θ-Dependence of QCD at Finite Isospin Density

Max Metlitski

Based on:

M. Metlitski, A. Zhitnitsky, Phys. Lett. B633 721, (2006)

M. Metlitski, A. Zhitnitsky, Nucl. Phys. B731 309, (2005)

θ-Parameter

$$L_\theta = i\theta \times \frac{g^2 G\widetilde{G}}{32\pi^2}$$

- θ-parameter: a handle on the topological properties of the theory
- θ-dependence often nontrivial and non-analytic

- Light fermions make θ-dependence a chiral property

$$L_m = \overline{\psi}\,\frac{1+\gamma^5}{2} M^\dagger \psi + \overline{\psi}\,\frac{1-\gamma^5}{2} M \psi$$

$$\psi \text{ ® } e^{i\theta\gamma^5/2N_f}\psi, \qquad M \text{ ® } e^{-i\theta/N_f} M$$

- $\theta = \pi$ - one quark mass negative

- Can address θ-dependence in the Chiral Lagrangian framework!

Finite Density

- Study topological properties at finite μ
- Understand interplay between μ and θ
- In $N_c = 3$ QCD, $\mu_B \gg \Lambda_{QCD}$ – understood
- Study $N_c = 3$ QCD, finite isospin density $\mu_I \ll \Lambda_{QCD}$

  (Similar picture in $N_c = 2$ QCD, finite baryon density $\mu_B \ll \Lambda_{QCD}$)
- Can use Chiral Lagrangian in this regime
- Good news for lattice:

  Positivity:

  $N_c = 2$, $N_f = 2 \bullet k$ at finite μ_B or μ_I, $\theta = 0$

  $N_c = 3$, $N_f = 2 \bullet k$ at finite μ_I, $\theta = 0$
- $\theta = \pi$, real but not positive

Chiral Lagrangian

- Full analytical control for μ_I, m « Λ_{QCD}
- Chiral Symmetry fixes μ dependence of low-energy Lagrangian

$$L = F^2 Tr(\nabla_\mu U \nabla_\mu U^\dagger) - \Sigma \,\mathrm{Re}\, Tr(MU)$$

$$\nabla_0 U = \partial_0 U - \frac{1}{2}\mu_I[\sigma^3, U], \quad \nabla_i U = \partial_i U$$

- Low lying excitations in vacuum ($N_f = 2$):

Triplet of pions: $$m_\pi^2(\theta) = \frac{m(\theta)\left|\langle\bar{\psi}\psi\rangle_0\right|}{4F^2}$$

$$m(\theta) = \frac{1}{2}\left((m_u + m_d)^2 \cos^2(\theta/2) + (m_u - m_d)^2 \sin^2(\theta/2)\right)^{1/2}$$

$$\frac{m_\pi^2(\theta = \pi)}{m_\pi^2(\theta = 0)} = \frac{|m_u - m_d|}{m_u + m_d}$$

Physics at Finite μ_I

$\theta = 0$ Normal Phase | Superfluid Phase: $U(1)_I$ broken

$\mu_I < m_\pi$ | $\mu_I > m_\pi$

$$n_I = \frac{1}{2}\left\langle \overline{\psi}\gamma^0\sigma^3\psi \right\rangle = 0 \qquad n_I = \frac{1}{2}\left\langle \overline{\psi}\gamma^0\sigma^3\psi \right\rangle = 4F^2\mu_I\left(1 - \frac{m_\pi^4}{\mu_I^4}\right)$$

$$i\left\langle \overline{u}\gamma^5 d \right\rangle = 0 \qquad i\left\langle \overline{u}\gamma^5 d \right\rangle = -\frac{1}{2}\left(1 - \frac{m_\pi^4}{\mu_I^4}\right)^{1/2}\left\langle \overline{\psi}\psi \right\rangle_0$$

