

**FIBONACCI'S SEQUENCE OBTAINED BY
A GENERAL RECURSIVE FORMULA**

By

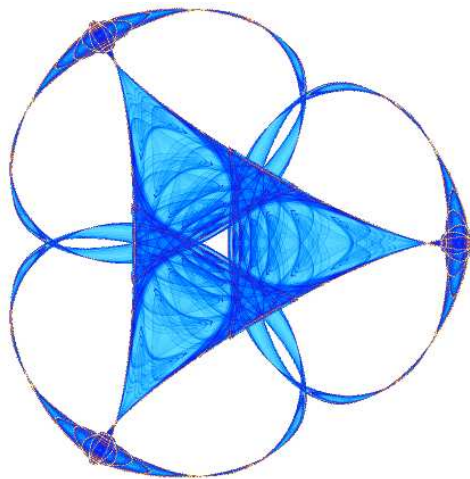
Ezio Marchi

and

Gustavo D. Olguín

IMA Preprint Series # 2201

(April 2008)



INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455-0436

Phone: 612-624-6066 Fax: 612-626-7370

URL: <http://www.ima.umn.edu>

FIBONACCI'S SEQUENCE OBTAINED BY A GENERAL RECURSIVE FORMULA

by

Ezio Marchi *) **) and Gustavo D. Olguín ***)

ABSTRACT

The most general linear difference equation has been introduced in the literature and solved theoretically by Marchi and Millán in [3]. However, the explicit formula has to be developed further for concrete applications. In some particular cases this is more suitable than that obtained with linear algebra and eigenvalues techniques. Here, we have applied our general recursive formula to two examples, namely to population dynamic where the population grows and the individuals do not die, and the second application is the classical Fibonacci's sequence. We obtain a combinatorial solution which one can compare with the Binet's formula.

KEY WORDS: Fibonacci's sequence. Recursive formula. Difference Equations

INTRODUCTION:

The Fibonacci's numbers are a sequence of numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$ where the ...rst two numbers are given and the other terms are obtained as the sum of the two previous numbers of the sequence. If we write the n -th element of the Fibonacci's sequence by f_n , then this sequence is completely defined for the equations: $f_0 = 0$; $f_1 = 1$ and for n greater or equal to 2

$$f_n = f_{n-1} + f_{n-2} \quad (1)$$

This equation is an example of the most general linear differential equation which the general form is:

$$x_{n+1} = a_k^{k+1} x_n + a_{k-1}^{k+1} x_{n-1} + a_{k-2}^{k+1} x_{n-2} + \dots + a_2^{k+1} x_2 + a_1^{k+1} x_1 + a_0^{k+1} x_0 + b_{k+1} \quad (2)$$

This last equation is the most general equation for the linear difference equations which does not appear in the literature.

The problem of this recursive sequence is the determination of x_n as a function of x_0 without give through the intermediate steps.

Let's remember that this topic is very useful since this issue is quite vast and it is applied to many topics mathematics ...elds for example to the numerical calculus, difference equations, biomathematics, etc.

*) Founder and First Director of the IMASL, Universidad Nacional de San Luis, Superior Resercher CONICET; Ejercito de los Andes 950, San Luis 5700, Argentina, email: emarchi@speedy.com.ar

**) Corresponding author.

***) Universidad Nacional de Cuyo, and CRICYT Mendoza, Argentina. Universidad Tecnológica Nacional, email: g_stavomza@yahoo.com.ar

Going back to the Fibonacci's sequence, we know that it is possible to obtain the solution by using the Binet 's formula which derived from using the simple step with linear algebra and eigenvalues techniques.

This formula is a compact way that gives the n-th term as:

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Such a formula is extraordinary, because it is de...ned in terms of an irrational number, even though all the Fibonacci's number are integers.

Now in this short paper we are going to derive among other things a general formula for the Fibonacci's numbers which is done from a combinatorial point of view.

Reference for Fibonacci's numbers might be Golberg [2], Poole [3], and Rosen [9], etc.

The equation (2) has been studied by Marchi-Millán with which they have derived the general solution. Therefore Fibonacci's numbers appear as a very particular case of it.

