Essays in Heterogeneous Agent Macroeconomics

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Robert Stephen Mrkonich

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Dedication

To Marko, Bobbi, Joey, Heinz, Libby, Joe, Arlene, Siegfried, Addie, Hallie, Brea, Ilsa, and Fiona.

Abstract

This dissertation consists of three chapters. The first chapter surveys the literature on the solving of heterogeneous agents models. From the Bewley/Huggett/Aiyagari models of the 1990s that focus on idiosyncratic risk to the addition of aggregate risk in models based on Krusell et al. (1998), heterogeneous agent models have become standard workhorses of macroeconomics. This is unlikely to change as the increased availability of household level data and improvements in computational speed have made previously infeasible models not only solvable but also able to be disciplined. This chapter will explore these advances while also keeping an eye on further extensions to the research frontier using cutting edge mathematical techniques.

The second chapter presents a model of international transmission of financial shocks where the country of origin is fundamental to the transmission of the shock. A country is defined by the quality of its financial markets, with financially-developed countries better able to insure against idiosyncratic shocks. Highly developed countries tend to accumulate larger positions in riskier, but more productive, capital flows, as seen in the data. When a financial shock occurs, the ability to insure is impaired, which lessens demand for risky foreign capital, which lowers production abroad. We interpret the Financial Crisis of 2008 as a change in the ability of financial market quality and calibrate the model to match the change in capital flows. Importantly, the calibrated model matches not only changes in capital flows, but also relative movements in interest rates as well as changes in debt flows.

The third chapter explores the dynamics of entrepreneurship in the United States. There is concern about the observed decline in entrepreneurship over the past 30 years and whether it will continue. This chapter argues the decline reflects a change in the timing of entrepreneurship decisions because of increased educational attainment and its associated cost. In the data, I observe a stark divergence. A greater number of older workers and fewer young people are choosing to become entrepreneurs. I use an overlapping generations model with endogenous education and entrepreneurship choices to explain the changing age composition of entrepreneurs in the U.S. I find that trends in education costs, the skill premium, and the compression of morbidity quantitatively explain the change in the age composition of entrepreneurs. In a series of sensitivity analyses, I establish that, to successfully match the observed rates, it is important to take into account each of these trends. An additional implication of the model is that efficiency increases sharply with a better educated workforce indicating that decreased entrepreneurship might not be as troubling a trend as previously thought.

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Chapter 1

Solving Heterogeneous Agent Models: A Literature Review

1.1 Introduction

The use of heterogeneous agent models has become a fundamental component of modern macroeconomics. To correctly evaluate the effects of policy changes, it is important to model heterogeneity in a variety of dimensions such as age, ability, wealth, and preferences. This heterogeneity allows for policy evaluations on a variety of fronts: how does the wealth/income distribution change? How does risk sharing and portfolio composition respond across those distributions? How do policies affect a population that is aging differently from the standard pyramidal age distribution? Although heterogeneous agent models are useful at answering these questions, they are much more difficult to work with than a representative agent model.

This chapter aims to track the developments of heterogeneous agent models from Bewley (1977), Huggett (1993), and Aiyagari (1994), which feature idiosyncratic risk, and their extensions, to models with aggregate risk that rely upon the work of Krusell et al. (1998). The focus of this paper will be on advances in the solving of these heterogeneous agent models. Prior to the technological advances in computation of the past 30 years, the use of representative agent models was prevalent. Representative agent models rely upon the work of Gorman (1951), which provides conditions on the utility function that allow for the aggregation of utility-maximizing individuals into a single representative agent. Most simply, when thinking of homothetic preferences, the resulting allocations can be broken into proportional bundles that are given to the heterogeneous agents that comprise the representative consumer. When compared to previous macroeconomic modeling strategies, representative agent models were much more difficult to solve. The difficulty in handling representative agent models put the use of heterogeneous agent models out of the realm of possibility. This did not, however, stem criticism of the practice of using a representative agent rather than heterogeneous agents. A prominent critique was from Kirman (1992). His paper states that the use of representative agent models "is not simply an analytical convenience as often explained, but is both unjustified and leads to conclusions that are usually misleading and often wrong." His paper focuses on appropriately addressing unemployment in macroeconomic models, and while some of the criticisms have been refuted, the general point that heterogeneous agent models have greater potential both theoretically and quantitatively in matching household choices and the data remains accurate.

In addition to the solving of heterogeneous agent models, another key improvement in the model's viability is the rapidly increasing availability of household and firm level micro data. The depth and breadth of this data allows for more precise calibration of models, as well as insight into the appropriate way of modeling household preferences and firm costs. Papers such as Guvenen et al. (2017) are able to access and link census data with social security numbers to provide incredible insight into household incomes over the past six decades. The resolution of this data also leads to new questions that were not possible to ask in the past. This generates new, more complicated models that require advancing both our computational abilities and the mathematic tools used to solve our problems.

Currently, heterogeneous agent models are solved in a variety of ways. One method is to solve by value function iteration. In this method, the value function is a fixed point solved for recursively after an initial guess. This method is usually reliable, but generally slow as you must solve over a grid and interpolate frequently. Other methods rely on various methods of approximating, among the most common of which are first order perturbation methods, such as (log)linearization and (in certain specific cases) linear quadratic approximation. For both of these methods, we have seen numerous speed improvements arise from a variety of methods such as the endogenous grid method and the use of adaptive sparse grids. This does not alter how the problem is solved, but rather improves upon the efficiency of our solution mechanism. In the end, the solution mechanism is reliant upon a search mechanism that typically finds local equilibria via a search method that approximates the equilibrium. In the case of models with multiple equilibria or models with many non-convexities this can lead to complicated and time consuming search methods. There has been an increasing focus on solving for all equilibria via global solution methods. This literature review will briefly discuss a new mathematical method for computing all global solutions that takes advantage of new (to economists at the least) mathematical techniques from algebraic geometry. In the end, there are many reasons to believe that new mathematical tools will provide more accurate solutions to existing problems as well as drastically expand the set of potential models we can solve.

The remainder of this chapter is dedicated to tracing these developments in the use of heterogeneous agent models. Initially, I will briefly discuss the widening availability of micro data that makes heterogeneous agent models even more important going forward. Following this, I will review the seminal works of the heterogeneous agent literature. Then, I will explore the different solution methods and tricks used to solve these models, including a look to the future and cutting edge solution techniques that have potential to solve currently infeasible models. A brief conclusion closes the chapter.

1.2 Increased Data Availability

In addition to the technical advances in computational ability that will be detailed below, there has also been a significant increase in the quality and quantity of data available. This allows for improved disciplining of current heterogeneous agent models, as well as inspires new models by sparking new research questions. As my focus is on heterogeneous agent macroeconomics, this section will be a brief review of improvements in household level data, but the increased use of heterogeneous agent models in non-macroeconomic fields such as industrial organization is most certainly helped by the increased availability of firm level micro-data. Furthermore, I will emphasize data sets available for the United States, though similar quality improvements are occurring globally.

The U.S. has many available household level data sets, including but not limited to the Panel Survey of Income Dynamics (PSID), the Survey of Consumer Finances (SCF), the National Longitudinal Survey of the Youth (NLSY), and the Current Population Survey (CPS). Prior to expounding on what has changed, it is important to understand what these surveys typically cover. Started in 1968, the PSID is the oldest continually running longitudinal survey of households. The PSID collects a variety of information on health, education, income, and savings decisions. Unlike the other surveys, however, it top codes the highest income levels. The SCF is older initially, starting in the wake of World War II before being discontinued in the early 1970s. It was reintroduced in 1983 and is now issued triennially. The survey collects basic demographic data such as health and education while focusing on employment history and savings and consumption decisions. Both the PSID and SCF focus on the head of household (called a primary economic unit in the SCF), while the PSID has kept detailed data on other family members throughout the course of its existence. The SCF, on the other hand, has adjusted to allow for more detailed information about other members activities over time. The NLSY is administered by the Bureau of Labor Statistics and is longitudinal data tracking two cohorts over time across a wide variety of indicators, such as education, health, and employment history. The initial NLSY cohort is from 1979, while the second is from 1997. The initial ages of the 1979 (NLSY79) cohort ranged from 14-22, while for the 1997 (NLSY97) cohort the age range is from 12-16. Both the NLSY and PSID are nationally representative samples, while the SCF oversamples the top of the wealth distribution, but provides representative population weights which translate the sample to a nationally representative one. Finally, the CPS is also maintained by the Bureau of Labor Statistics, and focuses on employment while also collecting demographic information relevant to the labor force.

What has changed in the usefulness of these data sets? First and foremost, time has passed. There are now over five decades worth of panel data from the PSID. The NLSY79 now has 40 years' worth of data, and its respondents are now entering retirement at a time of an impending entitlement crisis when data on end of life decision making is of paramount importance. As time continues to pass, the NLSY97 becomes more and more useful in concert with the NLSY79 in determining how education, health, and employment decisions have changed over time, as both are nationally representative longitudinal data sets. Currently, the oldest members of the NLSY97 cohort are in their mid-30s, which has limited its effectiveness in the past but is now leading to a highly useful look at how the financial crisis affected decision making among the youth at the time. In the end, the effectiveness of dynamic heterogeneous agent models is amplified with every additional year of data collection in these longitudinal surveys.

In addition to simply having enough relevant data points because of time, the scope of questions asked in previous versions of these surveys was much narrower than the current versions. Some of this increase in scope is natural. As an example, the 1992 Survey of Consumer Finances is the first to explicitly ask about student debt holdings. Student debt became much more accessible after the Higher Education Amendments of 1989 and 1992. In this sense, the SCF responded as quickly as possible to the changing circumstances of the time and adjusted the questionnaire. This is also the case in the wake of the financial crisis. After the crisis, over 75 new variables were added from the 2007 survey in the 2010 survey, and an additional 20 more questions were added to the 2013 survey. It is also not simply the volume of questions added, but also the range of potential answers to the questions that has expanded drastically, whether it be additional information on financial instruments, education, or employment sector. As a point of comparison, the codebook for the 2013 SCF is over 1000 pages long while the 1986 codebook is 138 pages. The same holds generally for the NLSY, which now includes a variety of additional questions that were not originally asked.

To really get a good understanding of the new breadth of these publicly available data sets it is useful to look at Rios-Rull and Kuhn (2016), which does a deep summary of the 2013 Survey of Consumer Finance. Their paper looks at the distribution of earnings, income, and wealth and tries to disentangle which assets contribute to the observed high degree of concentration at the tops of each of those distributions. The granularity of their results is astounding, with the ability to breakdown the household balance sheet into more than 20 different instruments while also conditioning on a variety of other control variables, such as sex, marital status, education, and more. This is not a frivolous exercise, as their conclusions provide important insight into understanding what we mean by the poor and the rich. Specifically, the earnings poor are very different from the income poor which in turn can be very different from the wealth poor. Additionally, they document only a "modest" increase in earnings inequality, a "weak" increase in income inequality, and a "substantial" increase in wealth inequality, while simultaneously documenting a weak correlation between income and wealth. They are able to explain this weak correlation because of the granularity of the information of household balance sheets and find that the main driver of inequality is business income and business wealth, and, in fact, these swamp capital income, which "even for the richest households [...] accounts for less than one third of income." This work highlights key differences in different types of inequality, which adjusts how models must be created to match these moments.

While publicly available data have improved greatly, the most interesting macroeconomic dataset is the one obtained by Guvenen et al. (2017). The authors have access to administrative data from the Social Security Administration for earnings histories of an incredibly large panel dataset. The data come directly from filed W-2 forms, and allow for an incredibly thorough understanding of the evolution of lifetime earnings across gender. The key takeaway from both Rios-Rull and Kuhn (2016) and Guvenen et al. (2017) is that our understanding of what our models must capture, as well as our ability to discipline them, is growing as quickly as our technical ability to solve models. While this literature review is less worried about taking the models to the data and more on solution mechanisms, it would be remiss to not mention that Fernandez-Villaverde et al. (2016) is an excellent resource for an in-depth description of how to properly calibrate models and how to take macroeconomic models to the data.

Having motivated the growing importance of heterogeneous agent models in practice, as well as the practicality of actually taking these models to the data, the next section will focus on the seminal papers in the development of heterogeneous agent modeling.

1.3 Seminal Papers

This section is concerned with the papers that have laid the foundation for other work in heterogeneous agent models. This leads to five key papers: Bewley (1977), Huggett (1993), Aiyagari (1994), Angeletos (2005), and Krusell et al. (1998). The first four deal with idiosyncratic risk only, while Krusell et al. (1998) obviously extends the heterogeneous agent modeling to include aggregate risk.

Bewley (1977) was the first paper to feature a general equilibrium model where agents face idiosyncratic income risk and do not have complete markets to insure that risk away. The best they can do is partially insure using a risk-free asset, such as an uncontingent bond. Bewley ties the research in his paper to Milton Friedman's permanent income hypothesis, though he acknowledges his differs in that Friedman's version is "directly or indirectly observable" while his paper focuses on setting the marginal utility of expenditure to a constant. Bewley comments that his model features households that "compensate for economic fluctuations by saving and dis-saving," giving the first hint of the idea of precautionary savings, which will become a focus of research in the later heterogeneous agent models listed above.

A large gap in time persists between this first foray in 1977 by Bewley, and the rise of heterogeneous agent models in the 1990s and beyond. Huggett (1993) is the next major work to use a heterogeneous agent economy. Huggett's use of heterogeneous agent modeling comes as a result of the shortcomings of representative agent economies in Mehra and Prescott (1985). In Mehra and Prescott (1985), interest rates are much too high relative to that observed in the data. Huggett instead switches from a representative agent framework to a heterogeneous agent framework with idiosyncratic endowment risk, as in Bewley (1977), with the ability to hold only a risk-free asset. Agents are allowed to borrow or save in that asset, but face a borrowing constraint of a year's income. This yields equilibrium interest rates that are significantly lower than those in a representative agent framework. This is the practical effect of a precautionary savings motive in an incomplete markets framework. While a representative agent with the same utility function might have the same precautionary savings motive, the existence of a representative agent means that fully state-contingent bonds are available among heterogeneous agents allowing for the use of a single agent instead. In terms of the desired outcome, the increased savings in this Huggett economy lead to an equilibrium interest rate, r^* that must satisfy $\beta * (1+r^*) < 1$, where β is the discount rate. In an economy with complete markets, $\beta * (1+r^*) = 1$, which implies that the equilibrium return will be lower in a Huggett economy. How much lower will depend on the size and persistence of the shocks and the nature of borrowing constraints.

Aiyagari (1994) extended the models of Bewley and Huggett to include production. Now, instead of facing idiosyncratic endowment risk, households face idiosyncratic labor income risk. The asset in the economy is no longer an uncontingent bond, but rather capital. Aiyagari (1994) builds off the model of Brock and Mirman (1972). In this paper, Aiyagari first puts into words the idea of precautionary saving, where households accumulate more assets (in his model, capital) today to help insure against future shocks and smooth consumption. This leads to a significantly different optimal savings rule than in the case of complete markets. In an Aiyagari economy, aggregate capital would be higher than the aggregate capital of a complete markets world. Aiyagari's formulation of the model is the basis for many of the models shown later, and is presented here now.

$$\max_{c_t, a_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1.1}$$

subject to

$$c_t + a_{t+1} = wl_t + (1+r)a_t \tag{1.2}$$

where $c_t \geq 0,\ a_t \geq -b$, almost surely. Here, uncertainty takes the form of idiosyncratic labor endowment shocks. Aiyagari's solution to this model is standard value function iteration using piece-wise linear interpolation over a grid that is finer at the lower end and coarser at the top. This is the standard starting point of computation for heterogeneous agent models, and will be a baseline for comparisons later in this chapter.

The next model to explore is another variation of Aiyagari (1994) with only idiosyncratic (as opposed to aggregate) risk, however in Angeletos (2005) this risk is not on labor income, but rather manifests itself as production risk. Specifically, Angeletos (2005) has a model of the form:

$$\max_{c_t^i, k_t^i, b_t^i} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t^i) \tag{1.3}$$

subject to

$$c_t + k_t^i + b_t^i = F(k_t^i, n_t^i, A_t^i) - \omega_t n_t^i + R_t b_t^i + \omega_t \tag{1.4}$$

Where the i superscript denotes the individual, and A_t^i is the idiosyncratic productivity shock to production the individual faces. This model is almost identical, except in how the uncertainty manifests itself. This, however, has strong implications for optimal capital holdings. In stark opposition to Aiyagari (1994), where more assets are accumulated than in a complete markets environment, in this model less capital is accumulated. These highly similar models provide starkly different policy recommendations. This is yet another reason why the increased resolution of data mentioned in the previous section is so important: There is a need to identify exactly what shocks individuals face, and more precise and frequent data being collected will help to inform our choice of models. As with the Aiyagari model, this formulation by Angeletos is used frequently as a starting point, especially in research on entrepreneurship such as Buera et al. (2015), Buera et al. (2011), and Buera and Moll (2012).

The final seminal paper that this review will touch on is Krusell et al. (1998). This paper updates the Aiyagari (1994) setting to include aggregate risk. In this case, the idiosyncratic risk is again on labor income rather than production risk, and there are aggregate productivity shocks. For comparison with the previous models, here the objective function is the same (maximize time-discounted expected utility of consumption), while the constraint now takes the form:

$$c + k' = r(\bar{k}, \bar{l}, z)k + w(\bar{k}, \bar{l}, z)l\epsilon + (1 - \delta)k \tag{1.5}$$

The two relevant risk variables are ϵ, which is idiosyncratic, and z which is aggregate. ϵ is either 0 or 1 and captures unemployment. z also takes two values, $\{z_b, z_g\}$, meaning the aggregate state of the economy is bad or good, and it is also a Markov process. As in an Aiyagari model, there is a borrowing constraint, here instead of an arbitrarily low number, it is a limited liability constraint as it is assumed capital must be non-negative. Because of aggregate uncertainty it is important to track the distribution of household over capital holdings and employment status, Γ. Note that the rate of return in the model only depends on total holdings of capital not the distribution of capital holdings.

The key contribution of Krusell et al. (1998) is the solution algorithm itself. The endogenous state variable, Γ, is of large dimension, and this makes numerically solving the problem increasingly difficult. The novel solution to this issue of increasing state space size is the assumption of boundedly rational agents that have beliefs about the path of Γ over time and the errors that are made because they are not fully rational are not of significant magnitude. Specifically, agents only have beliefs on a finite (say I) number of moments of the distribution Γ, $\mathbf{m}$. Let the law of motion for these beliefs be a function H_t, such that $\mathbf{m}' = H_t(\mathbf{m}, z, z')$. The Krusell-Smith algorithm is then:

1. Select the moments I that are relevant

2. Guess a form for H_t and its parameters

3. Solve consumer problem given H_t

4. Use consumers' decision rules to simulate the economy for a large number of agents, N over T (large) time periods.

5. Re-estimate the parameters for the assumed functional forms to obtain R^2 (check for stationarity of data when estimating).

6. If these parameters are sufficiently close to the initial guess, stop. If parameter values converge, but R^2 is not sufficiently high, increase the number of moments, I or change functional forms for H_t.

This algorithm is the main insight of the paper, and is still the most common method of solving heterogeneous agent models. However, much of the cutting edge research in computational economics is on finding faster and more accurate solutions to these problems. As will be shown in the next section, these advances improve upon the Krusell - Smith algorithm, but still lack the ability to solve models where the distribution cannot be estimated by a finite set of moments.

Having now seen the various formulations of heterogeneous agent models and their basic solution algorithms, the next section will show how computational advancements have allowed for us to solve more complex versions of these models as well as solve them more quickly.

1.4 Solving Heterogeneous Agent Models

This section focuses on improvements to the solving of heterogeneous agent models. This section is broken into subsections focusing on particular improvements, tricks (not meant in a pejorative way), and methods for solving these types of problems. Each section provides a representative example of the method and explains why it works and what its benefit is. This is then followed by a brief recap of a recent paper that updates the general idea. The last subsection will focus on potential avenues for future success in solving heterogeneous agent problems. Note that this listing is not comprehensive. A full covering of all the improvements to the seminal papers mentioned above would be hundreds of pages long (at the very least). This is meant to provide an idea of how to think about overcoming the limitations the field has faced by providing some relevant examples and recent breakthroughs as well as to provide a road map for future avenues of research.