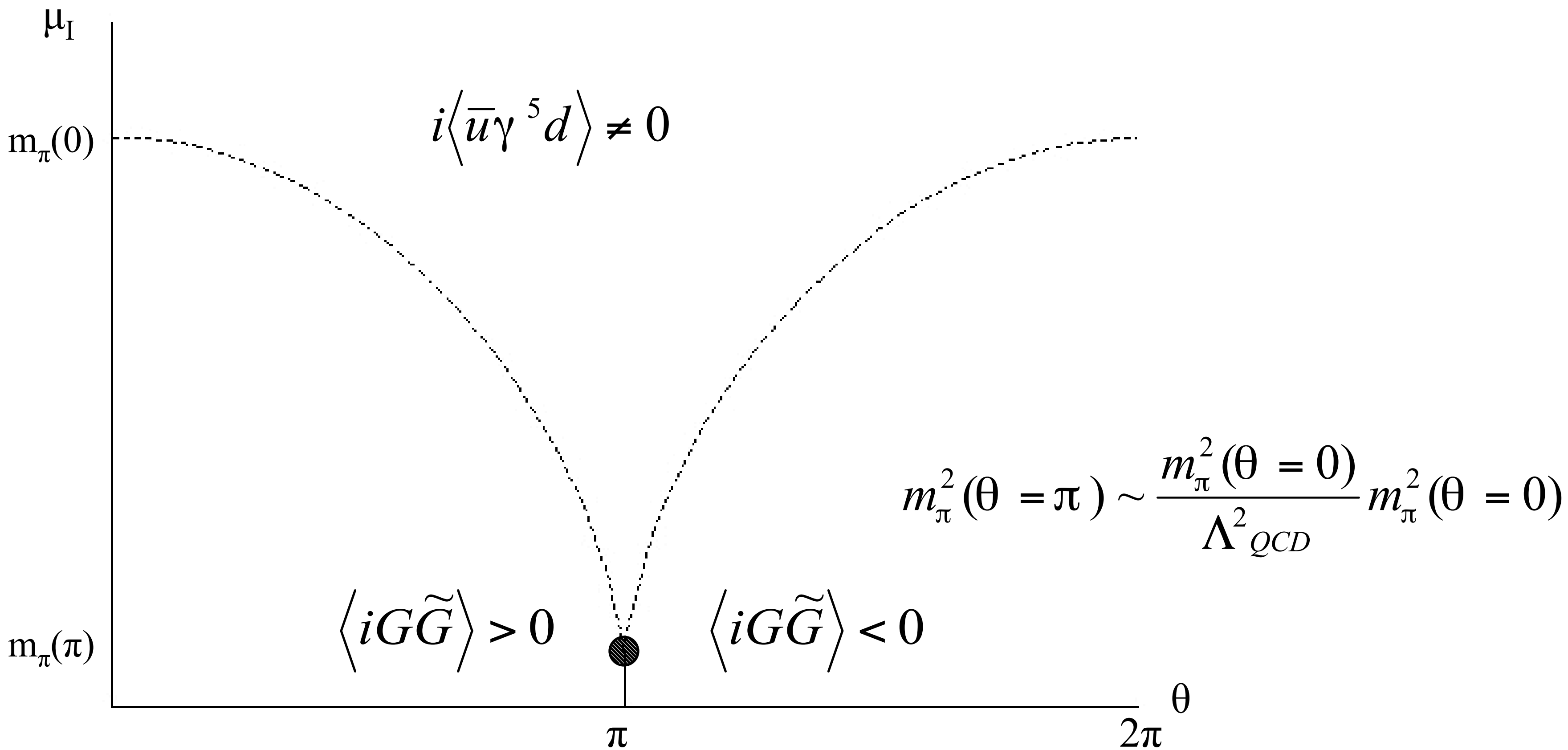
$$\left\langle \overline{\psi}\psi \right\rangle = \left\langle \overline{\psi}\psi \right\rangle_0 \qquad \left\langle \overline{\psi}\psi \right\rangle = \frac{m_\pi^2}{\mu_I^2}\left\langle \overline{\psi}\psi \right\rangle_0$$

$$i\left\langle \overline{u}\gamma^0\gamma^5 d \right\rangle = 0 \qquad i\left\langle \overline{u}\gamma^0\gamma^5 d \right\rangle = 4F^2\mu_I\frac{m_\pi^2}{\mu_I^2}\left(1 - \frac{m_\pi^4}{\mu_I^4}\right)^{1/2}$$

$\theta \neq 0$ $\mu_c = m_\pi$ ® $\mu_c = m_\pi(\theta)$

$$i\left\langle \overline{\psi}\gamma^5\psi \right\rangle \neq 0 \qquad \left\langle \overline{u}d \right\rangle \neq 0$$

θ-Dependence of $N_c = 3$, $N_f = 2$ Theory ($m_u = m_d$)



Dashen's Phenomenon disappears in the superfluid phase!

Topological Susceptibility

- $\Omega(\mu,\theta)$ – known
- Can compute correlators of $G\tilde{G}$ by differentiating!

$$\chi = \frac{\partial^2 \Omega}{\partial \theta^2} = \int d^4x \left\langle T \frac{g^2 G\tilde{G}}{32\pi^2}(x) \frac{g^2 G\tilde{G}}{32\pi^2}(0) \right\rangle_{conn}$$

- At $\theta=0$,

Normal Phase:

$$\chi(\mu) = -\frac{1}{4} m \langle \overline{\psi}\psi \rangle_0$$

Superfluid Phase:

$$\chi(\mu) = -\frac{1}{4}\frac{m_\pi^2}{\mu^2} m \langle \overline{\psi}\psi \rangle_0 \overset{!}{=} -\frac{1}{4} m \langle \overline{\psi}\psi \rangle(\mu)$$

Ward Identity

- Agreement between $\chi(\mu)$ and $\langle \bar{\psi}\psi \rangle(\mu)$ not coincidence!

$$\chi = \int d^4x \left\langle T \frac{g^2 G\tilde{G}}{32\pi^2}(x) \frac{g^2 G\tilde{G}}{32\pi^2}(0) \right\rangle_{conn} = -\frac{1}{N_f^2} \langle \bar{\psi} M \psi \rangle + O(M^2)$$