Now we will study in general the difference equation given by (2). Take the ...rst terms.

$$x_1 = a_0^1 x_0 + b_1$$

$$x_2 = (a_1^2 a_0^1 + a_0^2) x_0 + a_1^2 b_1 + b_2$$

$$x_3 = (a_2^3 a_1^2 a_0^1 + a_2^3 a_0^2 + a_1^3 a_0^1 + a_0^3) x_0 + (a_2^3 a_1^2 + a_0^3) b_1 + a_2^3 b_2 + b_3$$

The good observer will realize that any element of the sum of x_3 is obtained by giving "jumps" from 3 to 0 and then multiplying the corresponding elements, for example $3_i > 1, 1_i > 0; 3_i > 2, 2_i > 1, 1_i > 0$ determines the elements $a_1^3 a_0^1$ and $a_2^3 a_1^2 a_0^1$ respectively.

The coefficients b_i follow a similar law of formation. Thus, it is important to study the set of all possible "jumps".

A GENERAL FORMULA

Let for a given natural k and r less or equal to k be the set

$$E_r^k = \{ (l_1; \dots; l_r) : \sum_{i=1}^r l_i = k ; l_i \text{ positive integers} \}$$

and for an element $(l_1; \dots; l_r) \in E_r^k$ define

$$a^k(l_1; l_2; \dots; l_r) = a_{k_i l_1}^{k_i} a_{k_i(l_1+l_2)}^{k_i} a_{k_i(l_1+l_2+l_3)}^{k_i} \dots a_{k_i(l_1+\dots+l_{r-1})}^{k_i} \quad (3)$$

with $r \cdot k_i = \bar{k}$ and $\bar{k} = l_1 + \dots + l_r$

In the paper by Marchi and Millán [3], they have shown that the general solution is given by

$$x_k = \sum_{r=1}^k \sum_{(l_1; \dots; l_r) \in E_r^k} a^k(l_1; \dots; l_r) x_0 + \sum_{i=1}^k \sum_{r=1}^i \sum_{(l_1; \dots; l_r) \in E_r^{k-i}} a^k(l_1; \dots; l_r) b_i + b_k \quad (4)$$

This is indeed the solution of the difference equation given by (2).

The proof of this fact which depends only on the coefficient a^0 's and b^0 's has not been published yet.

We have used the induction principle and the main idea is to split the set E 's into smaller ones. Anyway we do not introduce here such proof and we will consider it in a future publication.

EXAMPLES

We will now present two examples of difference equations together with their corresponding solutions. The ...rst one is to consider all the $a^i = 1$ and the $b_i = 1$:

$$x_{n+1} = x_n + x_{n_i-1} + x_{n_i-2} + \dots + x_1 + x_0 + b_{n+1}; \quad n \geq 0 \text{ integer}$$

This model can be useful to interpret the grow of a population where individual do not die. As simple example which we do not wish to be exhaustive is the duplication of a cell, as it is described by Thom [12]

For this sequence we have that for all (l_1, \dots, l_r) and $\bar{k} \cdot k$:

$$a^r(l_1, \dots, l_r)^{\bar{k}} = 1; \quad b_i = 1$$

Therefore the answer of our problem will be

$$x_k = \sum_{r=1}^k E_r^{k-r} x_0 + \sum_{i=1}^{k_i-1} \sum_{r=1}^{k_i-i} E_r^{k_i-i-r} + b_k; \quad k = 1; 2; 3; \dots$$

Where the number of the elements of the sets E 's are just related with multisets presented in the book by Brualdi [1].

At the page 70 of this bibliography under the section 3.5 Combinations of Multiset the number of it is

$$E_r^{k-r} = \binom{k+r_i-1}{k+r_i-1} = \binom{k+r_i-1}{r_i-1}$$

The second example we now present the Fibonacci's sequence.

$$x_{k+1} = x_k + x_{k_i-1}; \quad x_{i-2} = 0; \quad x_0 = 1; \quad k = 0; 1; \dots \quad (6)$$

We emphasize that x_{i-2} is the ...rst element of the sequence. This is due to the fact that we have the sequence shifted.. If one does not make such as transformation then one has to go through a change of variable in (4).

