DIRECT AND INVERSE SCATTERING OF TIME HARMONIC ACOUSTIC WAVES IN AN INHOMOGENEOUS SHALLOW OCEAN

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I. Introduction

The goal of this research is to develop the theory and a computer simulation to study direct and inverse scattering of time harmonic acoustic waves in inhomogeneous oceans. The present paper will focus on a stratified ocean with a pressure release surface and rigid bottom. A local inhomogeneity (range dependent) is located in the ocean. Our result contains two components:

1. Direct scattering of the incident wave
2. Determination of the inhomogeneity shape from the scattered wave

The direct scattering problem can be formulated as follows: let u^i and u^s be the incident wave and scattered wave, respectively. For a given u^i, we want to solve

$$\nabla^2 u + k^2 n(x,z) u = 0, \quad -\infty < x < \infty, \ 0 \le z \le h, \tag{1.1}$$

$$u = 0 \text{ at } z = 0, \tag{1.2}$$

$$\frac{\partial u}{\partial z} = 0, \text{ at } z = h, \tag{1.3}$$

$$u^s(x,z) \text{ outgoing}, \tag{1.4}$$

where $u = u^i + u^s$ is the total field and the refraction index $n(x,z)$ has the form of $n(x,z) = n_0(z)$ for $|x| > a$. Here n_0 and a are some positive function and constant, respectively. The problem in equations (1.1)-(1.4) will be referred to as Problem 1.

We use a coupled finite element method and normal mode method to produce the propagating waves. There are many publications which address the coupled finite element

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and spectral method. Our method is closest to that in Kirsch and Monk [2]. A brief review of a related method can be found in the introduction of [2].

The inverse scattering problem for acoustic waves, which consists of recovering the inhomogeneity or the shape of the scatterer from the scattered field, forms the basis of a wide variety of areas in the engineering sciences such as remote sensing, nondestructive testing, imaging etc., and for this reason has been the object of study by scientists in a number of diverse disciplines. Recent progress in this field can be found in [1]. However, most of the efforts in this field are devoted to the free space cases of $\mathbf{R}^2$ and $\mathbf{R}^3$. In the homogeneous finite depth ocean, Gilbert and Xu [5] studied the shape determination problem from the "propagating" far-field pattern (ref. [3], [4]). In this paper we will present an algorithm which is a modification of Weston's nonlinear approach [5].

To determine the unknown inhomogeneous medium we make use of the Green's function (denoted by $G(x,z;\xi,\zeta)$) for the stratified medium to present a relation between the inhomogeneity $m(x,z) := n(x,z) - n_0(z)$ and the propagating far-field. Namely, we have

$$u(x,z) = k^2 \int_\Omega G(x,z;\xi,\zeta) m(\xi,\zeta) u(\xi,\zeta) d\xi d\zeta + u^i(x,z), \tag{1.5}$$

where $\Omega = \{(x,z) \mid |x| < a,\ 0 \le z \le h\}$.

By measuring the scattered wave in the near-field or far-field corresponding to an incident wave, we arrive at solving a minimization problem. This minimization problem is solved by a least-squares algorithm.

II. Coupled Finite Element and Normal Mode Methods for Scattering in an Inhomogeneous Finite Depth Ocean

2.1. Formulation of the Coupled Problem

In this section we study the numerical treatment of the direct scattering in Problem 1. We use the finite element method to approximate u in the neighborhood of the inhomogeneity. Then we match the near-field finite element solution with the far-field normal mode expansion solution by introducing an artificial boundary $\Gamma_a = \{(x,z) \mid |x| = a > 0,\ 0 \le z \le h\}$ and domain $\Omega_a = \{(x,z) \mid |x| < a,\ 0 \le z \le h\}$. Instead of solving Problem 1, we consider a coupled problem as shown in Figure 2.1. In $\Omega_a = [0,h] \times [-a,a]$, $a > 0$, we consider

$$\Delta u_i + k^2 n(x,z) u_i = 0, \qquad \text{in } \Omega_a, \tag{2.1}$$

$$u_i(x,0) = 0, \qquad |x| \le a, \tag{2.2}$$

$$\frac{\partial u_i}{\partial z}(x,h) = 0, \qquad |x| \le a, \tag{2.3}$$

$$\frac{\partial u_i}{\partial |x|} + iku_i = \lambda, \qquad 0 \le z \le h,\ |x| = a \tag{2.4}$$

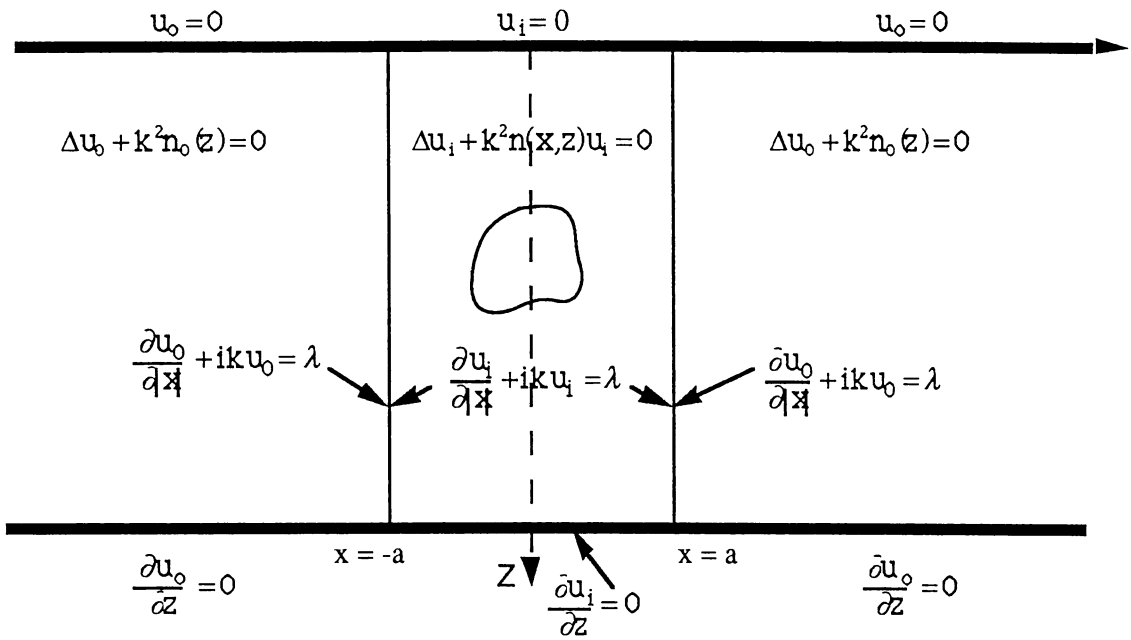

Figure 2.1 Coupling of finite element and normal mode regions.

