Lefschetz and non-Hard Lefschetz symplectic manifolds that are symplectic birational cobordant. We consider the merit of using this equivalence relation as a classification tool. The problem of classifying which symplectic manifolds are symplectic birational cobordant to a Kähler manifold remains open, but we provide discussion of the problem as well as a first conjecture. Lastly, we review and discuss a new example in the literature of a deformation equivalence between a Kähler and a non-Hard Lefschetz form; the first of its kind. Yunhyung Cho achieved this on the special class of compact six-dimensional simple Hamiltonian-S^1 manifolds that have diffeomorphic four-dimensional fixed componenets; we generalize the the result to compact ten-dimensional simple Hamiltonian-S^1 manifolds that have diffeomorphic eight-dimensional fixed components. We also discuss the outlook for extending the result to higher dimensions $4k + 2$ in this class.

Contents

Chapter 1

Symplectic Flexibility

On a compact symplectic or Kähler manifold (X^{2n}, ω), many different notions of cohomology exist. The de Rham cohomology $H^*_{dR}(X; \mathbf{F})$ - where possibly $F = \mathbf{R}$ or $\mathbf{C}$ - is the most basic, and has ramifications even without the assumption of a smooth structure. Somewhat more sophisticated are the symplectic harmonic cohomologies, introduced by Brylinski [7], which translate the spaces $\Omega^k_{hr}(X) := \ker(d : \Omega^k(X) \to \Omega^{k+1}(X)) \cap \ker(d^\wedge : \Omega^{k+1}(X) \to \Omega^k(X))$ to the cohomology level:

$$H^k_{hr}(X) = \frac{ker(d : \Omega^k(X; \mathbf{R}) \to \Omega^{k+1}(X; \mathbf{R})) \cap ker(d^\wedge : \Omega^k(X; \mathbf{R}) \to \Omega^{k-1}(X; \mathbf{R}))}{im(d : \Omega^{k-1}(X; \mathbf{R}) \to \Omega^k(X; \mathbf{R})) \cap \Omega^k_{hr}(X)}$$

Here, $d^\wedge = (-1)^{k+1} *_\omega d *_\omega$, where $*_\omega$ is the symplectic star operator associated to ω. The symplectic form ω also induces on the differential algebra $\Omega^*(X; \mathbf{R})$ a natural Lefschetz structure

$$\Omega^k(X; \mathbf{R}) = \bigoplus_{0 \le r \le max(0, k-n)} \left(\omega^r \wedge \mathcal{P}^{k-2r}\right)$$

where $\mathcal{P}^* = ker(\wedge\omega^{n-*+1} : \Omega^*(X;\mathbf{R}) \to \Omega^{2n-*+2}(X;\mathbf{R}))$. This structure is ideal to describe the action of $*_\omega$: from [35], we have

$$*_\omega(\omega^r \wedge p_{k-2r}) = (-1)^{\frac{(k-2r)(k-2r+1)}{2}} \frac{(k-2r)!}{(n-k+r)!}\omega^{n-k+r} \wedge p_{k-2r}$$

This cohomology group $H^*_{hr}(X;\mathbf{R})$ and the dimensions of its subspaces naturally depend upon the symplectic form ω and not just on the topology of X; in general, $h_{k,hr}(\omega) \leq b_k$ and all $0 \leq k \leq 2n$. In fact, Mathieu [25] proved that $h_{k,hr}(\omega) = b_k$ for all $0 \leq k \leq 2n$ if and only if each of the maps

$$\cup[\omega]^{n-k} : H^k_{dR}(X;\mathbf{R}) \to H^{2n-k}_{dR}(X;\mathbf{R})$$

are surjective (which, by Poincaré duality for this compact symplectic manifold (X,ω), is equivalent to isomorphism). In the '90s, a question was raised by Dusa McDuff and Boris Khesin, about whether or not a path of symplectic forms ω_t could be constructed on a compact symplectic (non-Kähler) manifold X, along which any of the symplectic harmonic numbers $h_{k,hr}(\omega_t)$ varied; a natural related question is whether or not a path of symplectic forms may be constructed on X along which at least one (but not all) has the Hard Lefschetz property (this is the same as asking if X supports a deformation equivalence between Hard Lefschetz and non-Hard Lefschetz forms; ordinarily, the answer to this question is not an obvious affirmative at all). Over the years, some partial answers to the first question have been achieved.

For closed symplectic 4-manifolds, Yan [36] proved some necessary conditions:

(Proposition 4.1, [36]) For the following statements, $(i) \implies (ii) \implies (iii) \implies (iv)$:

(*i*) *There exists a family* ω_t *of symplectic forms such that* $h_{3,hr}(\omega_t)$ *varies*

(*ii*) *There exist two symplectic forms* ω_1, ω_2 *such that* $ImL_{[\omega_1]} \neq ImL_{[\omega_2]}$*, where* $L_{[\omega_t]}$ *is the Lefschetz map with respect to* ω_i *(*$i = 1, 2$*).*

(*iii*) *There exists a symplectic form* ω *on* X *and a class* $c \in H^2_{dR}(X)$ *such that* ImL_c *is not a (strict) subspace of* $ImL_{[\omega]}$*.*

(*iv*) *There exists a symplectic form* ω *on* X *such that* $ImL_{[\omega]}$ *is not equal to the image of the cup product pairing* $H^1(X) \otimes H^2(X) \to H^3(X)$*.*

Unfortunately, none of conditions $(ii), (iii), (iv)$ are sufficient to guarantee (i), as the Kodaira-Thurston 4-manifold shows, by satisfying (iv) but not (i). However, Yan did establish his own set of sufficient conditions for (i):

Corollary 4.2, [36]: *Let* (X^4, ω) *be a closed symplectic manifold which satisfies the following conditions:*

- *The homomorphism* $L_{[\omega]} : H^1_{dR}(X) \to H^3_{dR}(X)$ *is trivial;*
- *The cup product* $H^1_{dR}(X) \otimes H^2_{dR}(X) \to H^3_{dR}(X)$ *is non-trivial.*

Then X *is flexible.*

Here, "flexible" is a term used to describe precisely the quality named in Khesin's and McDuff's question; in closed symplectic 4-manifolds, variance of symplectic harmonic numbers along a path ω_t of symplectic forms is equivalent to flexibility for $h_{3,hr}$. Indeed, for $k = 0, 1, 2, 4$, it is true that $b_k = h_{k,hr}$. For $k = 0$, this follows because $d^\wedge$ is a degree-(-1) operator; for $k = 4$, it follows because $*_\omega \omega^2 = 1$, which is d-closed; and for $k = 1$, it follows because $*_\omega \beta = \beta \wedge \omega$ for any (d-closed) 1-form β. For the case of $k = 2$, a simple proof is given in [34].
The case of dimension 4 has been extensively studied, and is not very interesting with regard to the question of flexibility; as we will see, either all or none of the symplectic forms on a given closed symplectic 4-manifold have the Hard Lefschetz property. The variance of $h_{3,hr}(\omega)$ is in general very tame in this dimension: if $b_1 = 1$, then by Poincaré duality, the nondegenerate cup product $H^2 \otimes H^1 \to H^3$ must be trivial, and hence, that $h_{3,hr} \equiv 0$, regardless of symplectic structure. For 4-dimensional nilmanifolds, the situation is quite cut-and-dried: from Yan, we have

Example 4.4, [36]: *Let (X, ω) be a closed 4-dimensional symplectic nilmanifold. It is known in [14] that $2 \leq b_1(X) \leq 4$. Then the following conditions hold:*

- *If $b_1(X) = 2$, then $h_{3,hr}(\omega) = 0$;*

- *If $b_1(X) = 3$, then $b_2(X) = 2$ or 4; for $b_2(X) = 2$, $h_3(\omega) = 0$; for $b_2(X) = 4$, $h_{3,hr}(\omega) = 2$;*

- *if* $b_1(X) = 4$, *i.e.,* $X = T^4$, *then* $h_{3,hr}(\omega) = 4$.

As noted and proved by Yan, Gompf's 4-dimensional construction in [18] is flexible:

Proposition 4.4, [36] (cf. [18]): *Let* G *be any finitely presentable group which admits a nontrivial skew structure. Then there is a closed, symplectic 4-manifold* (X, ω) *with* $\pi_1(X) = G$ *such that the following two conditions are satisfied:*

- *the Lefschetz map* $L_{[\omega]} : H^1(X) \to H^3(X)$ *is identically zero;*

- *there exists a class* $c \in H^2(X)$ *such that the map* $L_c : H^1(X) \to H^3(X)$ *is nontrivial.*

Draghici [12] also achieved a related result on a certain subset of Kähler 4-manifolds:

Theorem 1.1, [12]: *Let* M^4 *be a minimal complex surface of general type with* $b^+ > 1$. *Let* (g, J, ω) *be a Kähler structure on* M *and let* β *be a holomorphic* $(2, 0)$ *form on* M *with respect to the complex structure* J. *Denote by*

$$\alpha_t = (1 - t)[\omega] + t[Re(\beta)] \in H^2(M, \mathbf{R})$$

Then we have the following:

(i) $\alpha_0 \in \mathcal{K}$,

(ii) $\exists t_0 \in (0,1)$ such that for $t \in (t_0, 1)$, $\alpha_t \in \mathcal{S} - \mathcal{K}$,

(iii) $\alpha_1 \notin \mathcal{S}$.

Put simply, this theorem says that for some compact Kähler dimension-4 manifolds, there exists paths of diffferential 2-forms along which the Hard Lefschetz property is satisfied by some forms and failed by others, which - in particular - implies a variance in the symplectic harmonic numbers (cf. [25]).
More interesting and sophisticated advances have been made in dimension 6. Ibáñez, Rudyak, Tralle, and Ugarte achieved a few such results of their own, which they established as corollaries of a simple topological lemma:

(Lemma 2.10, [21]) *Let* $\mathbf{L}$ *be the space of all linear maps* $\mathbf{R}^k \to \mathbf{R}^l$. *Then the following holds:*

- *for every* r *the set*

$$\{A \in \mathbf{L} | rank A \leq r\}$$

is an algebraic subset of $\mathbf{L}$.

- *for every* $m \leq min\{k, l\}$, *the set* $\{A \in \mathbf{L} |\, rank A \geq m\}$ *is open and dense in* $\mathbf{L}$;

- *let* $A, B \in \mathbf{L}$ *be two linear maps such that* $rank A < rank B$. *Then the set*

$$\Lambda = \{\lambda \in \mathbf{R} | rank(A + \lambda B) \geq rank B\}$$

is an open and dense subset of $\mathbf{R}$.

Proof (cf [21]): The first statement is clear, since the set in question is defined precisely by the vanishing of he determinants of all minors of size s with $r < s \leq min\{k, l\}$. The second statement is equally clear, because in the Zariski topology on $\mathbf{L}$, non-empty open sets are dense, and the complement of our set is precisely

$$\{A \in \mathbf{L} | \text{ all minors of size} \geq m \text{ have vanishing determinant}\}$$

which is algebraic, and hence, closed. To prove the third statement, put $m := rank\, B$. By the first statement, the set

$$\{M \in \mathbf{L} | rank\, M \geq rank\, B\}$$

is open in $\mathbf{L}$, so for $\mu \neq 0$ small enough, $B + \mu A$ is an element of this set. Since

$$B + \mu A = \mu \left(A + \frac{1}{\mu} B \right)$$

it follows that $rank(A + \frac{1}{\mu} B) \geq rank(B)$, and hence, that $\mu \in \Lambda$, which is thus nonempty. Since

$$\mathbf{R} \setminus \Lambda = \{\lambda \in \mathbf{R} | rank(A + \lambda B) < rank\, B\}$$

is algebraic and closed (by the first statement), we conclude that Λ is open and dense in $\mathbf{L}$. □

From this lemma, Ibáñez, Rudyak, Tralle, and Ugarte achieved a couple of corollaries, the first of which gives some insight into the nature of the variance of h_{m+k}:

(Corollary 2.11, [21]): *Let ω_0 and ω_1 be two symplectic forms on a manifold X^{2m}. Suppose that for some $k > 0$, $h_{m-k}(\omega_0) = h_{m-k}(\omega_1)$, but $h_{m+k}(\omega_0) < h_{m+k}(\omega_1)$. Then, for every $\epsilon > 0$, there exists $\lambda \in (0, \epsilon)$ such that $h_{m+k}(\omega_0 + \lambda\omega_1) > h_{m+k}(\omega_0)$. Moreover, X is flexible provided that it is closed.*

(Corollary 2.12, [21]): *Let (X^{2m}, ω_0) be a closed symplectic manifold. Given k with $0 < k < m$, suppose that $h_{m-k,hr}(\omega) = b_{m-k}$ for every symplctic form on X. Furthermore, suppose that there exists $x \in H^2_{dR}(X)$ such that*

$$rank\{L^k_x : H^{m-k}_{dR}(X) \to H^{m+k}_{dR}(X)\} > h_{m+k,hr}(\omega_0)$$

Then X is flexible.

Define

$$\Omega_{sympl}(X) = \{\omega \in \Omega^2(X) | \omega \, is \, a \, symplectic \, form \, on \, X\}$$

and put $\Omega(b, k) = \{\omega \in \Omega_{sympl} | h_{k,hr}(\omega) = b\}$.

(Corollary 2.13, [21]): *Let X^{2m} be a manifold that admits a symplectic structure. Suppose that, for some $k > 0$, $h_{m-k,hr}(\omega)$ does not depend on the symplectic structure ω on X. Then the following three conditions are equivalent:*

- *the set $\Omega(b, m + k)$ is open and dense in $\Omega_{sympl}(X)$;*

- *the interior of the set $\Omega(b, m + k)$ in $\Omega_{sympl}(X)$ is non-empty;*

- *the set $\Omega(b, m + k)$ is non-empty and $h_{m+k,hr}(\omega) \leq b$ for every $\omega \in \Omega_{sympl}(X)$.*

While this lemma and these corollaries apply to compact symplectic manifolds of general dimension, the authors of [21] were able to use them to completely classify the compact symplectically flexible nilmanifolds in dimension 6. They accomplished this by brute force, running down the list of 34 six-dimensional compact nilmanifolds, whose associated Lie algebras had previously been described by S. Salamon [30]. Their chart of symplectic harmonic numbers for these nilmanifolds can be found in [21], revealing five nilmanifolds having a fourth harmonic number $h_{4,hr}$ that varied. All such examples were flexible, which - in retrospect - isn't surprising, since this follows from Corollary 2.11. We summarize the result here (cf. [21]):

If $\{\alpha_1, ..., \alpha_6\}$ *is a dual basis for the Lie algebra* $\mathfrak{g}$ *associated to a six-dimensional compact nilmanifold, then the Chevalley-Eilenberg structural constants* c_k^{ij}*, which uniquely identify the nilmanifold, may be read from the exterior derivatives of this basis:*

$$d\alpha_k = \sum_{1 \leq i < j < k} c_k^{ij} \alpha_i \wedge \alpha_j$$

Denoting $\alpha_l \wedge \alpha_m$ *by* lm*, we encode the structural constants for each flexible nilmanifold in six-tuples for the exterior derivatives of the basis elements:* $(0, 0, 12, 13, 23, 14-25)$*,* $(0, 0, 0, 12, 14, 15 + 23 + 24)$*,* $(0, 0, 0, 12, 13 + 14, 24)$*,* $(0, 0, 0, 12, 13, 23)$*, and* $(0, 0, 0, 0, 12, 13)$.

By contrast, the results of [21] answer in the negative for flexibility on 4-dimensional solvmanifolds whose de Rham cohomologies satisfy Nomizu isomorphism. Indeed, straightforward computation establishes

Algebra	h_3
(-24,-34,0,0)	0
(-23,13,0,0)	2
(-13,23,0,0)	2
(-23,0,0,0)	2
(0,0,0,0)	4

regardless of symplectic structure.

Generalizing the work of [21] to compact symplectic nilmanifolds of higher dimension was, at the time of its publication (2000), not a straightforward problem: not even in dimension 8 did there exist a neat classification a la Salamon's work of all compact symplectic nilmanifolds. However, in 2010, Guan [17] achieved a breakthrough, by demonstrating an inductive method that could potentially help determine symplectic flexibility in dimension 8 while bypassing the process of classifying these nilmanifolds via left-invariant bases of differential 1-forms.