1.4.1 Computational Languages and Hardware Improvement

This subsection is less focused on “tricks” and rather on improvements in basic computer functioning. Moore’s law states that computational power doubles roughly every two years. While there is evidence of this slowing down post-2015, the gains in computational speed from 1998, when the Krusell-Smith algorithm was introduced, to today are sufficiently large to merit notice. There are now over twenty billion transistors in a circuit chip, while in 1998 there were only five million. The corresponding computational power increase allows us to solve more complex models than those in Aiyagari (1994) and Krusell et al. (1998) in shorter order than it took to originally solve those models. In a recent Scientific American article, multiple experts answered the same question: “How do we keep increasing computational speed?” Each expert answered similarly: The speed of computer devices is limited by the physical speed of electron movement through matter, which is already a significant fraction of the speed of light. The speed of computation is not just the speed of computer devices, as people tend to forget about is the importance of the algorithm used to computational speed, with one expert noting the gains available from quantum computing and quantum algorithms are still unknown but potentially large.

Aruoba and Fernández-Villaverde (2015) looks into how effective different computing languages are for solving economics problems. In their paper, they implement an algorithm for value function iteration with grid search (the same method as in Aiyagari (1994)) across a variety of languages and on different operating systems. They find that Fortran and C++ are the quickest, with the caveat that the choice of compiler is very important. Julia is between 2.64 and 2.7 times longer in execution time than the best Fortran/C++ compilers, as compared to 9 to 11 times slower for MatLab. Python is significantly slower in its baseline form, but has the ability to be wrapped in C++, which reduces this gap to roughly 3-5 times slower than Fortran/C++. There is one caveat that ought to be noted: the authors did not optimize the algorithm for each language, they used the same algorithm for all. This might lead to misleading conclusions. The best comparison would be the most efficient implementation of value function iteration with grid search for each language. That being said, this is unlikely to change the general ranking of the languages in terms of speed and efficiency in execution, but instead would perhaps alter the size of the gains and losses.

The final important improvement is in the ability of economists to use these coding languages effectively. As mentioned above, the quality of the algorithm is likely much more responsible for the differences in execution time than many other factors. There is also a benefit to reduced coding time. Newer languages, such as Julia and Python, are much easier to use for less experienced coders, especially when compared to languages like Fortran. Furthermore, Thomas Sargent (winner of the Nobel Memorial Prize for Economics for his work in introducing dynamic programming to macroeconomics) and John Stachurski have created a series of free, publicly available lectures on quantitative macroeconomics that provides quality example code and lessons in both Python and Julia. The availability of resources and the quality of them have positive externalities in that they bring more bright minds into the fold, which, in turn, generates new ideas for algorithms, or allows those with complex mathematical knowledge practical tools to implement new solution mechanisms.

Regarding more complex languages, such as Fortran or C++, there are a variety of benefits of these languages to solving complex heterogeneous agent models beyond their raw speed. In C++, for example, the ability to create objects, classes, and functionals simplifies code and allows for easier updating of the model. Rather than having to change individual loops when adding a state variable, simply create a "state" class, and add the new member to the existing vector of members. If in loops throughout the code you have simply referred to the length of the vector of states, the loops will automatically update. This minimizes coding and debugging time, as well as leads to cleaner algorithm implementation. Sargent and Stachurski note that these higher level languages have a much larger cost in terms of time to learn the language, and for that reason they argue in favor of Python and Julia for newcomers. In many senses, they are likely correct. However, I argue that in the case of heterogeneous agent models, and especially those with aggregate shocks, the time investment in the language is fundamental to making these models solvable in practical time frames and in using the full power of the improved processors in computers.

1.4.2 Continuous Time Economics

The first true "trick" or method this review will be concerned with is the use of continuous time economics. Continuous time economics is not a new innovation, but it is a useful element in the toolkit for solving heterogeneous agent models. The main benefit is that it leads to significant algebraic simplification, which in turn leads to either an analytic solution or a simpler computer problem to solve. The remainder of this subsection will be devoted to two examples of the use of continuous time to solve complicated heterogeneous agent models, namely Moll (2014) and Sannikov and Brunnermeier (2012).

Moll (2014) looks at the impact of financial frictions on aggregate productivity in a entrepreneurship model. His model is along the lines of Angeletos (2005), though in continuous time. To give a basic idea of the environment, there exists a finite continuum of infinitely-lived entrepreneurs who each receive entrepreneurial ability shocks and must choose whether to produce or not at a given instance. Entrepreneurs have access to a sufficiently large supply of labor to meet any demand they might have, but face a collateral constraint when financing their business. The capital they use must be a proportion λ of the assets they currently have, and, as the production technology is constant returns to scale, this constraint always binds when the entrepreneur chooses to produce. This also implies, however, that the distribution of wealth is not stationary, but Moll points out that the share of wealth for each production type is, and that this is sufficient. The TFP process in the model is assumed to be mean reverting.

The solution to this problem is related to its continuous time properties. As the model features a linear savings decision rule and a mean-reverting TFP process, a single, straightforward differential equation is able to characterize the steady state analytically, and, more importantly, the transition paths can be solved numerically during transitions simply. The benefit of continuous time is readily apparent: the steady state is able to be solved for analytically, rather than numerically, in this problem, and the transition dynamics are much simpler to solve than the discrete time equivalent model.

With this all being said, there are some drawbacks to using continuous time economics in this case and in general. Computers work in discrete units, so in order to solve the transition dynamics numerically and approximation must be taken. Furthermore, the data used to discipline the model is in discrete time as well, so the translation of the data into parameters of the model must be carefully done.

Much of the above holds with the case of Sannikov and Brunnermeier (2012). In Sannikov and Brunnermeier (2012), they have an exceptionally complicated open economy model with a continuum of agents in each country. In addition, they have multiple assets, as well as financial frictions that introduce non-convexities. Yet, by specifying this model in continuous time, the equilibrium dynamics are again able to be described by a single state variable and a system of equations that determine the remaining endogenous variable values.

Finally, for a very recent example of the power of continuous time modeling in solving heterogeneous agent models, Achdou et al. (2017) recasts the Bewley/Huggett/Aiyagari model into continuous time. This allows the models to be characterized by a system of partial differential equations, and, in certain cases, solved analytically.

The Sannikov model is a nice segue to the next subsection which examines how it is possible to simplify state spaces using sharp mathematical tools and insights. Sannikov and Brunnermeier (2012) allows for simplification of a very difficult problem because of its continuous time formulation. But, as will be shown next, simplification is not unique to continuous time problems, and can be very powerful in the context of discrete time models as well.

1.4.3 State Space Simplification

Simplifying the state space is highly useful in heterogeneous agent problems as it reduces the dimensions of the problem, which can lead to significant gains in execution time. Including an additional state variable, say moving from one asset to two, will double the time for solving. If, instead of an additional asset, an additional shock is added the time increase could be much larger, depending on the number of possible values for that shock. Thus, investing time in simplifying the problem by finding a smaller number of variables to describe the state space can result in substantial gains in efficiency.

A good example of this is Mendoza et al. (2009), which looks at the global imbalances in financial flows in the early 2000s. The model is a straight forward update to Aiyagari (1994) and Angeletos (2005). Instead of looking at a single country, it features a finite number, $N \geq 2$ of countries. This means that instead of a single asset, there are $N+1$ assets: capital in each country and bond holdings. Rather than a shock solely on labor income, this model features $N+1$ stochastic shocks, one to wage income and one to the productivity of capital holdings in each of the N countries. The household budget constraint is of the following general form (this is not exactly as presented in their paper, but is simplified for ease of exposition and maintains the key elements of their problem):

$$c + \sum_{n=1}^{N} k_i' + b' = w + \sum_{i=1}^{N} R_k(z_i, k_i)k_i + R_b b \tag{1.6}$$

What would the state space be in this problem? It would be the idiosyncratic shock state, $s = (w, \mathbf{z})$, where w is the idiosyncratic wage and $\mathbf{z}$ is the vector of idiosyncratic investment shocks the household receives, and then the individual's bond holdings and capital holdings in the N countries. The authors, however, do not track all of these states. Instead, while they do care about policy functions for each type of asset holding, they do not care how much of each asset is held today. They simply care about the total wealth of the individual (net asset holdings) today as the portfolio choice does not depend on the composition of previous asset holdings. This simplifies the state down to the idiosyncratic shocks and a single asset. Suppose that each asset used a grid of 100 points, and the problem simplified from 3 assets (bonds and capital in two countries) to 1. The time savings during the optimization search are immense. However, note this works only because previous asset holdings do not have an impact on the portfolio choice. If, for example, there were adjustment costs, this simplification would not work.

This methodology of simplifying the state space is well suited to heterogeneous agent models with portfolio choice. An excellent example of this Chien et al. (2014), which looks at linking the quickly growing household finance literature with macroeconomic models. Their paper is an extension to their previous work that generalizes their algorithm to a wider variety of economies. In this model, households have differing access to assets in a world with both aggregate and idiosyncratic risk. Again, rather than tracking states for each asset, they simplify the state space to "rely on a single cross-sectional moment of the multiplier distribution." When approximating the multiplier distribution, they use finite histories as opposed to the Krusell et al. (1998) method of using various moments of the distribution. Their problem is still large and complex, but, through simplifying the state space, their generalization from their previous work "essentially adds one small step to the algorithm." In fact, their paper addresses three key issues of the Krusell et al. (1998) algorithm, namely that it does not handle multiple assets well, the computation of market clearing prices is slow, and guessing the state vector is not easy. Their algorithm using a recursive multiplier allows for computing state contingent prices easier as the recursive multiplier is sufficient to summarize the household's consumption share. The main drawback of this method is that it works only in the case of CRRA preferences. Regardless of this limitation, the ability to solve a model with both aggregate and idiosyncratic shocks, multiple assets, and substantial heterogeneity in beliefs is a major step forward for the literature. See also Judd et al. (2002) for additional information on how to solve multiple asset problems with substantial risk and no portfolio restrictions.

The past few subsections have focused on novel methods of simplifying the problem to make it solvable by a standard solution technique, such as value function iteration. The following subsections will focus on improving the accuracy of the solutions by better interpolation methods, and potentially speeding up solving by improving the parallelization of the algorithm, as well as exploring new types of solution methods.

1.4.4 Improvements to Value Function Iteration

This subsection will deal with a variety of methods for dealing with error arising from interpolation and issues with the "curse of dimensionality," which affects the feasibility of solving highly complicated problems. When using grid search to solve problems, it is fundamentally important to use quality interpolation to appropriately approximate the value and policy functions. However, interpolation is slow, both in terms of the execution of the code, as well as the need for a large grid over which to interpolate. Cai and Judd (2014) focuses on overcoming both the curse of dimensionality as well as using shape preserving interpolation techniques, such as shape preserving Chebyshev interpolation or Hermite interpolation, to be able to solve these problems appropriately. The authors find that using these interpolation techniques can reduce error by orders of magnitude. In fact, not worrying about shape preservation can lead to incorrect results, even in the case of the easiest, smoothest problems. Importantly, the algorithms they provide for shape-preserving interpolation have the same speed as interpolation methods that do not worry about shape preservation.

Regarding parallelization, the authors include multiple algorithms and show the significant gains to solving these problems, "the total CPU time [...] to solve the optimal growth problem is nearly 17 days" when solving the problem without parallelization. Using their parallelization code, they are able to achieve high parallel efficiency (up to 98%, and never lower than 90%). The showcase results for parallelizing across 50, 100, and 200 processors, which shows an almost linear increase in speed. When using the full 200 processors, the total solve time decreases from 17 days to roughly 2.25 hours. They note that while efficiency decreases from using 100 processors to 200 processors, the total wall clock time saved is almost two hours, which is what matters for the end user. For a further look at the differences in interpolation methods, see Aruoba et al. (2003).

Major innovations to the endogeneous grid method, yet another important method for solving heterogeneous agent models, have led to substantial improvements in both speed and accuracy. For example, Brumm and Scheidegger (2017) is able to solve a Krusell-Smith type problem more quickly, more accurately, and with many more state variables. This paper's method will be discussed in more depth below. Before diving into these innovations, the idea of endogenous grid points was first proposed by Carroll (2005). As he states, the method is "for solving numerical dynamic stochastic optimization problems that avoids root finding operations." The general idea is that, when solving problems, typically an exogenous grid is used. For example, there exists a grid over income. At each point on this grid, a root finder is used to obtain the optimal consumption value. This is generally a time-consuming process. With these point values, a value function is constructed using interpolation. Carroll's contribution was to realize that the root-finding occurs only because the grid of income values was not chosen in a smart way. The algorithm for the endogenous grid method holds assuming the derivative of the utility function is invertible on $(0, \infty)$. The algorithm follows:

1. Fix an endogenous grid for capital, $\{k_i\}$ $(k = y - c)$
2. Obtain c_i by inverting the Euler equation
3. Finally, for each set of (c_i, k_i), obtain $y_i = c_i + k_i$
4. Use the pairs of (y_i, c_i) to construct the value functions as it is clear they satisfy optimality

Barillas and Fernandez-Villaverde (2007) extended Carroll's method to multiple control variables. Barillas and Fernandez-Villaverde (2007) shows that the combination of VFI and the endogenous grid method leads to important gains. They note that VFI is useful in that it is good at handling discontinuities and nondifferentiabilities, and is able to be judged on its convergence and performance easily. The major drawback is that it is relatively slow. By combining Carroll's method with VFI, the authors find that they are able to maintain the positives of the endogenous grid by speeding the process up, while not sacrificing the accuracy of VFI. The other key advancement in this paper is the ability to use the endogenous grid point method with multiple choice variables, drastically extending the scope of possible solutions.

Another improvement to the use of grids in solving using value function iteration is the use of sparse grids, also known as Smolyak's method. As mentioned above, the "curse of dimensionality" occurs because of the use of tensor, or full, grids. As the problem extends across more dimensions, these number of times these grid points must be used increases exponentially, becoming computationally infeasible. Smolyak used a univariate quadrature rule to simplify the grids from full to sparse grids to significantly reduce computation time, though time cost still does increase quickly as dimensionality increases. The first example of using sparse grids to solve a dynamic economic model was Krueger and Kubler (2004). Following this paper, there have been many advances in the use of sparse grids. Judd et al. (2014) introduces a variety of improvements to this method that help with both computation speed and accuracy. These improvements include "a more efficient implementation of the Smolyak method" and an extension of the method which allows for varying the quality of the approximation. The main drawback of this method is that is is not parallelized.

Brumm and Scheidegger (2017) is a very recent example of yet another enhancement of Smolyak's method. The authors use adaptive sparse grids. As mentioned above, sparse grids grow slower than standard grids. The authors note that "adaptivity adds a second layer of sparsity, as grid points are added only where they are needed most." Beyond this, the authors have fully parallelized their solution method by "combining distributed and shared memory parallelization paradigms," making this incredibly useful for high-performance computing. Importantly, this method is both accurate and fast, and, in comparison to Judd et al. (2014) it is able to handle problems that are not smooth. This paper marks a significant advancement from the work of Krusell et al. (1998). This method is able to "compute global solutions for dynamic stochastic models with up to 100 dimensions," a truly staggering number, especially given the youth of true computational economics.

The discussion to this point has focused on solving problems using value function iteration. This is not the only way of solving heterogeneous agent problems. The following subsection will focus on these alternative methods and the benefits they bring to attacking these problems.

1.4.5 Non-VFI Solution Methods

This subsection is devoted to solution methods that do not rely on value function iteration. Value function iteration would generally be considered a "projection" method of solving. In Judd and Guu (1993), the authors introduce alternative solution methods which they termed a "perturbation"(or occasionally asymptotic) methods. The general idea behind these methods is to solve a more complicated model by looking at perturbations from a model with a known, exact solution. This is generally done by taking a Taylor series approximation around a deterministic steady state. Fernandez-Villaverde et al. (2016) explains why these methods have become much more common, noting that, despite the local nature of the approximation, they tend to hold globally as well, and have the additional benefit of being easily interpretable. As an example, think of Alvarez et al. (2005) which looks to address the constant risk premium present in most general equilibrium monetary models. Typically, a first-order approximation is used to solve models. The authors in this case use a second order Taylor polynomial. As they note, of course there will be no time varying risk after a first order approximation as risk is a second order concept. Finally, Fernandez-Villaverde et al. (2016) notes that these methods are much more accessible to economists with little computational experience as programs such as Dynare are able to solve high-order perturbations. The main costs of perturbation methods are fairly obvious. First, it is an approximation to an equilibrium, which will not be as precise as other solution methods. Second, depending on the size of the state space and length of the Taylor approximation, the time gains may not be sufficient to overcome the error introduced by approximating the solution.

Judd (2003) provides a method for applying a change of variables to the DSGE solutions obtained through perturbation techniques. This is highly useful as, in general, these models feature a lot of state variables, increasing the number of derivatives that need to be calculated drastically as the order of the Taylor approximation increases. This allows for an equally accurate solution that comes about from simply taking a lower order approximation and then instituting a change of variables. Note that this method is similar in spirit to the state space simplification methods mentioned above. Fernandez-Villaverde and Rubio-Ramirez (2006) updates this work by finding a "simple close-form relation between the parameters of the linear and log-linear solution of the model," and then generalizing this approach. Importantly, the authors find that by using this change of variables technique they are able to reduce error in the Euler equations by a factor of 3, which significantly reduces the cost of this technique.

The envelope condition method, introduced by Maliar and Maliar (2013) is a relatively new method that is also able to solve dynamic programming problems. This method simplifies root-finding and is similar in time and accuracy to Carroll (2005). Unlike Carroll (2005) and similar to value function iteration, this method still uses an exogenous grid. However, instead of solving by backward iteration, this method guesses the current value function and iterates forward, using the envelope condition to generate policy functions rather than the first order conditions of the problem. Also, the authors show that when using either the endogenous grid point method or the envelope condition method to approximate the derivative of the value function rather than the function itself yields significantly more accurate solutions. Arellano et al. (2016) extends this method to solving models with default.

This subsection was focused on alternative methods to solving dynamic models using tools that do not rely upon value function iteration. The next subsection will look at a novel method that relies upon a completely different type of math than is normally used to solve these problems.

1.4.6 A Look to the Future

From the previous subsections, it is clear that advanced mathematics have a lot to offer in solving computationally difficult heterogeneous agent problems. The technique discussed in this section is an extension to some computational work done by Kubler and Schmedders (2010). Canann (Forthcoming) is working on a particularly interesting application of algebraic geometry that has great potential in solving problems previously infeasible to compute. Furthermore, it has the ability to solve for all equilibria in non-linear, non-convex environments exactly, rather than by simply approximating them. The method is called the Modular Groebner Basis Approach (MGBA). A Groebner basis is a polynomial basis and is used in the solving of systems of polynomial equations. This method is able to solve any

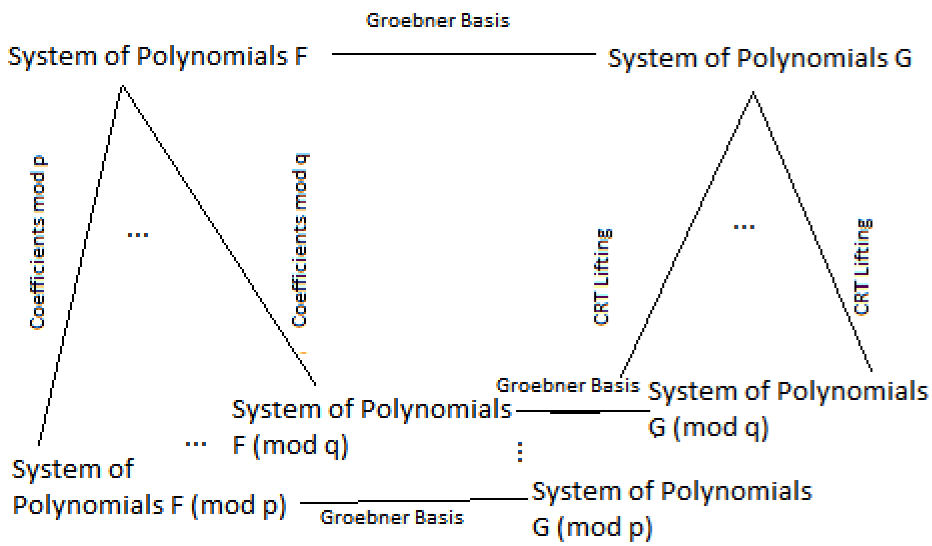

Figure 1.1: Modular Groebner Basis Approach Algorithm

model we are able to polynomialize using Buchberger's Algorithm, which is an extension of Gaussian elimination to systems of polynomials. In Gaussian elimination, the system of equations is triangularized and then solve for each variable. Buchberger's algorithm triangularizes systems of polynomials, but this process is more complex.