Equations (2.1)-(2.4) will be referred to as Problem 2. This problem defines an operator G_i:

$$u_i(x,z) = (G_i\lambda)(x,z), \qquad (x,z) \in \Omega \tag{2.5}$$

In $\Omega^e = [0,h] \times \{x \mid |x| > a\}$, we look for u_o such that

$$\Delta u_o + k^2 n_o(z) u_o = 0, \tag{2.6}$$

$$u_o(x,0) = 0, \qquad |x| \geq a, \tag{2.7}$$

$$\frac{\partial u_o}{\partial z}(x,h) = 0, \qquad |x| \geq a, \tag{2.8}$$

$$\frac{\partial u_o}{\partial |x|} + iku_o = \lambda, \qquad 0 \leq z \leq h, \ |x| = a \tag{2.9}$$

$$u_o \text{ is outgoing.} \tag{2.10}$$

Equations (2.6)-(2.10) will be referred to as Problem 3. It defines an operator G_e:

$$u_o(x,z) = (G_e\lambda)(x,z). \tag{2.11}$$

We define a function $u(x,z)$ in $R_b^2 = \{(x,z) \in R^2 \mid 0 < z < h\}$ by

$$u(x,z) = \begin{cases} (G_i\lambda)(x,z) + G_i\left(\frac{\partial u^i}{\partial |x|} + iku^i\right)(x,z), & \text{if } |x| < a \\ (G_e\lambda)(x,z) + u^i(x,z), & \text{if } |x| > a \end{cases} \tag{2.12}$$

Then $u(x,z)$ satisfies

$$\Delta u + k^2 n(x,z) u = 0 \qquad \text{in } R_b^2 \setminus \Gamma_a$$

and (1.2)-(1.4).

In order for $u(x,z)$ to be the solution to Problem 1, we need to choose λ such that $u(x,z)$ is smooth on $\Gamma_a = \{(x,z) \mid |x| = a,\ 0 \le z \le h\}$. From the definition of $u(x,z)$, we know that $\left(\frac{\partial}{\partial |x|} + ik\right)u$ is also continuous on Γ_a. If $u(x,z)$ is continuous on Γ_a, then $\frac{\partial u}{\partial |x|}$ is continuous on Γ_a. Therefore, $u(x,z)$ is a solution to Problem 1.

The requirement of $u(x,z)$ being continuous implies that

$$(G_i - G_e)\lambda = u^i - G_i\left(\frac{\partial u^i}{\partial |x|} + iku^i\right) \qquad \text{on } \Gamma_a. \tag{2.13}$$

For a given incoming wave u^i, Problem 1 is equivalent to solving (2.13) for λ on Γ_a. The solution to Problem 1 is obtained by substituting λ into (2.12).

The procedure for solving (2.13) is divided into the following four steps:

(1) Construct the operator G_i by the finite element method

(2) Construct the operator G_e by normal mode expansion

(3) Solve (2.13) by seeking a least square solution λ

(4) Use (2.12) to construct the numerical solution.

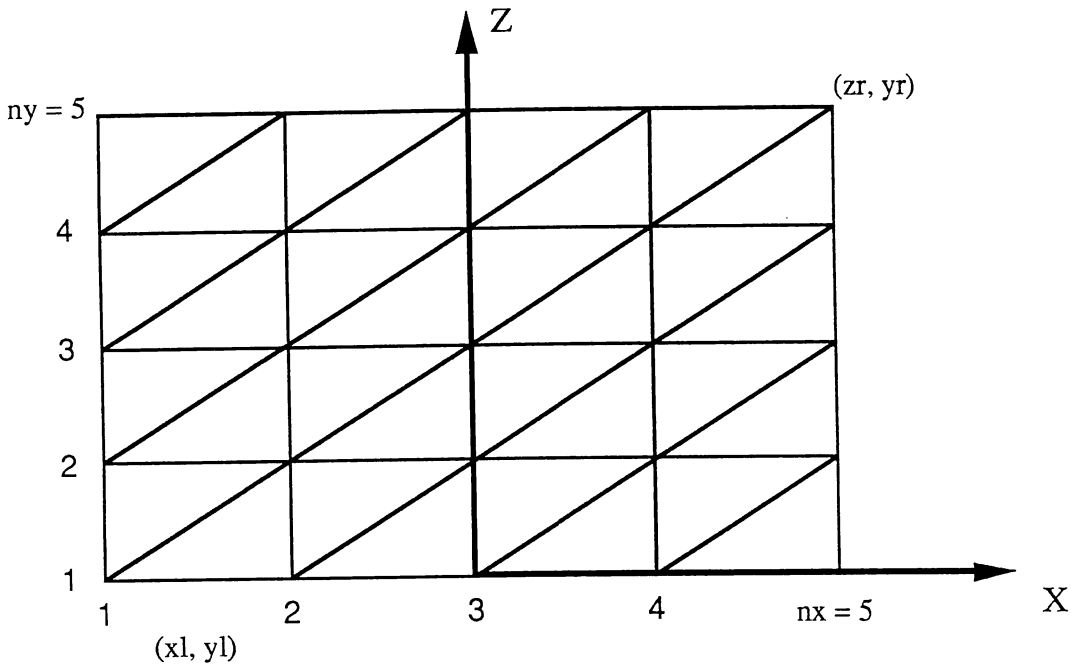

Figure 2.2 Finite element discretization of region containing inhomogeneity.