(Corollary 1, [17]): *For any compact nilmanifold X of real dimension $2n$ with a symplectic structure there is a new compact nilmanifold X_1 of real dimension $2n-2$ with a symplectic structure such that X_1 is a symplectic reduction of X.*

This means that if X is a compact symplectic 8-dimensional nilmanifold, then it must be a symplectic extension of a compact six-dimensional symplectic nilmanifold; specifically, the Lie algebra of the six-dimensional nilmanifold may be viewed as a symplectic subspace of the Lie algebra of the eight-dimensional nilmanifold. Guan's construction is performed as follows (cf. Extension Theorem,

[17]): let $\mathfrak{G}$ be an eight-dimensional real vector space and $\mathfrak{g}$ a six-dimensional subspace. Let $[\cdot,\cdot\cdot]$ be a Lie bracket on $\mathfrak{g}$, let σ a symplectic form on $\mathfrak{g}$, and let $A \in End(\mathfrak{g})$ be nilpotent. We gain a family $exp(tA)$ of automorphisms of $\mathfrak{g}$, and hence, a family of symplectic forms

$$\sigma_t = \sigma(e^{tA}\cdot, e^{tA}\,\cdot\,\cdot)$$

Put $\tau_t := \sigma_t(A\cdot,\cdot\cdot) + \sigma_t(\cdot, A\,\cdot\,\cdot)$ and $\bar{k}_t := \tau_t(A\cdot,\cdot\cdot) + \tau_t(\cdot, A\,\cdot\,\cdot)$. Since σ_t are all closed, so are τ_t and $\bar{k}_t$. We now construct an "extended" Lie bracket $<\cdot,\cdot\cdot>$ and symplectic form ω on $\mathfrak{G}$: let a, c span a complementary subspace of $\mathfrak{G}$ to $\mathfrak{g}$, and put

$$\begin{aligned} < g_1, g_2 > &= [g_1, g_2] + \tau_t(g_1, g_2)c \\ < a, g > &= Ag + k(g)c \;\; with\; dk = \bar{k} \\ < a, c > &= < g, c > = 0 \end{aligned}$$

We define $\omega(a, c) = 1$, $\omega(a, g) = \omega(c, g) = 0$, and $\omega(g_1, g_2) = \sigma(g_1, g_2)$. The Jacobi identity for $<\cdot,\cdot\cdot>$ is easy to verify, as is the closedness of ω, since we need only verify that

$$\omega(< x, y >, z) + \omega(< y, z >, x) + \omega(< z, x >, y) = 0$$

for all $x, y, z \in \mathfrak{G}$. $\square$

Guan notes in [17] that the symplectic extension of $\mathfrak{g}$, constructed in this way, exists if and only if $\bar{k}$ is exact. The Salamon classification of real nilpotent six-dimensional Lie algebras identifies all that are associated with compact nilmanifolds that support symplectic structures, of which there are twenty-six. What

remains, then, in studying flexibility in eight-dimensional symplectic nilmanifolds is to collect the symplectic extensions of the twenty-six 6-dimensional symplectic nilmanifolds, and to apply the brute force method to determine all that allow variable $h_{5,hr}, h_{6,hr}, h_{7,hr}$. While the idea is straightforward, the execution will be quite involved, since it will require a classification of which nilpotent elements of $End(\mathbf{R}^6)$ in Guan's construction lead to a symplectic extension; this is a highly nontrivial question, and might constitute some interesting future work.

Chapter 2

Symplectic Flexibility in Tsai-Tseng-Yau Cohomology

Another new avenue in which we can investigate symplectic flexibility is in the realm of the new Tsai-Tseng-Yau primitive and filtered symplectic cohomologies [33]. We will review the basic concept of these cohomologies.

Tseng, and Yau initially studied symplectic analogues

$$H^k_{d+d^\wedge}(X) = \frac{kerd \cap kerd^\wedge}{imdd^\wedge}, H^k_{dd^\wedge}(X) = \frac{ker(dd^\wedge)}{(imd + imd^\wedge)}$$

of the Bott-Chern and Aeppli cohomology groups (cf. [1]), before recognizing that their results reduced to special properties on the following "primitive" versions of these groups:

$$PH^k_{d+d^\wedge}(X) = \frac{kerd \cap kerd^\wedge \cap P^k}{imdd^\wedge \cap P^k}$$

$$PH^k_{dd^\wedge}(X) = \frac{ker(dd^\wedge) \cap P^k}{(imd + imd^\wedge) \cap P^k}$$

where P^k here means $\left(kerL^{n-k+1}\right) \cap \Omega^k$. They proved the following useful decompositions

$$PH^k_{dd^\wedge}(X) \cong ker(L^{n-k+1} : H^{k-1}_d \to H^{2n-k+1}_d) \oplus coker(L^{n-k+1} : H^{k-2}_d \to H^{2n-k}_d)$$

$$PH^k_{d+d^\wedge}(X) \cong ker(L^{n-k+1} : H^k_d \to H^{2n-k+2}) \oplus coker(L^{n-k+1} : H^{k-1}_d \to H^{2n-k+1}_d)$$

for all $k \leq n$. With Tsai, they later considered the following "Lefschetz" filtration of $\Omega^k(X)$ whose steps are

$$F^p\Omega^k(X) := \bigoplus_{max(0,k-n) \leq r \leq min(\lfloor \frac{k}{2} \rfloor, p)} L^k \wedge P^{k-2r}$$

for each $k \leq 2n$. Invoking some projection and reflection operators $\pi^p, *_r$ whose actions are

$$\pi^p \sum_{j=0} \frac{1}{j!} L^j B_{k-2j} = B_k + \omega \wedge B_{k-2} + ... + \frac{1}{p!}\omega^p \wedge B_{k-2p}$$

and

$$*_r \frac{1}{r!}\omega^r \wedge B_s = \frac{1}{r!}\omega^{n-r-s} \wedge B_s$$

(that is, π^p projects a form onto its "p-filtered" part and $*_r$ reflects terms of the Lefschetz decomposition of $\Omega^*(X)$ about the central vertical axis of the Lefschetz pyramid), Tsai, Tseng, and Yau defined two differential operators:

$$d_+ := \pi^p \circ d$$

and

$$d_- := *_r \circ d \circ *_r$$

The subscripts for these operators clearly correspond to their raising and lowering of form degree by 1, and they both square to 0 (by the uniqueness of the Lefschetz

decomposition). Importantly, they preserve filtration degree; it's clear that d_+ does this, and it becomes clear that d_- does this as well if one notes that, by [33, Definition 2.1], a form is p-filtered if $L^{p+1} *_r \beta = 0$. Indeed, if β is p-filtered, then

$$L^{p+1} *_r (d_-\beta) = L^{p+1} *_r *_r d *_r \beta = L^{p+1} d *_r \beta = dL^{p+1} *_r \beta = 0$$

where we have used the fact that $*_r$ is an involution and the fact that d, L commute. Taking cohomology of the differential complexes

$$0 \to F^p\Omega^0 \xrightarrow{d_+} F^p\Omega^1 \xrightarrow{d_+} ... \xrightarrow{d_+} F^p\Omega^{2n-1} \xrightarrow{d_+} F^p\Omega^{2n} \to 0$$

$$0 \to F^p\Omega^{2n} \xrightarrow{d_-} F^p\Omega^{2n-1} \xrightarrow{d_-} ... \xrightarrow{d_-} F^p\Omega^1 \xrightarrow{d_-} F^p\Omega^0 \to 0$$

one gets

$$F^p H^k_{d_+}(X) := \frac{ker d_+ \cap F^p\Omega^k}{im d_+ \cap F^p\Omega^k}$$

$$F^p H^k_{d_-}(X) := \frac{ker d_- \cap F^p\Omega^k}{im d_- \cap F^p\Omega^k}$$

Tsai, Tseng, and Yau managed to derive the following direct sum decompositions for $2p < k < n + p$,

$$F^p H^k_+(X) \cong ker(L^{p+1} : H^{k-2p-1}_d \to H^{k+1}_d) \oplus coker(L^{p+1} : H^{k-2p-2}_d \to H^k_d)$$

$$F^p H^k_-(X) \cong ker(L^{p+1} : H^{2n-k}_d \to H^{2n-k+2p+2}_d) \oplus coker(L^{p+1} : H^{2n-k-1}_d \to H^{2n+2p-k+1}_d)$$

Thus, Tsai, Tseng, and Yau have completely reduced the problem of computing symplectic and filtered cohomology to a problem of computing the ranks of some linear maps on finite-dimensional vector spaces. This is quite remarkable, given that Nomizu-type results in general only apply among a very limited class of compact symplectic manifolds.

Tseng and Yau discovered a few useful relations for their cohomologies:

$$H^k_{dR}(X) \cong H^{2n-k}_{d^\wedge}(X)$$

for all $k \leq n$ (via the symplectic star operator, which maps d-closed forms and *only* d-closed forms onto $d^\wedge$-closed forms as well as d-exact forms on $d^\wedge$-exact forms), and

$$H^k_{d+d^\wedge}(X) \cong H^{2n-k}_{dd^\wedge}(X)$$

for all $k \leq n$, as one might expect from the parallel with the Bott-Chern and Aeppli cohomology groups. This follows from the easily verified fact [32, Lemma 3.23] that

$$*_\omega \Delta_{d+d^\wedge} = \Delta_{dd^\wedge} *_\omega$$

Surprisingly, it holds for *all* compact symplectic manifolds that

$$\mathrm{H}^k_{d+d^\wedge}(X) \cong \mathrm{H}^{2n-k}_{d+d^\wedge}(X)$$

$$\mathrm{H}^k_{dd^\wedge}(X) \cong \mathrm{H}^{2n-k}_{dd^\wedge}(X)$$

for all $k \leq n$ via the map $\mathrm{L}^{n-k} = \cup[\omega]^{n-k}$. Furthermore, Tsai, Tseng, and Yau prove in [31] the following result, which will be useful later:

$$\ker(L^r : H^k_d \to H^{k+2r}_d) \cong \mathrm{coker}(L^r : H^{2n-k-2r}_d \to H^{2n-k}_d)$$

for all k, r that make sense.

It is important to notice that none of the theorems of Rudyak, Ibáñez, Ugarte, and Tralle required the assumption that X be a nilmanifold. All were valid at least for all closed symplectic manifolds. Moreover, the Tsai-Tseng-Yau decompositions

$$F^p H^k_+(X) \cong ker(L^{p+1} : H^{k-2p-1}_d \to H^{k+1}_d) \oplus coker(L^{p+1} : H^{k-2p-2}_d \to H^k_d)$$

$$F^p H^k_-(X) \cong ker(L^{p+1} : H^{2n-k}_d \to H^{2n-k+2p+2}_d) \oplus coker(L^{p+1} : H^{2n-k-1}_d \to H^{2n+2p-k+1}_d)$$

for $2p < k < n + p$ and

$$PH^k_{dd^\wedge}(X) \cong ker(L^{n-k+1} : H^{k-1}_d \to H^{2n-k+1}_d) \oplus coker(L^{n-k+1} : H^{k-2}_d \to H^{2n-k}_d)$$

$$PH^k_{d+d^\wedge}(X) \cong ker(L^{n-k+1} : H^k_d \to H^{2n-k+2}) \oplus coker(L^{n-k+1} : H^{k-1}_d \to H^{2n-k+1}_d)$$

for $k \leq n$ apply to all compact symplectic manifolds.

As far as I am aware, no attempt has yet been made in the literature to link the results of [33] and [21, 22]. Certainly, the Tsai-Tseng-Yau decompositions greatly ease the computation of the TY-primitive numbers and TTY-filtered numbers.

A first, simple result of ours is the following:

Proposition: *Let (X^4, ω) be a closed symplectic manifold. Then (X, ω) is TTY-filtered-inflexible and its TY-primitive-flexibility is equivalent to ordinary symplectic flexibility on X.*

Proof: We use the Tseng-Yau decompositions for $PH^0_*, PH^1_*, PH^2_*, F^{0,1}_+, F^{0,1}_-$, where $*$ can take the values of $dd^\wedge$ and $d + d^\wedge$. Recalling that the cokernel is the quotient of the codomain by the image, and that $h_{3,hr}(\omega)$ measures the dimension of this image, we utilize the rank-nullity theorem and compute the following dimensions:

$ph^0_{dd^\wedge}$	1
$ph^1_{dd^\wedge}$	b_1
$ph^2_{dd^\wedge}$	b_1+b_2-1-$h_{3,hr}(\omega)$
$ph^0_{d+d^\wedge}$	1
$ph^1_{d+d^\wedge}$	b_1
$ph^2_{d+d^\wedge}$	b_1+b_2-1-$h_{3,hr}(\omega)$
$f^{0,1}_+$	b_1
$f^{0,1}_-$	b_1

Since the first and third Betti numbers of X don't depend on ω, there is no continuous path of symplectic forms on X along which the TTY-filtered numbers vary. Furthermore, from previous remarks, $b_k = h_{k,hr}$ for $k = 0, 1, 2$, and 4, regardless of ω. By Corollary 2.11 in [21], the symplectic flexibility of X is thus equivalent to the variability of $h_{3,hr}(\omega)$, which - according to the table - is precisely equivalent to TY-primitive-flexibility. □

Remark: By Corollary 2.13 [21], the fact that $b_1 = h_{1,hr}(\omega)$ for all ω implies that the set $\Omega(b_3, 2+1)$ of symplectic forms on X with maximal third harmonic number is open in $\Omega^2(X)$. If nonempty, it is dense in Ω_{sympl}. That is, we may say that the TY-primitive numbers are stable near all symplectic structures with maximal third harmonic number; precisely those that have the Hard Lefschetz property. Furthermore, by Corollary 2.13 [21], the third harmonic number $h_{3,hr}(\omega)$, seen as a function on Ω_{sympl}, is upper semicontinuous at points in $\Omega(b_3, 2+1)$, a property that clearly is preserved in the $ph^k_{dd^\wedge}, ph^k_{d+d^\wedge}$ functions.

Example. *If $\mathfrak{g}^4$ is a solvable Lie algebra whose simply connected Lie group*

admits a cocompact lattice Γ*, then the resulting solvmanifold* $\mathfrak{g}/\Gamma$ *is both TY-primitive-inflexible and TTY-filtered-inflexible.* Straightforward computation using the structure equations for the dual spaces of left-invariant one-forms to these Lie algebras yields

	$ph^0_{dd^\wedge}$	$ph^1_{dd^\wedge}$	$ph^2_{dd^\wedge}$	$ph^0_{d+d^\wedge}$	$ph^1_{d+d^\wedge}$	$ph^2_{d+d^\wedge}$
(0,0,0,0)	1	4	5	1	4	5
(-23,0,0,0)	1	3	4	1	3	4
(-13,23,0,0)	1	2	1	1	2	1
(-23,13,0,0)	1	2	1	1	2	1
(-24,-34,0,0)	1	2	3	1	2	3

	$f^{0,1}_+$	$f^{0,1}_-$
(0,0,0,0)	4	4
(-23,0,0,0)	3	3
(-13,23,0,0)	2	2
(-23,13,0,0)	2	2
(-24,-34,0,0)	2	2

In the case of six-dimensional closed symplectic manifolds, we have an analogous result:

Proposition: *Let* (X^6, ω) *be a closed symplectic manifold of Lefschetz type. Then ordinary symplectic flexibility on* X *is equivalent to TY-primitive-flexibility.*

Proof: We use the Tseng-Yau decompositions for $PH^0_*, PH^1_*, PH^2_*, PH^3_*$, where $*$ can take the values of $dd^\wedge$ and $d + d^\wedge$, and compute:

$ph^0_{dd^\wedge}$	1
$ph^1_{dd^\wedge}$	b_1
$ph^2_{dd^\wedge}$	b_1+b_2-1-$h_{5,hr}(\omega)$
$ph^3_{dd^\wedge}$	$b_2 + b_3 - h_{4,hr}(\omega) - h_{hr,5}(\omega) - \dim(P^3 \cap L(H^1(X)))$
$ph^0_{d+d^\wedge}$	1
$ph^1_{d+d^\wedge}$	b_1
$ph^2_{d+d^\wedge}$	$b_1 + b_2 - 1 - h_{5,hr}(\omega)$
$ph^3_{d+d^\wedge}$	$b_2 + h_{3,hr}(\omega) - h_{5,hr}(\omega) - h_{4,hr}(\omega)$

Here, in addition to using the Tsai-Tseng-Yau kernel/cokernel formulas, we have applied the analysis in [22] of symplectic 3-harmonicity. Specifically, we have used:

Lemma 5.3, [22]: *Let (X, ω) be a symplectic manifold of dimension 6. Then*

$$h_{3,hr}(\omega) = h_{5,hr}(\omega) + \dim\ker(L : H^3_d(X) \to H^5_d(X))$$

Since $P^3(X) = \dim\ker(L : H^3_d(X) \to H^5_d(X))$ and since it is easily verified that

$$H^3_{hr}(X) = P^3(X) + L(H^1_d(X))$$

the lemma allows us to easily compute the third primitive harmonic numbers above. If we apply the assumption that (X, ω) is of Lefschetz type, that is,

$$L^2 : H^1_d(X) \to H^5_d(X)$$

is an isomorphism, then it is easily checked that the intersection $P^3 \cap L(H^1(X))$

vanishes. This yields

$$ph^3_{dd^\wedge} = b_2 + b_3 - h_{4,hr}(\omega) - h_{5,hr}(\omega)$$
$$ph^3_{d+d^\wedge} = b_2 + h_{3,hr}(\omega) - h_{4,hr}(\omega) - h_{5,hr}(\omega)$$

