Multiseparability and Superintegrability for Classical and Quantum Systems

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Abstract It has long been known that there are potentials on n-dimensional constant curvature spaces for which a given Hamiltonian system in classical mechanics, and Schrödinger equation in quantum mechanics, admits solutions via separation of variables in more than one coordinate system. Smorodinsky, Winternitz et.al., initiated the methodical search for such potentials in two and three dimensions, and there has been a considerable amount of work for various examples. Such a system is called *maximal* in dimension n if there exist $2n-1$ functionally independent integrals of motion. In some papers, these systems are called *superintegrable*.

In the first part of this paper we outline the basic ideas relating to the notion of superintegrable potentials. The energy observable is degenerate for potentials of this type and the corresponding intergrals of motion that arise from the simultaneous separability close quadratically under repeated commutation. We give examples of these systems and indicate how superintegrability can be of use, particularly in relation to bound states. Virtually all of the special functions of mathematical physics (in one and several variables) arise in this study and formulas expanding one type of special function as a series in another type emerge as a byproduct.

Finally, we describe how one can, in principle, classify all such systems and deduce the structure of the quadratic algebra. Many of the results reported

here were obtained in collaboration with E.G. Kalnins and G.S. Pogosyan.

1 Introduction

It has long been known that Schrödinger's equation with certain special potentials can admit (multiplicative) separation of variables in more than one coordinate system. This is intimately related to the notion of superintegrability, [1, 2, 3]. This subject has been studied by a number of authors, based on the use of the corresponding differential equations that that are implied by the requirement of simultaneous separability, [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Specifically, superintegrability means that for a Schrödinger equation in dimension n there exist $2n-1$ functionally independent quantum mechanical second-order observables (i.e., second-order self-adjoint operators that commute with the Hamiltonian). There is an analogous concept of superintegrability for classical mechanical systems. This relates to the corresponding additive separation of variables of the Hamilton-Jacobi equation. Furthermore, one observes that if we do have simultaneous separability then the resulting constants of the motion close quadratically under repeated application of the Poisson bracket, [11]. We also know that for spaces of constant curvature separable coordinate systems of the free motion are described by quadratic elements of the corresponding first order symmetries, [16, 17, 18].

Although concrete examples of superintegrable systems are easily at hand, a complete classification of all such systems has presented major difficulties. How can one be sure that all systems for free motion have been found? (For example, Rañada's classification [15] omits our system 5 below.) Once these are determined, how can one be sure that the most general additive potential term has been calculated?

Here we will present the background information to understand the problem, and its importance, and present a new approach to its solution, with details for two dimensional complex Euclidean space.

Consider an n-dimensional Riemannian manifold R_n. (In most of the following we will assume that R_n is a space of constant curvature, for that is the case where the most interesting and rich applications arise.) In local coordinates $q_1, \cdots, q_n$ the contravariant metric tensor is $\left(g^{jk}(\mathbf{q})\right)$. Let $V(\mathbf{q})$ be a potential function on R_n. The corresponding Hamilton-Jacobi equation is

$$\mathcal{H}(\mathbf{q},\mathbf{p}) = E \tag{1}$$

where

$$\mathcal{H}(\mathbf{q},\mathbf{p}) = \sum_{j,k=1}^{n} g^{jk}(\mathbf{q})p_j p_k + V(\mathbf{q}) = \sum_{j,k=1}^{n} g^{jk}(\mathbf{q})\frac{\partial S}{\partial q_j}\frac{\partial S}{\partial q_k} + V(\mathbf{q}), \tag{2}$$

and $S(\mathbf{q})$ is the action function, [19].

The quantum analog of this classical system is given by the Schrödinger equation

$$H\Psi(\mathbf{q}) = E\Psi(\mathbf{q}) \tag{3}$$

where in local coordinates

$$H = \Delta_n + V(\mathbf{q}), \qquad \Delta_n = \frac{1}{\sqrt{g}}\sum_{j,k=1}^{n}\frac{\partial}{\partial q_j}\cdot(\sqrt{g}g^{jk})\frac{\partial}{\partial q_k} \tag{4}$$

and $g = \det(g^{jk})^{-1}$.

Recall that a *complete integral* $S(\mathbf{q},\lambda_1,\cdots,\lambda_n)$ of the Hamilton-Jacobi equation solves the associated classical mechanical system [19, 20]. (A complete integral is a solution of (1) such that locally

$$\det\left(\frac{\partial^2 S}{\partial q_j \partial \lambda_k}\right) \neq 0.$$

Any solution of the Hamilton-Jacobi equation via (additive) separation of variables

$$S(\mathbf{q},\lambda_1,\cdots,\lambda_n) = \sum_{j=1}^{n} S^{(j)}(q_j,\lambda)$$

where $\lambda_1 = E, \lambda_2, \cdots, \lambda_n$ are the separation constants, yields a complete integral.

Similarly, in the quantum case, if the Schrödinger equation $H\Psi = E\Psi$ (multiplicatively) separates in the coordinates $\mathbf{q}$ then we can write

$$\Psi(\mathbf{q}) = \Pi_{j=1}^{n}\Psi^{(j)}(q_j,\lambda) \tag{5}$$

and this ansatz allows the decomposition of (3) into n ordinary differential equations, one for each of the factors $\Psi^{(j)}$. Many of the special functions of mathematical physics occur as solutions of these ordinary differential equations.

For orthogonal coordnates $\mathbf{q}$ on an n-dimensional constant curvature space (i.e., such that $g^{jk} = 0$ for $j \neq k$) one can show that the Hamilton-Jacobi equation is additively separable if and only if the Shrödinger equation is multiplicatively separable. (See [21] for a discussion of the relationship for general Riemannian manifolds.) We shall see that superintegrability is closely linked to symmetry properties of (1) and (3), and to separation of variables (special function) solutions of (3).

At this point it is useful to summarize briefly the history of the symmetry/special function approach to solving the Schrödinger equation (3). (Superintegrability is just one of the latest chapters.) Beginning with the introduction of the Schrödinger equation in the 1930's, and continuing until the 1960's, the main emphasis was on the study of Lie symmetry groups of unitary operators that commuted with the Hamiltonian H, hence mapped solutions of (3) to solutions. At the Lie algebra level, one looked for algebras of first-order differential operators L that commuted with H: $[L, H] \equiv LH - HL = 0$. (Again L maps solutions Ψ of (3) to solutions.) This led to studies of rotationally invariant potentials and the theory of spherical harmonics [22].

A related concept was that of *dynamical symmetry groups* or Lie algebras. The idea was to imbed H as an element of a Lie algebra of first and second-order differential operators. The representation theory of the Lie algebra could then be used to derive information about the eigenvalues and eigenvectors of H. The harmonic oscillator and the Morse potential were treated in this way. The so-called factorization method for solving the Schrödinger equation is related to this approach [23]. Among the special functions that arise and whose properties can be studied from this connection are Bessel functions and (more generally) hypergeometric functions. (More recently, q-analogs of the dynamical symmetry algebra approach have led to q-hypergeometric functions [24].)

Since the 1960's we have been in the "Cheshire Cat" era, [25]. In the most recent theories relating integrabilty, superintegrability and variable separation, the Lie groups and algebras have disappeared, but their *grin* persists, $\smile$. The focus here is on second order constants of the motion (symmetry operators that are built out of products of first order Lie symmetries for the zero potential problem) and their connection with variable separation for both the Hamilton-Jacobi and Schrödinger equations. One of the seminal papers in this regard was by Smorodinsky et al. [5]. Some other contributions are, for example, by Shapovalov, Kalnins and Miller, and Winternitz [26, 27, 28].These approaches also exploit the maximum symmetry of the physical system, but no longer in terms of Lie algebras of operators.

To examine these ideas, let us start with a classical system

$$\mathcal{H} = \sum g^{jk} p_j p_k + V(\mathbf{q}) \tag{6}$$

where the p_j are the momenta conjugate to the coordintes q_j. Recall that the *Poisson bracket* of two functions $f_h(\mathbf{q}, \mathbf{p})$, $h = 1, 2$ is the function

$$\{f_1, f_2\}(\mathbf{q}, \mathbf{p}) = \sum_{j=1}^{n} (\frac{\partial f_1}{\partial q_j}\frac{\partial f_2}{\partial p_j} - \frac{\partial f_1}{\partial p_j}\frac{\partial f_2}{\partial q_j}), \tag{7}$$

[20]. A *second-order constant of the motion* for (6) is a function

$$\mathcal{L} = \sum a^{jk}(\mathbf{q}) p_j p_k + W(\mathbf{q}), \qquad a^{jk} = a^{kj}, \tag{8}$$

such that $\{\mathcal{L}, \mathcal{H}\} = 0$.

Note that the null space of the map

$$T : \quad df(\mathbf{q}, \mathbf{p}) \to \{f, \mathcal{H}\}(\mathbf{q}, \mathbf{p})$$

is $2n-1$ dimensional. Thus (locally) there are $2n-1$ functionally independent constants of the motion (but not necessarily second-order). For the purposes of this paper we adopt the following definitions. We say that the classical system $\mathcal{H} = E$ is *superintegrable* or *maximal* if there are $2n-1$ functionally independent second-order constants of the motion:

$$\begin{aligned} \mathcal{L}_\ell &= \sum a^{jk}(\mathbf{q}) p_j p_k + W_\ell(\mathbf{q}) \\ \mathcal{L}_0 &= \mathcal{H}, \qquad \ell = 0, 1, \cdots, 2n-2 \\ \{\mathcal{L}_\ell, \mathcal{H}\} &= 0. \end{aligned} \tag{9}$$

We say that the quantum system $H\Psi = E\Psi$ is *superintegrable* or *maximal* if there are $2n-1$ linearly independent second-order symmetry operators:

$$\begin{aligned} L_\ell &= \sum \frac{1}{\sqrt{g}} \partial_{q_j} (\sqrt{g} a^{jk}(\mathbf{q}) \partial_{q_k} + W_\ell(\mathbf{q}) \\ L_0 &= H, \qquad \ell = 0, 1, \cdots, 2n-2 \\ [L_\ell, H] &\equiv L_\ell H - H L_\ell = 0. \end{aligned} \tag{10}$$

To clarify the connection between these ideas and variable separation we assume that the coordinates $\mathbf{q}$ are orthogonal, i.e., the covariant metric tensor is diagonal:

$$ds^2 = \sum g_{jk}\ dq_j dq_k = \sum H_j^2(\mathbf{q})\ dq_j^2,$$

so that the Hamilton-Jacobi equation is given by

$$\mathcal{H} = \sum_j H_j^{-2}(\frac{\partial S}{\partial q_j})^2 + V(\mathbf{q}) = E \tag{11}$$

Set $\frac{\partial S}{\partial q_j} = S_j = p_j$ and assume additive separation in the $\mathbf{q}$ coordinates, so that $\partial_j S_i = \partial_j \partial_i S = 0$ for $i \neq j$. The separation equations are postulated to be

$$S_i^2 - \sum_{j=1}^n u_{ij}(q_i)\lambda^j + f_i(q_i) = 0, \qquad i = 1, \cdots, n, \quad \lambda^1 = E. \tag{12}$$

Here $\partial_k u_{ij}(q_i) = 0$ for $k \neq i$ and $\det(u_{ij}) \neq 0$. We say that $U = (u_{ij})$ is a *Stäckel matrix*.