A more detailed discussion of each step will be presented in the sequel. Numerical test results are presented.

2.2 Algorithm for the Coupled Problem

2.2.1 A Finite Element Algorithm for Problem 2

A standard finite element algorithm is used for solving Problem 2. This algorithm gives the action of the operator G_i acting on a given function λ. We have used a similar algorithm for absorbing boundary conditions as Kirsch and Monk's. The theoretical discussion is omitted, only the numerical scheme is presented.

Approximate Ω by the union Ω_h of linear triangular elements as shown in Figure 2.2.

Let $S_h \subset H^1(\Omega)$ denote the finite element space, $G_i^h\lambda$, the usual finite element solution of Problem 2. Thus G_i is approximated by G_i^h: $H^{-\frac{1}{2}}(\Gamma_a) \to S_h$ and

$$\left(\nabla G_i^h\lambda, \nabla\phi_h\right) - k^2\left(nG_i^h\lambda, \phi_h\right) + ik\left\langle G_i^h\lambda, \phi_h\right\rangle - \left\langle \lambda, \phi_h\right\rangle = 0$$

for all $\phi_h \in S_h$ and $\lambda \in H^{-\frac{1}{2}}(\Gamma_a)$. Here $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\Gamma_a)$ or the dual bracket in $\left\langle H^{-\frac{1}{2}}(\Gamma_a), H^{\frac{1}{2}}(\Gamma_a)\right\rangle$.

2.2.2. Normal Mode Expansion Solution for Problem 3

Using a normal mode expansion, we can represent the solution of Problem 3 as

$$u_0(x,z) = (G_e\lambda)(x,z)$$
$$= \sum_{n=1}^{\infty} \frac{\lambda_n\left(\frac{x}{|x|}\right)}{ik(1+a_n)}\phi_n(z)e^{ika_n(|x|-a)}, \quad |x| > a, \tag{2.14}$$

where

$$\lambda(x,z) = \sum_{n=1}^{\infty} \lambda_n\left(\frac{x}{|x|}\right)\phi_n(z), \qquad |x| = a, \tag{2.15}$$

a_n, $\phi_n(z)$ are the n^{th} eigenvalue and normalized eigenfunction for the eigenvalue problem

$$\phi_n''(z) + k^2\left(n_0^2(z) - a_n^2\right)\phi_n(z) = 0 \tag{2.16}$$

$$\phi_n'(h) = 0, \quad \phi_n(0) = 0. \tag{2.17}$$

2.2.3. Regularization Solution of (2.13)

Discretizing equation (2.6) by using the finite element solution $G_i^h\lambda_p$ and normal mode expansion solution $G_e\lambda_J$ for $\lambda_J \in M_J := \left\{ f \,\middle|\, f(x,z) = \sum_{n=1}^{J} c_n(x)\phi_n(z), \ |x| = a \right\}$, we obtain a linear system

$$\left(P_J G_i^h - G_e\right)\lambda_J = P_J\left(u^i - G_i^h\left(\frac{\partial u^i}{\partial |x|} + iku^i\right)\right), \quad \text{on } \Gamma_a, \tag{2.18}$$

where P_J is the projection from $C(\Gamma_a)$ to M_J. We then approximate the solution of Problem 1 by

$$u_J^h(x,z) = \begin{cases} G_i^h\lambda_J^h(x,z) + G_i^h\left(\frac{\partial u^i}{\partial |x|} + iku^i\right)(x,z), & \text{if } |x| < a, \\ G_e\lambda_J^h(x,z) + u^i(x,z), & \text{if } |x| > a. \end{cases} \tag{2.19}$$

Observe that the corresponding homogeneous equation of (2.18) may have a non-trivial solution, unlike the case discussed by Kirsch and Monk [2]. In fact, for $u^i = 0$, Problem

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$$u_0(x,z) = (G_e\lambda)(x,z)$$
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where

$$\lambda(x,z) = \sum_{n=1}^{\infty} \lambda_n\left(\frac{x}{|x|}\right)\phi_n(z), \qquad |x| = a, \tag{2.15}$$

a_n, $\phi_n(z)$ are the n^{th} eigenvalue and normalized eigenfunction for the eigenvalue problem

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$$\left(P_J G_i^h - G_e\right)\lambda_J = P_J\left(u^i - G_i^h\left(\frac{\partial u^i}{\partial |x|} + iku^i\right)\right), \quad \text{on } \Gamma_a, \tag{2.18}$$

where P_J is the projection from $C(\Gamma_a)$ to M_J. We then approximate the solution of Problem 1 by

$$u_J^h(x,z) = \begin{cases} G_i^h\lambda_J^h(x,z) + G_i^h\left(\frac{\partial u^i}{\partial |x|} + iku^i\right)(x,z), & \text{if } |x| < a, \\ G_e\lambda_J^h(x,z) + u^i(x,z), & \text{if } |x| > a. \end{cases} \tag{2.19}$$

Observe that the corresponding homogeneous equation of (2.18) may have a non-trivial solution, unlike the case discussed by Kirsch and Monk [2]. In fact, for $u^i = 0$, Problem

1 may have non-trivial solutions. For example, let $n(x,z)=n_0^2$ for $|x|<1$, $n(x,z)=1$ for $|x|>1$, Problem 1 has the non-trivial solution

$$u(x,z)=\begin{cases} a_1 e^{\alpha x}\phi_n(z) & x<-1 \\ \left(a_2 e^{i\beta x}+a_3 e^{-i\beta x}\right)\phi_n(z) & |x|<1 \\ a_4 e^{-\alpha x}\phi_n(z) & x>1 \end{cases} \tag{2.20}$$

where n is an integer such that $k^2=\left(n-\frac{1}{2}\right)^2\pi^2-\alpha^2>0$ and $n_0^2=\frac{\left[\left(n-\frac{1}{2}\right)^2\pi^2+\beta^2\right]}{k^2}$, $\alpha=\beta\cot\beta$ for given $\beta>0$. In order to avoid the problem caused by the zero eigenvalue, we look for the solution to the regularized problem

$$\min_{\lambda\in M_J}\left\{\left\|\left(P_J G_i^h-G_e\right)\lambda-P_J\left(u^i-G_i^h\left(\frac{\partial u^i}{\partial|x|}+iku^i\right)\right)\right\|+\varepsilon\|\lambda\|^2\right\}, \tag{2.21}$$

where ε is a small positive number.