Now, from previous remarks, $b_k = h_{k,hr}$ for $k = 0, 1, 2, 6$, regardless of ω, so symplectic flexibility on X is defined by variance in at least one of $h_{3,hr}(\omega_t), h_{4,hr}(\omega_t), h_{5,hr}(\omega_t)$ along continuous paths ω_t of symplectic forms. We see that if $h_{5,hr}$ varies along a continuous path ω_t, then X exhibits TY-primitive flexbility for both operators $dd^\wedge, d + d^\wedge$ at level 2. If $h_{5,hr}$ does not vary along any continuous path ω_t, then let's suppose the variance comes from $h_{4,hr}$: in this case, X exhibits TY-primitive flexibility for the operator $dd^\wedge$ at level 3. If instead, neither $h_{4,hr}, h_{5,hr}$ vary along any continuous path ω_t, then variance must come from $h_{3,hr}$, in which case X will exhibit TY-primitive-flexibility at level 3 with respect to the $d + d^\wedge$ operator. Conversely, if X exhibits TY-primitive flexbility at level 2, then X also has symplectic 5-flexbility, and if the TY-primitive flexibility occurs at level 3 along some continuous path, then at least one of $h_{3,hr}, h_{4,hr}, h_{5,hr}$ must vary, inducing symplectic flexibility. Therefore, TY-primitive flexibility is equivalent to symplectic flexibility for Lefschetz-type (X^6, ω). $\square$

Remark: Since $h_{1,hr} = b_1$ is independent of symplectic form, the set $\Omega(b_5, 3 + 2)$ is open in Ω_{sympl}. If nonempty, it is dense, and the function $ph^2_{dd^\wedge}$ will be generically upper semicontinuous on Ω_{sympl}. Additionally, if we use the Tsai-Tseng-Yau decompositions for $F^{0,1}_\pm, F^{0,2}_\pm, F^{1,3}_\pm$, then we may compute the TTY-filtered numbers for (X^6, ω). We yield:

$f_+^{0,1}$	b_1
$f_-^{0,1}$	b_1
$f_+^{0,2}$	$b_1 + b_2 - 1 - \dim L(H_d^1(X))$
$f_-^{0,2}$	$b_1 + b_2 - 1 - \dim L(H_d^3(X))$
$f_+^{1,3}$	b_3
$f_-^{1,3}$	b_3

If (X, ω) is of Lefschetz type, then the map $L : H^1(X) \to H^3(X)$ is injective. Thus, $f_+^{0,2}$ reduces to $b_2 - 1$. On the other hand, by the rank-nullity theorem, we have

$$b_3 = \dim L(H^3(X)) + \dim ker(L : H^3(X) \to H^5(X))$$

Since the latter kernel is precisely $P^3(X)$, we use Lemma 5.3 from [22] to reduce $f_-^{0,2}$ to $b_1 + b_2 - b_3 + h_{3,hr}(\omega) - h_{5,hr}(\omega) - 1$. Summarizing, the TTY-zero-filtered numbers at level 2 for (X^6, ω) of Lefschetz-type are

$$f_+^{0,2} = b_2 - 1$$
$$f_-^{0,2} = b_1 + b_2 - b_3 + h_{3,hr}(\omega) - h_{5,hr}(\omega) - 1$$

From this, we see that if X^6 exhibits TTY-filtered flexibility, then it is certain symplectically flexible, but not necessarily conversely, as $h_{4,hr}$ could vary along continuous paths ω_t in $\Omega_{sympl}(X)$ with no effect on the filtered numbers.

By studying the symplectic harmonic numbers discovered in [22], and by using Corollary 2.11 [21] and Lemma 5.3 [22], we are able to fully classify the symplectically flexible 6-dimensional nilmanifolds:

Lie algebra structure	3-flexible	4-flexible	5-flexible
(0,0,12,13,14+23,24+15)	✓	-	-
(0,0,12,13,23,14-25)	-	✓	-
(0,0,0,12,14,15+23+24)	✓	✓	✓
(0,0,0,12,13+14,24)	-	✓	-
(0,0,0,12,13,14+23)	✓	✓	-
(0,0,0,12,13,24)	✓	-	-
(0,0,0,12,13,14)	✓	-	-
(0,0,0,12,13,23)	✓	✓	-
(0,0,0,0,12,14+25)	✓	-	-
(0,0,0,0,12,13)	-	✓	-

We establish the symplectic 3-flexibility of $(0,0,0,12,14,15+23+24)$ directly: consider the path

$$\omega_t = e^{13} + 4e^{15} + 6e^{23} - e^{16} - e^{25} + e^{34} + (1-t)(e^{26} - e^{45}), \text{ for } 0 \leq t \leq 1$$

We note that

$$\omega_t^3 = \left((1-t)^2 + 2(1-t) + 1\right) e^{123456} < 0$$

for all $0 \leq t \leq 1$, so this path consists of symplectic forms. Furthermore, we have

the family $\cup[\omega_t] : H^3_d(X) \to H^5_d(X)$ of linear maps:

$$\begin{bmatrix} t-1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & t-1 & 0 \\ 0 & t-1 & 0 \\ -4 & 1 & 1-t \\ -4 & 2 & t-1 \end{bmatrix},$$

For $t = 0$, this map has rank 3, and for $t = 1$, this map has rank 4. On the other hand, an elementary calculation shows that

$$\cup[\omega_t]^2 = \begin{bmatrix} 1-t & 2t-2 & 0 \\ 0 & 0 & t-1 \\ 0 & 0 & 2-2t \end{bmatrix},$$

which has rank 2 for $t = 0$ and rank 0 for $t = 1$. Thus, using Lemma 5.3, we calculate

$$h_{3,hr}(\omega_0) = h_{5,hr}(\omega_0) + 3 = 2 + 3 = 5$$
$$h_{3,hr}(\omega_1) = h_{5,hr}(\omega_1) + 4 = 0 + 4 = 4$$

This establishes the symplectic 3-harmonicity of symplectic nilmanifold $(0, 0, 0, 12, 14, 15+23+24)$. $\square$

Generalizing, we classify the TY primitive flexibility for the general case of closed symplectic manifolds (X^{2n}, ω) with the condition of being $(n-2)$**-Lefschetz type**; that is, for all $0 \leq s \leq n-2$, the maps

$$L^{n-s} : H^s_d(X) \to H^{2n-s}_d(X)$$

are isomorphisms. By using the TTY kernel/cokernel formulas, we calculate:

k	$ph^k_{dd^\wedge}$	$ph^k_{d+d^\wedge}$
0	1	1
1	b_1	b_1
$2 \leq k \leq n-2$	$b_k - b_{k-2}$	0
$n-1$	$b_{n-1} - b_{n-3}$	$b_{n-1} - h_{n+1,hr}(\omega)$
n	$b_n + b_{n-1} - b_{n-2} - h_{n+1,hr}(\omega)$	$b_{n-1} - b_{n-2} + h_{n,hr}(\omega) - h_{n+1,hr}(\omega)$

We thus conclude, naturally:

Theorem: *Let (X^{2n}, ω) be a compact symplectic manifold of $(n-2)$-Lefschetz type. Then its symplectic flexibility is equivalent to its TY-primitive flexibility.*

Proof: Analogous to previous proof. $\square$

Chapter 3

Symplectic Birational Transformations and the Hard Lefschetz Property: A Conjecture

By the Weierstrass Approximation theorem, the space $\mathcal{P}(\mathcal{I})$ of polynomials on a compact interval $\mathcal{I}$ is dense in $\mathcal{C}(\mathcal{I})$. This is a fundamental and useful fact in the study of analysis, and the centrality of this theorem motivates a similar question about the spaces $\mathcal{K}, \mathcal{HL}, \mathcal{S}$ of Kähler, Hard Lefschetz, and symplectic forms on a given compact manifold. Other than the coarse subset relation $\mathcal{K} \subset \mathcal{HL} \subset \mathcal{S}$, what relations between these spaces exist? Can a similar density relation be found?

Since on a compact manifold holding the Hard Lefschetz property is really a matter of several maps holding maximal rank, this is an open property; $\mathcal{HL}$ is an

open neighborhood of $\mathcal{K}$. It is also dense in $\mathcal{S}$, by Corollary 2.13, [21].

Furthermore, Theorem 1.1 of [12] (see page 10) showed that on some four-manifolds, the distance between the sets $\mathcal{K}$ and $\mathcal{Z}^2 - \mathcal{S}$ (and hence, that between $\mathcal{HL}$ and $\mathcal{Z}^2 - \mathcal{S}$) is - to some degree - small. This result, though limited in scope, nevertheless prompted a related question: what proportion of Hard Lefschetz symplectic forms are "close" in some sense to a Kähler form? Similarly, what proportion of symplectic forms are "close" in some sense to a Hard Lefschetz form? So far, the second question has been more tractable than the first.

In this discussion, it is of course of central importance to describe what we mean by "close". It was in the work of Guillemin and Sternberg that we found our answer. They had sought to adapt the algebro-geometric concept of birational equivalence for the sympletic category and they eventually settled on the following notion:

Definition 2.1, [19]: *Two symplectic manifolds (X, ω) and (X', ω') are (symplectically) birational cobordant if there are a finite number of symplectic manifolds (X_i, ω_i), $0 \leq i \leq k$, with $(X_0, \omega_0) = (X, \omega)$ and $(X_k, \omega_k) = (X', \omega')$, and if for each i, (X_i, ω_i) and (X_{i+1}, ω_{i+1}) are symplectic reductions of a semi-free Hamiltonian-S^1 symplectic manifold W_i (of 2 dimensions higher).*

Guillemin and Sternberg proved a useful factorization result for their concept of symplectic birational cobordism:

Theorem 2.2, [19] *A (symplectic) birational cobordism can be decomposed as a sequence of elementary ones, which are modeled on blow-up, blow-down, and*

Z-*linear deformation of symplectic structure.*

Here, a **Z**-linear deformation is a path of symplectic forms $\omega + t\kappa, t \in I$, where κ is a closed 2-form representing an integral class and I is an interval. We focused our attention on the symplectic blow-up as a model of symplectic birational cobordism, and will describe our investigation here. First, we will review the basic symplectic blow-up process, as originally explicated by McDuff in [26] and later by Cavalcanti in [9]: let (X^{2n}, ω) be a compact symplectic manifold with a compact symplectically embedded submanifold $M^{2d} \overset{\iota}{\hookrightarrow} X^{2n}$. We can choose a complex structure J for TX that restricts to one on the normal bundle $E \overset{\phi}{\to} M$ (here ϕ denotes the projection). Since $TX \to M$ is then a complex bundle, we can form its projectivization

$$\mathbf{CP}^{n-d-1} \to \tilde{M} \overset{p}{\to} M$$

and also form the tautological line bundle $\tilde{E}$ over $\tilde{M}$. We may form a diagram as follows:

$$\begin{array}{ccccc} \tilde{E}_0 & \longrightarrow & \tilde{E} & \xrightarrow{q} & \tilde{M} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle p} \\ E_0 & \longrightarrow & (E,\omega) & \xrightarrow[\pi]{} & (M,\sigma) \end{array}$$

Here, q and φ are the natural projection maps, and E_0 is the complement of the zero section in E and $\tilde{E}_0$ the complement of the zero section in $\tilde{E}$; it is easy to check that E_0 and $\tilde{E}_0$ are diffeomorphic via φ. Let V be a sufficiently small disc subbundle of $E \to$M, and if we equip E with its canonical symplectic structure,

then V will be symplectomorphic to a neighborhood of $M \subset X$ (we identify them from now on). Putting $\tilde{V} = \varphi^{-1}(V) \subset \tilde{E}$, we form the symplectic blow-up of X,

$$\tilde{X} = \overline{X - V} \cup_{\partial V} \tilde{V}$$

We can extend φ by the identity in the complement of V, producing the blow-down map $f : \tilde{X} \to X$. McDuff observed the following:

Lemma 1.1, [26] *There is a unique class $a \in H^2(\tilde{M})$ which restricts to the standard Kähler class on each fiber of $\tilde{M} \to M$ and pulls back to the trivial class in $\tilde{E}_0$. Moreover, $H^*(\tilde{E}) \cong H^*(\tilde{M})$ is a free module over $H^*(M)$ with generators $1, a, ..., a^{n-d-1}$.*

Theorem 1.1, [26] *If the codimension of M is at least 4, the fundamental groups of X and $\tilde{X}$ are isomorphic. Further, there is a short exact sequence*

$$0 \to H^*(X) \to H^*(\tilde{X}) \to A^* \to 0$$

where A^* is the free module over $H^*(M)$ with generators $a, ..., a^{n-d-1}$. Moreover, there is a representative α of a with support in the tubular neighborhood V such that, for ϵ small enough, the form $\tilde{\omega} = f^*(\omega) + \epsilon\alpha$ is a symplectic form in $\tilde{X}$.

By invoking the Leray-Hirsch theorem, McDuff also observed a relation among powers of the class a:

$$a^{n-d} = -c_{n-d} - c_{n-d-1}a + ... - c_1 a^{n-d-1}$$

where the c_j are the Chern classes of the normal bundle E. From [RT], this relation may also be written as

$$a^{n-d} = -f^*(t) - c_{n-d-1}a + ... - c_1 a^{n-d-1}$$

This enables one to completely describe multiplication in the cohomology ring of $\tilde{X}$ (cf. [Ca]): for any $v, w \in H^*(X)$ and $u_i \in H^*(M)$,

$$\begin{aligned} f^*(v) \wedge f^*(w) &= f^*(v \wedge w) \\ f^*(v)a &= \iota^*(v)a \\ au_1 \wedge au_2 &= a^2 u_1 \wedge u_2 \\ a^{n-d} &= -f^*(t) - c_{n-d-1}a + ... - c_1 a^{n-d-1} \\ f^*(t) \wedge u_1 &= f^*(t \wedge u_1) \end{aligned}$$

Working in this framework, Cavalcanti [9] made an interesting discovery: symplectic blow-up can in some cases reduce the failure of (X, ω) to achieve the Hard Lefschetz property.

Dimension 4 is not one of these cases. Indeed, let (X^4, ω) be a symplectic manifold and $(M^2, \sigma) \hookrightarrow (X^4, \omega)$ be a symplectically embedded surface. McDuff observed that in this case, with the codimension being less than 4, the commutative diagram of the symplectic blow-up allows one to identify M with $\tilde{M}$ and E with $\tilde{E}$. The end result is $\tilde{X} \approx X$, so that $\Omega_{sympl}(X) = \Omega_{sympl}(\tilde{X})$. Moreover, $H^*(\tilde{E})$, being generated over $H^*(M)$ by $1, a, a^2, .., a^{k-1}$, where $k = codim(M)/2 = 1$, contributes nothing to $H^*(\tilde{X})$. Thus, we may apply Corollary 2.13 of [21] to $\Omega_{sympl}(X) = \Omega_{sympl}(\tilde{X})$, and recognize that since h_1 does not depend on symplectic structure, the space of Hard Lefschetz symplectic structures on $X \approx \tilde{X}$ is open and dense in Ω_{sympl}, and is thus either empty or equal to Ω_{sympl}. Therefore, $\tilde{\omega}$ satisfies the Hard Lefschetz property if and only if ω does.

We shift our attention now to symplectic manifolds of dimension ≥ 6. Cavalcanti has:

Proposition 2.1, [9]: *Let $\iota : (M^{2d}, \sigma) \hookrightarrow (X^{2n}, \omega)$ be a symplectic embedding with M and X compact and $2d < n$. Let $\tilde{X}$ be the blown-up manifold. Then, for $i > 2d$,*

$$dim(ker(\tilde{\omega}^{n-i})) = dim(ker(\omega^{n-i}))$$

Theorem, [9]: *Let $\iota : M^{2d} \hookrightarrow X^{2n}$ be a symplectic embedding, M and X compact, and let $\tilde{X}$ be the blow-up of X along M. Then for ϵ small enough,*

- *If there is an element v in $ker(\omega^{n-2d})$ such that $\iota^* v \neq 0$, then*

$$dim(ker(\tilde{\omega}^{n-2d})) = dim(ker(\omega^{n-2d})) - 1$$

  otherwise these kernels have the same dimension, as long as ϵ is small enough.