Then (11) can be recovered from (12) provided $H_j^{-2} = (U^{-1})^{1j}$. The quadratic forms

$$\mathcal{L}_\ell = \sum_{j=1}^n (U^{-1})^{\ell j}(p_j^2 + f_j(q_j)) = \sum_{j=1}^n (U^{-1})^{\ell j} p_j^2 + W_\ell(\mathbf{q})$$

satisfy

$$\mathcal{L}_\ell = -\lambda_\ell, \qquad \mathcal{H} = \mathcal{L}_1 = \sum H_j^2 p_j^2 + V(\mathbf{q})$$

for a separable solution. Furthermore, we have

$$\{\mathcal{L}_\ell, \mathcal{L}_j\} = 0, \qquad \ell \neq j$$

Thus the $\mathcal{L}_\ell$, $\quad 2 \leq \ell \leq n$, are *constants of the motion* for the *Hamiltonian* $\mathcal{L}_1$.

An analogous construction, replacing (12) by n second-order linear ODE's for factors $\Psi^{(i)}(q_i)$ leads to second order linear partial differential operators $L_1 = H, L_2, \cdots, L_n$ such that

$$H\Psi = E\Psi, \qquad L_\ell \Psi = \lambda_\ell \Psi, \qquad \ell = 2, \cdots, n \tag{13}$$

and

$$\Psi(\mathbf{q}) = \Pi_{k=1}^{n}\Psi^{(i)}(q_i).$$

Then one can verify that

$$[L_k, H] = 0, \qquad [L_k, L_j] = 0.$$

How does one find all orthogonal separable coordinate systems $\mathbf{q}$ for a given space R_n for zero potential, $V \equiv 0$? This is a difficult problem in differential geometry. The answer is known for some constant curvature spaces. In real Euclidean 2-space there are four separable systems: cartesian, polar, parabolic and elliptic. For complex Euclidean 2-space, including real Euclidean space and real Minkowski space, there are six [4, 16, 18]: Cartesian, polar, parabolic, elliptic, hyperbolic and semi-hyperbolic. We describe these coordinate systems and their corresponding free particle constants of the motion L. (We adopt the basis $p_x, p_y, M = xp_y - yp_x$ for the Lie algebra $e(2, C)$ and define $p_\pm = p_x \pm ip_y$, $\smile$.) There is one orbit of constants of the motion, with representative Mp_+, that is not associated with variable separation [21]. The separable systems are:

Cartesian coordinates

$$x, y, \qquad L = p_x^2 \tag{14}$$

Polar Coordinates

$$\begin{array}{ll} x = r\cos\theta \ , & \\ y = r\sin\theta \ , & \qquad L = M^2 \end{array} \tag{15}$$

Parabolic Coordinates.

$$x_P = \frac{1}{2}(\xi^2 - \eta^2) \ , \qquad y_P = \xi\eta \ , \qquad L = Mp_y \tag{16}$$

Elliptic Coordinates (in algebraic form)

$$\begin{aligned} x_E^2 &= c^2(u-1)(v-1), \quad y_E^2 = -c^2uv, \\ L &= M^2 + c^2p_x^2 \end{aligned} \tag{17}$$

Hyperbolic Coordinates

$$\begin{aligned} x_H &= \frac{r^2 + r^2s^2 + s^2}{2rs}, \quad y_H = i\frac{r^2 - r^2s^2 + s^2}{2rs}, \\ L &= M^2 + p_+^2 \end{aligned} \tag{18}$$

Semi-Hyperbolic Coordinates

$$
\begin{aligned}
x_{SH} &= -\frac{1}{4}(w-u)^2 + \frac{1}{2}(w+u), \quad iy_{SH} = -\frac{1}{4}(w-u)^2 - \frac{1}{2}(w+u) \qquad (19) \\
L &= 2Mp_+ + p_-^2
\end{aligned}
$$

In real Euclidean 3-space there are 11 separable systems see Table 1, [4, 16]. On the real 2-sphere there are 2: spherical and ellipsoidal. For real n-dimensional Euclidean space and the n-sphere Kalnins and the author have a graphical procedure to classify and construct all possibilities, [17, 18]. On the 2-hyperboloid there are 9 separable systems, [18]. For the n-hyperboloid of two sheets there is again a graphical procedure to construct all possibilites, [18].

In each case above, the symmetries $\mathcal{L}_j$ are second order elements in the enveloping algebra of the symmetry Lie algebra of the corresponding manifold, e.g., the Lie algebra $e(n, C)$ for Euclidean n-space and $so(n+1, C)$ for the n-sphere, $\smile$.

We see that for zero potential, each of the constant curvature spaces listed above is separable in multiple coordinate systems. Indeed we can veryfy that the zero potential is superintegrable on each of these spaces. However, a potential $V \neq 0$ "breaks the symmetry" and reduces the number of separable systems, usually to zero. (See [21] for conditions that must be satisfied by a potential in order to permit separation in a given coordinate system.)

How does one determine which constants of the motion lead to variable separation?

Theorem 1 *Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system $\{q_i\}$ for the Hamilton-Jacobi equation $\mathcal{H}^1 = E$ on an n-dimensional Riemannian manifold are that there exist n quadratic forms $\mathcal{L}_k = \sum_{i,j=1}^n L_{ij}^{(k)} p_i p_j + W_k$ on the manifold such that:*

1. $\{\mathcal{L}_k, \mathcal{L}_\ell\} = 0, \quad 1 \leq k, i \leq n,$
2. *The set $\{\mathcal{L}_k\}$ is linearly independent (as n quadratic forms).*
3. *There is a basis $\{\omega_{(j)} : 1 \leq j \leq n\}$ of simultaneous eigenforms for the n matrices $\{L_{ij}^{(k)}\}$.*

Table 1 Separable coordinates	in 3-D real Euclidean space.
Coordinate System	Coordinates
I. Cartesian $x, y, z \in \mathbf{R}$	x, y, z
II. Cylindrical polar $\rho > 0,\ \varphi \in [0, 2\pi)$	$x = \rho\cos\varphi,\ \ y = \rho\sin\varphi,\ \ z$
III. Cylindrical elliptic $z \in \mathbf{R},\ e_1 < \mu_1 < e_2 < \mu_2$	$x^2 = \frac{(\mu_1-e_1)(\mu_2-e_1)}{(e_2-e_1)}, \quad y^2 = \frac{(\mu_1-e_2)(\mu_2-e_2)}{(e_1-e_2)}, \quad z$
IV. Cylindrical parabolic $\xi, x \in \mathbf{R},\ \eta \geq 0$	$x,\ \ y = \xi\eta, \quad z = \frac{1}{2}(\xi^2 - \eta^2)$
V. Spherical $r > 0,\ \theta \in [0, \pi], \quad \varphi \in [0, 2\pi)$	$x = r\cos\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$
VI. Prolate spheroidal $e_1 < u_1 < e_2 < u_2$, $\varphi \in [0, 2\pi)$	$x^2 = \frac{(u_1-e_2)(u_2-e_2)}{(e_1-e_2)}\cos^2\varphi, \quad y^2 = \frac{(u_1-e_2)(u_2-e_2)}{(e_1-e_2)}\sin^2\varphi,$ $z^2 = \frac{(u_1-e_1)(u_2-e_1)}{(e_2-e_1)}$
VII. Oblate spheroidal $e_1 < u_1 < e_2 < u_2$, $\varphi \in [0, 2\pi)$	$x^2 = \frac{(u_1-e_1)(u_2-e_1)}{(e_2-e_1)}\cos^2\varphi, \quad y^2 = \frac{(u_1-e_1)(u_2-e_1)}{(e_2-e_1)}\sin^2\varphi$ $z^2 = \frac{(u_1-e_2)(u_2-e_2)}{(e_1-e_2)}$
VIII. Sphero-conical $r \geq 0$ $e_1 < \rho_1 < e_2 < \rho_2 < e_3$	$x^2 = r^2\frac{(\rho_1-e_1)(\rho_2-e_1)}{(e_1-e_2)(e_1-e_3)}, \quad y^2 = r^2\frac{(\rho_1-e_2)(\rho_2-e_2)}{(e_2-e_1)(e_2-e_3)}$ $z^2 = r^2\frac{(\rho_1-e_3)(\rho_2-e_3)}{(e_3-e_2)(e_3-e_1)}$
IX. Parabolic $\xi, \eta \geq 0, \quad \varphi \in [0, 2\pi)$	$x = \xi\eta\cos\varphi, \quad y = \xi\eta\sin\varphi, \quad z = \frac{1}{2}(\xi^2 - \eta^2)$
X. Ellipsoidal $a_1 < u_1 < a_2 < u_2 < a_3 < u_3$	$x^2 = \frac{(u_1-a_1)(u_2-a_1)(u_3-a_1)}{(a_3-a_1)(a_2-a_1)}, \quad y^2 = \frac{(u_1-a_2)(u_2-a_2)(u_3-a_2)}{(a_1-a_2)(a_3-a_2)}$ $z^2 = \frac{(u_1-a_3)(u_2-a_3)(u_3-a_3)}{(a_1-a_3)(a_2-a_3)}$
XI. Paraboloidal $0 < \eta_1 < a_2 < \eta_2 < a_3 < \eta_3$	$x^2 = \frac{(\eta_1-a_3)(\eta_2-a_3)(\eta_3-a_3)}{(a_3-a_2)}, \quad y^2 = \frac{(\eta_1-a_2)(\eta_2-a_2)(\eta_3-a_2)}{(a_2-a_3)}$ $z^2 = \frac{1}{2}(\eta_1 + \eta_2 + \eta_3 - a_2 - a_3)$

If conditions (1)-(3) are satisfied then there exist functions $g_i(\mathbf{q})$ such that:

$$\omega_{(j)} = g_j dq_j, \quad j = 1, \cdots, n.$$

Theorem 2 *Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system $\{q_i\}$ for the Schrödinger equation $(\Delta_n + V)\Psi = E\Psi$ on an n-dimensional constant curvature space are that there exists a linearly independent set $\{L_1 = H = \Delta_n + V, L_2, \cdots, L_n\}$ of second-order differential operators on the manifold such that:*

1. $[L_k, L_\ell] = 0, \quad 1 \leq k, \ell \leq n,$
2. *Each L_k is in self-adjoint form,*
3. *There is a basis $\{\omega_{(j)} : 1 \leq j \leq n\}$ of simultaneous eigenforms for the $\{L_k\}$.*

If conditions (1)-(3) are satisfied then there exist functions $g_i(\mathbf{q})$ such that:

$$\omega_{(j)} = g_j dq_j, \quad j = 1, \cdots, n.$$

See [28, 21] for proofs and discussions of these theorems. The main point of the theorems is that, under the required hypotheses the eigenforms ω^ℓ of the quadratic forms L^{ij} are normalizable, i.e., that up to multiplication by a nonzero function, ω^ℓ is the differential of a coordinate. This fact permits us to compute the coordinates directly from a knowledge of the symmetry operators. For general Riemannian manifolds Theorem 1 remains true, but Theorem 2 is false unless separation is replaced by the more general concept of R-separation [21].