2.3 Numerical Experiments

The code, based on the algorithm above was tested on two examples with known exact solutions. The results are presented below.

Test Example 1.

$$n(x,z)=\begin{cases} 2 & \text{for } |x|\le 0.01,\ 0\le z\le 1 \\ 1 & \text{for } |x|>0.01,\ 0\le z\le 1 \end{cases}$$

$$u^i(x,z)=\sin[1.5\pi z]e^{ika_2 x}$$

where

$$k=26.5,\ a_2=\sqrt{1-\frac{1.5^2\pi^2}{k^2}}$$

Analytic scattered solution:

$$u_e^s(x,z)=\sin[1.5\pi z]\cdot u_2^s(x),$$

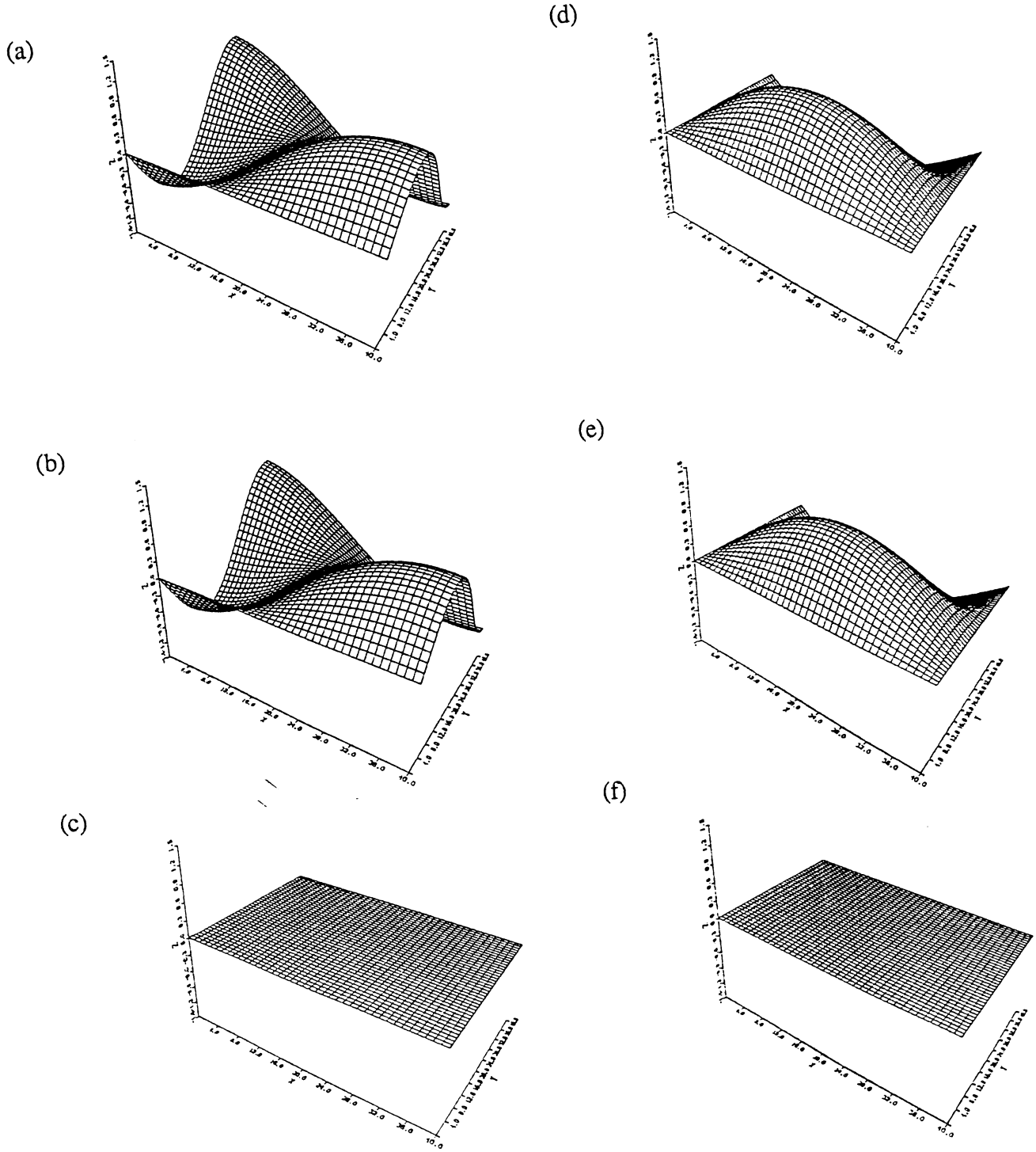


Figure 2.3 Comparison of analytic and numerical results for test examples 1 and 2. Items (a)-(c) show the analytic, numerical and error in the imaginary part of the total field in test example 1. Items (d)-(f) show the analytic, numerical and difference in the real part of the total field in example 2.

where

$$u_2^s(x)=\begin{cases} c_2 e^{ika_2 x}, & x>0.01 \\ c_3 e^{ik\sqrt{1+a_2^2}x}+c_4 e^{-ik\sqrt{1+a_2^2}x}-e^{ika_2 x}, & |x|<0.01 \\ c_1 e^{-ika_2 x}, & x<-0.01 \end{cases}$$

$$\begin{aligned} c_1 &= -5.955\times10^{-2}+i\ 0.2305 \\ c_2 &= -5.962\times10^{-2}+i\ 0.2429 \\ c_3 &= \quad 0.8179+i\ 0.1167 \\ c_4 &= 9.1789\times10^{-2}+i\ 0.1123 \end{aligned}$$

The analytic solution and computational solution as well as their difference are plotted in Figure 2.3.