- *If M satisfies the Hard Lefschetz property, and ϵ is small enough and $i \leq 2d$,*

$$dim(ker(\tilde{\omega}^{n-i})) \leq dim(ker(\omega^{n-i}))$$

  and for $d = 1$ (regardless of whether or not M satisfies the Hard Lefschetz property), if there are elements $v_1, v_2 \in ker(\omega^{n-1})$ such that $\iota^*(v_1 \wedge v_2) \neq 0$, then

$$dim(ker(\tilde{\omega}^{n-1})) \leq dim(ker(\omega^{n-1})) - 2$$

It was a result of Benson and Gordon [5] that no non-toroidal nilmanifold has the Hard Lefschetz property (nor, in particular, do they support a Kähler structure), but this new theorem shows that some of them may be related to

Hard Lefschetz manifolds via symplectic birational cobordism. A natural question arises: how many? Before we demonstrate some results, we prove a useful proposition that increases the utility of Cavalcanti's theorem:

Proposition: *Let (X^{2n}, ω) be a compact symplectic manifold. If the following conditions hold, then there exists a surface (M^2, σ) symplectically embedded in (X, ω) along which the symplectic blowing-up of X will decrease the dimensions of $ker(\cup[\omega]^{n-1})$ (resp. $ker(\cup[\omega]^{n-2})$) by at least two (resp. exactly one):*

- *There exists $[v], [w] \in ker(\cup[\omega]^{n-1})$ such that $[v \wedge w] \neq 0$.*
- *The element $[v \wedge w]$ lies in $ker(\cup[\omega]^{n-2})$.*

Proof: Fix such $[v], [w] \in H^1(X)$ as described. Since $[v \wedge w] \neq 0$, it defines a nonzero linear functional on $H^{2n-2}(X)$, so we obtain the following decomposition:

$$H^{2n-2}(X) = ker(\cup[v \wedge w]) \oplus \mathbf{R} \langle B \rangle$$

where B spans the space on which $\cup[v \cup w] : H^{2n-2}(X) \to \mathbf{R}$ does not vanish. Of course, we also have the decomposition

$$H^{2n-2}(X) = ker(\cup[\omega]) \oplus \mathbf{R} \left\langle [\omega]^{n-1} \right\rangle$$

so we may write

$$B = \xi[\omega]^{n-1} + B_K$$

where B_K is an element in $ker(\cup[\omega])$. Let's first consider the case when $B \notin ker(\cup[\omega])$. Since both $ker(\cup[\omega])$ and $ker(\cup[v \wedge w])$ are codimension-1 in $H^{2n-2}(X)$, their intersection is nonzero. Select C from this intersection and put

$$[M] := C + \frac{1}{\xi} B$$

We see that

$$[v \wedge w] \cup [M] = [v \wedge w] \cup C + \frac{1}{\xi}[v \wedge w] \cup B = \frac{1}{\xi}[v \wedge w] \cup B \neq 0$$
$$[\omega] \cup [M] = [\omega] \cup C + \frac{1}{\xi}[\omega](\xi[\omega]^{n-1} + B_K) = [\omega]^n > 0$$

Because $[M]$ pairs positively with $[\omega]$, there exists a symplectically embedded surface $(M^2, \sigma) \overset{\iota}{\hookrightarrow} (X^{2n}, \omega)$ represented by the class $[M]$. Furthermore, the restriction $\iota^*(v \wedge w)$ to M is non-exact, and by Cavalcanti's theorem, the symplectic blow-up $(\tilde{X}, \tilde{\omega})$ of X along M will cut the dimension of the Lefschetz kernel in $H^1(X)$ by at least two.

Now, let's consider the case when $B \in ker(\cup[\omega])$. In this case, the intersection of $ker(\cup[\omega])$ and $ker(\cup[v \wedge w])$ has dimension 1 less than $\dim ker(\cup[v \wedge w]) = \dim ker(\cup[\omega])$, so we select C from a complementary subspace to $ker(\cup[\omega])$ in $ker(\cup[v \wedge w])$, and put $[M] = B + C$. Of course,

$$[v \wedge w] \cup [M] = [v \wedge w] \cup B \neq 0$$
$$[\omega] \cup [M] = [\omega] \cup C \neq 0$$

We may, if necessary, scale C so that this last quantity is positive. Thus, as before, we obtain the desired symplectically embedded surface as before.

Lastly, we note that if our $[v \wedge w] \in ker(\cup[\omega]^{n-2})$, then by Theorem ([9]), the Lefschetz kernel in $H^2(X)$ decreases dimension by 1 in the symplectic blow-up $(\tilde{X}, \tilde{\omega})$ of X along M. $\square$

We note, therefore, that if $\dim ker(\cup[\omega]^{n-2}) = 1$ and $\dim ker(\cup[\omega]^{n-1}) = 2$ and the conditions of the proposition hold, then $(\tilde{X}, \tilde{\omega})$ has the Hard Lefschetz property. If these kernels have higher dimension, then we iterate the process through repeated applications of our proposition. In doing so, we establish:

Theorem: *There are at least eleven six-dimensional closed symplectic manifolds that are symplectic birational cobordant to Hard Lefschetz manifolds.*

Proof: The six-dimensional symplectic nilmanifolds have been classified by Salamon [30]. As explicated in [22], we may identify these nilmanifolds by the structure equations for their Lie algebras. We reproduce a piece of their analysis here:

b_1	b_2	Structure	$h_{4,hr}$	$h_{5,hr}$
2	3	(0,0,12,13,14,15)	3	0
2	3	(0,0,12,13,14+23,24+15)	2	0
2	3	(0,0,12,13,14,23+15)	2	0
2	4	(0,0,12,13,23,14)	4	0
2	4	(0,0,12,13,23,14-25)	2,3,4	0
2	4	(0,0,12,13,23,14+25)	4	0
4	7	(0,0,0,0,12,15)	3	2
4	7	(0,0,0,0,12,14+25)	3	2
4	8	(0,0,0,0,12,14+23)	6	2
4	8	(0,0,0,0,12,34)	7	2
4	9	(0,0,0,0,12,13)	7,8	2

For the first six nilmanifolds, we note from their structures that $H^1 = \mathbf{R}\langle[e^1],[e^2]\rangle$, but $[e^{12}] = 0$. Therefore, Cavalcanti's method does not directly apply to these cases. However, the way forward is not barred. Indeed, we note that

$$\begin{aligned}[\tilde{\omega}]^2 f^*[e^i] &= f^*[\omega^2 \wedge e^i] + \epsilon^2(\iota^* e^i)a^2 + 2\epsilon\iota^*(\omega \wedge e^i)a \\ &= \epsilon^2(\iota^* e^i)a^2\end{aligned}$$

Thus, $f^*[e^i] \in ker(\cup[\tilde{\omega}]^2)$ if and only if the restriction of e^i to a symplectic submanifold M^2 vanishes identically. We will choose M carefully so this does not happen.

For nilmanifolds $(0,0,12,13,14,15)$ and $(0,0,12,13,14,23+15)$, we calculate that $e^{2345}-e^{2346}$ is d-closed. Since in the first, a generic symplectic class $[\omega]$ has a $[e^{16}]$ component and since in the second, it has a $[e^{15}]$ component, we may choose the sign of the coordinates of these components so that a $[\omega]$ pairs positively with $[e^{2345}-e^{2346}]$. Thus, in both cases, there exists a symplectic submanifold M^2 representing $PD[e^{2345}-e^{2346}]$. The tangent space to this embedded surface is spanned by vector fields $\partial_1, \partial_5+\partial_6$ from basis $\partial_1, \partial_2, ..., \partial_6$ dual to $e^1, e^2, ..., e^6$. Thus,

$$[\tilde{\omega}]^2 f^*[e^1] = \epsilon^2(\iota^* e^1)a^2 \neq 0$$
$$[\tilde{\omega}]^2 f^*[e^2] = \epsilon^2(\iota^* e^2)a^2 = 0$$

In $(0,0,12,13,14,15)$ (and $(0,0,12,13,14,23+15)$ respectively), the 4-form f^*e^{1456} (respectively $f^*(e^{1456}-e^{2356})$) are d-closed. Since $\tilde{\omega} = f^*\omega + \epsilon a$ has an f^*e^{23}-component, $[\tilde{\omega}]$ can be made to pair positively with the classes $f^*[e^{1456}]$ and $f^*[e^{1456}-e^{2356}]$. Each is represented then by a $\tilde{\omega}$-symplectically embedded surface $\tilde{M}^2$, whose tangent space is spanned by ∂_2, ∂_3 (respectively $\partial_1+\partial_2, \partial_3-\partial_4$. Blowing up along these embedded surfaces, we yield that $(\tilde{\iota}^* f^* e^2)a^2 \neq 0$ in both cases. Thus, after two steps, we may shrink $ker(\cup[\omega]^2 : H^1 \to H^5)$ to zero. In these two manifolds, dimker$(\cup[\omega] : H^2 \to H^4) \leq 1$, so two steps is sufficient to yield a Hard Lefschetz manifold.

For nilmanifold $(0,0,12,13,23,14)$, the 4-form e^{2345} is d-closed, and generic

$[\omega]$ has an $[e^{16}]$ component, whose coordinate can be chosen positively. Let M be a symplectically embedded surface representing $PD[e^{2345}]$. Its tangent space is spanned by ∂_1, ∂_6, so

$$ker(\cup[\tilde{\omega}]^2) = \mathbf{R}\left\langle [e^2] \right\rangle$$

At this stage, we note that f^*e^{1356} is d-closed and pairs positively with a generic $f^*[\omega]$, which lies in the span of the classes $[e^{15}+e^{24}-e^{26}], [e^{15}+e^{24}-e^{34}], [e^{16}], [e^{25}]$. The tangent space to a symplectically embedded surface $[\tilde{M}] := PD[f^*e^{1356}]$ is spanned by ∂_2, ∂_4, and hence,

$$[\tilde{\tilde{\omega}}]^2 \tilde{f}^* f^*[e^2] = \epsilon^2(\tilde{\iota}^* e^2)a^2 \neq 0$$

Thus, after two steps, we shrink $ker(\cup[\omega]^2)$ (and $ker(\cup[\omega] : H^2 \to H^4)$) to zero, and $(0, 0, 12, 13, 23, 14)$ is symplectic birationally cobordant to Hard Lefschetz.

For $(0, 0, 12, 13, 14 + 23, 24 + 15)$, the 4-form $e^{1456} - e^{2356}$ is d-closed. Since a generic symplectic class $[\omega]$ has an $[e^{14}]$-component, we may say it pairs positively with $[e^{1456} - e^{2356}]$. Let M be a symplectically embedded surface representing $PD[e^{1456} - e^{2356}]$. Its tangent space is spanned by $\partial_1 + \partial_3, \partial_2 + \partial_4$, and hence,

$$[\tilde{\omega}]^2 f^*[e^1] = \epsilon^2(\iota^* e^1)a^2 \neq 0$$
$$[\tilde{\omega}]^2 f^*[e^2] = \epsilon^2(\iota^* e^2)a^2 \neq 0$$

Thus, $ker(\cup[\tilde{\omega}]^2) = 0$. Since dimker$(\cup[\omega] : H^2 \to H^4) = 1$, the nilmanifold $(0, 0, 12, 13, 14 + 23, 24 + 15)$ is only one step-symplectic birationally cobordant to a Hard Lefschetz manifold.

For the nilmanifolds $(0, 0, 12, 13, 23, 14 - 25)$ (respectively $(0, 0, 12, 13, 23, 14 + 25)$), we note that $e^{1345} + e^{2356}$ (respectively $e^{1345} - e^{2356}$) are d-closed. Since on

each manifold a generic symplectic class $[\omega]$ has an $[e^{14}]$ component, we can take $[\omega]$ to pair positively with the classes $[e^{1345} \pm e^{2356}]$. Let M be a symplectically embedded surface in each nilmanifold, representing $PD[e^{1345}+e^{2356}]$ (respectively $PD[e^{1345}-e^{2356}]$). The tangent space is spanned by $\partial_2 \pm \partial_1, \partial_6 + \partial 4$, and hence,

$$[\tilde{\omega}]^2 f^*[e^i] = \epsilon^2(\iota^* e^i)a^2 \neq 0$$

for $i = 1, 2$. For $(0, 0, 12, 13, 23, 14-25)$, we have $\dim\ker(\cup[\omega] : H^2 \to H^4) \leq 2$, while $\dim\ker(\cup[\omega]) = 1$, so each of these nilmanifolds is two step-symplectic birationally cobordant to a Hard Lefschetz manifold.

Summarizing, all of the six-dimensional symplectic nilmanifolds with $b_1 = 2$ are at most two-step symplectic birationally cobordant to a Hard Lefschetz manifold.

Now, we focus on the listed nilmanifolds with $b_1 = 4$. For $(0, 0, 0, 0, 12, 15)$, a generic symplectic class $[\omega]$ has nonzero $[e^{24}]$ component, so we can take it to pair positively with $[e^{1356}]$. Furthermore, linear algebra computations straightforwardly establish that

$$ker(\cup[\omega]^2 : H^1 \to H^5) = \mathbf{R}\left\langle \frac{C}{H}[e^1] + [e^2], [e^4] \right\rangle \text{ or } \mathbf{R}\left\langle \frac{-CG}{H}[e^1] + [e^4], \frac{C}{H}[e^1] + [e^2] \right\rangle$$

where C, G are the coordinates of the $[e^{16}], [e^{25}]$ components of $[\omega]$, respectively. The products of these spanning elements are

$$\frac{C}{H}[e^{14}] + [e^{24}] \quad \textit{and} \quad -[e^{24}] - \frac{CG}{H}[e^{14}],$$

respectively. The restrictions of both of these to a symplectically embedded surface representing $PD[e^{1356}]$ are nonzero, so Cavalcanti's method can directly be applied. We perform the symplectic blow-up along this surface and note that,

since ∂_2, ∂_4 span the tangent space to this surface, the four kernel spanning elements above do *not* lie in $ker(\cup[\tilde{\omega}]^2 : H^1(\tilde{X}) \to H^5(\tilde{X}))$. Since dimker$(\cup[\omega] : H^2 \to H^4) = 4$, it will take four steps to symplectically reduce $(0,0,0,0,12,15)$ to Hard Lefschetz.

For $(0,0,0,0,12,14+25)$, the generic symplectic class is of form $[\omega] = A[e^{13}] + B[e^{14}]+C[e^{15}]+D[e^{23}]+E[e^{24}]+F[e^{26}+e^{45}]+G[e^{34}]$ with $F \neq 0$ and $AF+CG \neq 0$. We have

$$ker(\cup[\omega]^2) = \mathbf{R}\left\langle \frac{C}{F}[e^1] + [e^4], [e^2] \right\rangle$$

The product of these spanning elements is $-[e^{24}]$, which pairs nontrivially with $[e^{1356}]$. By adjusting the sign of E if necessary, the symplectic class will pair positively with $[e^{1356}]$. Thus, we blow up the nilmanifold along a symplectically embedded surface representing $PD[e^{1356}]$. Since ∂_2, ∂_4 span the tangent space to this surface, blowing up removes both elements from the kernel. Again, the kernel at level 2 has dimension 4, so $(0,0,0,0,12,14+25)$ is four step-symplectic birationally cobordant to Hard Lefschetz.

For $(0,0,0,0,12,14+23)$, the generic symplectic class is of type

$$[\omega] = A[e^{13}] + B[e^{14}] + C[e^{15}] + D[e^{16} - e^{35}] + F[e^{24}] + F[e^{25}] + G[e^{26} + e^{45}] + H[e^{24}]$$

If $G = 0$ or $G \neq 0, D = 0$ or $G, D \neq 0$, the kernel at level 1 is

$$\begin{aligned} ker(\cup[\omega]^2) &= \mathbf{R}\left\langle [e^1], \frac{-F}{D}[e^2] + [e^3] \right\rangle \quad \text{or} \\ &= \mathbf{R}\left\langle \frac{C}{G}[e^1] + [e^4], [e^2] \right\rangle \quad \text{or} \\ &= \mathbf{R}\left\langle \frac{DF - GC}{-G^2}[e^1] + [e^4], \frac{-D}{G}[e^3] + [e^4] \right\rangle \end{aligned}$$

The 4-form e^{2456} is d-closed, and we can choose A so that $[\omega]$ pairs positively with $[e^{2456}]$. Let M be a symplectically embedded surface representing $PD[e^{2456}]$. Since ∂_1, ∂_3 span the tangent space to M, blowing up along M annihilates the kernels when $G = 0$ and when $G, D \neq 0$. When $G \neq 0, D = 0$, the kernel reduces to

$$ker(\cup[\tilde{\omega}]^2) = \mathbf{R}\left\langle f^*[e^2]\right\rangle$$

To reduce this kernel, we may blow up along a symplectically embedded surface representing the class $PD[f^*(e^{2356} - e^{1456})]$, which pairs positively with $[\tilde{\omega}]$, since this may be taken to have a nontrivial $f^*[e^{14}]$ component (put $F < 0$). Since dimker$(\cup[\omega] : H^2 \to H^4) = 2$, the nilmanifold $(0, 0, 0, 0, 12, 14 + 23)$ is two steps-symplectic birationally cobordant to a Hard Lefschetz manifold.