We expect a superintegrable system to separate in multiple coordinate systems, though the above remarks do not constitute a proof of this. Thus, one way to find superintegrable systems is to search for potentials $V(\mathbf{q})$ that permit separation in multiple coordinate systems.

2 Examples for the Euclidean plane

To illustrate the basic ideas we can consider the example of the Schrödinger equation with potential

$$V(x,y)=\frac{1}{2}\left(\omega^2(x^2+y^2)+\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right),$$

i.e.,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi-\left(\omega^2(x^2+y^2)+\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right)\Psi=-2E\Psi.$$

This equation separates in three coordinate systems: *Cartesian* coordinates (x,y); *polar* coordinates $x=r\cos\theta$, $y=r\sin\theta$, and *elliptical* coordinates

$$x^2=c^2\frac{(u_1-e_1)(u_2-e_1)}{(e_1-e_2)},\qquad y^2=c^2\frac{(u_1-e_2)(u_2-e_2)}{(e_2-e_1)}.$$

The bound state energies are given by

$$E_n=\omega(2n+2+k_1+k_2)$$

for integer n. The wave functions for each of these coordinate systems are:
1. *Cartesian* coordinates

$$\Psi_{n_1,n_2}(x,y)=2\omega^{\frac{1}{2}(k_1+k_2+2)}\sqrt{\frac{n_1!n_2!}{\Gamma(n_1+k_1+1)\Gamma(n_2+k_2+1)}}\,x^{(k_1+\frac{1}{2})}y^{(k_2+\frac{1}{2})}$$
$$e^{-\frac{\omega}{2}(x^2+y^2)}L_{n_1}^{k_1}(\omega x^2)L_{n_2}^{k_2}(\omega y^2)$$

where $n=n_1+n_2$, and the $L_n^k(x)$ are Laguerre polynomials, [29]
2. *polar* coordinates

$$\Psi(r,\theta)=\Phi_q^{(k_1,k_2)}(\theta)\omega^{\frac{1}{2}(2q+k_1+k_2+1)}\sqrt{\frac{2m!}{\Gamma(m+2q+k_1+k_2+1)}}$$
$$e^{(-\omega r^2/2)}r^{(2q+k_1+k_2+1)}L_m^{2q+k_1+k_2+1}(\omega r^2)$$

where $n=m+q$,

$$\Phi_q^{(k_1,k_2)}(\theta)=\sqrt{2(2q+k_1+k_2+1)\frac{q!\Gamma(k_1+k_2+q+1)}{\Gamma(k_2+q+1)\Gamma(k_1+q+1)}}$$

$$\times (\cos\theta)^{k_1+(1/2)}(\sin\theta)^{k_2+(1/2)}P_q^{(k_1,k_2)}(\cos 2\theta), \tag{20}$$

and the $P_q^{(k_1,k_2)}(\cos 2\theta)$ are Jacobi polynomials, [29].
3. *elliptical* coordinates

$$\Psi = e^{-\omega(x^2+y^2)}x^{k_1+\frac{1}{2}}y^{k_2+\frac{1}{2}}\prod_{m=1}^{n}\left(\frac{x^2}{\theta_m - e_1} + \frac{y^2}{\theta_m - e_2} - c^2\right)$$

where use has been made of the identity

$$\frac{x^2}{\theta - e_1} + \frac{y^2}{\theta - e_2} - c^2 = -c^2\frac{(u_1-\theta)(u_2-\theta)}{(\theta - e_1)(\theta - e_2)}.$$

The zeros θ_j satisfy the relations

$$\frac{k_1+1}{\theta_m - e_1} + \frac{k_2+1}{\theta_m - e_2} + \sum_{j\neq m}\frac{2}{(\theta_m - \theta_j)} - \omega = 0.$$

Associated with the separability of the Schrödinger equation in these coordinate systems there are second order symmetry operators. A basis for such operators is

$$L_1 = \partial_x^2 + \frac{(\frac{1}{4} - k_1^2)}{x^2} - \omega^2 x^2, \qquad L_2 = \partial_y^2 + \frac{(\frac{1}{4} - k_2^2)}{y^2} - \omega^2 y^2$$

$$M = (x\partial_y - y\partial_x)^2 + (\frac{1}{4} - k_1^2)\frac{y^2}{x^2} + (\frac{1}{4} - k_2^2)\frac{x^2}{y^2} - \frac{1}{2}.$$

(Note that $H = L_1 + L_2$.) The separable eigenfunctions already given are eigenfunctions of the symmetry operators L_1, M and $M + e_2L_1 + e_1L_2$ with eigenvalues

$$\lambda_c = -\omega(2n_1 + k_1 + 1), \qquad \lambda_p = (2q + k_1 + k_2 + 1)^2 + (1 + k_1^2 + k_2^2),$$

$$\lambda_e = 2(1-k_1)(1-k_2) - 2e_2\omega(k_1+1) - 2e_1\omega(k_2+1) - \omega^2 e_1 e_2 -$$
$$4\sum_{m=1}^{q}[e_2\frac{k_1+1}{\theta_m - e_1} + e_1\frac{k_2+1}{\theta_m - e_2}].$$

The algebra constructed by repeated commutators is (R is defined by the first relation)

$$[L_1, M] = [M, L_2] = R, \qquad [L_i, R] = -4\{L_i, L_j\} + 16\omega^2 M, \qquad i \neq j,$$

$$
\begin{aligned}
[M,R] &= 4\{L_1,M\} - 4\{L_2,M\} + 8(1-k_2^2)L_1 - 8(1-k_1^2)L_2,\\
R^2 &= \frac{8}{3}\{M,L_1,L_2\} + \frac{64}{3}\{L_1,L_2\} + 16\omega^2 M^2 - 16(1-k_2^2)L_1^2\\
&\quad -16(1-k_1^2)L_2^2 - \frac{128}{3}\omega^2 M - 64\omega^2(1-k_1^2)(1-k_2^2).
\end{aligned}
$$

These relations are quadratic.

In real Euclidean two-space there are precisely four potentials that have the multiseparation property, [14]. The second potential is

$$
V(x,y) = \omega^2(4x^2+y^2) - \frac{(\frac{1}{4}-k_2^2)}{y^2}.
$$

The corresponding Schrödinger equation is separable in two coordinate systems: *Cartesian* coordinates and *parabolic* coordinates

$$
x = \frac{1}{2}(\xi^2-\eta^2), y = \xi\eta.
$$

The third potential is

$$
V(x,y) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{1}{4\sqrt{x^2+y^2}}\left(\frac{(k_1^2-\frac{1}{4})}{\sqrt{x^2+y^2}+x} + \frac{(k_2^2-\frac{1}{4})}{\sqrt{x^2+y^2}-x}\right).
$$

The corresponding Schrödinger equation is separable in two coordinate systems: *polar*, *parabolic* and *modified elliptic* coordinates, where

$$
\xi^2 = c^2\frac{(u_1-e_1)(u_2-e_1)}{(e_1-e_2)}, \qquad \eta^2 = c^2\frac{(u_1-e_2)(u_2-e_2)}{(e_2-e_1)}.
$$

This last coordinate system can be written as

$$
x = \sqrt{\frac{(U_1-E_1)(U_2-E_1)}{4(E_1-E_2)}}, \qquad y = \sqrt{\frac{(U_1-E_2)(U_2-E_2)}{4(E_2-E_1)}} - 2\sqrt{E_1-E_2},
$$

where $E_1 = -e_1e_2$, $E_2 = -\frac{1}{4}(e_1+e_2)^2$ and $U_j = u_j^2 - u_j(e_1+e_2)$.

The fourth potential is

$$
V(x,y) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{B_1}{4}\frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}} + \frac{B_2}{4}\frac{\sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{x^2+y^2}}.
$$

Separation occurs here in *parabolic* and *parabolic coordinates of the second type*

$$x = \mu\nu, \ \ y = \frac{1}{2}(\mu^2 - \nu^2).$$

As an illustration of the utility of the notion of a quadratic algebra consider the last potential given. A basis for the quadratic algebra consists of L_1, L_2 and H with defining relations

$$[R, L_1] = -4L_2H + B_1B_2, \qquad [R, L_2] = 4L_1H + \frac{1}{2}(B_1^2 - B_2^2)$$

$$R^2 = 4L_1^2H + 4L_2^2H - 16\alpha^2H + (B_2^2 - B_1^2)L_1 - 2B_1B_2L_2 - 2\alpha^2(B_1^2 + B_2^2)$$

with $R = [L_1, L_2]$. If we look for eigenfunctions of the operators L_1, L_2 respectively, we have

$$L_1\varphi_m = \lambda_m\varphi_m, \qquad L_2\psi_n = \rho_n\psi_n.$$

If we write

$$L_1\psi_n = \sum_\tau C_{n\tau}\psi_\tau$$

then the quadratic algebra relations imply

$$[(\rho_n - \rho_\tau)^2 + 8E]C_{n\tau} = -[\frac{1}{2}(B_1^2 - B_2^2) - 16\alpha E]\delta_{n\tau}$$

$$\sum_\tau C_{n\tau}C_{\tau\sigma}(2\rho_\tau - \rho_n - \rho_\sigma) = (8E\rho_n + B_1B_2 + 16\alpha E)\delta_{n\sigma}.$$

These relations in turn imply that

$$C_{nn} = -\frac{\frac{1}{2}(B_1^2 - B_2^2) + 16\alpha E}{8E}$$

and $C_{nn+1} = C^*_{n+1n}$ are the only nonzero coefficents. Indeed they can essentially be determined by the relation

$$4\sqrt{-2E}\,(|C_{n,n+1}|^2 - |C_{n-1,n}|^2) = 8E\rho_n + B_1B_2 + 16\alpha E$$

where the eigenvalues λ_m and ρ_n are given by

$$\lambda_m = 2\alpha - \frac{B_1^2}{8E} - (2m+1)\sqrt{-2E}, \quad \rho_n = 2\alpha - \frac{(B_1 + B_2)^2}{16E} - (2n+1)\sqrt{-2E}$$

and the quantisation condition for E is

$$4\alpha - \frac{B_1^2 + B_2^2}{8E} = -(q+2)\sqrt{-2E}$$

for integer q.