In this case the coefficient for this sequence takes the form:

$$a_{k_i-1}^k = \begin{cases} 1 & \text{if } i=1 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases} \quad b_i = 0; \text{ for all } i.$$

In such an instance then we have that (5) takes a particular solution for the Fibonacci sequence. This is by means of a general formula. It is interesting how it is formed by the values of the coefficients. For this reason we are consider those coefficients in (4) that are non zero. They are related with the "jumps". Those contributing a positive value are those which possess the simple "jumps" with values of one or two as members of the multiplication. The simplest are the following:

For $k=1$ and $r = 1$, $l_1 = 1$ we have $\binom{1}{1}$ element. For $k=2$, $r=2$, there is the possibility $l_1 = l_2 = 1$ and then $\binom{2}{0}$ element. For $k = 3$ and $r = 2$ we have $l_1 = 2; l_2 = 1$ or $l_1 = 1$ and $l_2 = 2$. The total number is $\binom{2}{1}$: On the other hand we have for $k = 3$ and $r = 3$: $l_1 = l_2 = l_3 = 1$: This case appears with the number $\binom{3}{0}$:

In order to obtain a general expression for an arbitrary number of "jumps" we give explicitly some more numbers for k .

Let $k = 4$; then for $r=3$ we have $l_1 = 1; l_2 = 2; l_3 = 1; l_4 = 2; l_2 = 1; l_3 = 1$ or $l_1 = 1; l_2 = 1; l_3 = 2$

in this case the number of elements is $\binom{3}{1}$: For $k = 4$; and $r = 4$; it appears that $l_1 = 1; l_2 = 1; l_3 = 1; l_4 = 1$: The number of elements is $\binom{4}{0}$: For $k = 5$; and $r = 3$; it appears that $l_1 = l_2 = 2$ and $l_3 = 1; l_1 = 2; l_2 = 1$ and $l_3 = 2$; and $l_1 = 1; l_2 = l_3 = 2$: Consequently, we have $\binom{3}{2}$ elements. For $k = 5$; and $r = 4$; then we have $l_1 = l_2 = l_3 = 3; l_4 = 2; l_1 = l_2 = l_4 = 1; l_3 = 2; l_2 = l_3 = l_4 = 1; l_2 = 2$: All these elements are $\binom{4}{1}$: Finally $k = 5$ and $r = 4$ all $l_i = 1$ and then we have $\binom{5}{0}$: One might follow in this way as one wishes. In order to develop the number of x_k of Fibonacci's sequence one arrives to the general expression

$$x_{k+1} = \sum_{r=1}^{k+1} \binom{k+1}{r} x_{k+1-r} \quad k = 0; 1; 2; \dots \quad (7)$$

Where $[a]$; $a \geq 0$ is the lower non-negative number greater than the entire part of number a .

By the general formula expressing the solution given by Marchi and Millán, the last expression is the solution of the Fibonacci's sequence. However, there is a particular interest to prove directly that such expression is indeed the solution. For this we will apply the induction principle.

Take $k = 1$ in (6). Then

$$x_2 = \sum_{r=1}^2 \binom{2}{r} x_{2-r} = \binom{2}{0} x_2 + \binom{2}{1} x_1 = 2 + 1 = \sum_{r=1}^1 \binom{3}{r} x_{1-r} + x_0 = x_1 + x_0$$

Here, this is the first step of the induction principle. Now assume that the formula (7) it is true for an even $k = 2p$ where p is a non negative integer. We get: take k in (7) and $k-1, k-2$ then we write down their expressions

$$x_{k+1} = \sum_{r=1}^{k+1} \binom{k+1}{r} x_{k+1-r} = \binom{k+1}{0} a + \binom{k+1}{1} a + \binom{k+1}{2} b + \dots + \binom{k+1}{p+1} a$$

$$x_k = \sum_{r=1}^k \binom{k}{r} x_{k-r} = \binom{k}{0} a + \binom{k}{1} a + \binom{k}{2} b + \dots + \binom{k}{p} a \quad (8)$$

together with

$$x_{k-1} = \sum_{r=1}^{k-1} \binom{k-1}{r} x_{k-1-r} = \binom{k-1}{0} a + \binom{k-1}{1} a + \binom{k-1}{2} b + \dots + \binom{k-1}{p-1} a$$

Using the equality:-

$$\binom{n}{l} + \binom{n}{l+1} = \binom{n+1}{l+1} \quad (9)$$

with $k = n + 1$ and $l = 0$; the term with a in the bottom of x_{k+1} are eliminated with those corresponding to a and x_k and x_{k-1} : The same happens with the terms b and so until the last one.

The terms with a in x_{k+1} and x_k are the same and then eliminated.

Therefore the last term of x_{k+1} is $\binom{p+1}{p} = \binom{k-2+1}{k-2}$ and of x_k the last term is $\binom{p}{p}$: The analogous element in x_{k-1} is $\binom{p}{p-1}$: By the relation (9) they cancel out. That is to say the equation

$$x_{k+1} = x_k + x_{k-1}$$

is valid.