Canann (Forthcoming) develops a highly parallelizable algorithm, shown in Figure 1.1. The algorithm is broken into four main steps, all of which are able to be parallelized. The algorithm is as follows:

1. Given a set of primes, convert the system of polynomial equations into a set of polynomial equations modulo the set of primes

2. For each converted system, use Buchberger's algorithm to find the Groebner basis modulo the given prime, which generates a set of systems of polynomial equations

3. Using the Chinese Remainder Theorem, the algorithm lifts the coefficients of different sets of polynomial systems to the reals.

4. Check the solution to see if it is the correct solution by checking if the Groebner basis is the true Groebner basis of the original system.

This new method is useful in that it is able to be parallelized at each step and is able to solve any model that is polynomializable. The most significant drawback is that there is no upper bound to the number of primes that will be necessary to solve the problems. In terms of clock time, this process can still take extensive amounts of time. However this method is able to attack and solve problems that had not previously been solved, such as principal agent problems with non-convexities as in Renner and Schmedders (2015). Using algebraic geometry to transform problems via isomorphisms into new problems that are able to be solved has been shown to be worthwhile, and, with more research, might be the source of our next major computational breakthrough.

1.5 Conclusion

In this chapter, I have reviewed the use of heterogeneous agent models in macroeconomics. Advances in computational power and technical ability combined with increased data availability and higher resolution of that data has led to an increased importance of these models to be able to understand the macroeconomy. The value of these improvements is immense. It has led to clearer understanding of a variety of phenomena, from the evolution of the income, earnings, and wealth distributions over the past half century, to being able to compute models with as many as 20 nondifferentiabilities.

Historically, representative agent models were used as they were easier to work with, despite their less than ideal implications for individual behavior. With the aforementioned advances in both data and computational power, heterogeneous agent models become practical in two dimensions: The models are now able to be disciplined using actual data allowing them to inform policy, and these models can now be solved in time frames that make them relevant to concurrent discussions.

Chapter 2

Asymmetric International Transmission

2.1 Introduction

The Financial Crisis of 2008 originated in the United States and spread globally. While it started in the United States, the crisis did not actually hurt the United States the most, nor were all countries affected equally. When compared to the Scandinavian crisis in the early 1990s or the Mexican Peso Crisis in 1995, The 2008 crisis is markedly different in its global reach. In the case of the Mexican crisis, the impact was felt widely throughout Latin America, but the U.S. was unaffected. This paper looks at how financial crises spread and argues that the relative degree of financial market sophistication explains both the varying responses to and ability to cause financial crises. The degree to which financial markets are developed determines the ability of an economy to insure against non-financial shocks, which in turn determines the desire to hold risky capital assets relative to safe debt instruments. This leads to financially well-developed countries holding large shares of capital assets both domestically and abroad. Then, when a financial crisis hits, this affects the ability of the financial system to insure non-financial shocks, and, in response, capital holdings decrease, which transmits the crisis from the well-developed economy to the lesser-developed economies.

Using the capital flow database started by Lane and Milesi-Ferreti (2005), as well as data from the U.S. Treasury International Capital (TIC) database, we are able to examine capital flow patterns to show that financial flows in general, as well as flows from the United States specifically, are strongly correlated with output growth and are of an economically significant magnitude. We document significantly decreased capital flows during the 2008 financial crisis in not only the U.S. but other financially-developed economies as well. Using the aforementioned databases, we run preliminary regressions to establish some basic correlations between financial flow type and output and find that productive capital flows such as portfolio investment show a strong positive coefficient that is of economically significant magnitude. From this we hypothesize that financial development is key to both explaining the types of flows countries hold, as well as the transmission of the shocks.

The model used in this paper is most closely related to Mendoza et al. (2009). As in their paper, we allow for varying completeness of markets by imposing a restriction on the selling of state contingent bonds. Conveniently, this is equivalent to having only an uncontingent bond and transforming the idiosyncratic shocks faced by households in the economy into a linear combination of the raw shock value and its conditional expectation. In that sense, it is easy to see the main mechanism of the model: more developed financial systems translate directly into dampening the variance of the shocks faced in that economy. Thus, when a financial crisis distorts this ability, households now face more variance than before and, if they are risk averse, hold less capital. The more dependent on foreign capital an economy is, the stronger the contagion. As financially well-developed countries are not dependent on capital from less developed economies, the transmission of a financial crisis is fundamentally reliant on the country of origination and leads to asymmetric transmission.

The main contribution of this paper is to provide a model of international transmission in which the countries are not symmetric. This is, to our knowledge, the first model of transmission where no country is assumed to be a small open economy that generates such a result. This is important as many financial crises occur in countries that are not only financially important, but also economically large, making the assumption of a small open economy untenable.

Related Literature:

This paper fits into two main literatures, namely the literature on capital flows and, more specifically, global imbalances, as well as the literature on international transmission and contagion. The main mechanism in our paper is reliant upon the composition of capital flows, specifically on the idea that more financially developed countries hold larger positions in capital investments, such as foreign portfolio investment or foreign direct investment, as opposed to debt instruments. This ties in directly with the current literature on global imbalances which looks at the large purchases of U.S. debt instruments by foreign economies and is summarized thoroughly in Gourinchas and Rey (2014). Lane and Milesi-Ferreti (2005), in addition to creating the aforementioned database, establish the basic composition of flows, while Lane and Milesi-Ferretti (2014) discusses how flows have changed after the 2008 Financial Crisis. In that same vein, Milesi-Ferretti et al. (2010) examines bilateral linkages and finds limited importance of the size of gross vs. net external positions in explaining the transmission of the crisis. Our paper extends this and looks into the importance of composition of flows and finds instead that the size of risky capital flows is important. Along these lines, Fogli and Perri (2015) finds evidence that, in OECD countries, the precautionary savings motive is key to explaining their net foreign asset position.

Finally, from a modeling perspective, our paper closely follows Mendoza et al. (2009), which argues that differences in financial development help to explain trends in capital flows. They show that debt instruments represent a safe investment, while capital holdings, which are riskier, are held by those better able to bear the risk. As it reproduces the capital flows observed in the data, we use their model as a basis for ours and extend it to allow for examining contagion as well. Their paper is related to the work done in Caballero and Farhi (ated) and Caballero and Farhi (2013), which emphasizes the importance of U.S. debt as a safe asset and examines the policy consequences of a shortage of safe assets.

After the 2008 Financial Crisis many papers were published on the importance of financial linkages in spreading recessions. In most models, most economies are symmetric, meaning that, regardless of where the origination of the shock occurred, there is an effect on the other country. The only asymmetry that can exist is in the severity of the negative effect. In terms of mechanisms for transmission, there are two common avenues. First, having segmented financial markets can form a path for transmission. In Kalemli-Ozcan et al. (2013) some firms receive funding from international banks, which then are the vehicle for transmitting the negative shock. Second, collateral constraints are used by many papers to Perri and Quadrini (2011) use a model with collateral constraints where the value of the collateral is endogenous. While these papers have symmetric outcomes, Acemoglu et al. (2013) examines systematic risk and transmission using network theory, which, while not explicitly done in that particular paper, has the ability to make contagion starting node dependent. Our model is a large open economy model studying contagion. A paper with a similar scope is Aysun and Alpanda (2012) which uses a large open economy DSGE model with credit market frictions to examine the financial crisis, and finds that the baseline model cannot match the data. When adding in a global financial sector, contagion is strengthened significantly, though only when there exists friction in the international financial contracts.

Our model features a portfolio choice along the lines of both Devereux and Yu (2014) and Devereux and Yetman (2010), though, by introducing varying completeness of financial markets, our results are very different from theirs. Devereux and Yetman (2010) finds that financial linkages alone cannot produce transmission, and that leverage constraints are key in generating transmission. In addition, they find trading debt instruments only is insufficient for transmission, meaning that equity plays an important role. In Devereux and Yu (2014), financial linkages increase the occurrence but decrease the severity of crises. In our paper, the type of, not simply the existence of, financial linkages affects the incidence of transmission. Also, the impact of a crisis is related to the type of capital flow, meaning specifically that countries that are more dependent upon foreign financing of capital suffer from stronger contagion. These results are more in line with Amaral and Quintin (2010), who find that differences in financial development lead to large effects on output. Amaral and Quintin (2010) generates varying financial development through limited commitment, and we do so similarly by modeling financial development in a way that is equivalent to an economy with limited enforcement of contingent contracts.

From a data perspective, the importance of financial constraints is documented by Campello et al. (2009) who examine the results of a survey of over 1000 CFOs from across the U.S., Europe, and Asia. They find that credit constraints inhibited financing quality projects, both in the U.S. and abroad, and in many cases the results were amplified in Europe and Asia. This information is important to incorporate in our model, which we do through both our limited liability constraint as well as the constraint limiting the ability of bonds to span the idiosyncratic shocks present in the economy.

Most papers on contagion in the wake of the 2008 crisis focus on the role of banks in propagating the negative shock across borders. Meh and Moran (2008) focus on the importance of what they term the bank capital channel. In their model, bank capital helps to alleviate agency problems, which, when there is not enough bank capital, leads to insufficient loanable funds and macroeconomic misallocation that amplifies negative shocks. Other studies such as Kenourgios et al. (2011), which focuses on the interactions of two major financial centers, the U.S. and U.K., with four main emerging markets, Brazil, Russia, India, and China, during five recent financial crises, lend support to our claim that international transmission is asymmetric. They find that the emerging economies suffer more frequently from contagion than the U.S. or U.K.

The remainder of the paper is structured as follows: Section 2 provides insight into capital flows and the impact of the financial crisis across borders. Section 3 introduces the model and characterizes its solution. Section 4 details the parameterization of the model. Section 5 provides the main results. Section 6 briefly concludes.

2.2 Data

This section has two objectives: First, to document the large changes in U.S. capital flows following the financial crisis in 2008, and, second, to demonstrate the importance of foreign capital flows to capital formation and observed output abroad. To do this, two key data sources are used. First, Philip Lane and Gian Maria Milesi-Ferretti maintain a database of financial flows from 1970 to 2011 of over 180 countries. The database provides information on aggregated holdings as well as the composition of the holdings (debt, portfolio investment, direct investment, reserves, etc.). The second dataset is the U.S. Treasury International Capital (TIC) database, which provides data on U.S. financial holdings by country. Specifically, it tracks both which foreign countries hold U.S. securities and U.S. holdings of foreign securities by country, and it also includes information on both aggregate holdings as well as their composition. The Lane-Milesi-Ferretti database is useful for showing the general importance of financial flows, while the TIC database will be used to show the importance of U.S. capital flows to other countries.

Figure 2.1 shows how the U.S. balance sheet has changed over time. A number of things call for notice. Starting in 1980 and going through 2011, both U.S. asset holdings and liabilities have grown quickly and are now multiple orders of magnitude larger. As of 2011, financial flows are almost double the size of GDP after having previously been a small fraction. This is the fundamental motivation behind our project: in previous recessions, financial flows were not of sufficient magnitude to explain transmission across borders on their own. Now, with a crisis that originated in the financial sector, the connections between economies are of sufficient magnitude to merit deeper exploration. Globalization occurred rapidly throughout the past 30 years, and the growth of U.S. financial flows reflects this. Specifically, in the lead up to the financial crisis in 2008, we observed the steepest increase in financial flows, going from roughly \$5 trillion in 2000 to a little under \$20 trillion in 2007.

Our theory of contagion is contingent not only upon financial flows, but upon the type of financial flow in question. We argue that, as shown in Mendoza et al. (2009), the richness of financial markets determines the composition of flows and that countries like the U.S. typically purchase riskier capital assets abroad rather than safe debt instruments. Contagion occurs when these developed countries purchase significantly fewer productive capital assets. Figure 2.2 shows productive capital outflows (notationally, for the remainder of the paper any references to capital flows will refer to non-debt flows, while debt flows will refer to debt alone) from the United States from 1980 to 2011, calculated as the log difference in the stock of the sum of foreign direct investment and portfolio equity investment. The blue dotted line graphs these changes, while the red dashed line provides the average capital flow growth

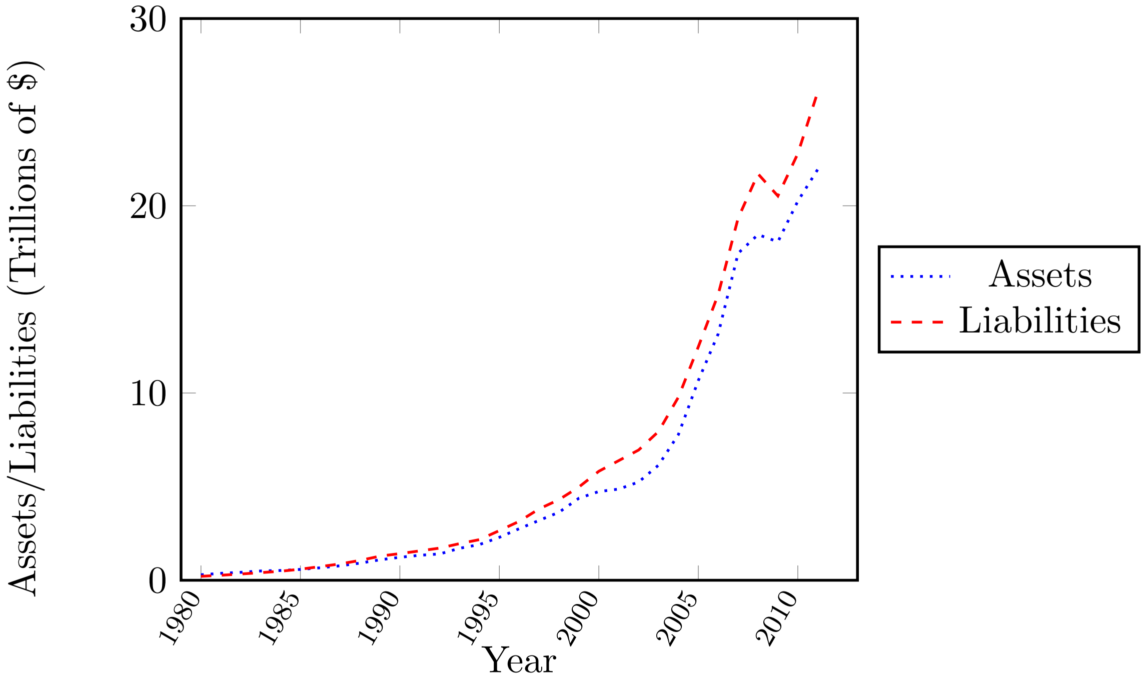

Figure 2.1: Total U.S. Assets And Liabilities (Constant 2011 $)

rate prior to the crisis. What is immediately noticeable is the decline in capital outflows during the crisis is a significant outlier, roughly 30 percentage points below the average capital outflow. As observed in Figure 2.1, we observe a real decline in asset holdings, and a sharp break from the trend growth.

Another key question is whether this decline is driven by the size of the U.S. economy or the depth of the U.S. financial market. Figures 2.3 and 2.4 attempt to illustrate that it is the quality of financial markets that driven this change rather than the size of the economy. Figure 2.3 shows the log differences of the capital holdings for the United Kingdom. Apart from the United States, the United Kingdom has among the deepest financial markets in the world, but they are not as large an economy. The behavior of capital outflows for the United Kingdom is remarkably similar to that of the United States. Both have average capital outflow growth of slightly over 10% in the lead up to the crisis and feature a stunning collapse during the crisis. Figure 2.4 shows the log differences of the capital holdings of Japan. Japan is among the largest economies during this time frame, both in terms of production and in terms of exposure to foreign trade, but does not have the same financial development as the United States or the United Kingdom. Here the break in capital flow growth is earlier, around roughly 1992. During the financial crisis the decline is nowhere near as pronounced as that for the United States. The United States is unique in its combination of the size of its economy and the depth of its markets, but, from these

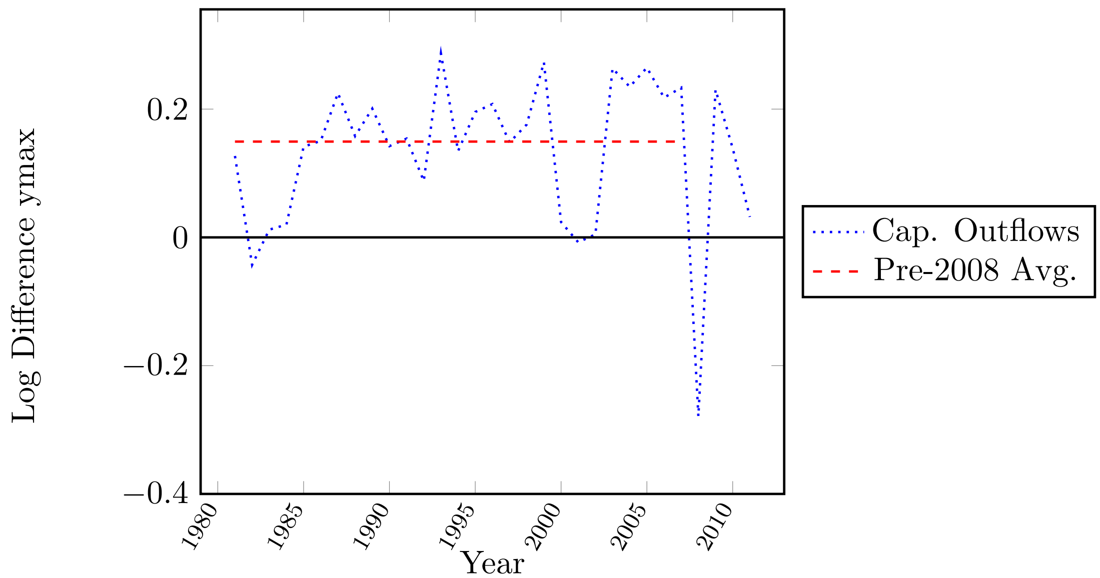

Figure 2.2: Total U.S. Capital Flows (Log Diff. of Stock)

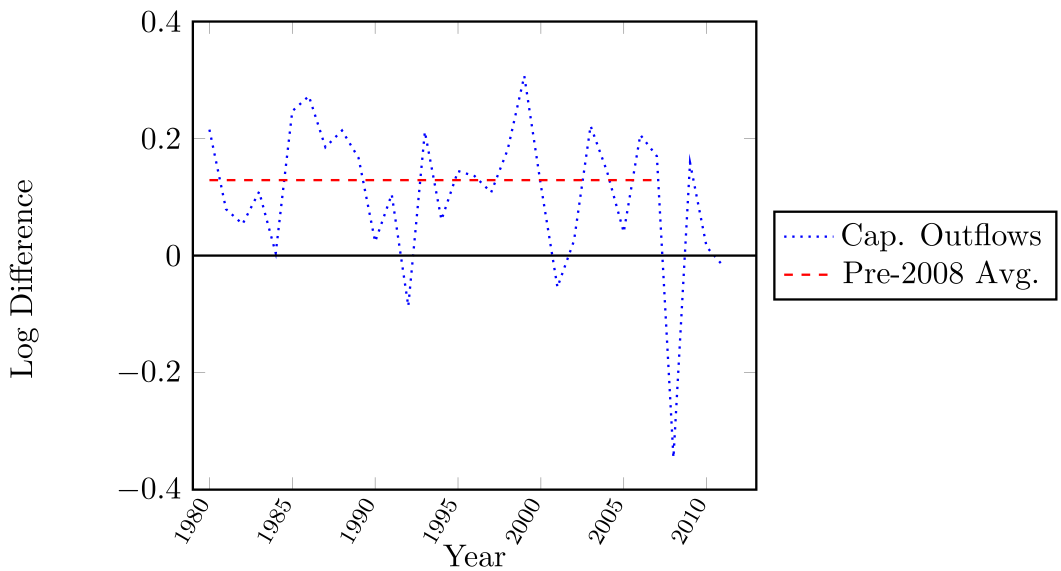

Figure 2.3: Total U.K. Capital Flows (Log Diff. of Stock)

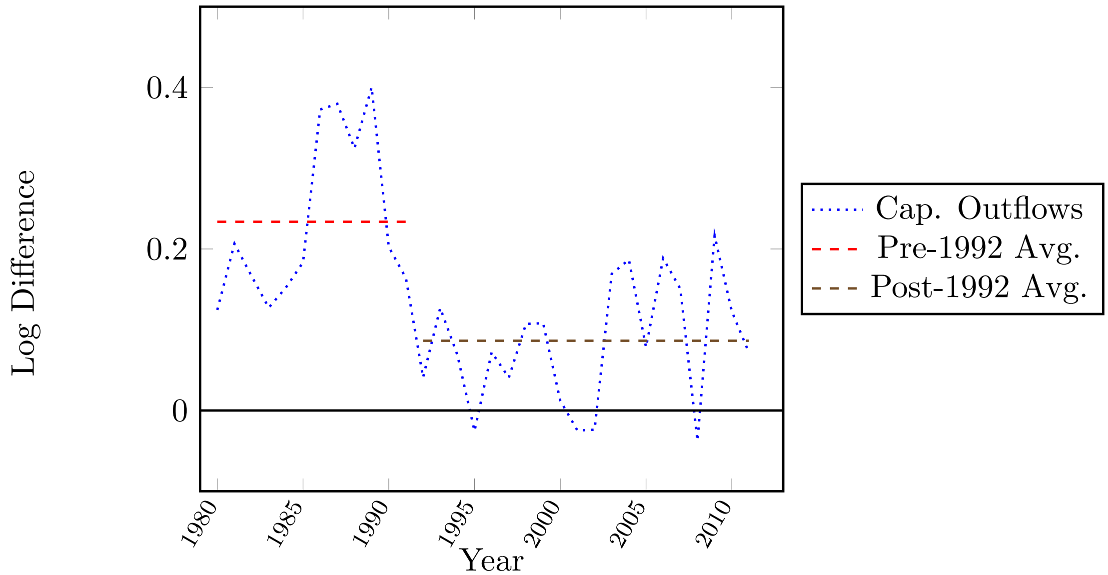

Figure 2.4: Total Japan Capital Flows (Log Diff. of Stock)

charts, we believe that the driving force behind the retrenchment of capital flows is the structure of its financial markets.