3 More examples: The real 2-sphere

These ideas work also for separable coordinates on the real two dimensional sphere, [14]. An important example is
The potential

$$V = \frac{1}{2}\left[\frac{k_1^2-\frac{1}{4}}{s_1^2}+\frac{k_2^2-\frac{1}{4}}{s_2^2}+\frac{k_3^2-\frac{1}{4}}{s_3^2}\right]$$

where $s_1^2+s_2^2+s_3^2=1$. The corresponding Schrödinger equation has the form

$$\left[(s_1\frac{\partial}{\partial s_2}-s_2\frac{\partial}{\partial s_1})^2+(s_1\frac{\partial}{\partial s_3}-s_3\frac{\partial}{\partial s_1})^2+(s_3\frac{\partial}{\partial s_2}-s_2\frac{\partial}{\partial s_3})^2\right]\Psi+$$

$$\left[\frac{(k_1^2-\frac{1}{4})}{s_1^2}+\frac{(k_2^2-\frac{1}{4})}{s_2^2}+\frac{(k_3^2-\frac{1}{4})}{s_3^2}\right]\Psi=-2E\Psi.$$

This equation admits solution via separation of variables in two coordinate systems: *spherical* coordinates

$$s_1=\sin\theta\cos\varphi,\quad s_2=\sin\theta\sin\varphi,\quad s_3=\cos\theta$$

and *elliptical* coordinates

$$s_i^2=\frac{(u_1-e_i)(u_2-e_i)}{(e_1-e_j)(e_i-e_j)},\qquad i,j,k=1,2,3\qquad i\neq j\neq k.$$

Indeed a basis for the second-order symmetries of Schrödinger's equation with this potential is

$$L_{ij}=(s_i\partial_{s_j}-s_j\partial_{s_i})^2+(\frac{1}{4}-k_i^2)\frac{s_j^2}{s_i^2}+(\frac{1}{4}-k_j^2)\frac{s_i^2}{s_j^2}-\frac{1}{2},\quad i\neq j.$$

These symmetries satisfy the quadratic algebra relations

$$[L_{ij},R]=4\{L_{ij},L_{jk}\}-4\{L_{ij},L_{ik}\}+8(1-k_i^2)L_{jk}-8(1-k_j)^2L_{ik},$$

$$R^2=-\frac{4}{3}\{L_{ij},L_{ik},L_{jk}\}+\frac{64}{3}\{L_{ij},L_{ik}\}+\frac{64}{3}\{L_{ij},L_{jk}\}+\frac{64}{3}\{L_{ik},L_{jk}\}-$$
$$16(1-k_k^2)L_{ij}^2-16(1-k_j^2)L_{ik}^2-16(1-k_i^2)L_{jk}^2+$$

$$\frac{128}{3}(1-k_i^2)L_{jk}+\frac{128}{3}(1-k_j^2)L_{ik}+\frac{128}{3}(1-k_k^2)L_{ij}, \quad i\neq j\neq k.$$

The eigenfunctions with bound state energy eigenvalues

$$E_p=\frac{1}{2}(2p+2+k_1+k_2+k_3)^2-\frac{1}{8}$$

in these coordinate systems are: *polar*

$$\Psi=(\sin\theta)^{-1}\Phi_n^{(k_2,k_1)}(\varphi)\Phi_m^{(2n+k_1+k_2,k_3)}(\theta)$$

where $p=m+n$, (here $\Phi_n^{(k_2,k_1)}$ is given by (20)) and *elliptical*

$$\Psi=(\prod_{\ell=1}^{3}s_\ell^{(k_\ell+\frac{1}{2})})\prod_{j=1}^{q}\left(\frac{s_1^2}{\theta_j-e_1}+\frac{s_2^2}{\theta_j-e_2}+\frac{s_3^2}{\theta_j-e_3}\right)$$

where

$$\frac{k_1+1}{\theta_m-e_1}+\frac{k_2+1}{\theta_m-e_2}+\frac{k_3+1}{\theta_m-e_3}+\sum_{j\neq m}\frac{2}{(\theta_m-\theta_j)}=0$$

and $q=p$. Here we have made use of the identity

$$\frac{s_1^2}{\theta_j-e_1}+\frac{s_2^2}{\theta_j-e_2}+\frac{s_3^2}{\theta_j-e_3}=\frac{\Pi_{\ell=1}^2(u_\ell-\theta_j)}{\Pi_{m=1}^3(\theta_j-e_m)}.$$

The separable eigenfunctions already given are eigenfunctions of the symmetry operators L_{12} and $e_3L_{12}+e_2L_{13}+e_1L_{23}$:

$$\lambda_S=(2n+k_1+k_2+1)^2,$$

$$\lambda_E=-2[k_1(e_2+e_3)+k_2(e_1+e_3)+k_3(e_2+e_1)+e_3k_1k_2+e_1k_2k_3+e_2k_1k_3]-$$
$$\frac{3}{2}(e_1+e_2+e_3)-4e_2e_3(k_1+1)\sum_{m=1}^{q}\frac{1}{\theta_m-e_1}-$$
$$4e_2e_1(k_3+1)\sum_{m=1}^{q}\frac{1}{\theta_m-e_3}-4e_1e_3(k_2+1)\sum_{m=1}^{q}\frac{1}{\theta_m-e_2}.$$

A second multiseparable potential on the sphere is

$$V=-\frac{\alpha s_3}{\sqrt{s_1^2+s_2^2}}+\frac{1}{4\sqrt{s_1^2+s_2^2}}\left[\frac{(k_1^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}+s_1}+\frac{(k_2^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}-s_1}\right].$$

Schrödinger's equation for this potential has the form

$$\left[(s_1\frac{\partial}{\partial s_2}-s_2\frac{\partial}{\partial s_1})^2+(s_1\frac{\partial}{\partial s_3}-s_3\frac{\partial}{\partial s_1})^2+(s_3\frac{\partial}{\partial s_2}-s_2\frac{\partial}{\partial s_3})^2\right]\Psi+$$

$$\left(-\frac{\alpha s_3}{\sqrt{s_1^2+s_2^2}}+\frac{1}{4\sqrt{s_1^2+s_2^2}}\left[\frac{(k_1^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}+s_1}+\frac{(k_2^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}-s_1}\right]\right)\Psi=-2E\Psi.$$

This equation admits solution via separation of variables in two coordinates systems: *spherical* and *elliptical coordinates of modified type*

$$s_1'=\cos f\, s_1+\sin f\, s_3,\qquad s_2'=s_2,\qquad s_3'=-\sin f\ s_1+\cos f\ s_3$$

where

$$s_i^2=\frac{(y_1-e_i)(y_2-e_i)}{(e_1-e_j)(e_i-e_j)},\qquad i,j,k=1,2,3\quad i\neq j\neq k,$$

and

$$\sin f=\sqrt{\frac{(e_2-e_1)}{(e_3-e_1)}},\qquad e_1=e_2+\frac{1}{4}(E_++E_-)^2,\qquad e_3=e_2+\frac{1}{4}(E_+-E_-)^2,$$

$$y_j=e_2+\frac{1}{4}(E_+^2+E_-^2)+\frac{1}{4}E_+E_-(Z_j+\frac{1}{Z_j}),\quad j=1,2.$$

Indeed if we use the variables

$$U_3^2=Z_1Z_2,\quad U_1^2=\frac{(Z_1+\Omega_-)(Z_2+\Omega_-)}{(\Omega_-^2-1)},\quad U_2^2=\frac{(Z_1+\Omega_+)(Z_2+\Omega_+)}{(\Omega_+^2-1)}$$

where

$$\Omega_+=\frac{E_+}{E_-},\Omega_-=\frac{E_-}{E_+},$$

then, putting $k_3=\sqrt{2(E-i\alpha)+1/4}$ and $\hat{E}=i\alpha+E$ and multiplying the Schrödinger equation by $(Z_1Z_2)^{-1}-1$, we see that the resulting equation has the form

$$H\Psi=(U_1\frac{\partial}{\partial U_2}-U_2\frac{\partial}{\partial U_1})^2+(U_1\frac{\partial}{\partial U_3}-U_3\frac{\partial}{\partial U_1})^2+(U_3\frac{\partial}{\partial U_2}-U_2\frac{\partial}{\partial U_3})^2)\Psi+$$

$$\left[\frac{(\frac{1}{4}-k_1^2)}{U_1^2}+\frac{(\frac{1}{4}-k_2^2)}{U_2^2}+\frac{(\frac{1}{4}-k_3^2)}{U_3^2}\right]\Psi+2\hat{E}\Psi=0$$

which is essentially the same form as for the first potential. The bound state quantisation condition has the form

$$E=\frac{1}{2}\left[2q+2+k_1+k_2+\sqrt{\frac{1}{4}+2(E-i\alpha)}\right]^2-\frac{1}{8}.$$

4 Features of superintegrability (n=2)

Based on the examples of the last two sections, we can point out some basic features of superintegrability in two dimensions.

1. The potential V permits separability of the Hamilton-Jacobi equation $\mathcal{H}=E$ and the Schrödinger equation $H\Psi=E\Psi$ in at least two coordinate systems, characterized by symmetry conditions $\mathcal{L}_1=\lambda_1, \mathcal{L}_2=\lambda_2$ in the first case and $L_1\Psi=\lambda_1\Psi, L_2\Psi=\lambda_2\Psi$ in the second.

2. One can obtain alternate spectral resolutions $\{\Psi_j^{(1)}\}, \{\Psi_k^{(2)}\}$ for the multiply-degenerate eigenspaces of H,

$$L_1\Psi_j^{(1)}=\lambda_1^{(1)}\Psi_j^{(1)}, \qquad L_2\Psi_k^{(2)}=\lambda_2^{(2)}\Psi_k^{(2)}.$$

   These alternate resolutions resolve the degeneracy problem.

3. The interbasis expansions

$$\Psi_k^{(2)}=\sum_j a_{jk}\Psi_j^{(1)}$$

   yield important special function identities. In many cases, these become expansions of one set of multivariable orthogonal polynomials in terms of another set.

4. The operators H, L_1, L_2 generate a quadratic algebra. Indeed, with $R=[L_1,L_2]$ we have that R^2 is a polynomial of order 3 in H, L_1, L_2, whereas $[L_1,R]$ and $[L_2,R]$ are polynomials of order 2 in H, L_1, L_2. A corresponding statement is true for algebra generated by the symmetries $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ under the Poisson bracket. (Note: This is a remarkable

property, and is false for general symmetries. Consider Euclidean 2-space with Hamiltonian $\mathcal{H} = p_x^2 + p_y^2$. The algebra of all symmetries of $\mathcal{H}$ is generated by $p_x, p_y, M = xp_y - yp_x$. Let

$$\mathcal{L}_1 = M^2 + p_x p_y, \quad \mathcal{L}_2 = p_x^2.$$

Then we have $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\} = 4Mp_xp_y$ and

$$\mathcal{R}^2 = \mathcal{F}(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) = 16\mathcal{L}_1\mathcal{L}_2(\mathcal{L}_0 - \mathcal{L}_2) - 16\mathcal{L}_2^{\frac{3}{2}}(\mathcal{L}_0 - \mathcal{L}_2)^{\frac{3}{2}}.$$

Since $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ are functionally independent $\mathcal{R}^2$ must be a function of these symmetries. However, although $\mathcal{F}$ is defined and bounded at the point $(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) = (0, 0, 0)$, it is not a polynomial, and not even analytic at this point. Thus it has no power series expansion about the origin.)