Test Example 2

$$n(x,z)\equiv 1, \quad \text{for all } (x,z)$$

$$u^i(x,z)=\sin[1.5\pi z]e^{ika_2 x},$$

$$k=26.5,\ a_2=\sqrt{1-\frac{1.5^2\pi^2}{k^2}}.$$

The total solution is $u(x,z)=u^i(x,z)$. The plots are shown in Figure 2.3.

Computational Example 1

$$n(x,z)=\begin{cases} 2, & x^2+(z-0.5)^2\le 0.05^2 \\ 1, & \text{elsewhere} \end{cases}$$

$$u^i(x,z)=\sum_{n=1}^{30}\phi_n(z_0)\varphi_n(z)e^{ika_n|x-x_0|},\ x_0=-6000,\ z_0=0.1$$

where $\phi_n(z)$, a_n are the nth eigenfunction and eigenvalue of problem (2.16), (2.17) for $h=1$, $k=24$. The near fields of the incident, total and scattered waves are plotted in Figure 2.4.

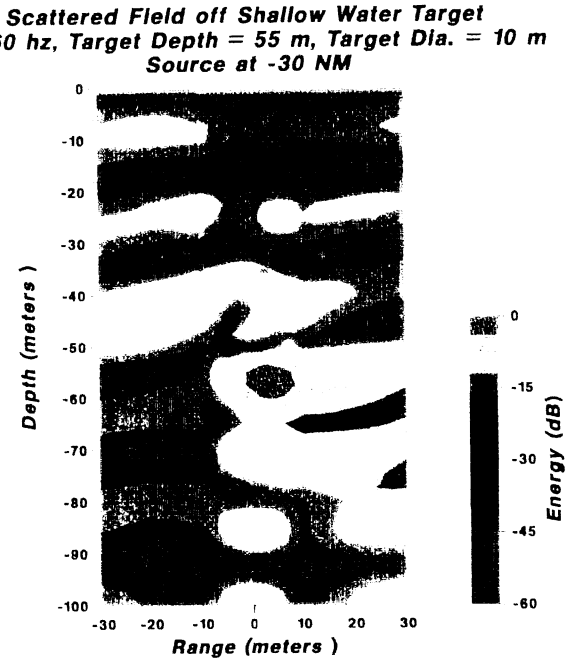


Figure 2.4 Energy in scattered field as described in computational example 1

Computational Example 2

$$n(x,z)=\begin{cases}2, & (x-0.005)^2+\dfrac{(y-.7)^2}{100}\le 0.01^2\\ 1, & \text{elsewhere}\end{cases}$$

$$u^i(x,z)=\phi_4(z)e^{ika_4x},\quad k=26.5$$

The near fields of the incident, total and scattered waves are plotted in Figure 2.5.

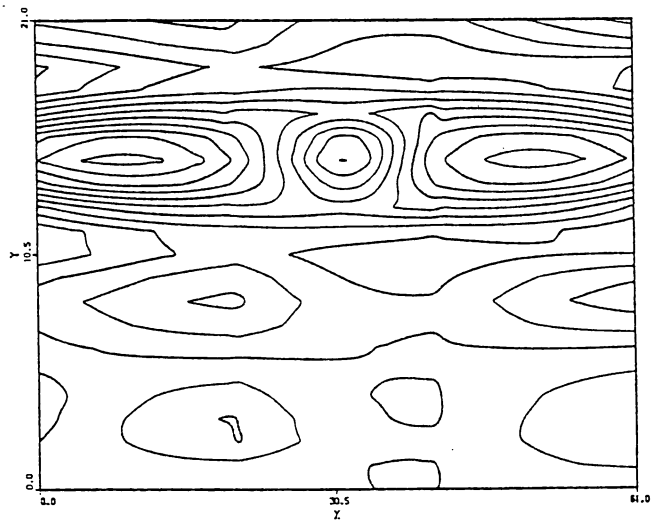
Computational Example 3

$$n(x,z)=\begin{cases}2, & x^2+(y-1)^2\le 0.05^2,\ |x|\le 0.02\\ 1, & \text{elsewhere}\end{cases}$$

$$u^i(x,z)=\sum_{n=1}^{18}\phi_n(z_0)\phi_n(z)e^{ika_n|x-x_0|},\ k=26.5,\ x_0=-5.0,\ z_0=0.025$$

The near fields of the incident, total, and the scattered waves are plotted in Figure 2.6.

(a)



(b)

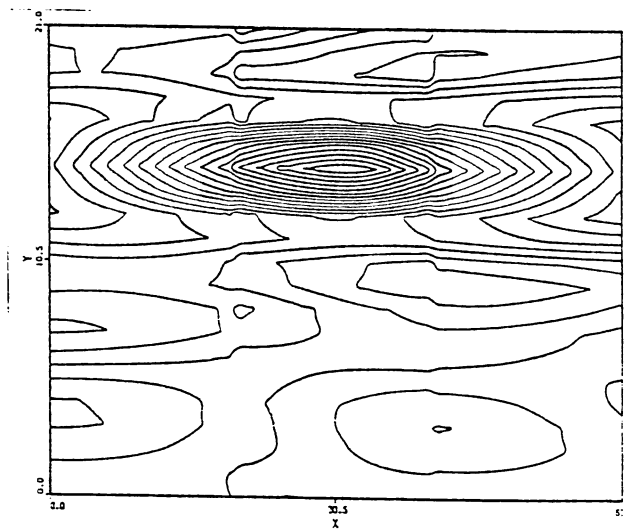


Figure 2.5 Real (Item (a)) and imaginary (Item (b)) parts of scattered field as described in computational example 2.

III. Determination of Inhomogeneity Shape From Scattered Field

3.1 Formulation of the Inverse Problem

Let $G^*(x,z;\xi,\zeta)$ be the Green's function satisfying the equation

$$\Delta G^* + k^2 n_*^2(x,z) G^* = -\delta(\underset{\sim}{x} - \underset{\sim}{\xi}) \tag{3.1}$$

with boundary condition

$$G^*(x,0;\xi,\zeta) = 0 \tag{3.2}$$

$$\frac{\partial G^*}{\partial z}(x,h;\xi,\zeta) = 0 \tag{3.3}$$

and outgoing radiation condition as $|x| \to \infty$.