The following charts attempt to document the importance of United States capital outflows to the rest of the world. The data for these charts comes from the TIC database, which is sparser in coverage in terms of countries than the Lane and Milesi-Ferretti database. For these charts, we look at the change in U.S. capital inflows into five countries: Germany, Canada, Mexico, Korea, and Japan. Figure 2.5 shows the log difference in U.S. capital holdings of foreign assets in Germany, Canada, and Mexico. Figure 2.6 shows the GDP growth rate in Germany, Canada, and Mexico. First, notice that both series are highly volatile, and that the volatility of capital flows is an order of magnitude higher than the volatility of GDP growth. In general, for these countries, it is difficult to see a relation between the financial flow retrenchment and the transmission of the global recession. All three countries do feature a decline around the recession, and it appears that the larger the change in U.S. outflows the larger the change in output. The result clearly holds best for Mexico, a less-developed economy than Germany or Canada.

It is easier to see the relationship between financial flows and output for Korea and Japan. Again, Figure 2.7 shows the log difference in the stock of U.S. capital outflows to Korea and Japan, while Figure 2.8 shows GDP growth for Korea and Japan. Here it is clear that during both of the most recent recessions for Korea, the East Asian Tiger Crisis in 1998 and the financial crisis in 2008, U.S. capital inflows decreased by a significantly larger amount than the standard fluctuations in flows and the GDP growth rate was significantly

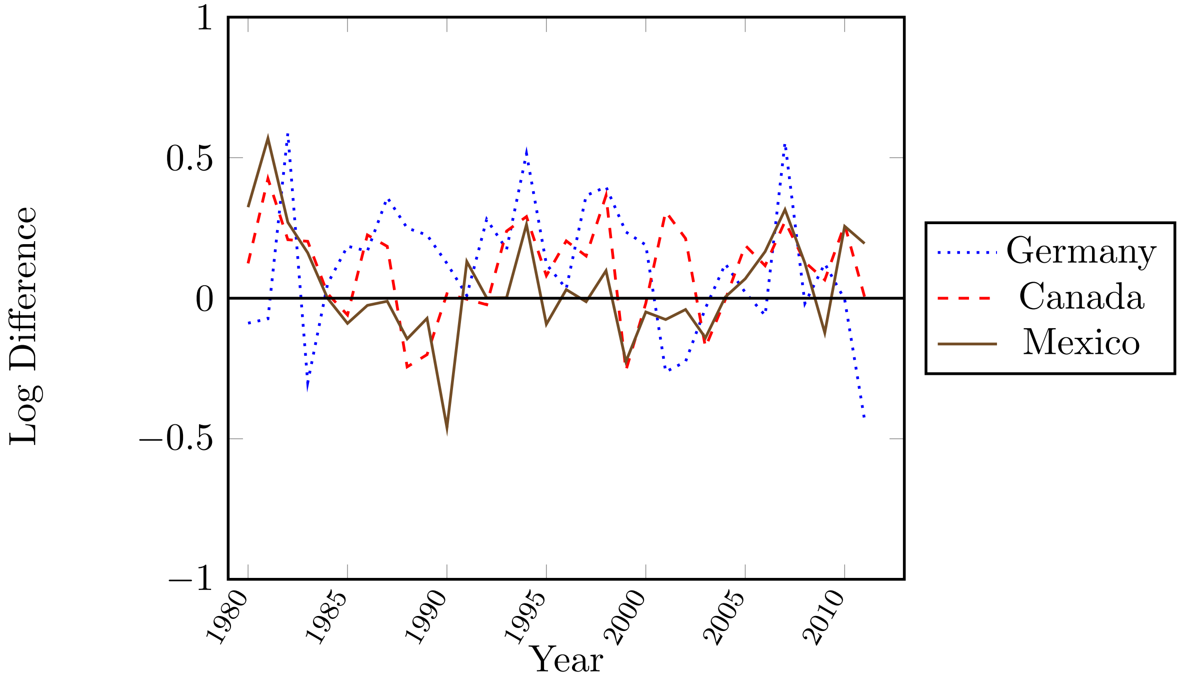

Figure 2.5: U.S. Capital Inflows (Log Diff. of Stock)

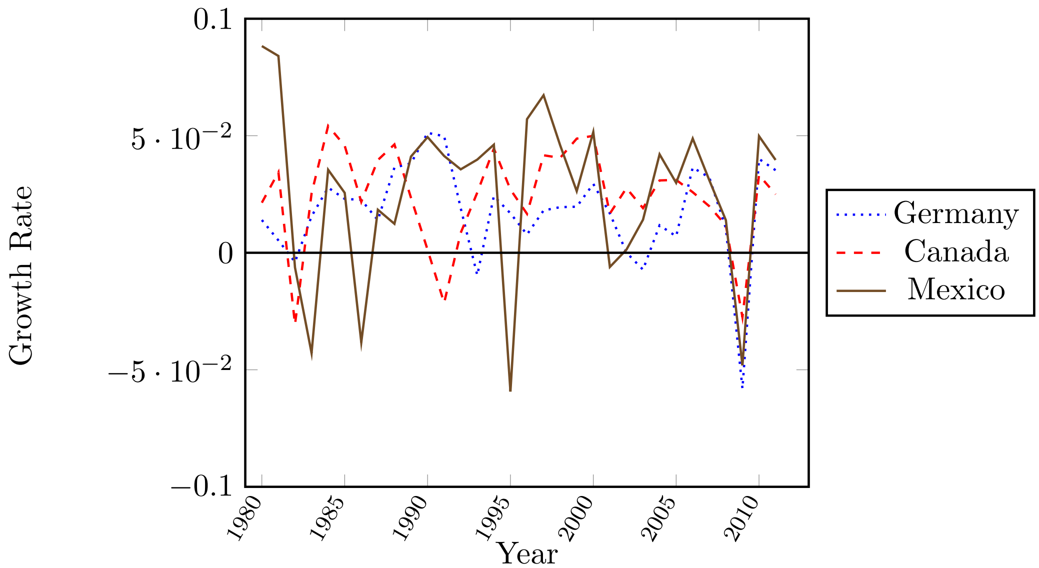

Figure 2.6: GDP Growth

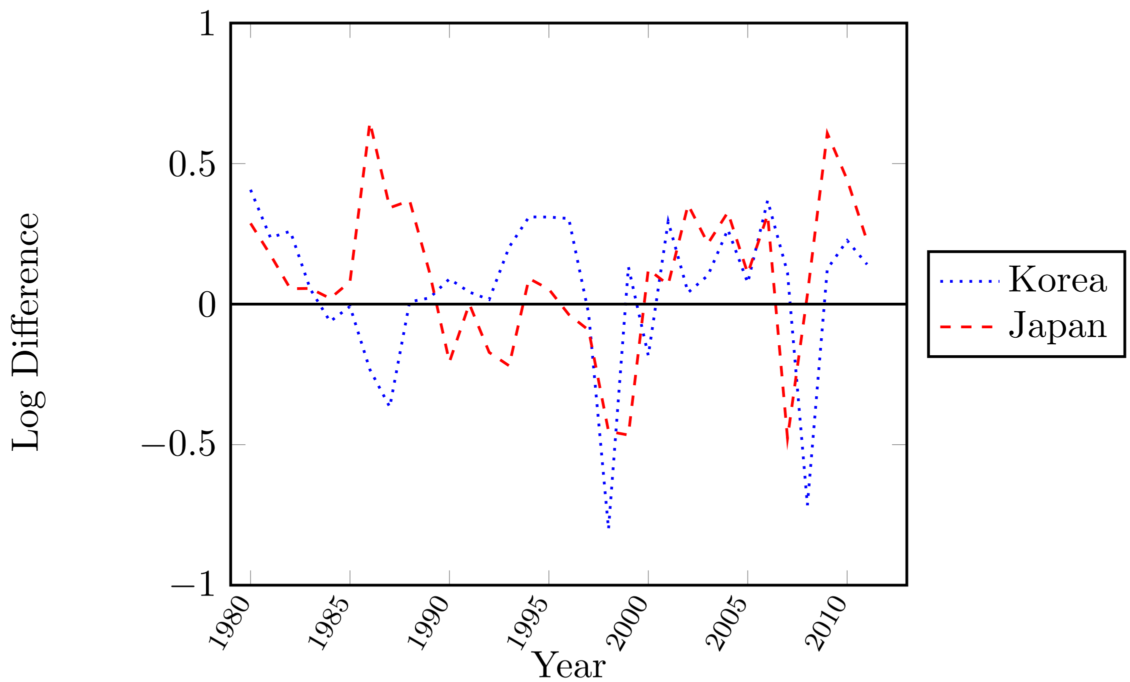

Figure 2.7: U.S. Capital Inflows (Log Diff. of Stock)

lower than the standard fluctuations of GDP growth. During the 2008 financial crisis, Japan also faces the largest decline in financial flows it has seen over the course of the available data, and has the largest decline in output seen in the time frame as well.

To briefly summarize the data to this point, U.S. capital outflows declined sharply in the wake of the financial crisis, and, from a cursory glimpse at the data for Canada, Germany, Mexico, Japan, and Korea, it is difficult to precisely tease out the correlation between U.S. capital flows and output growth in other countries. The following charts try to establish the importance of both the U.S. capital flows to output in other countries, as well as the importance of capital flows generally to the financing of capital stocks the overall relation between financial flows and output. The crisis originated in the United States, but affected all of the most sophisticated financial markets, so looking at the general importance of financial flows is instructive as well. The following two tables, Figures 2.9 and 2.10 show regression results for these two cases.

Figure 2.9 shows the importance of U.S. capital flows to GDP growth abroad for a sample of 42 countries from 1980 to 2011. The regression features many control variables including fixed effects, non-U.S. capital inflows and its lag, the lag of U.S. capital inflows, and other demographic and institutional quality controls. The key result is straightforward: U.S. capital flows have a positive correlation with output growth. This means that the more U.S. capital flows collapsed in the wake of the crisis, the more output was affected. While

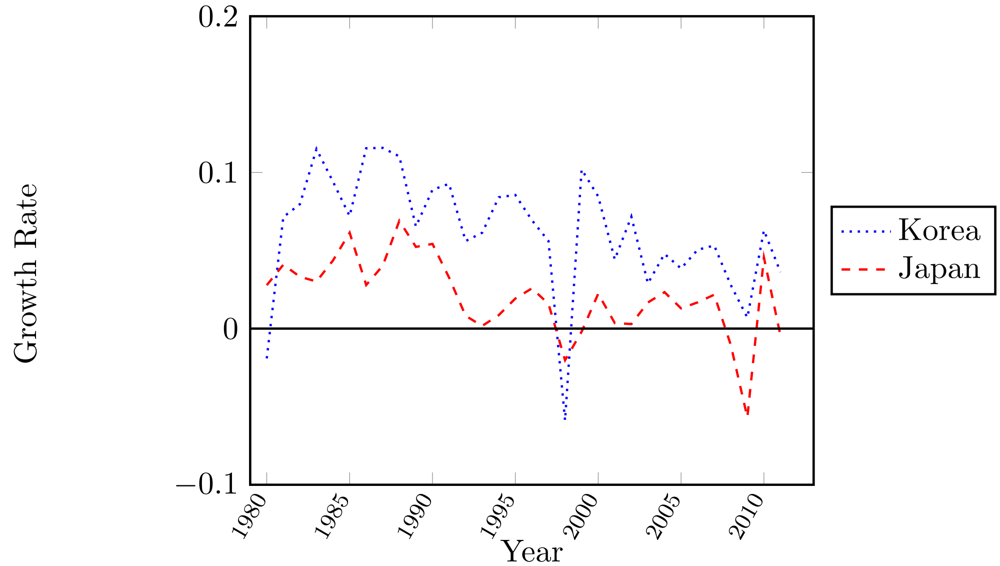

Figure 2.8: GDP Growth

VARIABLES	(1) GDP Growth
U.S. Capital Inflow Growth	0.00262*
	(0.00148)
Controls?	Yes
R-squared	0.323

Robust standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1

Figure 2.9: Regression Results: U.S. Capital Inflows Effect on Output

it would be premature to claim causality for this result, it is important to note that the magnitude of the coefficient is economically significant. From the previous graphs, we can see that U.S. capital inflows fell by almost 50% for Korea and Japan. This coefficient would imply a roughly 10% decline in output, which is double the observed decline. While not perfect, this result indicates that U.S. capital inflows are appropriately correlated and have the potential to be of a sizable magnitude in real economic terms.

Figure 2.10 shows regression results on the generic importance of capital flows to financing a country's capital stock as well as the relationship between capital flows and output growth. This table relies on the Lane and Milesi-Ferretti data set of 113 countries. Again, many control variables are used including fixed effects, lagged growth rates of the listed variables, and other demographic and institutional quality control variables. In the first column, we see the effect of capital inflows on the growth of the capital stock. Prior to the crisis, increases in capital inflows led to increases in the capital stock. After the crisis, the relationship flips. We interpret this as indicating that capital inflows serve as a substitute to private investment. In good times, foreign capital flows in and provides the needed investment, while in bad times, domestic investment is used. Note also that the coefficients on debt flows and labor force have the expected signs. Foreign debt holdings fell during the crisis so savings shifted from debt to financing the domestic capital stock. The coefficient on labor force growth indicates that labor is a substitute.

The second regression result shows the impact of capital flows on the growth rate of GDP. Prior to the crisis, capital inflows are strongly positively correlated with output growth. After the crisis, the relationship preserves the direction but becomes diminished in magnitude. Recall that capital flows dropped by an order of magnitude during the crisis, so the dampened overall coefficient does not imply a smaller overall effect. The key result, however, is the overall positive coefficient. On average, the larger the decline in capital inflows, the deeper the recession was during the crisis.

These two regression results indicate that financial flows are important both in general, and specifically, U.S. outflows are significant. The impact of the recession was asymmetric in that countries that were subject to sharper declines in inflows also faced deeper transmitted recessions. The initial charts show that the major driver of the decline in financial flows originated in more financially developed economies, rather than simply big economies with large trade exposure. From these facts the next section will introduce a model that can generate asymmetric transmission based upon the work of Mendoza et al. (2009).

2.3 Model

This section introduces and characterizes a model based upon the household problem from Mendoza et al. (2009). Our model differs in a few key dimensions: We allow for capital accumulation, we have different masses of individuals in each country, and, rather than having common stochastic processes across countries, we parameterize the income and productivity processes to match observed moments in the data. For the remainder of this section, we will introduce the environment of the model, and then detail the household problem and market clearing conditions. After defining the equilibrium object, this section concludes with a brief characterization of the model that provides insight into the main mechanism driving our results. For detailed information on the characterization as well as a definition

	(1) Cap. Stock Growth	(2) GDP Growth
Cap. Inf. Growth	0.0195	0.0176**
	(0.0148)	(0.00687)
Cap. Inf. Interaction	-0.0429	-0.0166
	(0.0306)	(0.0104)
Debt Inf. Growth	0.0112	0.0289***
	(0.0204)	(0.00749)
Debt Inf. Interaction	0.0361	-0.0474***
	(0.0493)	(0.0158)
Labor Force Growth	-0.183	0.128**
	(0.225)	(0.0503)
Cap. Stock Growth		-0.00540
		(0.00972)
Constant	0.191	0.133***
	(0.130)	(0.0449)
Controls?	Y	Y
Observations	1,632	1,632
R-squared	0.019	0.079
Number of Countries	113	113

Robust standard errors in parentheses
*** $p<0.01$, ** $p<0.05$, * $p<0.1$

Figure 2.10: Regression Results: Capital Inflows Impact on Capital Formation and Output

of the equivalent economy with the bond market constraint as opposed to the transformed idiosyncratic shocks presented below, please see the appendices for this chapter.

2.3.1 Environment

To analyze the importance of financial development on the transmission of crises, we use a straightforward model that has both global financial markets and portfolio choice. For purposes of exposition, assume the world economy has two countries, $i \in \{1, 2\}$, though this model can easily be generalized to N countries. Both countries have masses of infinitely-lived agents. In our model, country 1 will represent the United States and other developed nations, while country 2 is the rest of the world. As such the mass of agents will not be equal, with country 2 the larger of the two economies. It is assumed agents maximize expected lifetime utility $E \sum_{t=0}^{\infty} \beta^t U(c_t)$, where β is the intertemporal discount factor. It is assumed that the utility function is strictly increasing and strictly concave.

In each country, households face a portfolio choice problem. At the beginning of each period, households observe their idiosyncratic wage and productive capital shocks. This determines their wealth for the period, which they proceed to allocate across consumption, c, and two types of assets: one period, state-uncontingent bonds, b, and capital, k. Capital holdings are used in an idiosyncratic production function of the form zk^{ν} after a one period delay, where z is the idiosyncratic productivity shock. It is assumed that the production function is of decreasing returns to scale, meaning that $\nu < 1$.

Individuals face two types of idiosyncratic risk, exogenous income risk, y, and an idiosyncratic investment shock, z. Both income risk and the productivity shock are Markov processes. As all shocks are idiosyncratic, and there are no aggregate shocks, there is no aggregate uncertainty. As in Mendoza et al. (2009), the presence of both shocks allows for analyzing a portfolio choice between riskless and risky assets. The endowment shock is unavoidable and can only be insured up to the limit set by the sophistication of the financial markets in that country. On the other hand, it is possible to avoid the productivity shock completely by simply not purchasing any capital.

The key feature of our model is the financial environment in each country that will be fundamental to generating the capital flows and transmission we observe in the data. First, in both countries, households face a limited liability constraint, meaning that net assets must be greater than or equal to 0, which limits borrowing. Second, the quality of financial markets affects the idiosyncratic shocks households face. The variable $\phi \in [0, 1]$ denotes the quality of financial markets. When $\phi = 0$, the household faces the full force of its idiosyncratic shocks. When $\phi > 0$, the household gets a weighted average of the shocks it faces and the conditional expectation of the shock given last period's realization. This lowers variance while preserving expected return, so it captures the idea of efficient diversification.