For $(0, 0, 0, 0, 12, 34)$, a generic symplectic class is of the form

$$[\omega] = A[e^{13}] + B[e^{14}] + C[e^{15}] + D[e^{23}] + E[e^{24}] + F[e^{25}] + G[e^{34}] + H[e^{36}] + J[e^{46}]$$

with $J(AF - CD) + H(CE - BF) \neq 0$. When $H = 0$, the kernel at level 2 is

$$\mathbf{R}\left\langle \frac{C}{F}[e^1] + [e^2], [e^4]\right\rangle \quad \text{or} \quad \mathbf{R}\left\langle \frac{F}{C}[e^2] + [e^1], [e^4]\right\rangle$$

and when $H \neq 0$, the kernel is

$$\mathbf{R}\left\langle \frac{C}{F}[e^1] + [e^2], [e^3] + \frac{J}{H}[e^4]\right\rangle \quad \text{or} \quad \mathbf{R}\left\langle [e^1] + \frac{F}{C}[e^2], [e^3] + \frac{J}{H}[e^4]\right\rangle$$

When $H = 0$, we blow up along $[M] := PD[e^{1456}]$ or $[M] := PD[e^{2356}]$. Both annihilate the kernel after one step. When $H \neq 0$, we blow up along $[M] := PD[e^{1456}]$ or $[M] := PD[e^{2456}]$. These annihilate the kernel as well. Since dimker$(\cup[\omega] : H^2 \to H^4) = 1$, the nilmanifold $(0, 0, 0, 0, 12, 34)$ is one step-symplectic birationally cobordant to a Hard Lefschetz manifold.

Finally, for $(0,0,0,0,12,13)$, a generic symplectic class is of the form

$$[\omega] = A[e^{14}]+B[e^{15}]+C[e^{16}]+D[e^{23}]+E[e^{24}]+F[e^{25}]+G[e^{26}+e^{35}]+H[e^{34}]+J[e^{36}]$$

with $A(G^2 - FJ) - E(CG - BJ) + H(CF - BG) \neq 0$. Either $CF - BG \neq 0$, $CG - BJ \neq 0$, or $G^2 - FJ \neq 0$. In each of these cases, the kernel at level 1 is

$$\begin{aligned}\ker(\cup[\omega]^2 : H^1 \to H^5) &= \mathbf{R}\left\langle \left(\frac{CG-BJ}{CF-BG}\right)[e^3] + [e^2], -\left(\frac{G^2-FJ}{CF-BG}\right)[e^3] + [e^1] \right\rangle \\ &= \mathbf{R}\left\langle \left(\frac{G^2-FJ}{CG-BJ}\right)[e^2] + [e^1], \left(\frac{CF-BG}{CG-BJ}\right)[e^3] + [e^2] \right\rangle \\ &= \mathbf{R}\left\langle -\left(\frac{CF-BG}{G^2-FJ}\right)[e^1] + [e^3], \left(\frac{CG-BJ}{G^2-FJ}\right)[e^1] + [e^2] \right\rangle\end{aligned}$$

In each case, the product of the spanning elements is a nonzero constant multiple of $[e^{23}]$. We blow up along a symplectically embedded surface $[M] := PD[e^{1456}]$ (taking $D > 0$). For the third case, this is enough to annihilate the kernel. However, in the first two cases, the elements $f^*(\left(\frac{FJ-G^2}{CF-BG}\right)[e^3] + [e^1])$, $f^{ast}(\left(\frac{G^2-FJ}{CG-BJ}\right)[e^2] + [e^1])$ remain in the kernel if $FJ = G^2$ (otherwise, the kernels get annihilated in the first step). In this situation, we simply blow up along $[\tilde{M}] := PD[f^*e^{2345}]$ (taking $C > 0$). This accomplishes full annihilation of the kernel. Since $\dim\ker(\cup[\omega] : H^2 \to H^4) \leq 2$, we see that $(0,0,0,0,12,13)$ is at most two step-symplectic birationally cobordant to a Hard Lefschetz manifold.

This establishes the theorem, showing at least 11 six-dimensional symplectic nilmanifolds that are symplectic birationally cobordant to a Hard Lefschetz manifold exist. However, we conjecture that many more may yet be found in future explorations. □

Earlier, the question was posed: what proportion of symplectic non-Hard Lefschetz compact manifolds are in some sense "close" to a Hard Lefschetz manifold?

The notion of symplectic birational cobordism (exemplified above by the iterated symplectic blow-up) provides us a interesting partial answer. The related question - what proportion of Hard Lefschetz (or indeed, symplectic non-Hard Lefschetz) compact manifolds are in some sense "close" to a Kähler manifold? - is more difficult. Merely studying this question through the symplectic apparatus (symplectic forms, Lefschetz decomposition of forms and/or of cohomology classes, etc.) is insufficient; we need to analyze the question through different lenses.

For instance, if the symplectic blow-up of our manifold X along a symplectic submanifold M is to support a Kähler structure, then in particular, $\tilde{X}$ must be formal, and have no non-trivial Massey products. Cavalcanti determined that the same must be true of X:

Theorem 3.1, [**9**]: *Let $\iota : M^{2(n-k)} \hookrightarrow X^{2n}$ be a symplectic embedding with M compact and $\tilde{X}$ the blown-up manifold. If X has a nontrivial triple Massey product, so does $\tilde{X}$.*

Babenko and Taimanov conjectured that the same must be true when X and M are simply-connected:

Conjecture, [**8**]: *Let $Y \to X$ be a symplectic embedding of a nonformal simply-connected manifold into a simply-connected compact manifold X. Then the symplectic blow-up of X along Y is nonformal.*

In [10], it was shown that every non-toroidal nilmanifold has nontrivial Massey

product, and hence, none of these are symplectic birational cobordant (via symplectic blow-up) to a Kähler manifold. To find examples of such, we thus must extend our search to different categories of manifolds.

In studying this question, we must address the apparatus of (almost)-complex structures - not just the symplectic structure - on our manifolds. If $\tilde{J} \in \mathrm{End}(T\tilde{X})$ is compatible with a Hard Lefschetz symplectic form $\tilde{\omega} = f^*\omega + \epsilon\alpha$ that we yield from the symplectic blow-up of a compact, formal, symplectic manifold X along an embedded symplectic submanifold M, then can $\tilde{J}$ be integrable, or perhaps be no worse than a deformation of an almost complex structure $J_0 \in \mathrm{End}(T\tilde{X})$ that is integrable for $\tilde{\omega}$?

We believe a tractable approach to this last question is to study not the integrability of $\tilde{J}$ directly, but rather, to study the presence of relevant Kähler properties. For instance, from [13], we know that every compact Kähler manifold is C^∞-pure-and-full at the second stage:

Definition: *On a compact symplectic manifold (X, ω, J), the almost complex structure J defines a involution on $\Omega^{2k}(X)$ for any $k \geq 0$, and we thus get the decomposition of $\Omega^{2k}(X)$ into J-invariant and J-anti-invariant subspaces, a decomposition that descends to the cohomology and yields the subspaces H^+_J, H^-_J of $H^{2k}_{dR}(X)$. We call J C^∞-pure if $H^+_J \cap H^-_J = 0$ and C^∞-full if $H^{2k}_{dR}(X) = H^+_J + H^-_J$*

This concept has a complex analogue as well. For example, on compact complex surfaces (M, J) with b_1 even, we say that $H^1_{dR}(M; \mathbf{C})$ and $H^2_{dR}(M; \mathbf{C})$ are

complex-C^∞-pure-and-full when

$$H^2_{dR}(M;\mathbf{C}) = H_J^{(2,0),(0,2)}(M) \oplus H_J^{(1,1)}(M)$$

and

$$H^1_{dR}(M;\mathbf{C}) = H_J^{(1,0)}(M) \oplus H_J^{(0,1)}(M)$$

where $H_J^{(p,q)}(M) = \{[\alpha] | \alpha \in \mathcal{Z}_J^{(p,q)}\}$. Here, $\mathcal{Z}_J^{(1,1)}$ denotes the J-invariant closed 2-forms and $\mathcal{Z}_J^{(2,0),(0,2)}$ denotes the J-anti-invariant closed 2-forms, while $\mathcal{Z}_J^{(1,0)}$ gives the J-invariant closed 1-forms and $\mathcal{Z}_J^{(0,1)}$ gives the J-anti-invariant closed 1-forms.

Let $(\tilde{X}, \tilde{\omega}, \tilde{J})$ be the symplectic blow-up of (X, ω, J), with $\tilde{J}$ being $\tilde{\omega}$-compatible. From [16], we have the following:

Theorem 5.4, [16]: *Let (X, ω) be a symplectic manifold and $\iota : (M, \sigma) \to (X, \omega)$ be a symplectic embedding of a compact submanifold M. Consider the symplectic blow-up $\tilde{X}$ of (X, ω) along M. Let J be an ω-calibrated almost complex structure on X such that $J|_{TM} \in End(TM)$. Then there exists an almost complex structure $\tilde{J}$ on $\tilde{X}$ such that the blow-down map $f : (\tilde{X}, \tilde{J}) \to (X, J)$ is pseudo-holomorphic.*

and

Theorem 5.3, [16]: *Let (X, ω) be a symplectic manifold and $\iota : (M, \sigma) \to (X, \omega)$ be a symplectic embedding of a compact submanifold M. Consider the symplectic blow-up $\tilde{X}$ of (X, ω) along M. Let J and $\tilde{J}$, respectively, be almost complex structures on X and $\tilde{X}$ such that the blow-down map $f : (\tilde{X}, \tilde{J}) \to (X, J)$*

is pseudoholomorphic. If J is C^∞-pure-and-full and it is ω-calibrated, then $\tilde{J}$ is C^∞-pure-and-full.

(Here, "ω-calibrated" merely means "ω-compatible", as opposed to just "ω-tamed".) A quick study of the original paper shows that in these two results, the authors used the term "complex-C^∞-pure-and-full" to denote such purely at the second stage (though this terminology may be used for all levels of the complex de Rham cohomology). This tells us that if $\tilde{J}$ in our symplectic blow-up $(\tilde{X}, \tilde{\omega})$ is to be integrable, then it might be prudent for us to require (X, ω, J) to be complex-C^∞-pure-and-full (at the second stage).

If $\tilde{J}$ in our $(\tilde{X}, \tilde{\omega})$ is to be no worse than a deformation of an integrable complex structure J_0 that is compatible with $\tilde{\omega}$, then the question arises about the propagation of the complex-C^∞-pure-and-full property along such deformations. The picture is complex: in [3], Angella and Tomassini provided several examples of curves of almost-complex structures on compact symplectic manifolds that *satisfied or failed* (complex-)C^∞-pure and (complex-)C^∞-full properties at various stages. The general method (attributed to Junho Lee) was the following: let γ be a real 2-form and $V_\gamma \in \mathrm{End}(TX)$ such that $\gamma = g_{\tilde{\omega}}(V_\gamma \cdot, \cdot\cdot)$. Put $L_\gamma := \frac{1}{2} V_\gamma J_{\tilde{\omega}} \in \mathrm{End}(TX)$, and define the curve, for $t \in (-\epsilon, \epsilon)$ for $\epsilon > 0$ small enough, by

$$J_{t,\gamma} := (Id - tL_\gamma)\tilde{J}(Id - tL_\gamma)^{-1}$$

Example 1.5, [3]: Let (J, ω) be the standard Kähler structure on the complex

torus $\mathbf{T}^2$, and define the path

$$J_t := J_{t,l} := (id - tL)J(id - tL)^{-1} = \begin{bmatrix} 0 & 0 & \frac{tl-1}{1+tl} & 0 \\ 0 & 0 & 0 & -1 \\ \frac{1+tl}{1-tl} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

for t small enough, where

$$L = \begin{bmatrix} l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -l & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$l = l(x_2)$ being a smooth Z^4-periodic non-constant function on $\mathbf{R}^4$. For $t \neq 0$ small enough, it is simple to compute the level-1 J-invariant and J-anti-invariant cohomology groups

$$H^{(1,0)}_{J_t}(\mathbf{T}^2) = \mathbf{C}\left\langle dx^2 + idx^4 \right\rangle, \quad H^{(0,1)}_{J_t}(\mathbf{T}^2) = \mathbf{C}\left\langle dx^2 - idx^4 \right\rangle$$

However, the first Betti number of the complex torus $\mathbf{T}^2$ is 4, while the direct sum of the two subspaces is only 2-dimensional. Thus, for $t \neq 0$ small enough, while J is complex-C^∞-full at the first stage, J_t is not. $\square$

Example, [2]: Let $\eta\beta_5$ be the 10-dimensional nilmanifold considered in [AB], with the Darboux symplectic structure

$$\omega = \frac{i}{2}\sum_{j=1}^{5} \varphi^j \wedge \bar{\varphi}^j$$

The complex structure equations are

$$d\varphi^1 = 0$$
$$d\varphi^2 = 0$$
$$d\varphi^3 = 0$$
$$d\varphi^4 = 0$$
$$d\varphi^5 = -\varphi^{12} - \varphi^{34}$$

One computes

$$H^1_{dR}(\eta\beta_5; \mathbf{C}) = \mathbf{C}\left\langle \varphi^1, \varphi^2, \varphi^3, \varphi^4, \bar{\varphi}^1, \bar{\varphi}^2, \bar{\varphi}^3, \bar{\varphi}^4 \right\rangle$$

Hence, $\eta\beta_5$ is complex-C^∞-pure-and-full at the first stage. Consider the complex deformation defined by

$$\varphi^1_t = \varphi^1 + t\bar{\varphi}^1$$
$$\varphi^2_t = \varphi^2$$
$$\varphi^3_t = \varphi^3$$
$$\varphi^4_t = \varphi^4$$
$$\varphi^5_t = \varphi^5$$

for $t \in \mathbf{C}$ small enough. With respect to this transformation, we note that $\varphi^1 = \frac{1}{1-|t|^2}(\varphi^1_t - t\bar{\varphi}^1_t)$, $\varphi^2 = \varphi^2_t$, $\varphi^3 = \varphi^3_t$, $\varphi^4 = \varphi^4_t$, and $\varphi^5 = \varphi^5_t$. Thus, the complex

structure equations for this family are

$$\begin{aligned} d\varphi_t^1 &= 0 \\ d\varphi_t^2 &= 0 \\ d\varphi_t^3 &= 0 \\ d\varphi_t^4 &= 0 \\ d\varphi_t^5 &= \frac{-1}{1-|t|^2}\varphi_t^{12} - \varphi_t^{34} - \frac{t}{1-|t|^2}\varphi_t^{2\bar{1}} \end{aligned}$$

Therefore, for $t \in \mathbf{C}$ small enough, the J_t are complex-C^∞-pure-and-full at the first stage, for t small enough. Without too much trouble, one may also establish that the complex-C^∞-pure-and-fullness at the second stage also holds for $t \in \mathbf{C}$ small enough. $\square$

This last example provides further evidence for why it might be prudent to require our (X, J, ω) to be complex-C^∞-pure-and-full at the second stage: such will ensure that $(\tilde{X}, \tilde{J}, \tilde{\omega})$ has the same property, which will not necessarily bar the possibility that $\tilde{J}$ be no worse than a deformation of an $\tilde{\omega}$-compatible, integrable J_0.

Phrased another way, we are studying the existence on $(\tilde{X}, \tilde{\omega}, \tilde{J})$ of an integrable almost complex structure J_0 that is metrically related to $\tilde{J}$; that is, that they are compatible with the same symplectic form, $\tilde{\omega}$. In the case of $\tilde{X}$ being of real dimension 4, some information is known about the anti-invariant cohomology for such pairs:

Theorem 1.1, [13]: *Let (Y, J) be a compact complex surface. If $\tilde{J}$ is an almost complex structure on Y that is metric-related to J, $\tilde{J} \not\equiv \pm J$, then $h^-_{\tilde{J}} \in \{0, 1, 2\}$.*

In fact, the almost complex structures $\tilde{J}$ with $h^-_{\tilde{J}} = 0$ form an open and dense set with respect to the C^∞-topology in the space of almost complex structures metric-related to J.

On the other hand, if J_0 is too "close" to $\tilde{J}$, then J_0 might potentially not be integrable. Indeed, we have:

Theorem 5.4, [13]: *Suppose Y is a 4-manifold with an almost complex structure J such that $K^c_J(Y)$ is nonempty. Then for any almost complex structure J' in a sufficiently small neighborhood of J as in Corollary 5.3, we have*

- $K^c_{J'}(Y) \neq \emptyset$;
- $h^+_J(Y) \leq h^+_{J'}(Y)$;
- $h^-_J(Y) \geq h^-_{J'}(Y)$;

Here, "neighborhood" refers to an open set in the space $\mathcal{J}$ of almost complex structures on M that contains J and which satisfies certain constraints related to the symplectic Calabi-Yau equation (see Section 5, [13]). Draghici, Li, and Zhang further made the conjecture:

Conjecture 2.5, [13]: *On a compact 4-manifold, if $h^-_{J'} \geq 3$, then J' is integrable.*

If $\tilde{J}$ has a solution of the symplectic Calabi-Yau equation satisfying D-type constraints [13], then for any path through $\tilde{J}$, there is a small interval near $\tilde{J}$ such that the symplectic CY-equation is solvable with D_t-type constraints on this

interval. If J_0 is in this interval, then Theorem 5.4 implies that

$$h^-_{J_0} \leq h^-_{\tilde{J}} \in \{0, 1, 2\}$$

so $h^-_{J_0}$ is certainly not ≥ 3. This may or may not disqualify J_0 from integrability - depending on whether or not the conjecture is true or is in fact an equivalence statement. At any rate, it certainly seems desirable to - if possible - choose our J_0 outside such an interval containing $\tilde{J}$.