5. The quadratic algebra structure can be used to compute the interbase expansion coefficients.

5 Examples in higher dimensions

An extreme case, superintegrability in n dimensions, occurs for the Schrödinger equation

$$(\Delta_n + V_n(\mathbf{q}))\Psi = -M(M + G - 1)\Psi \tag{21}$$

where

$$V_n = -\frac{1}{4}\sum_{i=1}^{n}\frac{(\gamma_i - \frac{1}{2})(\gamma_i - \frac{3}{2})}{q_i^2} - \frac{1}{4}\frac{(\gamma_{n+1} - \frac{1}{2})(\gamma_{n+1} - \frac{3}{2})}{q_0^2} + \alpha, \tag{22}$$

$$\alpha = \frac{1}{4}\left[(1 - G)^2 - 1 - \frac{(n-3)(n+1)}{4}\right],$$

$G = \sum_{j=1}^{n+1}\gamma_j$, and Δ_n is the Laplace-Beltrami operator on the n-sphere. Here

$$q_0^2 + q_1^2 + \cdots + q_n^2 = 1.$$

One can transfer this Schrödinger equation with a scalar potential V_n to one with vector potential Λ_n through the use of a multiplier transformation ρ. Setting $\Phi(\mathbf{x}) = \rho(\mathbf{x})\Psi(\mathbf{x})$ for a nonzero scalar function ρ we find

$$\begin{aligned} H\Phi \equiv (\Delta_n + \Lambda_n)\Phi &= -M(M+G-1)\Phi \\ \Longleftrightarrow\ (\Delta_n + V_n(\mathbf{x}))\Psi &= -M(M+G-1)\Psi, \end{aligned} \tag{23}$$

provided

$$\rho^{-1} = x_1^{\gamma_1/2-1/4} \cdots x_n^{\gamma_n/2-1/4}(1-x)^{\gamma_{n+1}/2-1/4},$$

and

$$\Lambda_n = \sum_{j=1}^{n}[\gamma_j - \frac{1}{2} + (\frac{n+1}{2} - G)x_j]\partial_{x_j}.$$

Here

$$H = \sum_{i,j=1}^{n}(x_i\delta_{ij} - x_ix_j)\partial_{x_ix_j} + \sum_{i=1}^{n}(\gamma_i - Gx_i)\partial_{x_i}$$

in the coordinates

$$\begin{aligned} q_0^2 &= 1 - \sum_{i=1}^{n} x_i = 1 - x \\ q_1^2 &= x_1 \\ q_2^2 &= x_2 \\ &\vdots \\ q_n^2 &= x_n. \end{aligned}$$

In the paper [17] and the book [18] all separable coordinates for the equation $\Delta_n\Psi = \lambda\Psi$ are constructed, where Δ_n is the Laplace-Beltrami operator on S^n. It is shown that all separable coordinates are orthogonal and that for each separable coordinate system the corresponding separated solutions are characterized as simultaneous eigenfunctions of a set of n second order commuting symmetry operators for Δ_n. Moreover, the equation $(\Delta_n+V_n)\Psi = \lambda\Psi$ where the scalar potential takes the form

$$V_n = \sum_{i=1}^{n}\frac{\alpha_i}{q_i^2} + \frac{\alpha_0}{q_0^2}, \quad \alpha_0, \alpha_1, \ldots, \alpha_n$$

is separable in *all* the coordinate systems in which the Laplace-Beltrami eigenvalue equation is separable. These results can easily be extended to solutions of

$$(\Delta_n + \Lambda_n)\Phi = \lambda\Phi \tag{24}$$

through the mappings

$$\begin{aligned} \Delta_n + \Lambda_n &= \rho(\Delta_n + V_n)\rho^{-1} \\ S_{ij} &= \rho S'_{ij}\rho^{-1} \\ \Phi &= \rho\Psi. \end{aligned} \tag{25}$$

Indeed all separable solutions Ψ map to R-separable solutions Φ of (24), [21]. Finally, since $H = \Delta_n + \Lambda_n$ maps polynomials of maximum order m_k in x_k to polynomials of the same type, it follows that a basis of separated solutions can be expressed as *polynomials* in the x_i. The second-order symmetry operators for this operator can be chosen to be self-adjoint, so the basis of simultaneous eigenfunctions can be chosen to be *orthogonal* with respect to the inner product

$$(\Phi_1, \Phi_2) \equiv <\Psi_1, \Psi_2>= \int \cdots \int_{x_i>0, x<1} \Phi_1(\mathbf{x})\overline{\Phi_2}(\mathbf{x})\rho^{-2}(\mathbf{x})\ d\omega \tag{26}$$

$$= \int \cdots \int_{x_i>0, x<1} \Phi_1 \overline{\Phi_2}\ d\tilde{\omega},$$

$$d\tilde{\omega} = x_1^{\gamma_1 - 1} \ldots x_n^{\gamma_n - 1}(1-x)^{\gamma_{n+1}-1} dx_1 \ldots dx_n.$$

Thus every separable coordinate system for the Laplace-Beltrami eigenvalue equation on the n-sphere yields an orthogonal basis of polynomial solutions of equation (24), hence an orthogonal basis for all n-variable polynomials with inner product (26). For details about the bases that can occur and the interbasis expansion coefficients, see [30, 31]. Among the special functions that arise are polyspherical harmonics, products of Jacobi polynomials, Heun polynomials, Lame' polynomials, ellipsoidal polynomials, and Lauricella polynomials.

We will look at one more example, the generalized isotropic oscillator in Eucidean 3-space. This is the Schrödinger equation

$$H\Psi = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Psi + V(x,y,z)\Psi = E\Psi, \tag{27}$$

with potential

$$V(x,y,z)=\frac{\omega^2}{2}(x^2+y^2+z^2)+\frac{1}{2}\left[\frac{(k_1^2-1/4)}{x^2}+\frac{(k_2^2-1/4)}{y^2}+\frac{(k_3^2-1/4)}{z^2}\right] \qquad (28)$$

where the constant $k_i \geq 1/2$. For $k_i = 1/2$ we have the ordinary isotropic oscillator potential. The corresponding Schrödinger equation admits solutions via a separation of variables in eight coordinate systems: Cartesian, spherical, sphero-conical, cylindrical polar, cylindrical elliptic, prolate and oblate spheroidal, ellipsoidal. See [32] for the complete details. Here we consider only the quadratic algebra struture.

Note that a basis for the symmetries of Schrödinger's equation with the potential (28) consists of the six operators:

$$M_i = -D_{ii} - \frac{k_i^2-\frac{1}{4}}{x_i^2}, \qquad -\mathcal{H} = M_1+M_2+M_3 \qquad (29)$$

$$J_{ij} = L_{ij}^2 - (k_i^2-\frac{1}{4})\frac{x_j^2}{x_i^2} - (k_j^2-\frac{1}{4})\frac{x_i^2}{x_j^2} - \frac{1}{2} \quad i,j=1,2,3 \qquad (30)$$

where $L_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}$, $D_{ii} = -\partial_{x_i}^2 + \omega^2 x_i^2$ - is a diagonal components of the Demkov tensor [33] and we have the notation $x_1 = x$, $x_2 = y$, $x_3 = z$.

The commutators of the operators (3)-(4) can be closed to form a quadratic algebra as follows

$$[M_i, M_j] = 0, \quad [M_i, J_{jk}] = 0, \quad [M_i, J_{ij}] = Q_{ij} = Q_{[ij]}, \quad [J_{ij}, J_{ik}] = R_{[ijk]} = R$$

where Q_{ij} is totally antisymmetric and the totally antisymmetric quantity $R_{[ijk]}$ is denoted by R. Further commutators are calculated to be

$$[M_i, Q_{jk}] = 0, \quad [M_i, Q_{ij}] = 4\{M_i, M_j\}+16J_{ij}, \quad [M_i, R] = 4\{M_k, J_{ij}\}-4\{M_j, J_{ik}\},$$

$$[J_{ij}, Q_{ij}] = 4\{M_i, J_{ij}\} - 4\{M_j, J_{ij}\} - 8(k_j^2-1)M_i + 8(k_i^2-1)M_j,$$

$$[J_{ij}, Q_{ik}] = 4\{M_i, J_{jk}\} - 4\{M_j, J_{ik}\},$$

$$[J_{ij}, R] = 4\{J_{ij}, J_{jk}\} - 4\{J_{ij}, J_{ik}\} - 8(k_i^2-1)J_{jk} + 8(k_j^2-1)J_{ik},$$

where $\{A, B\} = AB + BA$. The expression for the commutators of the Q and R are

$$[Q_{ij}, Q_{ik}] = 4\{M_i, Q_{jk}\}, \quad [Q_{ij}, R] = -4\{J_{ij}, Q_{ik}\} - 4\{J_{ij}, Q_{jk}\}$$

All the commutators of the operators M_i, J_{mn}, Q_{pq} and R can be expressed in terms of quadratic symmetric products of themselves. The algebra therefore is closed quadratically. There are relations between the symmetric products of the generators of this algebra. The exhaustive list of these is as follows.

$$Q_{ij}^2 = \frac{8}{3}\{J_{ij}, M_i, M_j\} + \frac{64}{3}\{M_i, M_j\} + 16\omega^2 J_{ij}^2 - 16(1-k_j^2)M_i^2 - 16(1-k_i^2)M_j^2$$

$$-\frac{128}{3}\omega^2 J_{ij} - 64\omega^2(1-k_i^2)(1-k_j^2)$$

$$\{Q_{ij}, Q_{ik}\} = \frac{8}{3}\{J_{ij}, M_i, M_k\} + \frac{8}{3}\{J_{ik}, M_i, M_j\} - \frac{8}{3}\{J_{jk}, M_i, M_i\}$$

$$+32\omega^2(1-k_i^2)\{J_{ij}, J_{ik}\} - 32(1-k_i^2)M_j M_k - 64\omega^2(1-k_i^2)J_{jk}$$

$$\{Q_{ij}, R\} = \frac{8}{3}\{J_{ij}, J_{ij}, M_k\} - \frac{8}{3}\{J_{ij}, J_{ik}, M_j\} - \frac{8}{3}\{J_{ij}, J_{jk}, M_i\} - \frac{64}{3}\{J_{ij}, M_k\} - \frac{64}{3}\{J_{ik}, M_j\}$$

$$-\frac{64}{3}\{J_{jk}, M_i\} + 16(1-k_i^2)\{J_{jk}, M_j\} + 16(1-k_j^2)\{J_{ik}, M_i\} - 64(1-k_i^2)(1-k_j^2)M_k$$

$$R^2 = -\frac{4}{3}\{J_{ij}, J_{ik}, J_{jk}\} + \frac{64}{3}\{J_{ij}, J_{ik}\} + \frac{64}{3}\{J_{ij}, J_{jk}\} + \frac{64}{3}\{J_{ik}, J_{jk}\} - 16(1-k_k^2)J_{ij}^2$$

$$-16(1-k_j^2)J_{ik}^2 - 16(1-k_i^2)J_{jk}^2 + \frac{128}{3}(1-k_k^2)J_{ij} + \frac{128}{3}(1-k_j^2)J_{ik}$$

$$+\frac{128}{3}(1-k_i^2)J_{jk} + 64(1-k_i^2)(1-k_j^2)(1-k_k^2)$$

where $\{A, B, C\} = ABC + CAB + BCA$. Note that only five operators from (3)-(4) are functionally independent [3] and for all the coordinate systems that provide separable solutions for the Schrödinger equation the operators characterizing the separation are always combinations of the M_i and J_{ij}.