Writing

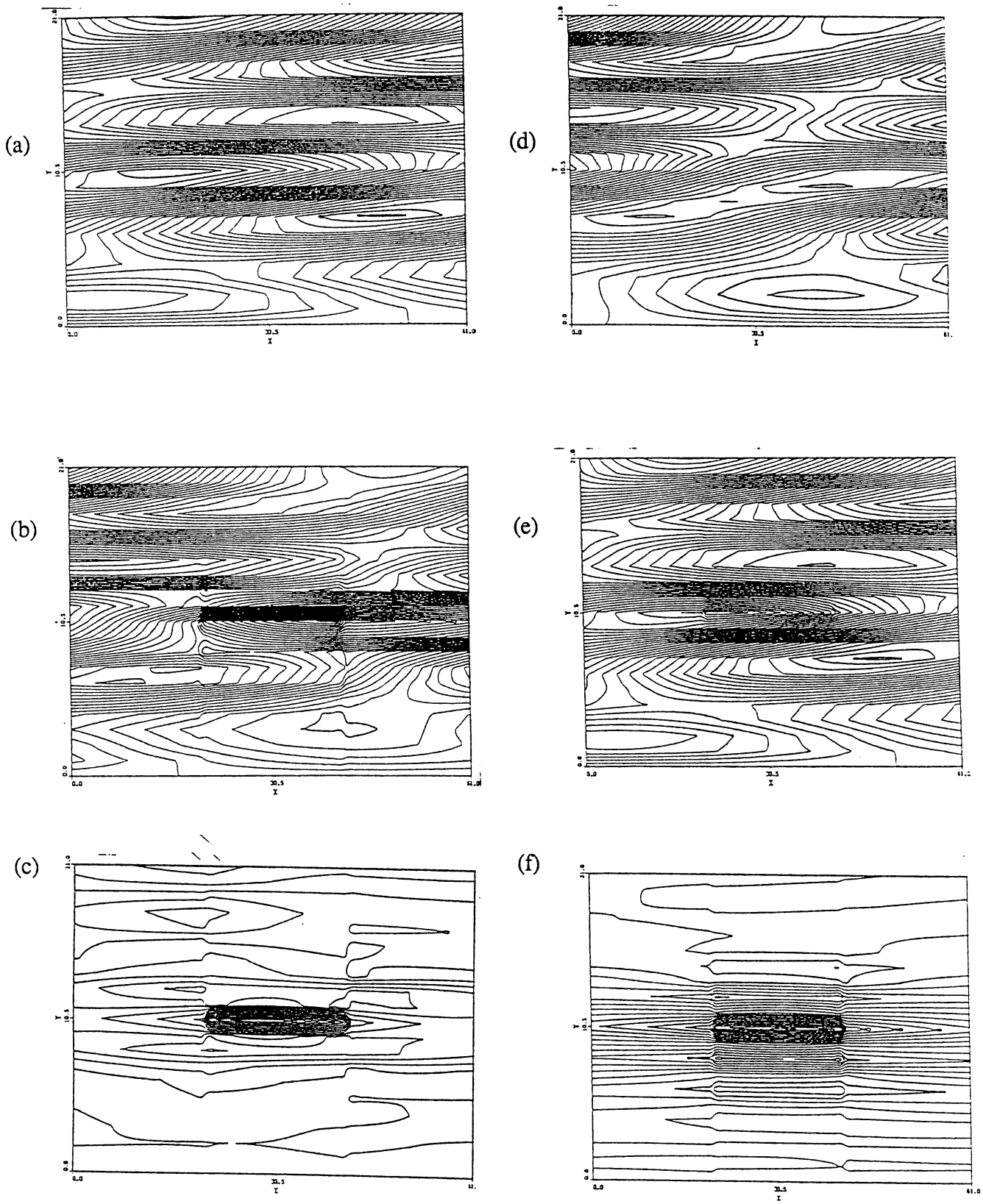


Figure 2.6 Incoming, total and scattered fields in computational example 3. Items (a)-(c) show the real part of the incoming, total and scattered fields, respectively, while items (d)-(f) show the corresponding imaginary parts of the solution.

$$m(x,z) = n^2(x,z) - 1, \quad m^*(x,z) = n_*^2(x,z) - 1,$$

and rewriting 1.1 in the form

$$\Delta u^s + k^2 n_*^2(x,z) u^s = k^2 \left[m_*(x,z) - m(x,z) \right] u^s - k^2 m(x,z) u^i, \tag{3.4}$$

we obtain the formula

$$u(x,z) = u_*(x,z) + k^2 \int_D G^*(x,z;\xi,\zeta) \left[m(\xi,\zeta) - m_*(\xi,\zeta) \right] u(\xi,\zeta) d\xi d\zeta \tag{3.5}$$

where $u_*(x,z)$ is the total field produced by the wave $u^i(x,z)$ incident upon the scatterer with the refraction index $n_*(x,z)$.

i.e.,

$$u_*(x,z) = u^i(x,z) + k^2 \int_D G^*(x,z;\xi,\zeta) n_*(\xi,\zeta) u^i(\xi,\zeta) d\xi d\zeta. \tag{3.6}$$

Since $n_*(x,z)$ is known, $u_*(x,z)$ can be produced numerically by the coupled finite element and normal mode methods.