To summarize, households face a portfolio choice in two countries that are defined by their financial completeness which affects the idiosyncratic shocks they face, and thus the composition of their portfolios. The timing of the model is as follows:

1. Households enter the period holding capital and bonds. Shocks are then observed, which determines current net asset holdings.
2. At the end of the period, households choose consumption (c) and their portfolio for tomorrow(k' and b') given their asset holdings.

The next subsection formalizes the household problem and how financial market depth manifests itself in this environment.

2.3.2 Household Problem

Let the idiosyncratic state be given by $s \equiv (y, z)$, where y is the income endowment the household receives this period, and z is a vector of productivity shocks with the same number of entries as countries in the economy. Specifically, an individual gets an idiosyncratic draw for portfolio productivity in each country. Let $g(s, s')$ be the associated conditional probability distribution. Define a to be assets available at the end of the period, then the household budget constraint is given by Equation 2.1.

$$a = c + k_1' + k_2' + b' \tag{2.1}$$

Assets are used to purchase today's consumption, as well as domestic capital, k_1', foreign capital, k_2', and bonds, b', all of which are used in the next period. As mentioned above, asset holdings are determined at the beginning of each period, when the idiosyncratic state, s, is realized. Assets for the period are described by the following law of motion:

$$a(s) = y + \tilde{z}_1 k_1^{\alpha} + \tilde{z}_2 k_2^{\alpha} + k_1 + k_2 + (1 + r)b \tag{2.2}$$

Where

$$\tilde{z} = (1 - \phi)z + \phi E\left[z|z_{-1}\right] \tag{2.3}$$

Assets are simply the realization of the endowment shock plus both the production and resale value of capital in each country, and the returns on any bonds that come due or the repayment of any debts that are owed. To clarify further, $\tilde{z}$ is the effective realization of the productivity shock. These realizations are determined by the level of financial development in the economy, ϕ_i. As is shown in the appendix, this is equivalent to an economy with state contingent assets and a limited enforcement constraint of the form: $a(s_j) - a(s_1) \geq (1-\phi)\left[(w_j + z_j k^{\nu}) - (w_1 + z_1 k^{\nu})\right]$, where the subscripts are elements of the finite support of the shocks with 1 corresponding to the worst realization of these shocks. This equation simply says that bonds can only make up for part of the difference from your current asset state and the worst asset state. Households have an incentive to claim to be in the worst asset state when they are not, but there is a cost ϕ to hiding assets. This constraint limits the ability of bonds to span the shocks. Our set-up with an uncontingent bond and transformed shocks has an equally easy to interpret meaning. The better the quality of the financial market, the more efficiently households are able to diversify and insure against the idiosyncratic shocks they face. Specifically, households receive a weighted average of the raw shock and the conditional expectation of the shock given last period's shock. This reduces variance, which, given the form of the households' utility function, makes households more willing to hold risky assets as ϕ increases.

The final component of the household problem is a limited liability constraint. Specifically, households must be solvent, regardless of the realization of the idiosyncratic state:

$$a(s_j) \geq 0 \tag{2.4}$$

This acts as a borrowing constraint, as borrowing in the risk free asset beyond the worst possible endowment realization must be offset by a corresponding position in capital. Having now explained the problem the household faces, we are able to state the full optimization problem, write our market clearing conditions, and define our equilibrium. The value function of an individual in country i is given by:

$$V^i(a,s';\phi^i) = \max_{c,k_1',k_2',b'} \left\{u(c) + \beta E\left[V^i(a',s';\phi^i)\right]\right\} \tag{2.5}$$

$$\text{subject to}$$

$$a = c + k_1' + k_2' + b'$$

$$a(s) = \tilde{y} + \tilde{z}_1 k_1^\alpha + \tilde{z}_2 k_2^\alpha + k_1 + k_2 + (1+r)b$$

$$\tilde{z} = (1-\phi^i)z + \phi^i E\left[z|z_{-1}\right]$$

$$a(s_j) \geq 0$$

2.3.3 Market Clearing

There are three markets that need to clear: the goods market, the capital market, and the bond market. Let $\Omega^i(s,k,b)$ is the joint distribution over shocks, capital, and bond holdings in country i.

The goods market clearing condition is given by:

$$\sum_i \int_{s,k,b} c\Omega^i(s,k,b) + \sum_i \int_{s,k,b} k'\Omega^i(s,k,b) = \sum_i \int y_i\Omega^i(s,k,b) + \sum i \int_{s,k,b} z_i k^\nu \Omega^i(s,k,b) \tag{2.6}$$

Total consumption and capital investment must equal the total production and sum of endowments for the world economy. The capital market clearing condition is:

$$\sum_i \int_{s,k,b} k^i(s,a)\Omega^i(s,k,b) = \sum_i K_i \tag{2.7}$$

The total capital in the world economy must equal the total capital held in the world economy. The bond market clearing condition is:

$$\sum_i \int_{s,k,b,s'} b^i(s,a,s')\Omega^i(s,k,b)g(s,s') = 0 \tag{2.8}$$

Net bond holdings in the world economy must equal zero. Now that we have market clearing, we are able to define our equilibrium.

2.3.4 Equilibrium Definition

Given the financial development, ϕ^i, of all countries, a general equilibrium with capital mobility is defined by sequences of:

1. agent's policies for consumption, capital, and bonds $\left\{c_t^i(s,a), k_t^i(s,a), b_t^i(s,a,s')\right\}_{t=0}^{\infty}$,
2. value functions, $\left\{V_t^i(s,a)\right\}_{t=0}^{\infty}$
3. prices, $\{\tau_t, r_t\}_{t=0}^{\infty}$,
4. and distributions, $\left\{M^i(s,k,b)\right\}_{t=0}^{\infty}$

Such that:

1. the policy rules solve the household problem (2.5) and $\left\{V_t^i(s,a)\right\}_{t=0}^{\infty}$ are the associated value functions,
2. and markets clear, meaning (2.6), (2.7), and (2.8) hold.

In the following section, we characterize the solution to the household problem

2.3.5 Characterization

While it is not possible to determine an analytic solution for this problem, the characterization provides insight into what the solution will look like. Specifically, showing the Euler equations for each country in a two country model provides insight into the main mechanism of the model. For the full algebraic derivation of the characterization, please see the appendix to the chapter. In this two country example it will be assumed that $\phi^1 > \phi^2$. The Euler equation for capital in country 1 is:

$$\beta E\left[u'(c')(\alpha\tilde{z}'_1 k_1'^{\alpha-1}+1)\right] + E\left[\mu(s')\alpha(\tilde{z}'_1 k_1'^{\alpha-1}+1)\right] = u'(c) \tag{2.9}$$

The Euler equation for capital in country 2 is:

$$\beta E\left[u'(c')(\alpha\tilde{z}'_2 k_2'^{\alpha-1}+1)\right] + E\left[\mu(s')\alpha(\tilde{z}'_2 k_2'^{\alpha-1}+1)\right] = u'(c) \tag{2.10}$$

The Euler equation for bonds is:

$$\beta E\left[u'(c')(1+R(A_1,A_2))\right] + E\left[\mu(s')(1+R(A_1,A_2)\right] = u'(c) \tag{2.11}$$

Before attacking the case where there is financial mobility, it is appropriate to examine the case of financial autarky. Define the gross rate of return on capital in country 1 to be $R_1(z,k) \equiv (\alpha \tilde{z'}_1 k_1'^{\alpha-1} + 1)$, and the gross rate of return in country 2 to be $R_2(z,k) = \alpha \tilde{z'}_2 k_2'^{\alpha-1} + 1$. If ϕ^1 is sufficiently high to ensure complete markets (in the case of iid shocks, this is $\phi^1 = 1$, while, with persistence, it must be greater than 1), then the expectations disappear and we are left with a certain outcome. This equates $R_1(z,k) = 1 + r$, and we know that $\beta(1+r) = 1$. Suppose now that country 2 has the polar opposite case of truly incomplete markets that cannot span any uncertainty, $\phi^2 = 0$. Then it must be that $\beta(1+r) < 1$, and $E\left[R_2(z,k)\right] = (1+r)$.

What happens then when we move to financial integration? The above tells us that the risk-free rate of return is lower in country 2 than country 1, so when moving to financial integration the risk free rate will move to an intermediate value between the two autarky rates to clear the bond market. Country 2 demands more risk-free bonds, while country 1 wants to borrow to finance more capital investment. Thus, the ordering of ϕ across countries determines the composition of flows, while the size of the gap between them determines the magnitude of the flows. These are the key insights to the mechanism in our model. If we interpret a financial crisis as a change in the ability of financial markets to span shocks, meaning a decrease in ϕ, we can replicate the observed pattern in flows. When the ϕ in the more developed country decreases, presuming it does not fall so much as to be less developed than the other country, the composition of the flows will stay the same, but the magnitude of the flows will decrease. This means the trade in bonds will collapse, and more importantly for transmission, financing of capital in the less developed country will decrease. With all this in mind, the next section will detail the computational algorithm and parameterization of the model.

2.4 Computational Algorithm and Parameterization

This section details the experiment we run and the computational algorithm and the parameterization of our model. The experiment we run is straight forward. We assume the economy was in steady state prior to the financial crisis. The financial crisis manifests itself as a sharp decrease in country 1's ϕ. We then solve for the new steady state at this lower ϕ, as well as for the transition path between the two steady states. The following subsection explains the computational algorithm for both solving the steady states and the transitions.

2.4.1 Computational Algorithm

First, we need to obtain the distribution and market clearing interest rate associated with each steady-state equilibrium. To solve for a given steady state:

1. Pick an interest rate.
2. Find policy functions for $k1$, $k2$, and b for each country.
3. Simulate economy to find net borrowing and capital holdings.
4. Check market clearing conditions (e.g. Net borrowing equal to zero, target aggregate capital, goods market clearing).
5. Repeat from step 1 until within error tolerance

Once we solve for the steady states, we obtain the associated distributions, call them Ω_{bef} and Ω_{aft}, and interest rates, r_{bef} and r_{aft} with subscript bef corresponding to the steady state *before* the crisis and subscript aft to the steady state *after* the crisis. Given these distributions and interest rates, we can solve for the transition dynamics using the following algorithm:

1. Guess r_i $i \in n$, where n is the number of transition periods
2. Using the sequence of $\{r_i\}$ and the new steady-state's value function, V_n, solve backwards for $V_{n-1}, \ldots, V_0$
3. Using the sequence of value functions, $\{V_i\}$, find the sequence of $r_0, r_1, \ldots, r_{n-1}$ which clear the bond market
4. If $\Delta V_i < \epsilon_i \; \forall i \in n$, end, otherwise go to (1).

Now that the algorithm has been specified, the next subsection provides the parameterization of the model.

2.4.2 Parameterization

Table 2.1 provides the parameter values we use in our numerical experiments. When parameterizing the model, we use standard values for as many parameters as possible. Specifically, we set the capital share parameter in our production function, $\alpha = 0.33$, and use a CRRA parameter of $\sigma = 2$ in our utility function. For our discount rate, we use a β of 0.95, which, in the case of complete markets, implies an interest rate of around 5%.

Parameter	Value
Discount Rate, β	0.95
Capital Share, α	0.33
CRRA Parameter, σ	2
Country 1 Prod. Shock Mean, z_1	0.08
Country 1 Prod. Shock SD, σ_1	0.20
Country 2 Prod. Shock Mean, z_2	0.10
Country 2 Prod. Shock SD, σ_2	0.25

Table 2.1: Parameter values

To discipline the stochastic processes governing the returns to capital, we look at stock market data and investment prospectuses from companies such as Morgan Stanley Capital International (MSCI) and Goldman Sachs. In general, when looking at the return on the U.S. stock market, we find that a mean of around 7-8% with a roughly 20% standard deviation is consistent across the various sources. For foreign returns, MSCI indicates a mean return of roughly 10-11% with a standard deviation between 22-25% for foreign capital.

For the income process, we first notice that the U.S. accounts for roughly half of the world's gross domestic product, while the per capita income is roughly 3 times the average from the rest of the world. This means that we set the size of the rest of the world to be 3 times that of the U.S. and make the income process have a mean that is one third of the U.S.'s mean.

Finally, for our numerical exercise, we need to pin down the quality of financial markets, ϕ, in each country. For simplicity, we will think of country 2 as completely financially-undeveloped, meaning that $\phi_2 = 0$. Regarding the U.S. financial market value, we look at Mendoza et al. (2009) and see that obtaining a precise value for this ϕ is difficult. For illustrative purposes, we will start with a ϕ_1 of 0.8 prior to the crisis. After the crisis, $\phi_1' = 0.6$, meaning that the crisis reduced market efficiency by 25% of its initial value. Now that the model has been parameterized, we present the solutions to both the steady states as well as the transition path between them.

2.5 Results

This section provides the results of our numerical exercise. In this exercise, we first calculate two steady states, and then solve for the transition dynamics between them. Generally speaking, we observe patterns consistent with the data documented above: the gross magnitude of bond flows sharply declines, and, in the immediate aftermath of the crisis, there is

	Country 1	Country 2
K_1	4.61	3.94
K_2	7.92	6.86
b_i	-10.29	10.61
A_i	6.15	18.55
ϕ_i	0.8	0.0

Table 2.2: Pre-Crisis Steady State

	Country 1	Country 2
K_1	6.96	4.71
K_2	11.60	7.93
b_i	-3.63	3.70
A_i	20.33	18.01
ϕ_i	0.6	0.0

Table 2.3: Post-Crisis Steady State

a correlation between the decline in foreign capital holdings of the well-developed country and the total asset holdings of the less-developed country, indicating the crisis has been transmitted.

2.5.1 Steady State

Tables 2.2 and 2.3 provide the steady state results for our model and calibration pre- and post-crisis. From Table 2.2, we notice that prior to the crisis most capital is held by country 1, and that these holdings are financed by large amounts of borrowing. The average overall asset holdings for Country 2 is roughly three times that of Country 1, as was targeted. The equilibrium interest rate is roughly 7%.

Table 2.3 shows that after the crisis occurs Country 1 drastically changes its portfolio composition. As there is less insurance available, we observe a strengthened precautionary savings motive that leads to significant increases in productive capital holdings and retrenchment in bonds for both countries. Again, the vast majority of capital is held by Country 1, though Country 2 has also significantly increased their holdings as well. As with the data, the gross magnitude of bond flows drops significantly after the crisis, and the equilibrium interest rate is now roughly 5%. This lower interest rate is a result of the increased demand for bonds, which raises their price. Unlike the data, asset holdings increases significantly, but this is unsurprising in the new steady state. The lack of insurance means that asset accumulation to insure against shocks. The next subsection will look at the transition dynamics in the wake of the crisis to see if the crisis was transmitted.

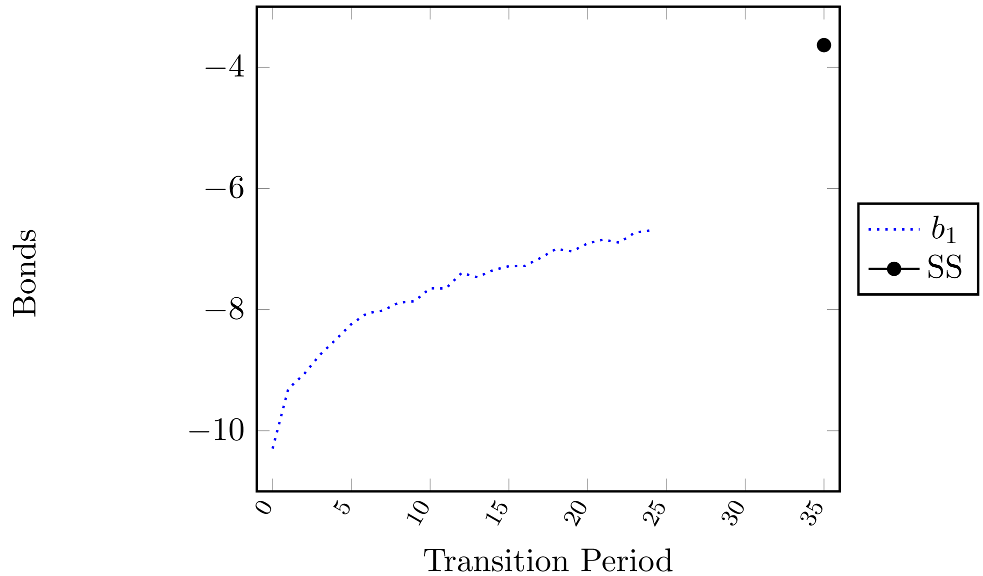

Figure 2.11: Bond Holdings (Total)

2.5.2 Transition Path

Figure 2.11 shows the transition path of bond holdings in the wake of the crisis. The blue line is the transition path of bond holdings in country 1. As net bonds in the global economy must be zero, the transition path of bond holdings for the rest of the world is simply the mirror image of this plot. The black dot is the new steady state value for bond holdings. The transition to this value is smooth, but takes time to get there. What we notice is an immediate decline of roughly 25% of the volume of bond flows in the global economy, which is of the same order of magnitude as observed in the data.

Figure 2.12 shows the transition path of Country 1's capital holdings. This shows that, initially, capital holdings decline prior to slowly but steadily recovering. This is a result of less insurance being available. Households want to initially reduce capital holdings until they can sufficiently increase their asset holdings because of an increased precautionary savings motive. This decline in overall financial flows clearly leaves Country 2 worse off, as the next image shows.

Figure 2.13 shows the transition paths of asset holdings for both countries. Notice that, after the crisis, Country 1 steadily accumulates assets and smoothly transitions to the new steady-state value of around 20 units. Country 2 on the other hand declines in asset holdings for a long while, even though in the long run they also end up with higher asset holdings in the new steady state. This is the transmission of the crisis. Foreign individuals are worse off both initially and for a long duration of the transition.