In the limiting case $k_i = \frac{1}{2}$, we obtain a quadratic algebra too. In this case

$$Q_{ij} = 2\left(L_{ij}D_{ij} + D_{ij}L_{ij}\right), \qquad R = \{L_{ik}, \{L_{ij}, L_{kj}\}\},$$

and instead of operators $\{M_i, J_{ij}, Q_{ij}, R\}$ we can consider as a basis for the symmetries the Demkov tensor - D_{ij} and the components of orbital momentum - L_{ij}. In this regard we arrive at the Lie algebra corresponding to the symmetries of the isotropic oscillator [33].

6 Some questions and a new approach

The preceding examples motivate some questions:

1. Is there a uniform procedure to find superintegrable potentials on a constant curvature space?

2. Is there a way of proving completeness, i.e., of demonstrating that all superintegrable potentials for a space have been found?

3. Is superintegrability always associated with multivariable separation?

4. What is the relationship between superintegrability and the existence of quadratic algebras?

First, let us consider the issue of constructing the superintegrable potentials on a given space. We will look at three approaches to the problem.

1. Use what is known from separation of variables theory.

   This is the most common approach to finding superintegrable potentials. One classifies the separable systems for the zero potential, determines a general expression for a potential separating in each one of these systems, and then tries to identify potentials that simultaneously separate in several systems. Although the approach is useful for finding examples, there are serious problems in obtaining exhaustive lists. Separable coordinates occur in equivalence classes: two coordinate systems related by a symmetry in the motion group of the space are consided as equivalent. A potential may separate in an ellipsoidal coordinate system, but the ellipsoidal coordinates may not be centered at the origin or aligned with the Cartesian coordinate axes. Such systems are difficult to uncover and many of the lists of superintegrable potentials contain gaps, even for 2-dimensional Minkowski space and hyperboloids. Furthermore, for $n > 2$ there is no proof that superintegrability necessarily implies multiseparability.

2. Derive superintegrable potentials on a space as restrictions of free particle systems on higher dimensional spaces. (Split off ignorable coordinates.)

A very simple example will give the basic idea. Consider the free particle Hamiltonian on Euclidean four-space:

$$\mathcal{H}_4 = p_1^2 + p_2^2 + p_3^2 + p_4^4, \quad ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

There are 4 separable coordinate systems for the Hamilton-Jacobi equation that have two angular variables θ_1, θ_2. The simplest of these is

$$x_1 = r_1 \cos\theta_1, \quad x_2 = r_1 \sin\theta_1, \quad x_3 = r_2 \cos\theta_2, \quad x_4 = r_2 \sin\theta_2.$$

The other 3 systems are of the form $u_1, u_2, \theta_1, \theta_2$ where $r_j = f_j(u_1, u_2),\ j = 1, 2$. In the new coordinates we have

$$\mathcal{H}_4 = p_{r_1}^2 + p_{r_2}^2 + \frac{p_{\theta_1}^2}{r_1^2} + \frac{p_{\theta_2}^2}{r_2^2}, \quad ds^2 = dr_1^2 + dr_2^2 + r_1^2 d\theta_1^2 + r_2^2 d\theta_2^2.$$

Note that the θ_j are *ignorable coordinates*, i.e., the metric tensor in the new coordinates is independent of the θ_j. Now restrict to the symplectic submanifold such that $p_{\theta_j} = \alpha_j, \quad d\theta_j = 0$ where the α_j are constants. Then the Hamiltonian becomes

$$\mathcal{H}_4 = p_{r_1}^2 + p_{r_2}^2 + \frac{\alpha_1^2}{r_1^2} + \frac{\alpha_2^2}{r_2^2}, \quad ds^2 = dr_1^2 + dr_2^2.$$

Locally, we can regard the Hamiltonian as defined in Euclidean 2-space and with the potential $V = \frac{\alpha_1^2}{r_1^2} + \frac{\alpha_2^2}{r_2^2}$. The restricted Hamiltonian-Jacobi equation remains separable in 4 cordinate systems $\{u_1, u_2\}$, hence this is a superintegrable system. Boyer, Kalnins and Winternitz have applied this idea to the restriction of a free Hamiltonian on Hermitian hyperbolic spaces to hyperboloids in order to derive previously unknown superintegrable systems, [34, 35]. This is a powerful method, but there is no evident way to prove by this approach that all superintegrable systems have been obtained.

3. New way: use integrabilty conditions. This last approach is the one that we shall advocate in the remainder of this paper.

7 Complex Euclidean 2-space treated in detail

We consider, in detail, the case

$$\mathcal{H} = p_1^2 + p_2^2 + V(x, y) \tag{31}$$

where $p_1 = p_x, p_2 = p_y$ and all variables are complex. Let us assume that in addition to the classical Hamiltonian we have two quadratic constants of the motion

$$\mathcal{L}_h = \sum_{k,j=1}^{2} a_{(h)}^{kj}(x, y) p_k p_j + W_{(h)}(x, y) \equiv \ell_h + W_{(h)}, \quad h = 1, 2 \tag{32}$$

which must satisfy

$$\{\mathcal{H}, \mathcal{L}_h\} = 0$$

with $\{\}$ the usual Poisson bracket. We require that the set $\{d\mathcal{H}, d\mathcal{L}_1, d\mathcal{L}_2\}$ is linearly independent, so that $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ is a maximal set of functionally independent constants of the motion. It is clear that $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\}$ is a constant of the motion, so it and $\mathcal{R}^2$ must be expressible as an analytic function of $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$:

$$\mathcal{R}^2 = \mathcal{F}(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2), \qquad \mathcal{H} \equiv \mathcal{L}_0. \tag{33}$$

Note that $\mathcal{R}$ has the form

$$\mathcal{R} = \sum_{k,l,m=1}^{2} c^{klm} p_k p_l p_m + \sum_{k=1}^{2} d^k p_k, \tag{34}$$

but that it doesn't follow that $\mathcal{R}^2$ is necessarily a polynomial as a function of $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$. We will find conditions that guarantee that $\mathcal{F}$ is a third-order polynomial in its arguments.

Using the identity

$$\{\mathcal{K}, \mathcal{G}\} = \sum_{h=0}^{2} \{\mathcal{K}, \mathcal{L}_h\} \frac{\partial \mathcal{G}}{\partial \mathcal{L}_h} \tag{35}$$

for a continuously differentiable function $\mathcal{G}(\mathcal{L}_h)$, we find the relations

$$\{\mathcal{L}_1, \mathcal{R}\} = \frac{1}{2} \frac{\partial \mathcal{F}}{\partial \mathcal{L}_2}, \quad \{\mathcal{L}_2, \mathcal{R}\} = -\frac{1}{2} \frac{\partial \mathcal{F}}{\partial \mathcal{L}_1}. \tag{36}$$

Thus, the constants of the motion $\{\mathcal{L}_1, \mathcal{R}\}, \{\mathcal{L}_2, \mathcal{R}\}$ are easily computed once $\mathcal{F}$ is known. Further, if F is a polynomial in the invariants, then so are $\{\mathcal{L}_1, \mathcal{R}\}$, and $\{\mathcal{L}_2, \mathcal{R}\}$.

We first determine the conditions that the function

$$\mathcal{L} = \sum_{j,k=1}^{2} a^{jk}(x,y) p_k p_j + W(x,y), \quad a^{jk} = a^{kj}, \tag{37}$$

must satisfy to be a constant of the motion. The requirement is $\{\mathcal{H}, \mathcal{L}\} = 0$ where

$$\{f, g\} = \sum_{j=1}^{2} \left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j}\right), \quad (x_1, x_2) = (x, y), \tag{38}$$

and

$$\mathcal{H} = p_1^2 + p_2^2 + V(x,y). \tag{39}$$

The conditions are thus

$$\begin{array}{ll} \dfrac{\partial a^{11}}{\partial x} = 0 \qquad 2\dfrac{\partial a^{12}}{\partial x} + \dfrac{\partial a^{11}}{\partial y} & = \ 0 \\ \dfrac{\partial a^{22}}{\partial y} = 0 \qquad \dfrac{\partial a^{22}}{\partial x} + 2\dfrac{\partial a^{12}}{\partial y} & = \ 0, \end{array} \tag{40}$$

and

$$\frac{\partial W}{\partial x} - a^{11}\frac{\partial V}{\partial x} - a^{12}\frac{\partial V}{\partial y} = 0, \quad \frac{\partial W}{\partial y} - a^{12}\frac{\partial V}{\partial x} - a^{22}\frac{\partial V}{\partial y} = 0. \tag{41}$$

The solution for the terms quadratic in the p_j is

$$a^{11} = \alpha_1 y^2 + \alpha_2 y + \alpha_3' \tag{42}$$

$$a^{12} = -\alpha_1 xy - \frac{1}{2}\alpha_2 x - \frac{1}{2}\alpha_4 y + \frac{1}{2}\alpha_5 \tag{43}$$

$$a^{22} = \alpha_1 x^2 + \alpha_4 x + \alpha_3'', \tag{44}$$

where the α_k are constants. The requirement that $\partial_x W_y = \partial_y W_x$ leads from (41) to the second order partial differential equation for the potential

$$\begin{aligned} & \tfrac{1}{2} \quad (2\alpha_1 xy + \alpha_2 x + \alpha_4 y - \alpha_5)(V_{xx} - V_{yy}) + (\alpha_1[y^2 - x^2] + \alpha_2 y - \alpha_4 x + \alpha_3)V_{xy} \\ & = \quad (-3\alpha_1 y - \frac{3}{2}\alpha_2)V_x + (3\alpha_1 x + \frac{3}{2}\alpha_4)V_y, \end{aligned} \tag{45}$$

where $\alpha_3 = \alpha_3' - \alpha_3''$. This is a fundamental equation in our approach. We denote the solution space of this equation by

$$[\alpha_1, \cdots, \alpha_5]. \tag{46}$$

Let us now return to our assumption that the Hamilton-Jacobi equation admits two constants of the motion:

$$\mathcal{L}_h = \sum_{j,k=1}^{2} a_{(h)}^{jk} p_k p_j + W_{(h)}, \qquad h = 1, 2.$$

These two operators together with $\mathcal{H}$ are assumed functionally independent. The constant of the motion $\mathcal{L}_1$ leads to the condition (46) on the potential V; whereas $\mathcal{L}_2$ leads to the second condition

$$[\beta_1, \cdots, \beta_5]. \tag{47}$$

The potential must lie in the intersection of the solution spaces (46,47) for these two conditions. It follows that the equations

$$V_{xx} - V_{yy} = AV_x + BV_y, \qquad V_{xy} = CV_x + DV_y \tag{48}$$

must hold, where

$$\begin{aligned}
A\mathcal{E} &= \frac{3}{2}H_{12}(x^2+y^2) - 3H_{14}xy + 3H_{13}y - \frac{3}{2}H_{24}x + \frac{3}{2}H_{23} \\
B\mathcal{E} &= \frac{3}{2}H_{14}(x^2+y^2) - 3H_{12}xy - 3H_{13}x + \frac{3}{2}H_{24}y + \frac{3}{2}H_{34} \\
2C\mathcal{E} &= -3H_{14}y^2 + (-\frac{3}{2}H_{24} + 3H_{15})y + \frac{3}{2}H_{25} \\
2D\mathcal{E} &= 3H_{12}x^2 + (-\frac{3}{2}H_{24} - 3H_{15})x - \frac{3}{2}H_{45} \\
2\mathcal{E} &= -H_{12}xy^2 + H_{14}x^2y - H_{12}x^3 + H_{14}y^3 - 2H_{13}xy + H_{24}(x^2+y^2) \\
&+ H_{15}(x^2-y^2) + (H_{34} - H_{25})y + (H_{45} - H_{23})x - H_{35},
\end{aligned} \tag{49}$$

and $H_{k\ell} = -H_{\ell k} = \alpha_k\beta_\ell - \alpha_\ell\beta_k$.