In the case that k is odd, then $k = 2p + 1$; where p is a non-negative integer.

We have the expression (8) in this case is similar to the expression for even k . The only term that we have to be careful with one are the last one in their expressions. For x_{k+1} the last term is $\binom{p+1}{p+1}$; analogously the last term in x_k is $\binom{p+1}{p}$ and then the corresponding term in x_{k-1} is $\binom{p}{p}$:

Therefore the first part cancels out we the remaining two. Therefore (6) holds true also for odd k .

Thus for the induction principle we have proved that (6) is valid.

This establishes a new formula for Fibonacci's numbers which has a combinatorial character. This matter is surely related with the material explained in the book by Brualdi [1]. Perhaps in the future we are going to use it for our studies in this matter.

Thus we have the general formula (5) is rather powerful and general for a combinatorial point of view that makes easier than those given by linear algebra computing the corresponding eigenvalue, for example see Poole [8].

FINAL REMARKS

In this paper we have observed that the general formula may be applied to many problems as for instance how it is applied to population dynamics.

Another application was presented in the paper by Marchi and Vila [5]. As it is well known the Ising model with nearest-neighborhood interacting has attracted the attention of many investigations, in particular there are several methods for solving it. For example the matrix method and the generating function method. However, when more than the nearest-neighbor interactions are assumed in the physical problem, then their solutions require approximations.

In the same paper, we developed a new recursive method which allows us to obtain the exact analytical solution when higher-neighbor interactions are taken into account. We applied an argument which allows as far as the third neighbor interaction but we indicated the method for possible higher interaction.

We present a new method for evaluating the partition function in the thermodynamic limit, finally an application for this method to the computation of the recursive matrix is worked out.

Another subject we have done was the integration or the exact solution of the rate equation for the three level laser. The system of the two non linear differential equations describing a three level laser was solved exactly. It related the number of the photons and the population inversion. The function describing this physical quantities are expanded in a power series and recurring relations are obtained. This was solved and the solution agrees with experimental results.

Further, we point out that the equation (7) has been generalized by Marchi and Morillas [6] for any type of polynomial. In our knowledge we do not know whether

this has been applied yet.

Finally, it would be very interesting to take limit and to pass to the continuous.

All this material is related with the solution of recursive equations. Perhaps there would be some relations with the functional integrals. This could be interesting.

ACKNOWLEDGMENTS

We acknowledge a suggestion indicating to us the material about multisets by Dr D. Jaume

- [1] R.A. Brualdi: Introductory Combinatorics. Prentice Hall. 1999
- [2] Goldberg, S.: Introduction to Difference Equations. J. Wiley. 1967
- [3] Marchi, E. and L. Millán: On the Exact Solution of Differential Equations of Hodgkin-Huxley model type. Proc. First Int. Conference Mathematical Modeling. San Diego-La Jolla, AMS, vol.2, 1977, pp447-258.
- [4] Marchi, E. L. Millán and E. Crinó: Exact Solution of the Rate Equation for the three Energy Laser. IEEE Journal of Quantum Electronics. Vol Q 5-16, No 4. April 1980.
- [5] Marchi, E.; Vila, J.: Recursive Method in One Dimensional Ising Model. J. Phys. A: Gen 13(1980)pp2465-2477.
- [6] Marchi, E. and Morillas P.: The Solution of a General Polynomials Recursive Formula. Ciencia Abierta, Santiago of Chile. [http://cec.uchile.cl/~cabierta/revista/13-\(2001\)](http://cec.uchile.cl/~cabierta/revista/13-(2001)).
- [7] Pérez C.F., F. y V. Hernandez and J.M. Vargas Montaner: Ecuaciones diferenciales y en diferencias. Sistemas Dinámicos. Thomson 2003.
- [8] Poole D.: Algebra Lineal. Thomson. Spain, (2004).
- [9] Ross, D.: Laser Light and Oscillators, New York, Acad. Press (1969)
- [10] Siegman A. E., Introduction to Laser and Masers. New York. Mc Graw Hill (1971).
- [11] Takahashi, T: Ecuaciones Diferenciales con Aplicaciones. Grupo Editorial Iberoamérica, Ciudad de México 1994.
- [12] Thom R.: Stabilité Structurale et Morphogénese. Benjamin W.A. Inc 1972.
- [13] Thompson C.J.: Mathematical Statical Mechanics. New York, Mc. Millan 1972.