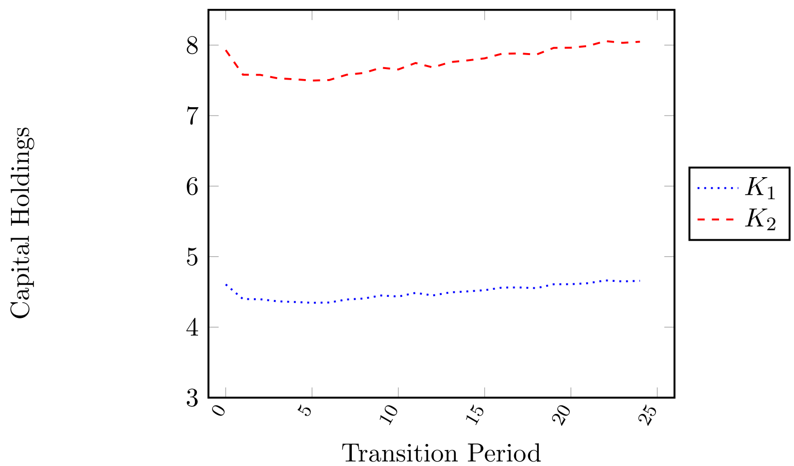

Figure 2.12: Country 1 Capital Holdings

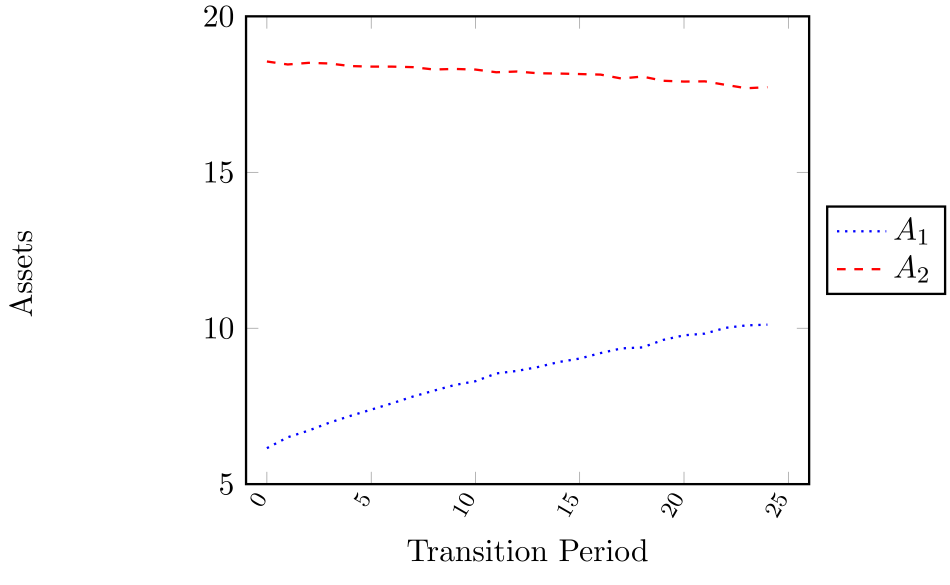

Figure 2.13: Asset Holdings (Total)

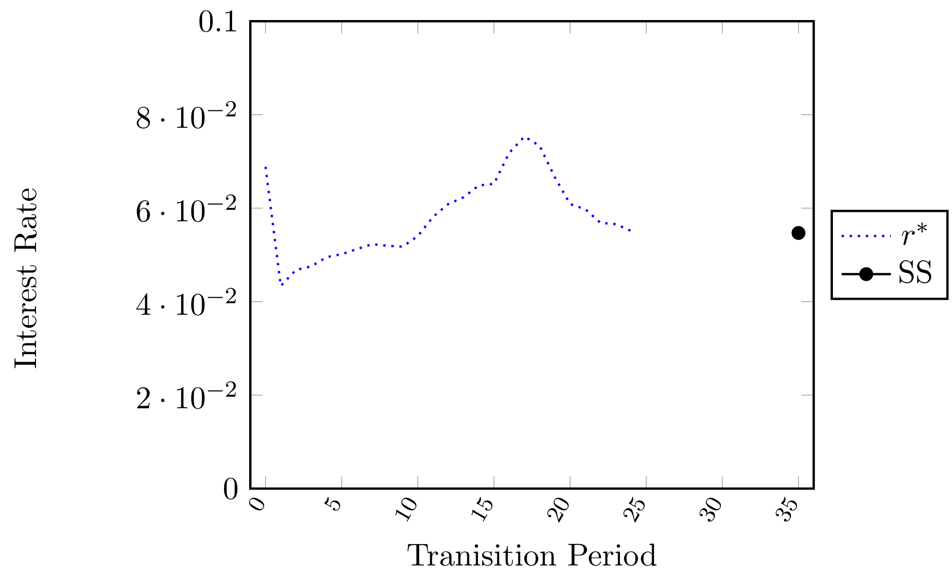

Figure 2.14: Equilibrium Interest Rate

Figure 2.14 shows the transition path of the equilibrium interest rate. This graph indicates that demand for bonds drastically increases following the crisis. The lack of insurance increases the demand for safe assets in Country 1 and correspondingly increases the price of bonds initially. The price falls and interest rate rises as country 1 accumulates assets throughout the transition, overshooting the new steady state value initially, before finally smoothly transitioning to the new steady state value. This occurs because demand for borrowing increases as Country 1 households are able to finance more capital accumulation.

To synthesize the results of these dynamics, the picture painted is broadly that observed in the data. Bond flows collapsed precipitously, while interest rates simultaneously plummeted. Productive capital flows from the financially well-developed economy drop in the wake of the crisis, and we observe a persistent decline in the overall wealth of the lesser-developed country, which is interpreted as the transmission of the crisis.

2.6 Conclusions

Our paper examines the asymmetric transmission of financial crises across borders. We interpret financial crises as disruptions in the ability of financial markets to insure against idiosyncratic shocks. When this occurs in a country like the United States, this changes the portfolio choice and leads to a retrenchment in foreign capital holdings. This decrease in foreign capital transmits the crisis, and, furthermore, hurts foreign countries more the more they rely upon capital inflows.

When looking at data from the Lane and Milesi-Ferretti database as well as the U.S. Treasury International Capital database, we see preliminary evidence that confirms this relationship. Specifically, we observe that financial flows from financially well-developed countries, like the U.S. and the U.K., fell sharply, while flows from large economies, such as Japan, did not behave similarly. We also look into the importance of these capital outflows to the productive ability of those receiving them. For capital flows in general, and U.S. capital flows in particular, there is evidence of an economically significant impact on both output and domestic capital formation.

From this evidence, we propose a model that features financial markets of varying depth based upon Mendoza et al. (2009). In the model, when financial markets no longer intermediate risk as well, countries hold fewer productive assets, leading to less capital in foreign economies, and thus lower production. This transmission is stronger the more reliant upon foreign capital the country suffering from contagion was. We then show in a simple numerical example that the model is able to produce changes of similar magnitude to that observed in the data.

Our paper is, as far as we know, the first to feature asymmetric transmission, meaning that the originating country of the shock is fundamental to the spread of the shock, and that the countries that suffer from transmission do not suffer equally. In fact, it is possible that foreign countries suffer more than the initially affected country. This leads to many possible future research directions. Are more developed economies truly unaffected by crises abroad? Or are they simply not negatively affected because demand for safe assets increases which leads to cheap credit in developed economies? From a technical standpoint, turning the quality of financial markets into a stochastic process would require outside of the box thinking and development of new solution techniques. In the case of financial market quality being subject to aggregate shocks, the interest rate would depend on the distribution of assets, which, as shown in Sager (2015), is not currently solvable.

To conclude, we have presented a model that explains asymmetric international transmission of financial crises. When looking at the data, we find evidence that financial market quality is fundamental to the composition of financial flows, and that capital inflows drop sharply during transmitted crises. Our model shows that, when taking into account financial market depth, we are able to generate transmission of similar magnitude to that in the data.

Chapter 3

Education Costs, Entrepreneurship, and Demographics

3.1 Introduction

Entrepreneurship in the United States has declined steadily over the past 30 years. As shown in Figure 3.1, from 1983 to 2013, there was a decline of over 30% in new firm entrants per labor force participant. There is significant worry that this trend will continue. In addition to this decline in entrepreneurship, my paper documents the changing age composition of entrepreneurs, specifically the decreasing trend in entrepreneurship among those in their thirties, and the drastic rise in the fraction of households in their sixties that are entrepreneurs. From these facts, my paper contends that the observed decline in overall entrepreneurship is a result of this transition to a different age profile of entrepreneurs, and that there is reason to believe the decline will stabilize.

Using the Survey of Consumer Finances, I show how the rate of entrepreneurship by cohort has changed over time. The rate for those in their 30s has fallen from roughly 10.5% down to roughly 6% from 1983 to 2013, while, the rate for those in their 60s has increase from roughly 7.5% to over 10%. I contend these changes in the cohort rates of entrepreneurship are a result of the increased educational attainment of the population. Specifically, the drastic rise in post-secondary education costs makes it more difficult to self-finance entrepreneurial ventures, which, in turn, leads to entrepreneurship starting later in life.

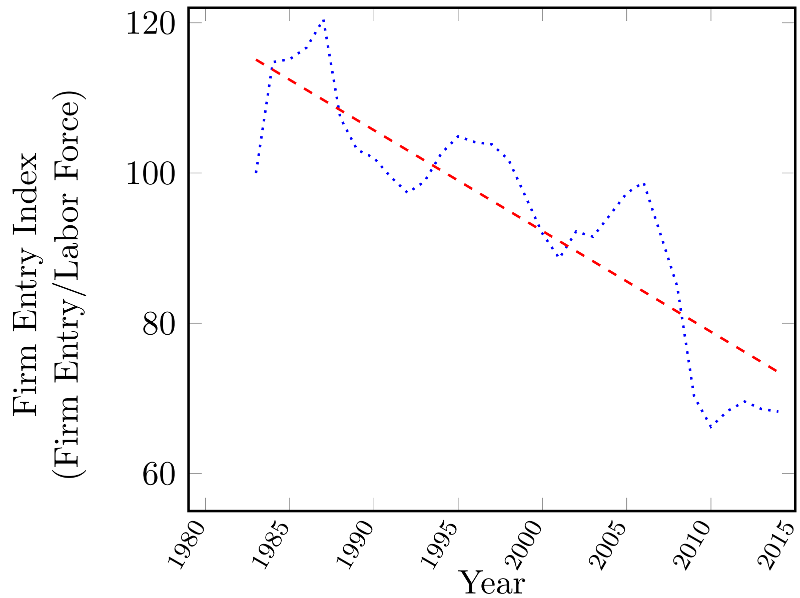

Figure 3.1: Decline in Firm Entry, 1983-2013 (Source: US Census BDS)

To understand what is driving the change in the rate of entrepreneurship by cohort and the implications for this change, I develop an overlapping generations model that includes an initial education choice as well as an entrepreneurship choice to match the age specific rates in the data in order to predict future cohort specific rates. In my model, college has two benefits designed to mimic those observed in the data. First, effective labor hours are increased by attending college, which creates a wage premium. Second, as the cost of college has increased faster than the skill premium, attending college must also improve entrepreneurial ability. In the model, this affects the mean and standard distribution of the productivity shock to the entrepreneurial production function.

The model works by combining the above benefits and costs with a collateral constraint. Going to college is beneficial in two ways, but has important side effects. First, it lowers the initial asset level upon entering the labor market making the collateral constraint bind more tightly, decreasing the rate of entrepreneurship among the young. Furthermore, it takes time to accumulate assets to a level that makes entrepreneurship appealing. When calibrating the model to the data, I am able to match the observed rates of education and entrepreneurship by cohort from the early 1980s. I then run several experiments in an attempt to replicate the rates of entrepreneurship and education observed today. Holding constant the productivity processes facing entrepreneurs, I find that my model can replicate the observed education choices of the population, as well as the decline in young entrepreneurs when taking into account the change in education cost, availability of student loans, and skill premia. However, it is much more difficult to match the rate of entrepreneurship among older cohorts. In an attempt to explain the behavior of older entrepreneurs, I incorporate several salient changes into my model. Namely, I lengthen the working life of households, which takes into account the observed compression of morbidity over the time period. Compression of morbidity is the technical term for the idea that "70 is the new 50," meaning that people live longer, healthier lives, which is changing the quality and duration of retirement. When taken in combination, the four key trends of increased skill premia, increased student debt holdings, increased college costs, and the compression of morbidity, lead to an equilibrium that matches the observed educational and entrepreneurial profile of the United States.

Another key result from my model is that the decline in entrepreneurship is efficient, with implied total factor productivity rising by almost two-thirds from 1983 to 2013. In addition, this improved efficiency helps to explain the tepid entrepreneurship response for older cohorts in the model. The improved efficiency arises from the increased supply of effective labor that comes from increased education. The most productive entrepreneurs are constrained not only by the collateral constraint, but also by the supply of labor which is now larger. With more effective labor available and increased demand from the best entrepreneurs, the general equilibrium wage increases, leading to fewer entrepreneurs with higher productivity. This affects the number of older entrepreneurs significantly, as not only do they need a higher productivity shock, but also more assets in order to choose to be an entrepreneur. However, older cohorts are decumulating assets, which leads to the low levels in older entrepreneurship in the model. This result also provides a partial answer to the motivation for this paper: is the observed decline in entrepreneurship worrisome? In my model, the answer would be no, it is a sign of improved efficiency of the economy. Fewer, larger, more productive firms with a better educated workforce is not a troubling sign for the economy.

Related Literature:

This paper fits into two main strands of literature, namely the work done on entrepreneurship in the United States and on the impact of rising education costs and student debt. Regarding student debt, Ionescu (2009) looks at the impact of student loan policies on college enrollment and finds that the amendment the Higher Education Act (HEA) implemented in 1986 improved access to post-secondary education by making financing for college easier to obtain for low income households. This ties directly to the above argument that newly educated households have tighter collateral constraints as the cost of college increases. In addition, Lochner and Monge-Naranjo (2015) also study the availability of college loans and the ability to pay back student debt afterwards. They show that despite increased costs, the returns to college are sufficiently large to generate a rising demand for college loans. Finally, the Federal Reserve Bank of New York maintains a database on student debt, and Chakrabarti et al. (2017) show that especially among the youngest cohorts, student debt and automotive debt have been the only growing components, and there are now more student loan borrowers with higher balances than ever.

The decline in entrepreneurship has been the subject of much research. Regarding the trend of the startup rate, Pugsley et al. (2015) examine how much of the long term decline in firm entry is a result of the aging of the population in the U.S. They argue that fewer firms enter because the growth rate of the working age population has declined, meaning there are relatively fewer young people available to become entrepreneurs. My paper focuses on cohort-specific rates of entrepreneurship, which controls for the size of the cohort population. In this sense, my findings extend the results of Pugsley et al. (2015) to show that not only has the size of the working age population changed, but the rate of individuals choosing to become entrepreneurs has decreased as well. In addition, Stangler and Spulber (2013) indicates that while demographics have indeed been shifting, the absolute number of Americans in the historical peak age for becoming entrepreneurs is at its highest ever, which indicates the entrepreneurship decision must have changed.

The troubling nature of the decline in entrepreneurship is based upon work by King and Levine (1993), among others, who show that entrepreneurship is important to economic growth. Understanding the characteristics of entrepreneurs is important to understanding this trend. To that end, the Kauffman Foundation described entrepreneurs in a series of releases in 2009 based on a survey of roughly 500 successful entrepreneurs in high growth fields. Wadhwa et al. (2009b) and Wadhwa et al. (2009a) summarize these results. A relevant result is that education is important to successful entrepreneurship (over 90% of those surveyed had at least a bachelor's degree, when, in comparison, under 40% of the overall U.S. population has at least a bachelor's degree). Levine and Rubinstein (2013) also examines the characteristics of new entrepreneurs using the NLSY79 and finds that entrepreneurs tend to be both smarter and engaged in riskier behavior as youths than non-entrepreneurs. This indicates that a college education has a positive effect on entrepreneurial ability, which, as mentioned above, manifests itself as potentially different stochastic processes in my model.

The importance of collateral constraints on entrepreneurial ventures is widely documented in the literature, starting with the seminal paper by Evans and Jovanovic (1987). More recently, Robb and Robinson (2014) use the Kauffman Firm Survey to examine funding for new firms and see that personal financing through external debt (e.g. bank loans) is the most important, which coincides with the aforementioned summaries on entrepreneurs above. For further reference on this literature, Kerr and Nanda (2009) is an excellent survey of papers dealing with the importance of financing constraints on entrepreneurship, and finds consensus on the importance of self-financing entrepreneurial ventures which validates the use of collateral constraints when modeling entrepreneurial decisions.

From a modeling standpoint, my paper is closely related to the work on entrepreneurship choice models with collateral constraints such as Buera et al. (2015), Buera et al. (2011), Buera and Moll (2012), and Cagetti and Nardi (2005). My model is an extension of the model used in Moll (2014). As previously mentioned, instead of having infinitely lived households as in Moll (2014), an overlapping generations model is used. Furthermore, there is an education choice in the first period of life for each generation. Most importantly, in Moll (2014) there are fixed measures of entrepreneurs and workers. If an entrepreneur chooses not to produce that period, they do not enter the labor market. In my model, if an agent chooses not to be an entrepreneur they enter the labor market and obtain employment as workers.

There is recent work on the intersection between student debt and entrepreneurship. Krishnan and Wang (2015) is among the first to try to empirically tease out a causal relationship between student debt and declining entrepreneurship using the Survey of Consumer Finances. Ambrose et al. (2015) find that increased student loan debt is detrimental to small business formation. Baum (2015) provides an overview of the current state of research. My paper's contribution to this joint literature is a calibrated dynamic stochastic general equilibrium model that is used to forecast future changes in entrepreneurial activity.

The structure of the paper is as follows. Section 2 describes the data. In section 3, I present the model and characterize its solution. In section 4, I detail the parameterization of the model. Section 5 provides the main numerical results of the model. Finally, Section 6 provides a brief conclusion.

3.2 Data

The primary data set used in my analysis is the Survey of Consumer Finances, which has been administered triennially starting in 1983. Unfortunately, the SCF is not a panel data set, but otherwise it is ideally suited to answering questions on entrepreneurship as it asks detailed questions on both the financial and employment status of the respondents, which are the heads of the primary economic unit, meaning the respondent, their spouse, and any dependents currently living in the same domicile. The SCF is especially useful for looking at entrepreneurship as it oversamples at the top end of the wealth distribution, where many entrepreneurs exist, to give a clearer picture of entrepreneurial characteristics.

Before examining entrepreneurship, it is important to define an "entrepreneur" in the context of this data set. The Survey of Consumer Finances does not directly ask respondents if they are an entrepreneur, though it does ask many questions about respondents' occupations. Following Cagetti and Nardi (2005), I define an entrepreneur as a respondent who satisfied all three of the following criteria: Is self-employed, a business owner, and an active business manager. This definition eliminates wealthy respondents who invest in businesses as a form of diversification without taking an active management interest, in addition to eliminating managers of businesses who do not have an ownership share.

Figures 3.2 and 3.3 show the rates of entrepreneurship across 10 year age cohorts. Figure 3.2 shows the rates for entrepreneurs in their 30s and 40s and the fitted values for these times series. According to Fairlie et al. (2015), the historical peak age for entrepreneurship is 40 years old, so the 30 year old cohort shows the rate of entrepreneurship in the lead up to this age, while the 40 year old cohort shows the rates after this peak age. There is a pronounced change for the 30 year old cohort, falling from around 10.5% in 1983 to roughly 5.5% in the most recent survey. When looking at the 40 year old cohort, it is relatively stable with perhaps a slight downward trend.

In Figure 3.3, the rates of entrepreneurship for people in their 50s and 60s and their fitted values are shown. With the exception of a noticeable bump for 50 year olds around the 2000 dot-com boom, the rate of entrepreneurship of 50 years olds has been very stable over the course of the surveys. The noticeable trend is the increase in entrepreneurship in the 60 year old cohort. Going from around 7% on average in the 1980s to over 11% in the late 2000s. To briefly summarize, there are noticeable trends in the age composition of entrepreneurs. Relative to the early 1940s, fewer thirty-somethings are becoming entrepreneurs, while many more sixty-somethings are either starting or maintaining their entrepreneurial ventures today.