From the fundamental equations (48) we can compute all of the third partial derivatives of V. Indeed

$$V_{xxx} = (A_x + BC + C_y + C^2 + A^2)V_x + (B_x + DB + D_y + CD + AB)V_y$$

$$
\begin{aligned}
& + \quad (A+D)V_{yy} \\
V_{xxy} &= (C_x + DC + AC)V_x + (D_x + D^2 + BC)V_y + CV_{yy} \\
V_{xyy} &= (C_y + C^2)V_x + (D_y + CD)V_y + DV_{yy} \qquad (50) \\
V_{yyy} &= (-A_y + C_x + DC)V_x + (-B_y - AD + D_x + D^2 + BC)V_y \\
& + \quad (C-B)V_{yy}.
\end{aligned}
$$

Thus if the potential V is subject to the two conditions (46,47), then V can depend on at most 3 parameters, in addition to a trivial additive constant. We can choose these parameters to be $V_x(x_0, y_0), V_y(x_0, y_0), V_{yy}(x_0, y_0)$ for any fixed regular point (x_0, y_0). Then $V_{xx}(x_0, y_0)$ and all higher derivatives can be computed by successive differentiation of relations (50). We require that our potential be *nondegenerate*, i.e., that it depend on 3 arbitrary parameters.

Then, the conditions $\partial_x V_{xxy} = \partial_y V_{xxx}, \partial_y V_{xxy} = \partial_x V_{xyy}, \partial_y V_{xyy} = \partial_x V_{yyy}$ for the fourth partial derivatives lead to the *integrability conditions*

$$
\begin{aligned}
\partial_x(2C - B) &= \partial_y(2D + A) \quad (satisfied\ identically) && (51) \\
C_{xx} - C_{yy} - A_{xy} &= 2CC_y - DA_y - 2CD_x + AA_y - AC_x \\
& + \quad CB_y + BC_y && (52) \\
D_{xx} - D_{yy} - B_{xy} &= -2DD_x - CB_x + 2DC_y - BB_x \\
& - \quad BD_y + DA_x + AD_x. && (53)
\end{aligned}
$$

Note that if we have another constant of the motion $\mathcal{L}_3$ associated with a nondegenerate potential, then $\mathcal{L}_3$ must be a linear combination of $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$. Indeed, if $\mathcal{L}_3$ is not a linear combination of the basis functions, then the potential V must satisfy an equation (45) that is linearly independent of the equations associated with $\mathcal{L}_1, \mathcal{L}_2$. This means an additional constraint on the solution space and that V can depend on at most two parameters, which is a contradiction.

We will use the conditions (52,53) to classify the possible potentials V and the corresponding constants of the motion $\mathcal{L}_1, \mathcal{L}_2$. For this we note that it is only the three-dimensional subspace spanned by $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ that matters; we can choose any basis for this subspace. Hence we can replace the conditions (46, 47) by linear combinations of themselves without changing the potential. Moreover, to simplify the results we note that we can always subject the coordinates (x, y), and $\mathcal{L}_1, \mathcal{L}_2$ to a simultaneous Euclidean motion, i.e., we regard all translated and rotated potentials as members of the same equivalence class.

Multiplying both sides of (52) and (53) by $\mathcal{E}^3$ we obtain polynomial identities in x and y. Equating the coefficients of the various powers $x^n y^m$ we obtain conditions on the parameters H_{jk}. The simplest non-trivial condition, which is associated with the coefficient of a fifth order power in either of the equations, is

$$2H_{15}(H_{14}^2 - H_{12}^2) + H_{24}(H_{14}^2 + H_{12}^2) - 4H_{14}H_{12}H_{13} = 0. \tag{54}$$

We exploit these and the remaining conditions, and Euclidean motions to classify the possibilities for the $\mathcal{L}_j$. The full conditions (52) and (53), expressed in terms of the parameter H_{ij}, take several pages to list and are complicated to solve directly. (Indeed a symbol manipulation program was an important aid to our computations.) However, by dividing the problem up into special cases and using Euclidean motions, we can simplify the conditions and obtain a full solution. In the listing that follows we use the fact that the constants of the motion can each be expressed as a quadratic element in the enveloping algebra of the Euclidean group in the plane with basis elements

$$p_x, \ p_y, \ M = xp_y - yp_x,$$

to which a potential term $W(x, y)$ is added. (Strictly speaking, conditions (52) and (53) are only necessary conditions for existence of nondegenerate potentials. However, in our case-by-case study we have found that they are also sufficient: all solutions of these equations lead to nondegenerate potentials.)

Suppose $I \equiv H_{12}^2 + H_{14}^2 \neq 0$. Via an appropriate coordinate rotation through complex angle θ we obtain a new set of equations (48) in the rotated coordinates where the new parameters H'_{12}, H'_{14} are related to the original ones by

$$H'_{12} = H_{12}\cos\theta + H_{14}\sin\theta, \quad H'_{14} = H_{14}\cos\theta - H_{12}\sin\theta, \tag{55}$$

and for which $I = I'$. Thus, by an appropriate choice of θ, we can assume $H_{12} = 0$. Then, by an appropriate Euclidean translation that leaves H_{12}, H_{14} unchanged, it follows from (49) that we can assume $H_{13} = H_{24} = 0$. Then (54) implies $H_{15} = 0$, so $H_{35} = H_{23} = 0$. Further, the fourth order integrability conditions give $H_{34}H_{45} = 0$, and under appropriate translations, we can take $H_{34} = H_{45} = 0$.

Case (1) $H_{12}^2 + H_{14}^2 \neq 0$

$$[1,0,0,0,0], \quad [0,0,0,1,0] \tag{56}$$

Here,

$$\begin{aligned} L_1 &= 4M^2 + W^{(1)}, \qquad L_2 = -2Mp_y + W^{(2)} &(57)\\ V(x) &= \frac{\alpha}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{x^2+y^2}}[\frac{\beta}{\sqrt{x^2+y^2}+x} + \frac{\gamma}{\sqrt{x^2+y^2}-x}]. &(58)\end{aligned}$$

This potential allows separation in parabolic or polar coordinates.

We obtain the remaining cases by continuing in this way; for details see [36]. The results are:

Case (2a)

$$[1,0,2c^2,0,0], \quad [0,-1,ic,i,-c], \tag{59}$$

where,

$$\begin{aligned} L_1 &= M^2 + c^2p_x^2 + W^{(1)}, \qquad L_2 = Mp_+ + \frac{ic}{2}p_+^2 + W^{(2)}\\ V(x) &= \frac{\alpha z}{(c^2-z^2)^{\frac{1}{2}}} + \frac{\beta}{\sqrt{(c-z)(c+\bar{z})}} + \frac{\gamma}{\sqrt{(c+z)(c+\bar{z})}}. &(60)\end{aligned}$$

The corresponding Hamilton-Jacobi and Schrödinger equations for this system separates in elliptical coordinates, as well as shifted elliptical coordinates.

Case (2b)

$$[1,0,2,0,2i], \quad [0,-1,2i,i,-2], \tag{61}$$

where,

$$\begin{aligned} L_1 &= M^2 + p_+^2 + W^{(1)}, \qquad L_2 = (M+2ip_+)^2 + p_+^2 + W^{(2)}\\ V(x) &= \frac{\alpha}{z^2} + \frac{\beta}{\sqrt{z^3(\bar{z}+2)}} + \frac{\gamma}{\sqrt{z(\bar{z}+2)}}. &(62)\end{aligned}$$

This system separates in terms of hyperbolic coordinates and displaced hyperbolic coordinates.

Case (3a) $H_{15}^2 + H_{13}^2 \neq 0$

$$[1,0,0,0,0], \quad [0,0,1,0,0] \tag{63}$$

Here,

$$\begin{aligned} L_1 &= M^2 + W^{(1)}, \qquad L_2 = p_x^2 + W^{(2)} \qquad (64)\\ V(x) &= \alpha(x^2+y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}. \end{aligned}$$

$$(65)$$

This potential permits separation in polar, elliptic and cartesian coordinates.
Case (3b) $H_{15}^2 + H_{13}^2 = 0$

$$[1,0,0,0,0], \quad [0,0,2,0,\pm 2i] \tag{66}$$

Here,

$$\begin{aligned} L_1 &= M^2 + W^{(1)}, \qquad L_2 = p_+^2 + W^{(2)}\\ V(x) &= \alpha\frac{x^2+y^2}{(x+iy)^4} + \frac{\beta}{(x+iy)^2} + \gamma(x^2+y^2). \end{aligned}$$

$$(67)$$

(There is a similar solution where the term p_+^2 in L_2 is replaced by p_-^2.) The potential permits separation in hyperbolic and polar coordinates.
Case (3c) $H_{15}^2 + H_{13}^2 = 0$

$$[1,0,c^2,0,0], \quad [0,0,2,0,\pm 2i] \tag{68}$$

Here,

$$\begin{aligned} L_1 &= M^2 + c^2p_x^2 + W^{(1)}, \qquad L_2 = p_+^2 + W^{(2)}\\ V(x) &= \frac{\alpha z}{\sqrt{z^2-c^2}} + \frac{\beta\bar{z}}{\sqrt{z^2-c^2}(z+\sqrt{z^2-c^2})^2} + \gamma z\bar{z} \end{aligned}$$

$$(69)$$

The potential permits separation in hyperbolic and elliptic coordinates.
Case (4) $H_{12} = H_{13} = H_{14} = H_{15} = 0,\ \alpha_2 \neq 0,\ H_{24} \neq 0$

$$[0,1,0,0,0], \quad [0,0,0,1,0] \tag{70}$$

Here,

$$L_1 = -2Mp_x + W^{(1)}, \qquad L_2 = -2Mp_y + W^{(2)} \tag{71}$$

$$V(x) = \frac{\alpha}{\sqrt{x^2+y^2}} + \beta\frac{(\sqrt{x^2+y^2}+x)^{\frac{1}{2}}}{\sqrt{x^2+y^2}} + \gamma\frac{(\sqrt{x^2+y^2}-x)^{\frac{1}{2}}}{\sqrt{x^2+y^2}}. \tag{72}$$

Separation of variables is possible in two types of parabolic coordinates, the usual parabolic coordinates and the interchanged parabolic coordinates $x = \mu\nu, y = \frac{1}{2}(\mu^2 - \nu^2)$.