The inverse problem associated with scattering an incident wave $u^i(x,z)$ by the unknown inhomogeneity now can be formulated according to the measurements made in the near or far-field. For measurements made in the near-field at the set of points (x_ℓ, z_ℓ), $\ell = 1, 2, \cdots, L$, we obtain from (3.5) the relations

$$k^2 \int_D G^*(x_\ell, z_\ell; \xi, \zeta) \left[m(\xi,\zeta) - m_*(\xi,\zeta) \right] u(\xi,\zeta) d\xi d\zeta = b_\ell, \tag{3.7}$$

where b_ℓ is the complex number corresponding to the difference in the measured value of $u(x_\ell, z_\ell)$ and the calculated value of $u_*(x_\ell, z_\ell)$:

$$b_\ell = u(x_\ell, z_\ell) - u_*(x_\ell, z_\ell) \tag{3.8}$$

Next consider the case when measurements are made in the far-field. The asymptotic form of $G^*(x,z;\xi,\zeta)$ as $|x| \to \infty$ is needed. Decompose the Green's function $G^*(x,z;\xi,\zeta)$ in the form

$$G^*(x,z;\xi,\zeta) = G(x,z;\xi,\zeta) + \tilde{G}(x,z;\xi,\zeta), \tag{3.9}$$

where

$$G(x,z;\xi,\zeta) = \frac{2}{h} \sum_{n=1}^{\infty} \sin\left[\left(n - \frac{1}{2} \right) \frac{\pi z}{h} \right] \sin\left[\left(n - \frac{1}{2} \right) \frac{\pi \zeta}{h} \right] e^{ika_n |x-\xi|},$$

$$a_n = \left[1 - \left(n - \frac{1}{2}\right)^2 \frac{\pi^2}{k^2 h^2}\right]^{\frac{1}{2}}, \tag{3.10}$$

observe that $\tilde{G}(x,z;\xi,\zeta)$ is continuous on (x,z) and (ξ,ζ) since it satisfies the integral equation

$$\tilde{G}(x,z;\xi,\zeta) = k^2 \int_D G\left(x,z;\hat{\xi},\hat{\zeta}\right) G\left(\hat{\xi},\hat{\zeta};\xi,\zeta\right) m_*\left(\hat{\xi},\hat{\zeta}\right) d\hat{\xi} d\hat{\zeta}$$
$$+ k^2 \int_D G\left(x,z;\hat{\xi},\hat{\zeta}\right) \tilde{G}\left(\hat{\xi},\hat{\zeta};\xi,\zeta\right) m_*\left(\hat{\xi},\hat{\zeta}\right) d\hat{\xi} d\hat{\zeta} \tag{3.11}$$

Since a_n is purely imaginary as $n > N := \left[\frac{1}{2} + \frac{kh}{\pi}\right]$, $G(x,z;\xi,\zeta)$ has the asymptotic form

$$G(x,z;\xi,\zeta) = \frac{2}{h} \sum_{n=1}^{N} \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi z}{h}\right] \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi \zeta}{h}\right] e^{-ika_n \frac{x}{|x|}\xi} e^{ika_n|x|} + O\left(\frac{1}{|x|}\right), \quad \text{as } |x| \to \infty. \tag{3.12}$$

Hence, $\tilde{G}(x,z;\xi,\zeta)$ has the asymptotic form

$$\tilde{G}(x,z;\xi,\zeta) = \frac{2}{h} \sum_{n=1}^{N} \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi z}{h}\right] e^{ika_n|x|} \tilde{f}_n\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right), \tag{3.13}$$

where

$$\tilde{f}_n\left(\frac{x}{|x|}\right) = \int_D \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi \hat{\xi}}{h}\right] e^{-ika_n \frac{x}{|x|}\hat{\xi}} G^*\left(\hat{\xi},\hat{\zeta},\xi,\zeta\right) m_*\left(\hat{\xi},\hat{\zeta}\right) d\hat{\xi} d\hat{\zeta},$$

and $G^*\left(\hat{\xi},\hat{\zeta},\xi,\zeta\right)$ has the asymptotic form

$$G^*(x,z;\xi,\zeta) = \frac{2}{h} \sum_{n=1}^{N} \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi z}{h}\right] e^{ika_n|x|} \tilde{f}_n^*\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right), \text{ as } |x| \to \infty \tag{3.14}$$

where

$$f_n^*\left(\frac{x}{|x|}\right) = \sin\left[\left(n-\frac{1}{2}\right)\frac{\pi\zeta}{h}\right] e^{-ika_n\frac{x}{|x|}\xi} + \tilde{f}_n\left(\frac{x}{|x|}\right).$$

From (3.6) and (3.5)

$$u^s(x,z) = \frac{2}{h}\sum_{n=1}^{N}\sin\left[\left(n-\frac{1}{2}\right)\frac{\pi z}{h}\right] e^{ika_n|x|}\left\{\left(f_n^*\left(\frac{x}{|x|}\right), k^2 n_* u^i\right) + \left(f_n^*\left(\frac{x}{|x|}\right), k^2(m-m_*)u\right)\right\} + O\left(\frac{1}{|x|}\right),$$

$$\text{as } |x| \to \infty \tag{3.15}$$

where $(\cdot,\cdot)$ is the integral over D.

The vector

$$F\left(\frac{|x|}{x}, z\right) := \left(f_1\left(\frac{|x|}{x}\right)\phi_1(z),\ f_2\left(\frac{|x|}{x}\right)\phi_2(z), \cdots,\ f_n\left(\frac{|x|}{x}\right)\phi_n(z)\right),$$

where

$$\phi_n(z) = \frac{2}{h}\sin\left[\left(n-\frac{1}{2}\right)\frac{\pi z}{h}\right],$$

$$f_n\left(\frac{|x|}{x}\right) = \left(f_n^*\left(\frac{|x|}{x}\right),\ k^2 n_* u^i + k^2(m-m_*)u\right),$$

is called the propagating far-field pattern of the scattered field $u^s(x,z)$ (cf. [4]).