Concluding this discussion, we form the following conjecture:

Conjecture: *Let (X^{2n}, ω, J) be a formal symplectic manifold and $(M^{2d}, \sigma) \overset{\iota}{\hookrightarrow} (X, \omega)$ be a symplectically embedded submanifold. Let $(\tilde{X}, \tilde{\omega}, \tilde{J})$ be the symplectic blow-up of X along M, and suppose it has the Hard Lefschetz property and admits $\tilde{\omega}$-compatible, integrable almost complex structures. Suppose the following conditions hold:*

- *X and $\tilde{X}$ are formal;*
- *$J \in End(TX)$ restricts to an element $J|_{TM} \in End(TM)$;*
- *$\tilde{J}$ is complex-C^∞-pure-and-full at every stage;*

Then there is an integrable complex structure J_0 that is metrically related to $\tilde{J}$ and that may be realized as a deformation of $\tilde{J}$ via a J. Lee-type path of a.c. structures.

If this conjecture can be proven, it will provide a first step in answering our question on whether or not a given compact symplectic Hard Lefschetz (respectively, non-Hard Lefschetz) manifold is symplectic birationally cobordant to a compact Kähler manifold.

Chapter 4

Deformation Equivalence Between $\mathcal{K}$ and $\mathcal{SHL}$

Here we discuss the type of symplectic "flexibility" known as deformation equivalence. As shown, it is not a trivial matter whether a symplectic non-Hard Lefschetz form and a symplectic Hard Lefschetz (much less Kähler) form can be related along a path; on most compact symplectic nil- or solvmanifolds, for example, it is usually the case that *all* symplectic forms are Hard Lefschetz or none of them are (see [21], [22]).

However, Yunhyung Cho has recently found a class of six-dimensional symplectic manifolds supporting some symplectic flexibility. Specifically, he has proven:

(Theorem 1.3, [11]): *There exists a compact Kähler manifold* (X, ω, J) *with* $dim_{\mathbf{C}} X = 3$ *such that*

- *X is simply connected,*
- *$H^{2k+1}(X) = 0$ for every integer $k \geq 0$;*
- *X admits a symplectic form $\sigma \in \Omega^2(X)$ of non-Hard Lefschetz type, and*
- *σ is a deformation equivalent to the Kähler form ω.*

To our knowledge, this is the first example in the literature showing deformation equivalence between a Kähler form and a non-Hard Lefschetz form. The class of manifolds that Cho uses are six-dimensional compact simple Hamiltonian-S^1 symplectic manifolds (X, ω): that is, they support a Hamiltonian action of S^1 whose fixed point set has precisely two components. It is a well-known fact that these components constitute symplectic submanifolds of (X, ω), and therefore may have dimensions $0, 2$, or 4. When at least one component has dimension less than 4, ad hoc methods can be used to establish that all symplectic forms on X have the Hard Lefschetz property. Therefore, Cho focuses most of his attention on the case when both components have dimension 4 and are in fact diffeomorphic via the flow of the gradient vector field associated to the Hamiltonian function $H : X \to [0, 1]$. He provides a complete characterization of the Hard Lefschetz property on (X, ω) in terms of cohomological conditions on the components. Precisely:

Proposition 1.8, [11]: *Let (X, ω) be a six-dimensional simple Hamiltonian S^1-manifold with a moment map $H : X \to [0, 1]$. Assume that all fixed components are of dimension four so that $X_{min} = H^{-1}(0) \cong H^{-1}(1) = X_{max}$ are diffeomorphic. Then (X, ω) satisfies the Hard Lefschetz property if and only if*

- *$(X_{min}, \omega_0 + \omega_1)$ satisfies the Hard Lefschetz property, and*

- $[\omega_0] \cdot [\omega_1] \neq 0$ *in* $H^4(X_{min}; \mathbf{R}) \cong H^4(X_{max}; \mathbf{R})$,

Cho's approach - outlined in the next section - seems rich for generalization. Indeed, we achieve the following:

Theorem: *There exists a compact Kähler manifold* (X, ω, J) *with* $dim_{\mathbf{C}} X = 5$ *such that*

- X *is simply connected,*
- $H^{2k+1}(X) = 0$ *for every integer* $k \geq 0$*;*
- X *admits a symplectic form* $\sigma \in \Omega^2(X)$ *of non-Hard Lefschetz type, and*
- σ *is a deformation equivalent to the Kähler form* ω*.*

Along the way, we achieve a characterization of the Hard Lefschetz property on our X in terms of cohomological conditions on the fixed components. Since the odd cohomology of our X will vanish, the condition of ω having the HLP is equivalent to the nondegeneracy of the Hodge-Riemann forms on $H^2_{dr}(X)$ and $H^4_{dR}(X)$:

Proposition: *Let* (X, ω) *be a ten-dimensional simple Hamiltonian* S^1*-manifold with a moment map* $H : X \to [0, 1]$*. Assume that all fixed components are of dimension eight so that* $X_{min} = H^{-1}(0) \cong H^{-1}(1) = X_{max}$*. Then* (X, ω) *satisfies the Hard Lefschetz property if and only if the following is true:*

- *The bilinear forms $Q_{max^2}, Q_{max\cdot min}$ are nondegenerate on H^2_{max}.*

- *If $n_G = 0$, then $e \in K_{\omega^3_{max}}$ and Q_G is nondegenerate on $K_{\omega^3_{max}}$ or $e \notin K_{\omega^3_{max}}$ and $det(Q_G|_{K_{\omega^3_{max}}}) \neq -det(Q_G|_{H^2_{min}})$. If $n_G = 1$, then $H^2_{max} = K_G \oplus K_{\omega^3_{max}}$ and Q_G is nondegenerate on $K_{\omega^3_{max}}$.*

Now, to begin, we shall review some basic facts about equivariant cohomology, since this is the key to studying the relationship between the de Rham cohomology rings of a Hamiltonian S^1-manifold X and of the components of its fixed point set X^{S^1} (for a more detailed treatment, see [11]).

Let X be a compact symplectic manifold on which S^1 acts in a Hamiltonian fashion, and let X^{S^1} be the fixed point set. Let ES^1 denote a contractible space on which S^1 acts freely. The S^1-equivariant cohomology ring on X is defined to be

$$H^*_{S^1}(X) = H^*_{dR}(X \times_{S^1} ES^1)$$

We call the quotient $BS^1 = ES^1/S^1$ the classifying space of S^1, and we note that the projection map $X \times ES^1 \xrightarrow{\pi} BS^1$ fibres as an X-bundle. The Leray-Hirsch theorem tells us then that $H^*_{S^1}(X)$ has the structure of an $H^*_{dR}(BS^1)$-module over $H^*_{dR}(X)$:

$$H^*_{S^1}(X) = \bigoplus (H^*_{dR}(X) \otimes H^*_{dR}(BS^1))$$

Since $BS^1 \approx \mathbf{CP}^\infty \supset ... \supset \mathbf{CP}^2 \supset \mathbf{CP}^1$, the ring $H^*_{dR}(BS^1)$ is isomorphic to the polynomial ring $\mathbf{R}[u]$, where u is a positive generator of degree 2 for which $\langle u, [\mathbf{CP}^1] \rangle = 1$.

By the Kirwan Injectivity Theorem [23], the inclusion $X^{S^1} \overset{\iota}{\hookrightarrow} X$ induces an injective "restriction" homomorphism

$$\iota^* : H^*_{S^1}(X) \to H^*_{S^1}(X^{S^1}) \cong \bigoplus_{F \subset X^{S^1}} H^*(F) \otimes H^*(BS^1)$$

where F are the connected components of X^{S^1}; a similar restriction to $H^*_{S^1}(F)$ from $H^*_{S^1}(X^{S^1})$ exists for each $F \subset X^{S^1}$. What this tells us is that every equivariant class on X is uniquely determined by its restriction(s) to the components of the fixed point set X^{S^1}. Indeed, we have a decomposition for each equivariant class on X in terms of its restrictions to the components of X^{S^1}: let $\alpha \in H^k_{S^1}(X)$. Then

$$\alpha|_F \in \bigoplus_{i=0}^{j} H^*(F) \otimes H^i(BS^1)$$

for some smallest positive integer j, which Cho refers to as the $H^*(BS^1)$-degree of $\alpha|_F$.

From further work by Kirwan, that X has a Hamiltonian S^1-action makes X *equivariantly formal*, which ensures that the inclusion $X \overset{f}{\hookrightarrow} X \times_{S^1} ES^1$ induces a ring homomorphism $f^* : H^*_{S^1}(X) \to H^*(X)$ that is actually surjective, with the kernel being $u \cdot H^*_{S^1}(X)$, where $\cdot$ connotes scalar multiplication in the ring; this means that every de Rham class on X is associated with at least one equivariant class on X, and that any two such equivariant classes differ only by an element of the ring $H^*(BS^1)$. By composing f^* with the restriction homomorphisms discussed earlier, we obtain injective mappings $H^*_{S^1}(F) \to H^*_{dR}(X)$, enabling us to fully commute between the cohomologies of the fixed point set and of X itself. This means that we can construct a basis for the de Rham cohomology of X

by using special basis elements from the cohomologies of the fixed components. Indeed, we have the McDuff-Tolman basis theorem:

Theorem 3.5, [28] (cf. [11]): *Let (X, ω) be a closed symplectic manifold equipped with a Hamiltonian circle action with a moment map $H : X \to \mathbf{R}$. As a Morse-Bott function, through every critical point of H passes both a stable and unstable manifold defined with respect to the gradient vector field of H. For each connected fixed component $F \subset X^{S^1}$, let k_F be a Morse index of F with respect to H. For any given cohomology class $Y \in H^i(F)$, there exists a unique class $\tilde{Y} \in H_{S^1}^{i+k_F}(X)$ such that*

(1) $\tilde{Y}|_{F'} = 0$ *for every* $F' \in X^{S^1}$ *with* $H(F') < H(F)$,

(2) $\tilde{Y}|_F = Y \cup e_F^-$ *with* $Y = Y \otimes 1 \in H_{S^1}^*(F)$, *and*

(3) *for all fixed components* $F' \neq F$, *the* $H^*(BS^1)$*-degree of* $\tilde{Y}|_{F'} \in H_{S^1}^*(F')$ *is less than the index* $k_{F'}$.

We call such a class $\tilde{Y}$ ***the canonical class with respect to*** *Y. If we fix a basis S_F of the $\mathbf{R}$-vector space $H^*(F)$ for each fixed component $F \subset X^{S^1}$, then the set*

$$\mathcal{B} = \{\tilde{Y} | Y \in \bigcup_{F \subset X^{S^1}} S_F\}$$

is a basis of $H_{S^1}^*(X)$ as an $H^*(BS^1)$-module. Moreover, $f^*(\mathcal{B})$ is a basis of $H^*(X)$ as an $\mathbf{R}$-vector space.

Note that, here, e_F^- denotes the equivariant Euler class of the negative normal bundle ν_F^- in X to the component F, which may be seen as a subbundle of the normal bundle ν_F whose fiber over each point p is a subspace of T_pX that is tangent to the unstable manifold of X at F.

Using the McDuff-Tolman basis theorem, we may choose some convenient bases for the second and fourth de Rham cohomology groups of our compact simple Hamiltonian-S^1 manifold X^{10}:

$$H^2_{dR}(X) = \mathbf{R}\left\langle f^*\tilde{\beta}, f^*\tilde{\beta}_i | \{\beta_i\}_i \text{is a basis for} H^2_{dR}(X_{min}), \beta = 1 \in H^0_{dR}(X_{max})\right\rangle$$

$$H^4_{dR}(X) = \mathbf{R}\left\langle f^*\tilde{\xi}_i, f^*\tilde{\delta}_i | \{\delta_i\}_i \text{is a basis for} H^4_{dR}(X_{min}), \{\xi_i\}_i \text{is a basis for} H^2_{dR}(X_{max})\right\rangle$$

Before proving our main theorem, we will establish our proposition characterizing the Hard Lefschetz property on (X^{10}, ω) in terms of cohomological conditions on the diffeomorphic eight-dimensional components of X^{S^1}. We do this by determining conditions equivalent to the nondegeneracy of the Hodge-Riemann forms HR_2, HR_4 on $H^2_{dR}(X)$ and $H^4_{dR}(X)$, respectively. The McDuff-Tolman theorem provides us a suitable basis, and the Atiyah-Bott-Berline-Vergne localization theorem simplifies the integral computations:

Theorem 3.6, Atiyah-Bott-Berline-Vergne localization theorem, [11]: *Let X be a compact manifold with a circle action. Let $\alpha \in H^*_{S^1}(X; \mathbf{R})$. Then as an element of $\mathbf{R}(u)$, we have*

$$\int_X \alpha = \sum_{F \subset X^{S^1}} \int_F \frac{\alpha|_F}{e_F}$$

where the sum is taken over all fixed points, and e_F is the equivariant Euler class of the normal bundle to F.

Since $H^*_{S^1}(X) = H^*(X) \otimes H^*(BS^1)$, the integration of α over X is performed on the ordinary de Rham cohomology factor of α in $H^*(X) \otimes H^0(BS^1)$. Thus, for example, given $\alpha, \beta \in H^2_{dR}(X)$, we have

$$HR_2(\alpha, \beta) = \int_X \tilde{\alpha}\tilde{\beta}\tilde{\omega}^3$$

where $\tilde{\omega}$ is an equivariant extension of ω; that is, $f^*[\tilde{\omega}] = [\omega]$ (such can always be constructed on compact simple Hamiltonian-S^1 X; see [11] for details). Naturally, we need to know the restrictions of $[\tilde{\omega}]$ to the components of the fixed point set, in order to complete our calculations. We have:

Proposition 3.7, [4]: *Let (X, ω) be a Hamiltonian-S^1-manifold with a moment map $H : X \to \mathbf{R}$. Let $[\tilde{\omega}]$ be the equivariant symplectic class with respect to H. Then for any fixed component $F \subset X^{S^1}$, we have*

$$[\tilde{\omega}]|_F = -H(F) \otimes u + \omega|_F \otimes 1 \in H^*_{S^1}(F) \cong H^*(F) \otimes H^*(BS^1)$$

Thus, in our case, with $H(X) = [0, 1]$, we have $[\tilde{\omega}]|_{X_{min}} = \omega|_{min} \otimes 1$ and $[\tilde{\omega}]|_{X_{max}} = -1 \otimes u + \omega|_{max} \otimes 1$. Additionally, Cho observed in his Lemma 4.4, that the equivariant Euler class for the normal bundle over X_{max} may be given by

$$e_{X_{max}} = -u \otimes 1 - e \otimes u$$

where e is the Euler class of a principal S^1-bundle $H^{-1}(1-\epsilon) \to H^{-1}(1-\epsilon)/S^1 \cong X_{max}$ for a sufficiently small ϵ, and that the equivariant Euler class for the normal bundle over X_{min} may be given by

$$e_{X_{min}} = 1 \otimes u^2 + c_1(\nu_{X_{min}}) \otimes u$$

With that, we may begin computing.