Figure 3.4 shows the change in the rate of entrepreneurship when tracking a single group over the course of the available surveys, which provides insight into when households are choosing to begin and end their entrepreneurial ventures. The data points show the change in the rate of entrepreneurship for the 40 (60) year old cohort relative to the value of the 30 (50) year old cohort from three surveys prior. To provide a concrete example, the 1992 data points are the change in the rate of entrepreneurship for the 40 (60) year old cohort in the 1992 survey minus the rate of the 30 (50) year old cohort in 1983. The top (blue dotted) line shows a general upward trend, meaning that more entrepreneurs are entering in their 40s

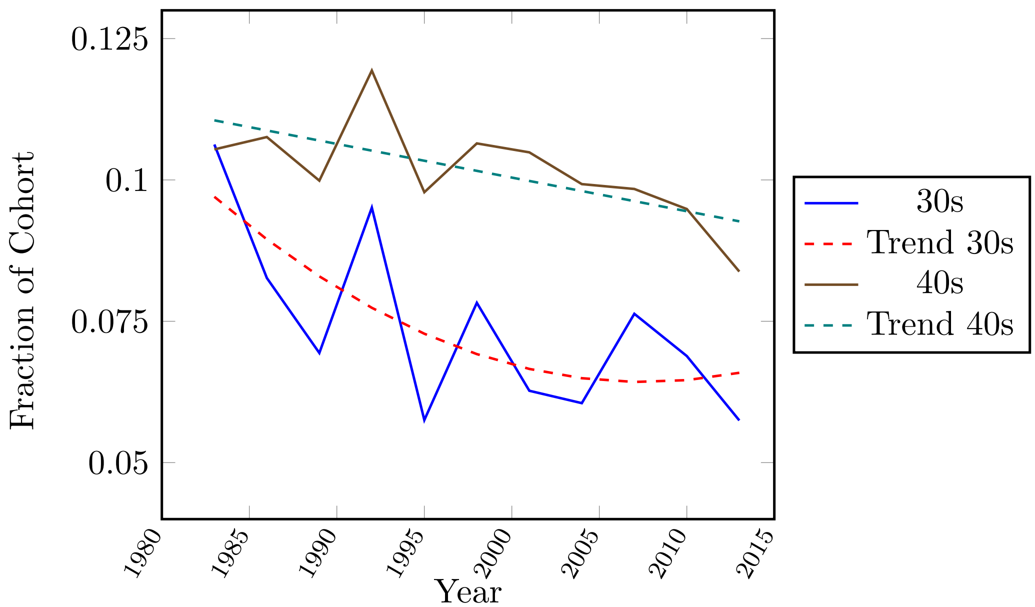

Figure 3.2: Rate of Entrepreneurship by Cohort

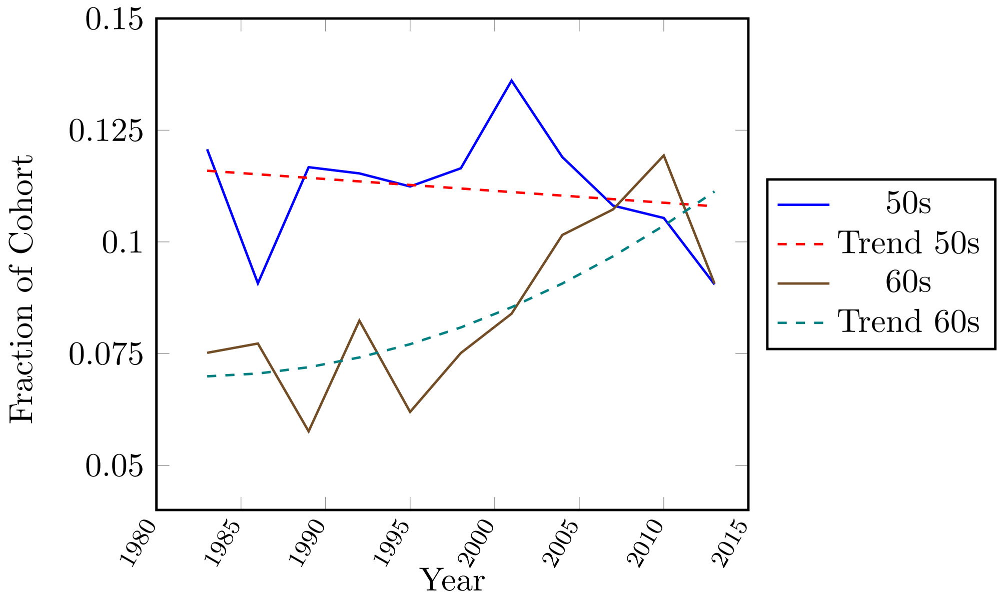

Figure 3.3: Rate of Entrepreneurship by Cohort

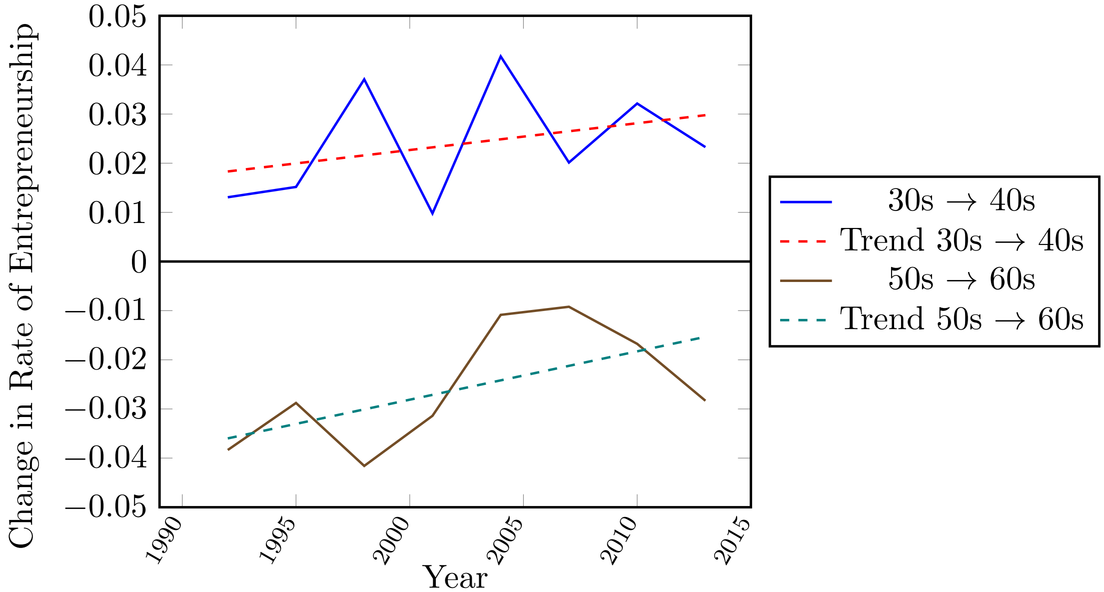

Figure 3.4: Changes in Rates of Entrepreneurship by Cohort

relative to the past. Similarly, the bottom (red dashed) line shows that fewer entrepreneurs are leaving in their 60s. Taken together, Figures 2 and 3 show that the entrepreneurial choice decision facing households has changed significantly since 1983 and has affected the timing of entrepreneurial entry.

Figure 3.5 shows the change in educational attainment from 1983-2013 by showing the fraction of both the population as a whole and of entrepreneurs that have a post-secondary degree, meaning a college degree or higher. Both lines show a strong upward trend, rising from around 22% to just under 40% for the whole population, while entrepreneurs increase from around 32% to 52%. Throughout, entrepreneurs have a higher average education level, which coincides with the literature that indicates education is important to entrepreneurial success. Importantly, the educational attainment gap between entrepreneurs and the population as a whole never narrows. If a skill premium alone accounted for the benefit obtained from post-secondary education, then entrepreneurship would be less appealing than the increased wage obtained from working leading to a reduced fraction of educated entrepreneurs, but this is not the case. Again, this indicates that college is both important to entrepreneurial ventures and that this benefit may have concurrently increased with the skill premium.

From the College Board's Trends in College Pricing database, I am able to track the change in the real cost of college from 1983-2013. Figure 3.6 documents the increasing cost of college in the U.S., both for public and private institutions during this time frame.

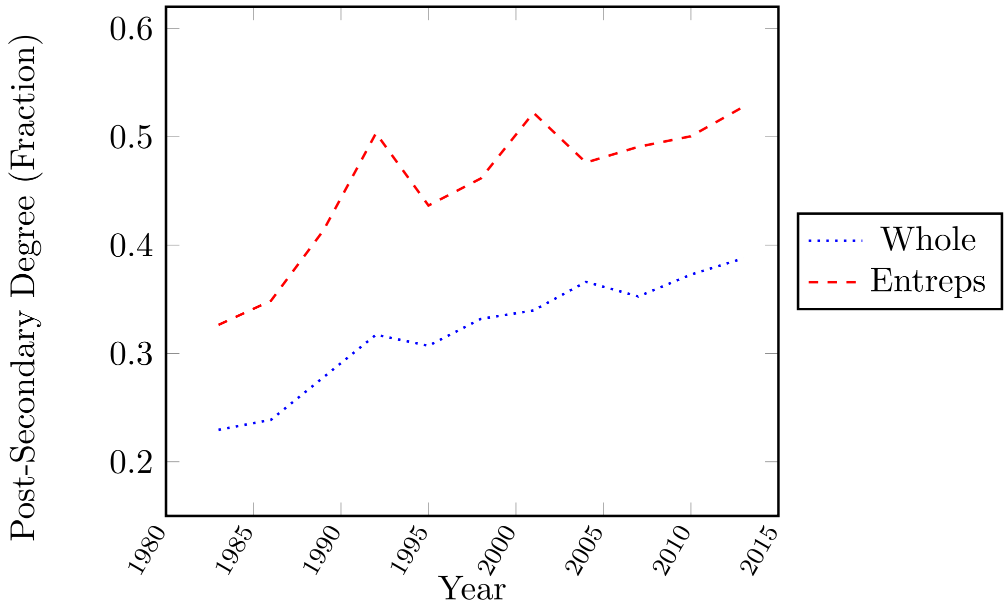

Figure 3.5: Education (Fraction with a Post-Secondary Degree)

The cost of college includes tuition, room and board, and other fees and expenses and is expressed in 2016 dollars. From this figure, we see that college costs have more than doubled during this time frame. It is important to note that, if the data set is extended back to the early 1970s, prices were relatively stable up until the late 1980s, which coincides with the observations from Ionescu (2009) and Lochner and Monge-Naranjo (2015) regarding the HEA amendments likely having a significant impact on college demand and pricing.

Figures 3.7 and 3.8 show how education loan holdings have changed over time. The Survey of Consumer Finance started explicitly collecting data on student loans in 1992, after the Higher Education Amendments of 1986 and 1992 went into effect and significantly widened the availability of education loans. Figure 3.7 shows the unconditional average amount of student debt in 2016 dollars. The average real value of student loans has almost quintupled in the past 20 years. Figure 3.8 shows that not only is the magnitude increasing, the fraction of the population exposed to student loan debt is as well. Figure 3.8 is the distribution of student loan holdings for individual years of the Survey of Consumer Finances, with the x-axis showing age cohorts and the y-axis is the fraction of each cohort that have outstanding student loan debt. From this chart, it is apparent that the incidence of student loans has increased from roughly 20% to over 40% in under 20 years. Furthermore, student loans persist longer into the life of households. Now over 20% of people between 35 and 40 have outstanding student loan debt, as compared to roughly 10% in the past. One caveat to note is that, for the very oldest cohorts that have outstanding student debt it is possible that it is inherited debt from their child if that child defaulted on their loans, though this

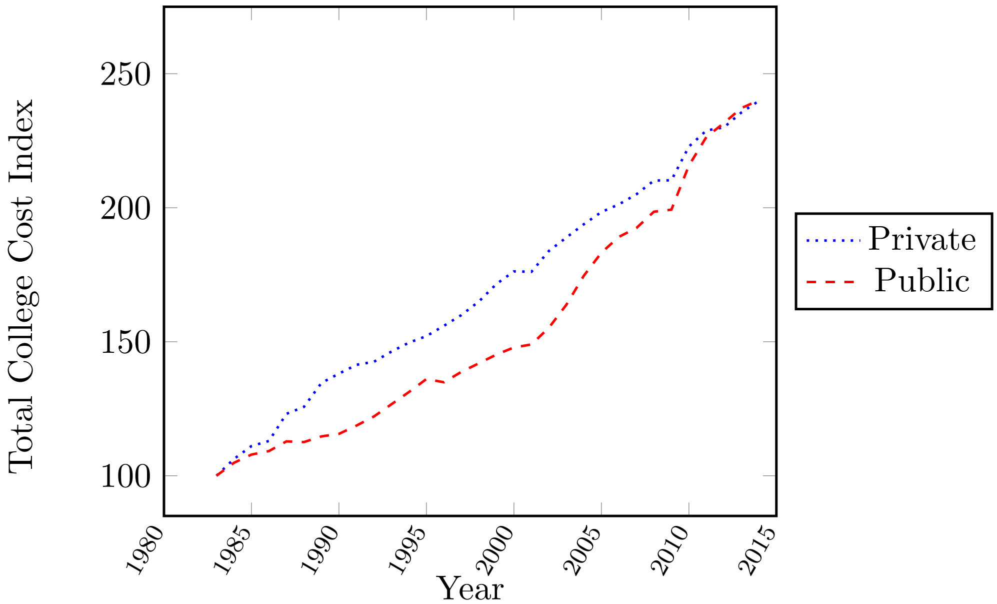

Figure 3.6: Increase in Real College Costs

is highly unlikely in the case of outstanding student debt for those in their 40s and early 50s given the current average first child age of roughly 26 years old in the United States. From these two charts, it is clear that education and its associated costs have become a more important component of household balance sheets for much of the household's prime working years.

To briefly summarize the key data points, the age composition of entrepreneurs has been changing over the past 30 years, with entrepreneurs both starting and ending their ventures later in life. At the same time, the population became better educated at increasingly higher costs which increasingly manifested itself on household balance sheets, and persisted into middle age which historically was the optimal time to begin an entrepreneurial venture. In addition, the importance of education for entrepreneurs has not diminished during this time, indicating that, in addition to an increasing skill premium over time, the returns to entrepreneurship for educated entrepreneurs may also have improved. In the following section, I present a model which will attempt to explain these data observations and allow for better predictions of entrepreneurial behavior in the future.

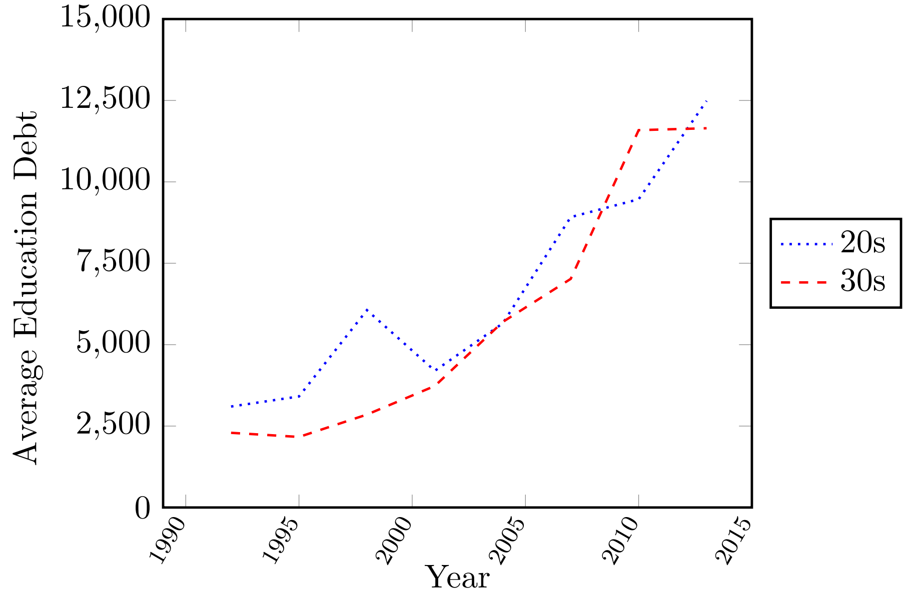

Figure 3.7: Average Education Debt by Cohort ($2016)

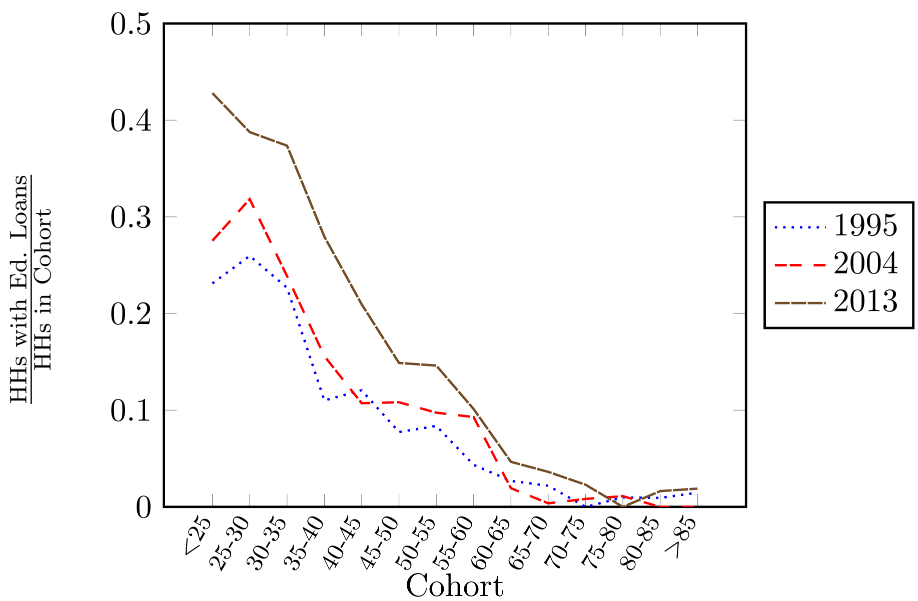

Figure 3.8: Distribution of Education Loans

3.3 Model

3.3.1 Environment

To evaluate the effects of increasing education costs on entrepreneurial decisions, I use an overlapping generations model with heterogeneous agents. Each period a unit mass of households are born. Each household lives for $N+1$ periods consisting of an initial period followed by N periods of working life. Upon birth, these households receive an idiosyncratic initial endowment, ω from a log-normal distribution, Ω. In subsequent periods, each household receives an idiosyncratic productivity shock z to their entrepreneurial production technology. This productivity shock is assumed to follow a Markov process, $g(z(X), z'(X))$ that is dependent up their education choice X. As all shocks are idiosyncratic and the law of large numbers holds, there is no aggregate uncertainty. In each period, the state of the economy is the joint distribution, $d_t(a, z, X)$, of households over assets, shocks, and education. Let $\phi_t(a, X)$ be the corresponding marginal distributions over assets, and let $\psi_t(z, X)$ be the marginal distributions over productivity shocks.

In the first period of life, households make their education choice, $X \in \{0, 1\}$. There is a fixed cost, ψ, paid in the first period to attend college. There are two benefits to education: an increase to the endowment of effective labor hours, $\bar{L}(X)$, as well as the aforementioned changing of the process of productivity shocks for entrepreneurial technology.

The timing of the model is as follows:

- First period of life:
  - Endowment shock realized
  - Household choose consumption, c, assets for next period, a', and education, X
- Subsequent periods:
  - Enter period with assets, a, and education, X, observe productivity shock z
  - Make entrepreneurship choice, e, and earn income
  - Choose consumption and next period's assets

3.3.2 Household Problems

When presenting the problems faced by households I will use superscripts to denote the period the household was born, and subscripts to denote the current period. For a household born in period t, the value function in period t is:

$$V_t^t(\omega) = \max_{c_t^t, a_{t+1}^t, X} \left\{ u(c_t^t) + \beta \sum_{z_{t+1}} V_{t+1}^t(z_{t+1}(X), a_{t+1}, X) g(z_t(X), z_{t+1}(X)) \right\} \tag{3.1}$$

It is assumed that the period utility function is continuous, differentiable, strictly increasing, strictly concave, and is the same for all households and periods. β is the rate of time preference, and it is assumed that productivity shocks have a finite support. In the first period of life, households receive an idiosyncratic initial endowment, ω. They use these resources to choose between consumption today, c_t^t, education, $X \in \{0, 1\}$, and assets to carry into tomorrow, a_{t+1}^t. When the household chooses to enroll in post-secondary education (meaning $X = 1$), they face a fixed cost, ϵ, that is only incurred during this initial period of life. This leads to the following law of motion for assets:

$$a_{t+1}^t = \omega - \psi X - c_t^t \tag{3.2}$$

Additionally, it is assumed that both consumption and assets must be non-negative, meaning there is a limited liability constraint. This assumption corresponds with households financing their own college education. This is reasonable for modeling the college choice facing households in the early 1980s, prior to the Higher Education Amendments of 1986 and 1992 which increased the availability of student loans. In my numerical exercises, I will relax this constraint to introduce the concept of student debt. Regardless of the situations, I need to ensure assets are non-negative upon ending working life, as markets are incomplete. In this case, the equilibrium rate of return is insufficient to compensate for the discount rate. In the case of a worker that knows they will always be a worker, and thus faces no uncertainty in their lifetime, this would lead them to decumulate assets as much as possible over the course of their life, necessitating either this limited liability constraint, or, in the case where I allow student debt, a penalty for ending working life with outstanding debt.