- Ward Identity – consequence of chiral anomaly

  insensitive to IR terms (μ, T)

- Both sides depend non-trivially on μ
- Correction,

$$O(M^2) = \frac{1}{N_f^2} \int d^4x \left\langle T \bar{\psi} M \gamma^5 \psi(x) \bar{\psi} M \gamma^5 \psi(0) \right\rangle_{conn}, \qquad \frac{O(M^2)}{\chi} \sim \frac{m_\pi^2}{m_{\eta'}^2}$$

Gluon Condensate

- Conformal anomaly:

$$\Theta^{\mu}_{\mu} = \frac{\beta(g)}{2g} G^{a}_{\mu\nu} G^{a\mu\nu} + \bar{\psi} M \psi, \quad \beta_{QCD} \approx -\left(\frac{11}{3} N_c - \frac{2}{3} N_f\right)\frac{g^3}{16\pi^2}$$

- Thermodynamics (know $\Omega(\mu)$):

$$\left\langle \Theta^{\mu}_{\mu} \right\rangle = \varepsilon - 3p$$

$$p = -\Omega, \quad \varepsilon = \Omega + \mu n, \quad n = -\frac{\partial \Omega}{\partial \mu}$$

- Independently know,

$$\left\langle \bar{\psi}\psi \right\rangle = \frac{\partial \Omega}{\partial m}$$

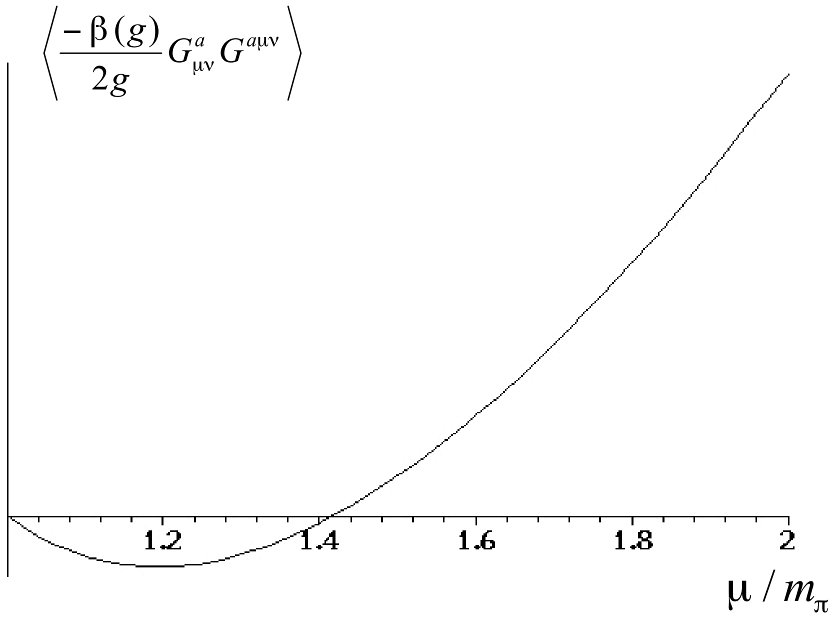
- Gluon Condensate:

$$\left\langle \frac{-\beta(g)}{2g} G^{a}_{\mu\nu} G^{a\mu\nu} \right\rangle_{\mu} - \left\langle \frac{-\beta(g)}{2g} G^{a}_{\mu\nu} G^{a\mu\nu} \right\rangle_{\mu=0} = 4F^2\left(\mu^2 - m_\pi^2\right)\left(1 - 2\frac{m_\pi^2}{\mu^2}\right)$$

More on Gluon Condensate

- Non-monotonic!
- Decrease for $\mu \sim m_\pi$
  - $\varepsilon \sim m_\pi n \gg p$
- Increase for $m_\pi \ll \mu \ll \Lambda_{QCD}$
  - $\varepsilon \sim p$ – interactions win over
- Size of correction:

$$\Delta\left\langle G^2 \right\rangle \sim \Lambda_{QCD}^2 \mu^2 << \Lambda_{QCD}^4 \sim \left\langle G^2 \right\rangle_{\mu=0}$$



- Dependence on quark mass (at $\mu = 0$) reproduces SVZ low-energy theorem:

$$\left\langle \frac{-\beta(g)}{2g} G^a_{\mu\nu} G^{a\mu\nu} \right\rangle_{m,\mu=0} - \left\langle \frac{-\beta(g)}{2g} G^a_{\mu\nu} G^{a\mu\nu} \right\rangle_{m=0,\mu=0} = -3\, m \left\langle \bar{\psi}\psi \right\rangle_0$$

Conclusion

- Chiral Lagrangian gives a lot of info for $\mu_{B,}$ μ_I, m « Λ_{QCD}

  - Gluon Condensate
  - Topological Susceptibility
  - Self-consistency of Ward Identities

  } Can be checked on the lattice!

  - Intricate θ dependence (non-analyticity at fixed μ)

  - Exciting physics for θ ~ π

    - Very small critical μ
    - Dashen's transition splitting (triple point?)