At level 2, we have

$$\begin{aligned} HR_2(f^*\tilde{\beta}_i, f^*\tilde{\beta}_j) &= \int_X \tilde{\beta}_i\tilde{\beta}_j\tilde{\omega}^3 \\ &= \int_{X_{min}} \frac{\beta_i\beta_j(\omega_{min}^3 \otimes 1)}{e_{X_{min}}} + \int_{X_{max}} \frac{\phi(\beta_i)\phi(\beta_j)(\omega_{max} \otimes 1 - 1 \otimes u)^3}{e_{X_{max}}} \end{aligned}$$

In the first integral on the right hand side, we may multiply numerator and denominator by the equivariant element

$$1 \otimes u^3 - c_1(\nu_{X_{min}}) \otimes u^2 + c_1(\nu_{X_{min}})^2 \otimes u - c_1(\nu_{X_{min}})^3 \otimes 1$$

The denominator simply becomes a scalar in $H^*(BS^1)$, while the numerator becomes an equivariant class whose ordinary cohomology parts have degree higher than $8 = dim(X_{min})$; this integral vanishes. In the second integral on the right hand side, we multiply numerator and denominator by the equivariant element

$$1 \otimes u^4 - e \otimes u^3 + e^2 \otimes u^2 - e^3 \otimes u + e^4 \otimes 1$$

and yield

$$\begin{aligned} HR_2(f^*\tilde{\beta}_i, f^*\tilde{\beta}_j) &= \int_{X_{max}} \phi(\beta_i)\phi(\beta_j)(\omega_{max} \otimes 1 - 1 \otimes u)^3 \sum_{k=0}^{4} (-1)^k e^k \otimes u^{4-k} \\ &= -u^{-5} \int_{X_{max}} \phi(\beta_i)\phi(\beta_j)(-3\omega_{max}^2 \otimes u^5 - 3\omega_{max}e \otimes u^5 - e^2 \otimes u^5) \\ &= \int_{X_{max}} \phi(\beta_i)\phi(\beta_j)(\omega_{max}^2 + \omega_{max}\phi^*\omega_{min} + (\phi^*\omega_{min})^2) \end{aligned}$$

Here, we have used the Duistermaat-Heckman theorem, which shows that $e =$

$\phi^*[\omega_{min}] - [\omega_{max}] \in H^2(X_{max}; \mathbf{R}) \cong H^2(X_{min}; \mathbf{R})$. Next, we have:

$$\begin{aligned} HR_2(f^*\tilde{\beta}_i, f^*\tilde{\beta}) &= 0 + \int_{X_{max}} \frac{\phi(\beta_i)\hat{\beta}(-e \otimes 1 - 1 \otimes u)(\omega_{max} \otimes 1 - 1 \otimes u)^3}{-e \otimes 1 - 1 \otimes u} \\ &= -u^{-5} \int_{X_{max}} \phi(\beta_i)(\omega_{max} \otimes 1 - 1 \otimes u)^3 \\ &= \int_{X_{max}} \phi(\beta_i)\omega_{max}^3 \end{aligned}$$

Note that here we have used the properties of the McDuff-Tolman canonical classes. Lastly, we have:

$$\begin{aligned} HR_2(f^*\tilde{\beta}, f^*\tilde{\beta}) &= 0 + \int_{X_{max}} \frac{\hat{\beta}^2(-e \otimes 1 - 1 \otimes u)^2(\omega_{max} \otimes 1 - 1 \otimes u)^3}{-e \otimes 1 - 1 \otimes u} \\ &= \int_{X_{max}} (-e \otimes 1 - 1 \otimes u)(\omega_{max} \otimes 1 - 1 \otimes u)^3 \\ &= -\int_{X_{max}} e\omega_{max}^3 \end{aligned}$$

The matrix for HR_2 then takes the form

$$HR_2 = \left[\begin{array}{c|c} \int_{X_{max}} \phi(\beta_i)\phi(\beta_j)(\omega_{max}^2 + \omega_{max}\phi^*\omega_{min} + (\phi^*\omega_{min})^2) & \int_{X_{max}} \phi(\beta_i)\omega_{max}^3 \\ \hline \int_{X_{max}} \phi(\beta_j)\omega_{max}^3 & -\int_{X_{max}} e\omega_{max}^3 \end{array}\right]$$

Now, put $G := \cup([\omega_{max}]^2 + [\omega_{max}]\phi^*[\omega_{min}] + \phi^*[\omega_{min}]^2) : \phi^*(H^2_{dR}(X_{min})) \to H^6_{dR}(X_{max})$, and let's assume that it is *not* injective; let n_G denote the dimension of its kernel K_G. If $n_G \geq 2$, then we may choose a basis for $H^2_{dR}(X_{min})$ that is the union of a basis for $(\phi^*)^{-1}(K_G)$ and a basis for $(\phi^*)^{-1}(C)$, where $K_G \oplus C = \phi^*(H^2_{dR}(X_{min}))$. Let n_C denote the dimension of C. Then the matrix for HR_2 takes the form

$$\left[\begin{array}{c|c|c} 0 & 0 & \int_{X_{max}} \phi(\beta_i)\omega_{max}^3 \\ \hline 0 & \int_{X_{max}} \phi(\beta_{n_G+i})\phi(\beta_{n_G+j})G & \int_{X_{max}} \phi(\beta_{n_G+i})\omega_{max}^3 \\ \hline \int_{X_{max}} \phi(\beta_j)\omega_{max}^3 & \int_{X_{max}} \phi(\beta_{n_G+j})\omega_{max}^3 & -\int_{X_{max}} e\omega_{max}^3 \end{array}\right]$$

We note that since $n_G \geq 2$, the minors of the elements of the last column are all 0, and hence, $detHR_2 = 0$. Thus, $n_G < 2$ is necessary for the nondegeneracy of HR_2.

Let's examine the two cases. First, assume $n_G = 0$. Since the cup product $H^2_{min} \otimes H^6_{min} \to \mathbf{R}$ is nondegenerate, we see that the bilinear form $Q_G : H^2_{min} \otimes H^2_{min} \to \mathbf{R}$ must be nondegenerate, otherwise $ker(G) \neq 0$. Thus, $det(Q_G) \neq 0$. If $e \in K_{\omega^3_{max}}$, then we choose β_1 so that $\phi(\beta_1) = e$ and complete with $K_{\omega^3_{max}} = \mathbf{R}\left\langle \{\phi(\beta_i)\}_{2 \leq i \leq b_{2,min}} \right\rangle$ to a basis for H^2_{min}. The determinant of HR_2 in this case reduces to

$$det(HR_2) = \left(\int_{X_{max}} \omega^4_{max} \right)^2 det\left(Q_G|_{H^2_{min} - \mathbf{R}\langle \omega_{max} \rangle} \right)$$

Thus, for nondegeneracy of HR_2, Q_G must be nondegenerate on the indicated subspace of H^2_{min}. If $e \notin K_{\omega^3_{max}}$, then we choose a basis for the following decomposition $H^2_{max} = K_{\omega^3_{max}} \oplus \mathbf{R}\langle e \rangle$, and the determinant for HR_2 reduces to

$$det(HR_2) = \left(- \int_{X_{max}} e\omega^3_{max} \right) \left(det(Q_G|_{H^2_{min}}) + det(Q_G|_{K_{\omega^3_{max}}}) \right)$$

For nondegeneracy of HR_2 to follow, the latter sum of determinants must not vanish.

Now, we examine the case of $n_K = 1$. Let $K_G = \mathbf{R}\langle v \rangle$. If $v \in K_{\omega^3_{max}}$, then put β_1 such that $\phi(\beta_1) = v$ and complete to a basis of H^2_{min} compatible with

$$H^2_{min} = K_{\omega^3_{max}} \oplus \mathbf{R}\langle \omega_{max} \rangle$$

In this case, the first row and column of HR_2 zero out, leaving $det(HR_2) = 0$. Therefore, for nondegeneracy of HR_2, if $n_K = 1$, then we require $K_G \cap K_{\omega^3_{max}} = 0$.

It follows that we may consider $H^2_{min} = K_G \oplus K_{\omega^3_{max}}$ and choose our basis this way. The determinant reduces to

$$\begin{aligned} det(HR_2) &= \left(-\int_{X_{max}} e\omega^3_{max}\right)\cdot 0 - \left(\int_{X_{max}} \phi(\beta_{b_{2,min}})\omega^3_{max}\right)^2 det(Q_G|_{K_{\omega^3_{max}}}) \\ &= -\left(\int_{X_{max}} \phi(\beta_{b_{2,min}})\omega^3_{max}\right)^2 det(Q_G|_{K_{\omega^3_{max}}}) \end{aligned}$$

Therefore, for nondegeneracy of HR_2, if $n_K = 1$ and $K_G \cap K_{\omega^3_{max}} = 0$, then we must require nondegeneracy of Q_G on $K_{\omega^3_{max}}$.

Summarizing: for nondegeneracy of HR_2, it is necessary that one of the following is true:

- $n_G = 0$ and $e \in K_{\omega^3_{max}}$ and Q_G is nondegenerate on $K_{\omega^3_{max}}$ or $e \notin K_{\omega^3_{max}}$ and $det(Q_G|_{K_{\omega^3_{max}}}) \neq -det(Q_G|_{H^2_{min}})$.

- $n_G = 1$ and $H^2_{max} = K_G \oplus K_{\omega^3_{max}}$ and Q_G is nondegenerate on $K_{\omega^3_{max}}$.

Tracing back through the arguments, it is clear that these are sufficient conditions for nondegeneracy of HR_2 as well.

At level 4, we have

$$\begin{aligned} HR_4(f^*\tilde{\delta}_i, f^*\tilde{\delta}_j) &= \int_X \tilde{\delta}_i\tilde{\delta}_j\tilde{\omega} \\ &= \int_{X_{min}} \frac{\delta_i\delta_j(\omega_{min}\otimes 1)}{e_{X_{min}}} + \int_{X_{max}} \frac{\phi(\delta_i)\phi(\delta_j)(-1\otimes u + \omega_{max}\otimes 1)}{-1\otimes u - e\otimes 1} \\ &= -u^{-5}\int_{X_{max}} \phi(\delta_i)\phi(\delta_j)(-u^{-5}) \\ &= \int_{X_{max}} \phi(\delta_i)\phi(\delta_j) \end{aligned}$$

$$\begin{aligned} HR_4(f^*\tilde{\delta}_i, f^*\tilde{\xi}_j) &= 0 + \int_{X_{max}} \frac{\phi(\delta_i)\xi_j(-1\otimes u - e\otimes 1)(-1\otimes u + \omega_{max}\otimes 1)}{-1\otimes u - e\otimes 1} \\ &= \int_{X_{max}} \phi(\delta_i)\xi_j(-1\otimes u + \omega_{max}\otimes 1) \\ &= \int_{X_{max}} \phi(\delta_i)\xi_j\omega_{max} \end{aligned}$$

$$\begin{aligned} HR_4(f^*\tilde{\xi}_i, f^*\tilde{\xi}_j) &= 0 + \int_{X_{max}} \frac{\xi_i\xi_j(-1\otimes u - e\otimes 1)^2(-1\otimes u + \omega_{max}\otimes 1)}{-1\otimes u - e\otimes 1} \\ &= \int_{X_{max}} \xi_i\xi_j(-1\otimes u - e\otimes 1)(-1\otimes u + \omega_{max}\otimes 1) \\ &= \int_{X_{max}} -\xi_i\xi_j e\omega_{max} \end{aligned}$$

Thus, the matrix for HR_4 is of the following form:

$$HR_4 = \left[\begin{array}{c|c} \int_{X_{max}} \phi(\delta_i)\phi(\delta_j) & \int_{X_{max}} \phi(\delta_i)\xi_j\omega_{max} \\ \hline \int_{X_{max}} \phi(\delta_j)\xi_i\omega_{max} & -\int_{X_{max}} \xi_i x_j e\omega_{max} \end{array}\right]$$

Let's suppose for the sake of contradiction that the map $\cup[\omega_{max}] : H^2_{dR}(X_{max}) \to H^4_{dR}(X_{max})$ is not injective. Then there exists $v \neq 0 \in H^2_{dR}(X_{max})$ such that $v\cup[\omega_{max}] = 0$. We can complete $\{v\}$ to a basis for $H^2_{dR}(X_{max})$, and set $\xi_{b_{2,max}} := v$. Then the entire $(b_{4,min} + b_{2,max})$th column and row of the matrix at left zero out, implying the matrix has non-full rank. Therefore, injectivity of $\cup[\omega_{max}] : H^2_{max} \to H^4_{max}$ is necessary for nondegeneracy of HR_4.

Let's suppose, then, that this injectivity holds, and determine whether or not it is sufficient to guarantee nondegeneracy of HR_4. Then we choose $\phi(\delta_{b_{4,min}+i}) := \xi_i \cup [\omega_{max}]$ for $1 \leq i \leq b_{2,max}$ and $\{\phi(\delta_i)\}_{1\leq i\leq b_{4,min}}$ to be a basis for the cup-orthogonal complement to $[\omega_{max}]\cup H^2_{max}$ in H^4_{max}. The matrix for HR_4 then takes

the form

$$HR_4 = \left[\begin{array}{c|c|c} \ldots \int_{X_{max}} \phi(\delta_i)^2 \ldots & \ldots\ldots & \ldots\ldots \\ \hline \ldots\ldots & \ldots \int_{X_{max}} \xi_i^2 \omega_{max}^2 \ldots & \ldots \int_{X_{max}} \xi_i^2 \omega_{max}^2 \ldots \\ \hline \ldots\ldots\ldots & \ldots \int_{X_{max}} \xi_i^2 \omega_{max}^2 \ldots & \ldots - \int_{X_{max}} \xi_i \xi_j e \omega_{max} \ldots \end{array}\right]$$

which is nonsingular if and only if both the upper left submatrix and the lower-right $b_{2,max}$-by-$b_{2,max}$ submatrix are both nonsingular. We row-reduce this latter submatrix by exerting the action $R_{i+b_{4,min}} \mapsto R_{i+b_{4,min}} + (-1)R_{i+b_{4,min}-b_{2,max}}$ for all $1 \leq i \leq b_{2,max}$ on the entire matrix for HR_4. The resulting submatrix has the form

$$\left[\begin{array}{c|c} \ldots \int_{X_{max}} \xi_i^2 \omega_{max}^2 \ldots & \ldots \int_{X_{max}} \xi_i^2 \omega_{max}^2 \ldots \\ \hline \ldots\ldots & - \int_{X_{max}} \xi_i \xi_j \omega_{max} \phi^* \omega_{min} \end{array}\right]$$

This submatrix is block-upper triangular, and is hence nonsingular if and only if Q_{max^2} and $Q_{max \cdot min}$ are both nondegenerate on H^2_{max}. Thus, the map $\cup[\omega_{max}]\phi^*[\omega_{min}] : H^2_{dR}(X_{max}) \to H^6_{dR}(X_{max})$ being an isomorphism is a necessary condition for HR_4 to be nondegenerate (note that the nondegeneracy of Q_{max^2} also implies the injectivity of $\cup[\omega_{max}]^2 : H^2_{max} \to H^4_{max}$). Thus, the nondegeneracy of HR_4 forces the nondegeneracy of the cup product on the cup-orthogonal complement of $[\omega_{max}] \cup H^2_{max}$ (but this is already a natural consequence of the nondegeneracy of the cup product on H^4_{max}), and of Q_{max^2} and $Q_{max \cdot min}$ on H^2_{max}. A careful tracing-through of the arguments thus laid out shows that these conditions are also sufficient to guarantee nondegeneracy of HR_4. Therefore, the fourth Hodge-Riemann form, HR_4, is nondegenerate if and only if the following hold:

- The bilinear form Q_{max^2} is nondegenerate on H^2_{max}.
- The bilinear form $Q_{min \cdot max}$ is nondegenerate on H^2_{max}.

□

In summary: this completes the proof of our proposition characterizing the Hard Lefschetz property on compact simple Hamiltonian-S^1 manifolds of dimension 10. Now, we use it to prove our main theorem.

Cho's method - which we will show is readily generalizable - is to take a compact symplectic 4-dimensional manifold (resp. 8-dimensional manifold, in our case) (N, σ) with an integral cohomology class $e \in H^2(N; \mathbf{Z})$. With it, he constructs a 6-dimensional (resp. 10-dimensional) compact simple Hamiltonian-S^1 manifold $(\tilde{M}(N, e, 2\epsilon)_{-\epsilon,\epsilon}, \tilde{\omega})$ like so: the integral class e determines a unique complex line bundle ξ and an associated principal S^1-bundle $P \to N$ whose first Chern class $c_1(P)$ is e. Let $M(N, e, \epsilon) := P \times (-\epsilon, \epsilon)$ be the product symplectic manifold with

$$\omega_\sigma = \pi^*\sigma + d(t \cdot \theta)$$

where $\pi : P \to N$ is the quotient map by the S^1-action, θ is a connection 1-form on P, and t is the parameter of $(-\epsilon, \epsilon)$, ϵ being small enough to guarantee nondegeneracy of ω_σ. Now, the S^1-action on P induces an S^1-action on $M(N, e, \epsilon)$ that is Hamiltonian with respect to ω_σ, with moment map H being the projection onto the real parameter, $H(p, t) = t$.

We now employ the Lerman symplectic cut method. Consider the product

$$(M(N, e, \epsilon) \times \mathbf{C}^2, \omega_\sigma + \frac{1}{i}\sum_{j=1}^{2} dz_j \wedge d\bar{z}_j)$$

with the product symplectic structure. We note that it admits a T^3-action:

$$(w_1, w_2, w_3) \cdot (p, t, z_1, z_2) = (w_1 w_2 w_3 p, t, w_2^{-1} z_1, w_3 z_2)$$

The two-dimensional sub-torus $T_2 := 1 \times S^1 \times S^1$ exerts its own action on $M(N, e, \epsilon)$, with moment map being given by

$$H_{T_2}(p, t, z_1, z_2) = (t - |z_1|^2, t + |z_2|^2)$$

Note that a point in our product has non-trivial isotropy group in T_2 if and only if $z_1 = z_2 = 0$, so the action of T_2 is free on the level set $H_{T_2}^{-1}(u, v)$ where $u \neq v$. By taking the quotient

$$\tilde{M}(N, e, \epsilon)_{-\epsilon,\epsilon} := H_{T_2}^{-1}(-\epsilon, \epsilon)/T_2$$

with the reduced symplectic form $\tilde{\omega}_\sigma$, we yield a compact 6-dimensional (resp. 10-dimensional) simple Hamiltonian-S^1 manifold (cf. Section 2, [11]).