For measurements made in the far-field in the direction $x > 0$ or $x < 0$ for $z \in [0,h]$, we can obtain from (3.15) the relations

$$k^2 \int_D f_n^*\left(\frac{x}{|x|}, \zeta\right)\left[m(\xi,\zeta) - m_*(\xi,\zeta)\right]u(\xi,\zeta)\,d\xi\,d\zeta = b_n\left(\frac{x}{|x|}\right), \quad n = 1,2,\cdots,N. \tag{3.16}$$

Here N is the number of propagating modes and b_n is the coefficient of the nth normal mode of the difference between the measured data and the calculated far-field associated with index of refraction $n_*(x,z)$:

i.e.,

$$b_n\left(\frac{x}{|x|}\right) = \int_0^h \frac{2}{h}\sin\left[\left(n-\frac{1}{2}\right)\frac{\pi z}{h}\right]\left[u^s(x,z)e^{-ika_n|x|}\right]dz - \left(f_n^*\left(\frac{x}{|x|}\right), k^2 n_* u^i\right). \tag{3.17}$$

Observe that the inverse problem based on far-field measurement has a particular difficulty caused by the fact that only a finite number of modes can propagate. This loss of information cannot be recovered by mathematical procedure. Some a priori knowledge of the refraction index is crucial for the purpose of recovering the obstacle shape. Discussions of this problem can be found in [3].

The inverse problem we consider in this paper can be stated as follows:

(1) Near field measurement: find the real continuous function $m(x,z)$ vanishing on the boundary of D such that (3.7) and (3.5) are satisfied.

(2) Far-field measurement: find the real continous function $m(x,z)$ vanishing on the boundary of D such that (3.15) and (3.5) are satisfied.

In this paper we will present only the numerical results for the inverse problem from near-field measurement. The obstacle shape determination from far-field measurement is discussed by Gilbert and Xu [4] using a Colton-Monk type method [6]. The result for Problem 2 will be presented in a separate paper.

3.2 Minimization Algorithm for Solution of the Linear Functional Equations.

The inhomogeneity shape is determined by solving for $m(\xi,\zeta)$ in (3.7). Since $u(\xi,\zeta)$ is a function of $n(\xi,\zeta)=1+m(\xi,\zeta)$, (3.7) is a nonlinear equation of $m(\xi,\zeta)$. Following Weston, we simplify the problem by introducing a function

$$w=(m-m_*)u$$

and consider the linearized equations

$$Gw(x_\ell,z_\ell) := k^2\int_D G^*(x_\ell,z_\ell;\xi,\zeta)w(\xi,\zeta)d\xi d\zeta = b_\ell \tag{3.18}$$

with constraints (cf. [5]).

$$\text{Im}\ \overline{w}\left[u_*+Gw\right]=0 \tag{3.19}$$

$$\frac{\left|u_*+Gw\right|}{|w|}\geq c>0 \quad \text{for some constant } c. \tag{3.20}$$

The inhomogeneity is given through

$$m-m_*=\frac{w}{\left[u_*+Gw\right]}. \tag{3.21}$$

In this paper, we present a minimization algorithm to approximate the solution w.

Define the functional

$$J(w)=\sum_{\ell=1}^{L}\left|k^2\int_D G^*(x_\ell,z_\ell;\xi,\zeta)w(\xi,\zeta)d\xi d\zeta-b_\ell\right|^2+\left\|\mathrm{Im}\ \overline{w}\left[u_*+Gw\right]\right\|_{L^2(D)}. \tag{3.22}$$

and solve for the minimizer w^*

$$J(w^*)=\min_{w\in C(M)}J(w) \tag{3.23}$$

where

$$C(M)=\left\{u|\ u \text{ is continuous in D},\ |u|\le M,\ u|_{\partial D}=0\right\},$$

M is a positive constant. It is clear that if $M=\infty$, then $J(w^*)=0$. (w^* may not be unique.)

Let

$$\overline{w}u_*=\phi(x,z)-i\psi(x,z), \tag{3.24}$$

(3.21) implies

$$J(\phi,\psi)=\sum_{\ell=1}^{L}\left|V(\phi+i\psi)(x_\ell,z_\ell)-b_\ell\right|^2+\left\|(\psi,\phi)W\begin{pmatrix}\psi\\ \phi\end{pmatrix}-\psi\right\|_{L^2(D)} \tag{3.25}$$

where

$$W=\begin{pmatrix}L_I & -L_R\\ L_R & L_I\end{pmatrix}$$

and

$$L_If:=k^2\int_D\left(\mathrm{Im}\ G^*\right)\left[u_*(x,z)\overline{u}_*(\xi,\zeta)\right]^{-1}fd\xi d\zeta \tag{3.26}$$

$$L_Rf:=k^2\int_D\left(\mathrm{Re}\ G^*\right)\left[u_*(x,z)\overline{u}_*(\xi,\zeta)\right]^{-1}fd\xi d\zeta \tag{3.27}$$

$$V(\phi+i\psi)(x_\ell,z_\ell):=k^2\int_D G^*(x_\ell,z_\ell;\xi,\zeta)\overline{u}^{-1}(\xi,\zeta)\left[\phi(\xi,\zeta)+i\psi(\xi,\zeta)\right]d\xi d\zeta. \tag{3.28}$$

We assume it is known that the inhomogeneity is contained in a box $D=[-d,d]\times[h_1,h_2]$. A pair of functions (ψ,φ) is approximated by

$$\psi_{KJ}(x,z)=\sum_{k=1}^{K}\sum_{j=1}^{J}\alpha_{kj}\sin\frac{k\pi(x+d)}{2d}\sin\frac{j\pi(z-h_1)}{h_2-h_1},$$

$$\varphi_{KJ}(x,z)=\sum_{k=1}^{K}\sum_{j=1}^{J}\beta_{kj}\sin\frac{k\pi(x+d)}{2d}\sin\frac{j\pi(z-h_1)}{h_2-h_1}.$$

The discretization of $J(\varphi,\psi)$ gives a quadratic function of $\alpha_{k,j}$, $\beta_{k,j}$. A minimization subroutine is then used to produce $\alpha_{k,j}$, $\beta_{k,j}$. Finally, from (3.21) the refraction index $n(x,z)$ is recovered by

$$n(x,z)=1+m_*(x,z)+\frac{1}{|u_*|^2}\varphi[1+L_R\varphi-L_I\psi]^{-1}.$$

The convergence of this algorithm will be presented in a separate paper. The numerical result will be presented at this conference.

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