Case (5) $(H_{34} - H_{25})^2 + (H_{45} - H_{23})^2 \neq 0$

$$[0, 1, \alpha_3, \pm i, 0], \quad [0, 0, \pm i, 0, 1] \tag{73}$$

Here we choose the typical case

$$L_1 = 4iMp_- + p_+^2 + W^{(1)}, \quad L_2 = p_-^2 + W^{(2)} \tag{74}$$

$$V(x) = \alpha(x - iy) + \beta(x + iy - \frac{3}{2}(x - iy)^2) + \gamma(x^2 + y^2 - \frac{1}{2}(x - iy)^3). \tag{75}$$

The possible separable coordinates are semihyperbolic coordinates corresponding to operator $Mp_- + p_+^2$ and shifted semihyperbolic coordinates with operator $Mp_- + \delta p_-^2 + p_+^2$. This corresponds to the standard coordinates shifted via the transformation $x \to x + \delta, y \to y + i\delta$.

Case (6a) $H_{45} = 0$

$$[0, 0, 1, 0, 0], \quad [0, 0, 0, 1, 0] \tag{76}$$

Here,

$$L_1 = p_x^2 + W^{(1)}, \qquad L_2 = -2Mp_y + W^{(2)} \tag{77}$$

$$V(x) = \alpha(4x^2 + y^2) + \beta x + \frac{\gamma}{y^2}. \tag{78}$$

The possible separable coordinates are cartesian and parabolic.

Case (6b) $H_{45} \neq 0$

$$[0, 0, 1, 0, \pm i], \quad [0, 0, 0, 1, 0] \tag{79}$$

Here we take

$$L_1 = 2p_yp_+ + W^{(1)}, \qquad L_2 = Mp_y + W^{(2)} \tag{80}$$

$$V(x) = \frac{\alpha}{\sqrt{x+iy}} + \beta x + \gamma \frac{2x+iy}{\sqrt{x+iy}}. \tag{81}$$

There is the possibility of separability in parabolic coordinates $\{Mp_y\}$ or displaced parabolic coordinates $\{(M + \delta(p_x \pm ip_y))p_y\}$ for suitable δ.

This is the complete list of superintegrable systems in complex Euclidean 2-space. It includes real Euclean space and Minkowski space as special cases. Now we demonstrate that there is a quadratic algebra associated with each nondegenerate potential. Because we are working in two dimensions there can only be three functionally independent constants at most. Consequently all Poisson brackets must be functionally dependent on $\mathcal{H} = \mathcal{L}_0, \mathcal{L}_1$ and $\mathcal{L}_2$. We want to show that in fact $\mathcal{R}^2 = \{\mathcal{L}_1, \mathcal{L}_2\}^2 = \mathcal{F}(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ is a polynomial in these variables.

Note that for arbitrary $\mathcal{L}_1, \mathcal{L}_2$, the $\mathcal{F}$ is in general *not* a polynomial. Consider the example:

$$\mathcal{L}_0 = p_x^2 + p_y^2, \quad \mathcal{L}_1 = M^2 + p_xp_y, \quad \mathcal{L}_2 = p_x^2.$$

Then we have $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\} = 4Mp_xp_y$ and

$$\mathcal{R}^2 = \mathcal{F}(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) = 16\mathcal{L}_1\mathcal{L}_2(\mathcal{L}_0 - \mathcal{L}_2) - 16\mathcal{L}_2^{\frac{3}{2}}(\mathcal{L}_0 - \mathcal{L}_2)^{\frac{3}{2}}.$$

Here, although $\mathcal{F}$ is defined and bounded at the point $(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) = (0, 0, 0)$, it is not analytic at this point. Thus it has no power series expansion about the origin. We conjecture that this is an illustration of the general problem: if $\mathcal{F}$ is not a polynomial, then there are branch points or cuts at $(0, 0, 0)$.

We will show, however, that for nondegenerate potentials the associated $\mathcal{F}$ *is* a polynomial. First, we can verify that this is true when the potential is turned off, i.e., if we consider only the functions

$$\ell_h = \sum_{j,k=1}^{2} a_{(h)}^{jk} p_k p_j, \ i = h, 2 \quad \ell_0 = p_x^2 + p_y^2,$$

where $\mathcal{L}_h = \ell_h + W^{(h)}$. Let $\mathcal{R} = \{\ell_1, \ell_2\}$. Then for each of the cases listed above it is straightforward to check that $\mathcal{R}^2 = \mathcal{P}_3(\ell_0, \ell_1, \ell_2)$ where $\mathcal{P}_3$ is a homogeneous third order polynomial in its arguments. [1] It follows that

$$\mathcal{R}^2 = \mathcal{F}(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) = \mathcal{P}_3(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) + \mathcal{F}_4(\mathbf{s}, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2), \tag{82}$$

where $\mathcal{F}_4$ is a fourth, second and zeroth order polynomial in the momenta p_x, p_y, and $\mathcal{F}_4(\mathbf{0}, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) = 0$. Here, the parameters in the potential are denoted by $\mathbf{s} = (V_x^0, V_y^0, V_{yy}^0)$, evaluated at some fixed point (x_0, y_0) and $\mathcal{F}_4$ is a polynomial function of these parameters.

From (36) we have

$$\{\ell_1, \mathcal{R}\} = \frac{1}{2}\frac{\partial \mathcal{P}_3}{\partial \ell_2}(\ell_0, \ell_1, \ell_2),$$

$$\{\ell_2, \mathcal{R}\} = -\frac{1}{2}\frac{\partial \mathcal{P}_3}{\partial \ell_1}(\ell_0, \ell_1, \ell_2),$$

hence

$$\{\mathcal{L}_1, \mathcal{R}\} = \frac{1}{2}\frac{\partial \mathcal{P}_3}{\partial \mathcal{L}_2}(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) + \frac{1}{2}\frac{\partial \mathcal{F}_4}{\partial \mathcal{L}_2}(\mathbf{s}),$$

$$\{\mathcal{L}_2, \mathcal{R}\} = -\frac{1}{2}\frac{\partial \mathcal{P}_3}{\partial \mathcal{L}_1}(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) - \frac{1}{2}\frac{\partial \mathcal{F}_4}{\partial \mathcal{L}_1}(\mathbf{s}),$$

where the $\partial\mathcal{F}_4/\partial\mathcal{L}_h(\mathbf{s})$ have only terms of orders two and zero in the momenta. It follows that the $\partial\mathcal{F}_4/\partial\mathcal{L}_h(\mathbf{s})$ must be expressible as linear combinations of the $\mathcal{L}_h$. This shows that the commutators $\{\mathcal{L}_h, \mathcal{R}\}$ can be expressed as polynomials in $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$. It is then a simple matter to verify that $\mathcal{F}$ itself is a polynomial in $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$.

A complete list of the quadratic algebra relations for each of the cases studied above is given in [36]. In view of relations (36) it is sufficient to give the relation $R^2 = F(L_0, L_1, L_2)$ for each case. An example is

Case (1) $\quad [1, 0, 0, 0, 0], \quad [0, 0, 0, 1, 0]$

$$\mathcal{R}^2 = 16\mathcal{L}_1^2\mathcal{H} - 16\mathcal{L}_2^2\mathcal{L}_1 - 32(\beta+\gamma)\mathcal{L}_2^2 + 64\alpha(\beta-\gamma)\mathcal{L}_2 + 16\alpha^2\mathcal{L}_1 - 256\beta\gamma\mathcal{H} - 32\alpha^2(\beta+\gamma).$$

[1] Moreover, it is straightforward to verify that the cases corresponding to nondegenerate potentials are the *only* cases where $\mathcal{P}_3$ is a homogeneous third order polynomial in its arguments. Thus the possible quadratic algebras generated by second order elements in the Euclidean Lie algebra correspond one-to-one with nondegenerate potentials.

There are analogous quantum algebras for superintegrable systems arising from the potentials we have already computed. The only difference is that the Poisson bracket is now replaced by the commutator bracket $[A, B] = AB - BA$ and the operators H, L_1 and L_2 are the obvious (formally self-adjoint) symmetry partial differential operators.

$$H = \partial_x^2 + \partial_y^2 + V(x, y), \quad L_h = \sum_{k,j=1}^{2} \partial_k (a_{(h)}^{kj}) \partial_j + W_{(h)}(x, y), \quad h = 1, 2. \quad (83)$$

Just as for the Hamilton-Jacobi case, if we have another constant of the motion L associated with a maximal potential, then L must be a linear combination of H, L_1, L_2. Indeed, if L is in self-adjoint form, then the conditions that $[H, L] = 0$ are identical with (40), (41). Thus, if L is not a linear combination of the basis functions, then the potential V must satisfy an equation (45) that is linearly independent of the equations associated with L_1, L_2. This means an additional constraint on the solution space and that V can depend on at most two parameters, which is a contradiction.

Furthermore the proof of the existence of quadratic algebra relations goes through almost unchanged for the operator case: $[L_1, L_2]^2 = R^2$ and $[L_1, R], [L_2, R]$ can be expressed as (symmetric) polynomials in the operators H, L_1, L_2. To make the prior construction go through, one need only note that since R^2 is a formally self-adjoint 6th order differential symmetry operator, the 5th order terms are fixed linear functions of the 6th order terms. The expressions $\{A, B\} = AB + BA$ and $\{A, B, C\} = ABC + CAB + BCA$ are operator symmetrizers. The detailed results are contained in [36]. An example is

Case (1)

$$[L_2, R] = 8L_2^2 + 8HL_2 + 8\alpha^2,$$

$$[L_1, R] = 8\{L_2, L_1\} + 16(1 + 2\beta + 2\gamma)L_2 + 32\alpha(\gamma - \beta),$$

$$\begin{aligned} R^2 &= 16L_1^2 H - \frac{8}{3}\{L_2, L_2, L_1\} - 16(2\beta + 2\gamma + \frac{11}{3})L_2^2 - \frac{176}{3} HL_1 \\ &+ 64\alpha(\beta - \gamma)L_2 + 16\alpha^2 L_1 + (-\frac{32}{3} + 96\gamma + 96\beta + 256\beta\gamma)H \\ &- \frac{32}{3}\alpha^2(3\beta + 3\gamma - 1). \end{aligned}$$

8 Conclusions

We have used the concept of a "nondegenerate potential" to add structure to the study of superintegrable classical and quantum mechanical systems in $E_{2,C}$. We have shown how to classify all such systems in a straightforward manner. Furthermore:

1. Each system is associated with a pair of constants of the motion in the classical case, and a pair of symmetry operators in the quantum case, that generate a quadratic algebra.

2. There is a one-to-one correspondence between superintegrable systems and free-field symmetry operators that generate quadratic algebras.

3. Superintegrability implies multiseparability.

We have analogous results for superintegrable systems on the complex 2-sphere, [37]. The real sphere and the real hyperboloid are special cases. Again we find and fill gaps in the known list of superintegrable systems. The next major challenge is to extend this analysis to higher dimensional systems.

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