Cho noted that if one chooses (N, σ, J) to be Kähler and e to be of type $(1, 1)$, then even if the reduced symplectic form $\tilde{\omega}_\sigma$ yielded from the symplectic cut method fails to be Kähler, it will be no worse than deformationally equivalent to $\tilde{\omega}_{\sigma'}$, where σ' is another Kähler form on N. Moreover, two deformationally equivalent symplectic forms on N will lift to deformationally equivalent forms on $X := M(N, e, \epsilon)_{-\epsilon,\epsilon}$ (Lemma 2.3, [11]). We will construct a deformation of a Kähler form σ on N, lift it to a deformation on X, and finally, we will use our characterization theorem to identify a symplectic form in the latter deformation that fails the Hard Lefschetz property. This will establish the deformation equivalence between a Kähler form and a non-Hard Lefschetz form that we seek. Cho used this technique to prove his main theorem, and we shall be able to use the same general method, since - if one takes careful note - the proposition below (used crucially in his proof) is applicable outside the case of $\dim N = 4$.

Proposition ([27], Proposition 5.8 (i), p. 156]). *Let $I \subset \mathbf{R}$ be an interval and let (N, σ) be a four-dimensional compact symplectic manifold and $e \in H^2(N; \mathbf{Z})$ an integral cohomology class of N. Let $\{\sigma_t\}_I$ be a one-paratmetrized family of symplectic forms on N such that $[\sigma_t] - [\sigma_s] = (s-t)e$ for all $s, t \in I$. Let P be a principal bundle whose first Chern class is e. Then there exists an S^1-invariant symplectic form ω on the manifold $P \times [-1, 1]$ with a moment map equal to the projection $P \times [-1, 1] \to [-1, 1]$ and each reduced symplectic form is σ_t for all $t \in [-1, 1]$.*

Proof: Let $\phi_t(p) = p \cdot e^{2\pi i t}$ denote the obvious circle action on P and let $X : P \to TP$ denote the vector field which generates this action, namely $X(p) = p \cdot (2\pi i) \in T_p P$. Now let $\tau_\lambda \in \Omega^2(N)$ be a family of symplectic forms which satisfies $\tau_t - \tau_s = (s-t)e$. Then the closed form

$$\dot{\tau_\lambda} = \frac{d}{d\lambda}\tau_\lambda$$

represents the cohomology class c_I and hence there exists a connection 1-form on P whose curvature (multiplied by $i/2\pi$) is the pull-back $\pi^* \dot{\tau_\lambda}$. More explicitly, this means that there exists an invariant 1-form $\alpha_\lambda \in \Omega^1(P)$ (a connection form multiplied by $1/2\pi i$) such that

$$\alpha_\lambda(X) = 1, \qquad d\alpha_\lambda = -\pi^*(\dot{\tau_\alpha}).$$

Now the formula

$$\omega = \pi^* \tau_\lambda + \alpha_\lambda \wedge d\lambda$$

defines an S^1-invariant symplectic form on $P \times I$. A moment's thought shows that the moment map of the circle action on $P \times I$ is the projection $P \times I \to I$. $\square$

It is instructive to note that this proposition, though stated for manifolds N of dimension 4, does not actually require this condition to be true. The same result holds for N of any dimension; in particular, it holds when $\dim N = 8$. What this proposition then implies is that if twe find a family of symplectic forms $\{\omega_t\}_{-1\leq t\leq 1}$ on N with $[\sigma_t] - [\sigma_s] = (s-t)e$ for all $s, t \in [-1, 1]$, then we may choose $\epsilon = 1$ such that our manifold $(\tilde{M}(N, e, 1+\delta)_{-1,1}, \tilde{\omega}_\sigma)$ is symplectic for sufficiently small $\delta > 0$. Therefore, if we find

- a compact symplectic 8-manifold (N, ω)
- an integral class $e \in H^2(N; \mathbf{Z})$, and
- a family of symplectic forms $\{\sigma_t\}_{-1\leq t\leq 1}$ on N satisfying $[\sigma_t] - [\sigma_s] = (s-t)e$ for all $s, t \in [-1, 1]$ such that at least one of HR_2, HR_4 for ω is degenerate,

then the corresponding manifold $\tilde{M}(N, e, 1+\delta)_{-1,1}, \tilde{\omega}_\sigma)$ would not satisfy the hard Lefschetz property.

Proof of Main Theorem

Proof. We will construct (X, ω, J) with $N := K3^{[4]}$, a Hilbert scheme of the $K3$ surface. By Beauville, this is a holomorphic symplectic manifold of dimension 8, and is, in particular, a simply connected compact Kähler manifold. Let ϕ be a Kähler form on N; after a rescaling, perhaps, we may assume that $e := [\phi] \in H^2(N; \mathbf{Z})$. Since N admits a hyperKähler structure, there exists a holomorphic symplectic form $\rho \in \Omega^{2,0}(N)$. Consider $\sigma = \rho + \bar{\rho}$, the real closed 2-form in $\Omega^{2,0}(N) \oplus \Omega^{0,2}(N)$. We note that $\sigma^4 = 6\rho^2 \wedge \bar{\rho}^2 > 0$ on N, so that σ is symplectic and real on N; indeed, we may normalize the Beauville-Bogomolov-Fujiki form q

so that the integral of $\rho^2 \wedge \bar{\rho}^2$ over N is 1. In this case, we have

$$cq(\alpha, \beta) = 2 \int_N \alpha \wedge \beta \wedge \rho\bar{\rho} - \frac{1}{2} \frac{\int_N \alpha\rho\bar{\rho}^2 \int_N \beta\bar{\rho}\rho^2}{\int_N \rho^2\bar{\rho}^2}$$

where c is a positive constant that depends on the topological type of N. Now, we will construct a family of symplectic forms $\{\sigma_t\}_{-\epsilon \leq t \leq \epsilon}$ for ϵ small enough on N. Let $\sigma_t = \sigma - t\phi$ for $-\epsilon \leq t \leq \epsilon$. Note that $[\sigma_t] - [\sigma_s] = (s-t)e$ for all s, t. We have

$$\sigma_t^4 = 6\rho^2 \wedge \bar{\rho}^2 + 12t^2\rho \wedge \bar{\rho} \wedge \phi^2 + t^4\phi^4$$

Integrating, we see that

$$\int_N \sigma_t^4 = 6 + 12t^2 q(\phi) + t^4 \int_N \phi^4$$

which is positive everywhere on N, and the σ_t are hence symplectic for $-\epsilon \leq t \leq \epsilon$. Indeed, this follows from the following decomposition, in which we take $a := e = [\phi]$:

Proposition 22, [31]: *Let (X, ρ) be a 2n-dimensional, irreducible symplectic variety with normed Beauville-Bogomolov form $q_X = q_{X,\rho}$, and an ample class $a \in H^2(X, \mathbf{R})$. Restricting q_X gives a real quadratic form $H^2(X, \mathbf{R}) \to \mathbf{R}$ and $H^2(X, \mathbf{R})$ splits into the following q_X-orthogonal subspaces of dimensions $3, b_2 - 3$:*

$$V_+ := \mathbf{R}\langle \rho + \bar{\rho}, i\rho - i\bar{\rho}, a \rangle$$
$$V_- := \mathbf{R}\left\langle d \in H^{1,1}(X) \cap H^2(X, \mathbf{R}) | q_X(d, a) = 0 \right\rangle$$

such that q_X is positive definite on V_+, negative definite on V_-.

To prove the existence of a non-Hard Lefschetz symplectic form on N, we intend to show that HR_4 is degenerate. To reiterate, the fourth Hodge Riemann form HR_2 for (X, ω) is nondegenerate if and only if the bilinear forms

$Q_{max^2}, Q_{max\cdot min}$ are nondegenerate on H^2_{max}. It is enough to show that $Q_{max\cdot min}$ degenerates on H^2_{max}. We use the decomposition above, with $a := e = [\phi]$. The product

$$[\omega_{max}]\phi^*[\omega_{min}] = [\sigma + \epsilon\phi][\sigma - \epsilon\phi] = [\rho^2 + 2\rho \wedge \bar{\rho} + \bar{\rho}^2 - \epsilon^2\phi^2]$$

when paired with the element $[\rho + \bar{\rho}]$ is

$$[\omega_{max}]\phi^*[\omega_{min}][\rho + \bar{\rho}] = [3\rho^2 \wedge \bar{\rho} + 3\rho \wedge \bar{\rho}^2 - \epsilon^2\phi^2(\rho + \bar{\rho})]$$

By type considerations, this vanishes upon cupping with a as well as with any element $d \in V_-$. Clearly, it also vanishes when cupped with $[i\rho - i\bar{\rho}]$, and scaling a by $\sqrt{6/2\epsilon^2 q(a)}$ if necessary, we see that its product with $[\rho + \bar{\rho}]$ is zero as well. therefore, $Q_{max\cdot min}$ is degenerate on H^2_{max}, and HR_4 is degenerate as well. That is, there exists a non-HLP reduced symplectic form in our deformation of the (Kähler up to deformation) reduced symplectic form ω_σ obtained from Kähler σ.

Finally, we must prove the first and second statements of the theorem. Since X is a sphere bundle over the simply-connected N, the simple-connectedness of X follows easily from examining a long exact sequence of homotopy groups:

$$\cdots \to \pi_2(N) \to \pi_1(S^2) \to \pi_1(X) \to \pi_1(N) \to \pi_0(S^2) \to \cdots$$

It is known that the first Betti number b_1 of compact irreducible hyperkähler varieties is always 0. Furthermore, it is known from [20] that $b_2(N) = 23$, and that this implies $b_3 = 0$. Using the Leray-Serre spectral sequence, we can express the cohomology of X as a module over $H^*(N)$:

$$H^*(X) = \bigoplus H^*(S^2) \otimes H^*(N)$$

From this, using the Poincare polynomial, we compute

$$b_1(X) = b_1(N) = 0$$

$$b_3(X) = b_1(N) + b_3(N) = 0$$

$$b_5(X) = b_3(N) + b_5(N) = 2b_3(N) = 0$$

$$b_7(X) = b_5(N) + b_7(N) = b_3(N) + b_1(N) = 0$$

$$b_9(X) = b_7(N) = 0$$

Here, we have used the Poincare duality theorem, due to the compactness of N. Thus, all odd Betti numbers of X are indeed 0. This completes the proof. $\square$

Bibliography

[1] D. Angella. On the Bott-Chern and Aeppli Cohomology eprint: arXiv:1507.07112v2 [math.CV]. August 18, 2015.

[2] L. Alessandrini, and G. Bassanelli. Small deformations of a class of compact non-Kähler manifolds In volume 4 of *Proceedings of the American Mathematical Society*, pages 1059–1062. 1990.

[3] D. Angella, and A. Tomassini. On the Cohomology of Almost-Complex Manifolds eprint: arXiv:1211.6276v1 [math.DG]. November 27, 2012.

[4] M. Audin. Torus actions on symplectic manifolds (Second revised edition) In volume 93 of *Progress in Mathematics*. Birkhäuser, Basel, 2004.

[5] C. Benson, and C.S. Gordon. Kähler and Symplectic Structures on Nilmanifolds, In volume 27, No. 4, of *Topology*, pages 513–518, 1998.

[6] M. Britze. On the Cohomology of Generalized Kummer Varieties : eprint: http://kups.ub.uni-koeln.de/1049/

[7] J.-L. Brylinski. A differential complex for Poisson manifolds : In volume 28, number 1, of *Journal of Differential Geometry*, pages 93–114, 1988.

[8] I. K. Babenko, and I. A. Taimanov. On nonformal simply connected symplectic manifolds In volume 41 of *Siberian Mathematical Journal*, pages 204–217. 2000.

[9] G. R. Cavalcanti. The Lefschetz Property, Formality, and Blowing Up in Symplectic Geometry In volume 359, No. 1, of *Transactions of the American Mathematical Society*, pages 333–348. January 2007.

[10] L. Cordero, M. Fernández, and A. Gray. Symplectic manifolds with no Kähler structure In volume 25 of *Topology*, pages 375–380. 1986.

[11] Y. Cho. Hard Lefschetz Property of Symplectic Structures on Compact Kähler Manifolds In *Transactions of the American Mathematical Society*. May 6, 2016.

[12] T. Draghici. The Kähler Cone Versus the Symplectic Cone In volume 42 of *Bull. Math. Soc. Sc. Math. Roumanie*, No. 1. 1999.

[13] T. Draghici, T.-J. Li, and W. Zhang. On the J-Anti-Invariant Cohomology of Almost Complex 4-Manifolds eprint: arXiv:1104.2511v1 [math.SG]. April 13, 2011.

[14] M. Fernández, M. Gotay, and A. Gray. Symplectic manifolds with no Kähler structure volume 25 of *Topology*, pages 375–380. 1986.

[15] M. Fernández, V. Muñoz, and L. Ugarte. Weakly Lefschetz symplectic manifolds eprint: arXiv:math/0404479v2 [math.SG]. February 8, 2005.

[16] A. Fino, and A. Tomassini. Blow-ups and cohomology of almost complex manifolds In volume 30 of *Differential Geometry and its Applications*, pages 520–529. 2012.

[17] D. Guan. Toward a Classification of Compact Nilmanifolds with Symplectic Structures In *International Mathematics Research Notices*. March 2010.

[18] R. Gompf. A new construction of symplectic manifolds In volume 142 of *Annals of Mathematics*, pages 527–595. 1995.

[19] V. Guillemin, and S. Sternberg. Birational equivalence in the symplectic category In volume 97 of *Inventiones mathematicae*, pages 485–522. 1989.

[20] D. Guan. On the Betti Numbers of Irreducible Compact Hyperkähler Manifolds of Complex Dimension In *Mathematical Research Letters*. February 2001.

[21] R. Ibáñez, Y. Rudyak, A. Tralle, and L. Ugarte. On Symplectically Harmonic Forms on Six-dimensional Nilmanifolds eprint: arXiv:math/0002053v1 [math.SG]. February 7, 2000.

[22] R. Ibáñez, Y. Rudyak, A. Tralle, and L. Ugarte. Symplectically Harmonic Cohomology of Nilmanifolds eprint: arXiv:math/0111074v1 [math.SG]. November 7, 2011.

[23] F. C. Kirwan. Cohomology of quotients in symplectic and algebraic geometry In volume 31 of *Mathematical Notes*, Princeton University Press, Princeton, NJ. 1984

[24] A. Malcev. On a class of homogeneous spaces In volume 13 of *Izv. Akad. Nauk. Armyan. SSSR Ser. Mat.*, pages 201–212. 1949.

[25] O. Mathieu. Harmonic cohomology classes of symplectic manifolds In volume 70 of *Comment. Math. Helv.*, pages 1–9. 1990.

[26] D. McDuff. Examples of Simply-Connected Symplectic Non-Kählerian Manifolds In volume 20 of *Journal of Differential Geometry*, pages 267–277. 1984.

[27] D. McDuff, and S. Salamon. Introduction to symplectic topology *Oxford Mathematical Monographs*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York. 1995. MR1373431.

[28] D. McDuff, and S. Tolman. Topological properties of Hamiltonian circle actions In *IMRP Int. Math. Res. Pap.*, 72826, pages 1–77. MR2210662. 2006.

[29] Y. Rudyak, and A. Tralle. On Thom spaces, Massey products and non-formal symplectic manifolds In volume 10 of *International Mathematical Research Notices*, pages 495-513. 2000.

[30] S. Salamon. Complex structures on nilpotent Lie algebras In volume 157 of *Journal of Pure and Applied Algebra*, pages 311-333. 2001.

[31] M. Schwald. Fujiki Relations and Fibrations of Irreducible Symplectic Varieties eprint: arXiv:1701.09069v1 [math.AG]. Jan 31, 2017.

[32] L.-S. Tseng, and S.-T. Yau. Cohomology and Hodge Theory on Symplectic Manifolds In volume 91 of *Journal of Differential Geometry*, pages 383-416. 2012.

[33] C.-J. Tsai, L.-S. Tseng, and S.-T. Yau. Cohomology and Hodge Theory on Symplectic Manifolds: III eprint: arXiv:1402.0427v2 [math.SG]. May 2, 2014.

[34] C. Voisin Hodge structures on cohomology algebras and geometry In volume 341 of *Mathematische Annalen*, pages 39-69. 2008.

[35] A. Weil *Introduction à l'Étude des Variété Kählériennes* Publications de l'Institut de Mathematique de l'Université de Nancago VI, Hermann, Paris, 1958. MR 0111056, Zbl 0205.51701.

[36] D. Yan Hodge structure on symplectic manifolds In volume 120 of *Advances in Mathematics*, pages 143-154. 1996.

[37] T. Yamada Examples of Compact Lefschetz Solvmanifolds In volume 25 of *Tokyo Journal of Mathematics*, 2002.