After the first period of life, the household state is defined by their college education choice, the assets they saved from the previous period, and the productivity shock for the current period, $t + i$ where $i \in \{1, 2, \ldots, N\}$. The value function for a given i is:

$$V^t_{t+i}(z_{t+i}, a^t_{t+i}, X) = \max \left\{ u(c^t_{t+i}) + \beta \sum_{z_{t+i+1}} V^t_{t+i+1}(z_{t+i+1}, a_{t+i+1}, X) g(z_{t+i}, z_{t+i+1}) \right\} \tag{3.3}$$

The main difference in the problem is in the law of motion for assets. Prior to choosing consumption and next period's asset holdings, the household must earn income either by entering the labor market or starting an entrepreneurial venture, $e^t_{t+i} \in \{0, 1\}$, where $e = 1$ indicates an entrepreneur. From this, households either earn labor income or entrepreneurial profits. Labor income is simply the prevailing wage in the market, w_{t+1}, multiplied by the number of effective labor hours supplied by the household, ℓ^t_{t+i}. The endowment of effective labor hours each household has available depends on its education decision, with more effective hours if the household had attended college.

$$\underbrace{\ell^t_{t+i}}_{\text{Labor Supply}} \leq \underbrace{\bar{\ell}^t_{t+i}(X)}_{\text{Labor Endowment}} \tag{3.4}$$

For convenience define entrepreneurial profits, Π, to be $f(z_{t+i}, k^t_{t+i}, \ell^t_{d,t+i}) - w_{t+i}\ell^t_{d,t+i} - (r_{t+i} + \delta)k^t_{t+i}$, where $f(z, k, \ell_d)$ is the entrepreneurial production technology and is assumed to satisfy the Inada conditions. Entrepreneurial profits are simply the difference between production and factor payments. It is assumed that factors of production are rented in perfectly competitive markets and that capital is subject to a depreciation rate given by δ. There is one additional source of income, namely the returns on assets brought into the period. Now that income has been determined, households choose between consumption and savings for next period. This leaves us with the law of motion for assets for all periods $t + i$, $i \in \{1, 2, \ldots, N\}$:

$$\begin{aligned} a^t_{t+i+1} = (e^t_{t+1}) & \overbrace{\left[f(z_{t+i}, k^t_{t+i}, \ell^t_{d,t+i}) - w_{t+i}\ell^t_{d,t+i} - (r_{t+i} + \delta)k^t_{t+i} \right]}^{\text{Entrepreneurial Profits}} \\ & + (1 - e^t_{t+i}) \underbrace{\left[w_{t+i}\ell^t_{t+i} \right]}_{\text{Labor Income}} + (1 + r_{t+i})a^t_{t+i} - c^t_{t+i} \end{aligned} \tag{3.5}$$

There is one final constraint that governs the consumption savings choice. When starting an entrepreneurial venture, households face a collateral constraint in the vein of Evans and Jovanovic (1987), where the amount of capital an entrepreneur can raise for his venture is proportional to the assets they hold:

$$k_{t+i}^{t} \leq \lambda a_{t+i}^{t} \tag{3.6}$$

Using a collateral constraint that is linear in wealth is useful as will be shown when characterizing the solution to the problem. Furthermore, it is possible to interpret λ as a measure of market completeness, where $\lambda = \infty$ is representative of perfect capital markets, and $\lambda = 1$ is representative of pure self-financing.

3.3.3 Market Clearing

There are three markets in the model that need to clear: the goods market, the capital market, and the labor market. By Walras's law, the combination of each household's budget constraint and the market clearing conditions for labor and capital make the goods market clearing condition redundant. The labor market clearing condition is sufficient to pin down the first of the two endogenous prices in the model, specifically the wage. For the labor market to clear, we need the total number of effective labor hours demanded by entrepreneurs to equal the total amount of effective labor hours supplied in each period. For notational convenience define $\ell_{d,t} = \sum_{i=1}^{N} \ell_{d,t}^{t-i}$ as the sum across all generations of labor demand in a given period, and $\bar{\ell}_t(X) = \sum_{i=1}^{N} \bar{\ell}_t^{t-i}(X)$ is the sum over all generations and is contingent on the education choice from the first period of life. Then, labor market clearing is given by:

$$\int \ell_{d,t} dG(a,z,X) = \int \bar{\ell}_t(X) dG(a,z,X) \qquad \forall t \tag{3.7}$$

The second price, the rate of return on assets each period, will be determined by the capital market clearing condition. Each period the total amount of capital used by entrepreneurs must equal the total amount of assets available.

$$\int k_t dG(a,z,X) = \int a_t dG(a,z,X) \qquad \forall t \tag{3.8}$$

3.3.4 Equilibrium Definition

For the remainder of the paper I will assume that each new-born generation receives its initial endowment from the same log-normal distribution, Ω. This, combined with the lack of aggregate risk, allows us to define a stationary equilibrium, where the exact time period is no longer important. As before, the subscripts denote generation, with $i = 0$ being the newest born generation, and $i = N$ being the oldest. For this model, a stationary equilibrium is:

1. Prices: w and r

2. Value function and policy functions for consumption, assets, and education for the initial generation: $V_0(\omega), c_0(\omega), a_0'(\omega)$ and $X(\omega)$.

3. Value functions and policy functions for consumption, assets, and entrepreneurial decisions for generations $i = 1, \cdots, N$:

$$\{V_i(z,a,X)\}_{i=1}^{N}, \{c_i(z,a,X)\}_{i=1}^{N}, \{a_i'(z,a,X)\}_{i=1}^{N}, \{e_i(z,a,X)\}_{i=1}^{N},$$
$$\{k_i(z,a,X)\}_{i=1}^{N}, and\, \{\ell_{d,i}(z,a,X)\}_{i=1}^{N}$$

Such that, given the initial endowment distribution, Ω, the cost of education, ψ, a stochastic productivity process for entrepreneurial shocks, z, the rate of depreciation, δ, the collateral constraint parameter, λ, and sequences of education-dependent effective labor hour endowments for each generation, $\left\{\bar{\ell}_i(X)\right\}_{i=1}^{N}$

1. Taking prices as given, the generation 0 value and policy functions for household education choice, consumption, and asset holdings solve the first period household problem given above by equations (3.1) and (3.2)

2. Taking prices as given, the value and policy functions for household allocations, entrepreneurial decisions, and capital and labor demand solve the remaining generations' household problems given above by equations (3.3), (3.5), and (3.6)

3. Both the capital and labor markets clear according to (3.7) and (3.8)

3.3.5 Characterization

This section derives the key analytical result of my model and provides insight into how changes to skill premia and education costs will affect rates of entrepreneurship across cohorts. The following steps lead to a productivity cutoff rule that depends on both age and education as the skill premium a household earns varies over their working life. For the remainder of this section, I will suppress time notation for ease of presentation, as in each period of working life the household faces a static income maximization problem. This means for any working household:

$$e^* = \begin{cases} 1 & f(z, k^*, \ell_d^*) - w\ell_d^* - (r+\delta)k^* > w\ell^* \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

The easier side of the cutoff rule is the optimal labor supply choice. As labor is not in the utility function, households will supply all of their labor endowment for the period, $L^* = \bar{L}(X)$. The more complicated component comes from entrepreneurial profits.

As was done above, let $\Pi = f(z, k, \ell) - w\ell_d - (r+\delta)k$. Let the production function be $f(z, k, \ell) = (z(X)k)^{\alpha}\ell^{1-\alpha}$. Full algebraic details of the results that follow are available in the appendix to the paper. Entrepreneurs will maximize their profit by finding their optimal labor demand, which is:

$$\ell_d^* = \left(\frac{(1-\alpha)}{w}\right)^{\frac{1}{\alpha}} z(X)k \tag{3.10}$$

Substituting this into the profit function and simplifying yields:

$$\Pi = \left(\frac{(1-\alpha)}{w}\right)^{\frac{1-\alpha}{\alpha}} z(X)k - w\left(\frac{(1-\alpha)}{w}\right)^{\frac{1}{\alpha}} z(X)k - (r+\delta)k$$

Notice that this is linear in k, so our optimal k^* is either 0 or our constrained value λa. Simplifying further by substituting λa into (3.6) yields the following:

$$\Pi = \max\{\pi z - r - \delta, 0\}\,\lambda a \tag{3.11}$$

Where $\pi = \alpha\left(\frac{(1-\alpha)}{w}\right)^{\frac{1-\alpha}{\alpha}}$ is a constant. When household income is maximized by simply not producing as an entrepreneur, entrepreneurs switch to being laborers, and instead earn a wage. This will define a cutoff level of productivity, $z^*(X)$, for each level of education. Households will be indifferent to labor or entrepreneurship when labor income equals entrepreneurial profits, which yields the following cutoff condition that is conditional on the education choice of the household:

$$\underbrace{w\bar{\ell}(X)}_{\text{Labor Income}} = \underbrace{(\pi z^*(X) - r - \delta)\,\lambda a}_{\text{Entrepreneurial Income}} \tag{3.12}$$

This is the key result. Notice first that the cutoff productivity is lower the higher the current level of assets. By paying for college in the first period, assets brought forward into the next period decrease, and increases the time to accumulate the necessary assets to make entrepreneurship viable. This effect will be accentuated as college education costs increase overtime, which will be the main mechanism to decrease the rate of entrepreneurship for the young. Education also changes the effective labor hour endowment, increasing the productivity cutoff in any given period. When looking at the data, the skill premia has increased not only on the whole, but also at a cohort level. As was mentioned before, if the only skill premia increased for each cohort, then we would expect fewer educated entrepreneurs at all ages, which is not the case in the data, which indicates the returns to entrepreneurship have changed over time as well. From this equation, the main mechanisms of the model are apparent, but to estimate the value of change in returns to entrepreneurship, as well as make predictions about how entrepreneurial decisions will be made in the future, a calibrated version of the model is required.

3.4 Calibration

When calibrating my model, I am attempting to match 1983 before feeding in changes to education costs and skill premia. I use $N = 5$ generations, where I interpret each generation as 10 years. Thus, in the first generation of existence, the agents are in their twenties when making their post-secondary education decisions. The remaining four generations correspond to their thirties through sixties. I assume a CRRA utility function, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, and the previously mentioned Cobb-Douglas production technology, $f(z, k, \ell) = (z(X)k)^{\alpha}\ell^{1-\alpha}$. I separate my parameters into two sets, those directly estimated from the data, either using my data set (such as the skill premia across age cohorts) or those reliant upon extensive previous work (e.g. the capital share), and those that I use to match some relevant moments from the data, specifically the fraction of the population that attains a post-secondary degree, as well as the fraction of each cohort that are entrepreneurs.

Table 3.1 presents the parameters I use that are derived from the data or are commonly used values from previous work. I assume an annual discount rate of 0.98, which implies a beta of 0.82 over ten years. The depreciation rate of 46% corresponds to an annual depreciation rate of 6%, as estimated in Stokey and Rebelo (1995). Gollin (2002), I use a capital share of 0.33. For the collateral constraint parameter, λ, I follow Evans and Jovanovic (1987) who use $\lambda = 1.5$ for the 1980s. Finally, $\sigma = 1.5$ as is standard in the entrepreneurship literature, such as Buera et al. (2011) and Cagetti and Nardi (2005).

Using the SCF, I estimate the skill premium for each cohort, which generates values for my effective labor hours starts at around 1.75 for those in their thirty year old cohort,

Parameter	(Annualized) Value
Discount Rate, β	0.98
Depreciation Rate, δ	0.06
Capital Share, α	0.33
Collateral Constraint Parameter, λ	1.5
CRRA Parameter, σ	1.5

Table 3.1: Fixed Parameter Values

to around 2.3 for those in their fifties. Recall that, for the purposes of this paper, the skill premium is for all post-secondary education including post-graduate degrees not only college degrees. Finally, in my base case, I assume that the productivity shocks to the entrepreneurial activity are iid, regardless of education status. In the United States, under one-third of small business start-ups survive for ten or more years according to the Small Business Administration. Given this information and my period length of ten years, I do not want to assume any persistence to an individual entrepreneurial venture.

After estimating the above parameters from the data, there are five remaining parameters available to calibrate the model. These five parameters are: the cost of education, the means of the two productivity processes, and the standard deviation of the two productivity processes. The five moments I want my model to match are the fraction of the first cohort that attends college, and the fraction of the following four cohorts that are entrepreneurs. Table 3.2 reports the calibration results of these parameters. The fraction attending college is slightly lower than that observed in the data, but the model closely matches the data other than for entrepreneurs in their 40s. The calibrated productivity process values have no intrinsic meaning, but are interesting in terms of the relative differences between the processes for educated and uneducated entrepreneurs. The calibrated means of the two processes are close, with the mean for college educated slightly above the mean for the uneducated, while the calibrated standard deviations are significantly different. The standard deviation for the college process is roughly one and a half the size of the standard deviation for the uneducated process. In this sense, college makes you slightly better as an entrepreneur on average, but results in bigger boom or bust ideas. Regarding the other calibrated parameter, the relative value for education costs is roughly one and a half times the mean net asset value of the initial log-normal distribution. In the SCF, the mean net worth of the 20 year old cohort in 1983 was roughly \$18,000, implying a total college cost of roughly \$27,000, or a yearly cost of just under \$7,000. The College Board's Trends in College Pricing data set puts the cost of a public college education in 1983 at just above \$7,000 per year. This result gives me further confidence in my model's calibration.

Moment	Data	Model
College Frac.	0.22	0.19
30 yr. old Entrep. Frac.	0.11	0.13
40 yr. old Entrep. Frac.	0.11	0.16
50 yr. old Entrep. Frac.	0.12	0.13
60 yr. old Entrep. Frac.	0.08	0.07

Table 3.2: Calibration Results for 1983

3.5 Numerical Exercises

3.5.1 Decomposing the Effects of Education and the Compression of Morbidity

In this section I use my model to evaluate the impact of four trends on educational attainment and the rates of entrepreneurship across age cohorts. Specifically, I use the data to update key exogenous variables in my model: The cost of post-secondary education, the ability to borrow to finance college, and the cohort specific skill premia. For the increase in the college education cost, I adjusted the cost of education to reflect the observed increase documented in Figure 3.6. In the initial calibration, I assumed college needed to be self-financed, while now I relax the borrowing constraint. This is meant to reflect the significant increase in availability in college debt that occurred after the passage of the previously mentioned Higher Education Amendments of 1986 and 1992. In the model, I allow households to have a negative net asset position that corresponds to the data shown in Figure 3.7after the initial education choice that they must clear prior to the end of their working life. Finally, I have adjusted the cohort-specific skill premia in accordance with the values in the 2013 wave of the SCF.

The fourth key trend to incorporate is the compression of morbidity observed from the mid-1980s to the mid-2000s. The above three trends all would lead to a decrease in entrepreneurship across all demographics, so it is important to understand what is changing the end of life decisions of households. People now live longer, healthy lives. The idea that "70 is the new 50," technically known as the compression of morbidity, has led to longer retirements. The working life of the average American has changed significantly in the past 40 years, as the average life expectancy has risen to almost 80 year olds. Freedman et al. (2013) examines trends in quality of life changes for older Americans. They find that, over the 1980s and 1990s, those in 65-84 year old cohort were progressively healthier and had fewer activity limitations, but that these gains have flattened out post-2005. This has translated into an expectation of a longer retirement To reflect this in my model, I extend

Moment	Data	Model
College Frac.	0.38	0.33
30 yr. old Entrep. Frac.	0.06	0.07
40 yr. old Entrep. Frac.	0.10	0.10
50 yr. old Entrep. Frac.	0.12	0.08
60 yr. old Entrep. Frac.	0.11	0.11

Table 3.3: Numerical Exercise: All Trends

the retirement of households by a generation, meaning households still retire at 70, but now live an extra generation.

The following tables show the importance of the combination of the above trends. For all the comparative statics, I hold the stochastic processes for educated and uneducated entrepreneurs constant and then solve for the new steady-state general equilibrium to find how post-secondary education and entrepreneurship choices have changed. The main results are presented in Table 3.3.

First, the model explains roughly 85% of the observed increase in educational attainment (19% to 33% in the model, 22% to 38% in the data). The rates of entrepreneurship for each cohort match well, but for the 50 year old cohort being slightly lower than observed in the data. This is a combination of the peak of the cohort specific skill premia, as well as general equilibrium effects of the model that are more readily apparent after showing the importance of each trend to the main result. The following tables will show what happens if only 3 of the four trends are present, and from these the main mechanisms of the model will become apparent which will, in turn, explain why all four trends are necessary.

For the first decomposition, I will eliminate the increased length of retirement associated with the compression of morbidity. These results are presented in Table 3.4. First, notice the lack of compression of morbidity did not affect the education decision. The trends in costs, debt holdings, and cohort specific skill premia have effects that dominate those of compression of morbidity on the college decision. However, without the elongated retirement associated with that trend, households do not accumulate as many assets, which leads to lower entrepreneurship rates at all ages, with especially noticeable declines in entrepreneurship among the oldest age cohorts. Unsurprisingly, entrepreneurial activity declines most significantly at the end of the life cycle. Households now start their working career in greater debt and need to work longer to build up enough assets, but, once out of debt, a strong motive to decumulate assets at the end of life dominates any further asset accumulation. Interestingly, without the compression of morbidity, the model actually overshoots the decline among younger cohorts as well. Beyond the increased incidence of

Moment	Data	Model
College Frac.	0.38	0.33
30 yr. old Entrep. Frac.	0.06	0.04
40 yr. old Entrep. Frac.	0.10	0.05
50 yr. old Entrep. Frac.	0.12	0.02
60 yr. old Entrep. Frac.	0.11	0.01

Table 3.4: Numerical Exercise: No Compression of Morbidity

Moment	Data	Model
College Frac.	0.38	0.28
30 yr. old Entrep. Frac.	0.06	0.10
40 yr. old Entrep. Frac.	0.10	0.10
50 yr. old Entrep. Frac.	0.12	0.14
60 yr. old Entrep. Frac.	0.11	0.08

Table 3.5: Numerical Exercise: No Skill Premia Increase

debt and the time it takes to accumulate assets, households smooth their saving over the course of the lifecycle and with a shorter retirement the need to build up savings is reduced and leads to more consumption and less asset accumulation at younger ages.

Table 3.5 shows how the results change if the cohort-specific skill premia are not updated. In this case, the returns to education have decreased, and so too has the fraction of households that choose to obtain an education. With fewer households attending college and lower skill premia, the opportunity cost of entrepreneurship is lower for many households, and higher rates of entrepreneurship are observed for all cohorts. Notice that for younger cohorts, increased student debt accounts for a little under half the observed drop in entrepreneurship (13% to 10%) for 30 year olds. As will be see in Table 3.6, the skill premium will have an even smaller effect, so the combination of the two effects is clearly important in changing the entrepreneurship decision.

Table 3.6 examines the impact of eliminating student debt while still accounting for the other trends. The increased availability of student debt is key to understanding much of the increase in educational attainment. The results from Table 3.5 show that the previous returns to college were enough to generate a roughly 10% increase in demand when student debt was available, and Table 3.3 shows that with both debt and increased returns, but it is clear that the improvement in the skill premium is still an important component of the decision to attend college and to become an entrepreneur. It is also the case that the increase in the skill premia leads to more entrepreneurship for the oldest cohort. In a sense, the increasing of the skill premia, while altering the trade-off for that period, also makes

Moment	Data	Model
College Frac.	0.38	0.25
30 yr. old Entrep. Frac.	0.06	0.11
40 yr. old Entrep. Frac.	0.10	0.15
50 yr. old Entrep. Frac.	0.12	0.10
60 yr. old Entrep. Frac.	0.11	0.14

Table 3.6: Numerical Exercise: No Student Debt

it easier to accumulate assets when young which makes entrepreneurship in future periods more viable.

From these decompositions, it is clear that all four trends are important to understanding the decline in entrepreneurship and its link to educational attainment. The model also implies that some of this decline in entrepreneurship is efficient. One side effect of increased college attendance is that, while there are fewer entrepreneurs, there is an increase in capital in the economy. The mechanism producing this increased efficiency is as follows: With the increase in effective labor hours that comes from attending college, there is more labor available for the most productive entrepreneurs to use, which in turn leads them to accumulate more assets to have larger firms and demand more workers. This increased demand for workers leads to a higher equilibrium wage (in the model, the general equilibrium wage increased by around 15%), pushing out formerly effective entrepreneurs and concentrating production to fewer, but larger and more productive, firms. This is reflected in the implied total factor productivity of the model. The implied total factor productivity in 2013 is 1.65 times the value from 1983. These results indicate decreased entrepreneurship in and of itself is not necessarily a worrying trend. The economy is more efficient, has a more effective labor force, and has more capital. These results on efficiency hold in both the decompositions as well as the robustness checks in the next section.

3.5.2 Robustness Checks

In this section, I hope to show that the results obtained by the model are robust to a variety of specifications of estimated parameters, as well as to show that the model's matching of the data is difficult to achieve, even under extreme assumptions.

To evaluate the robustness of the mechanism responsible for later life entrepreneurship, I first tested how sensitive the model is to changes in exogenous parameters. Simply changing parameters such as the CRRA parameter, σ, the depreciation rate, δ, or the discounting rate, β, yields minimal change in the observed rates of entrepreneurship. The results of these

Moment	Data	Model
College Frac.	0.38	0.33
30 yr. old Entrep. Frac.	0.06	0.06
40 yr. old Entrep. Frac.	0.10	0.07
50 yr. old Entrep. Frac.	0.12	0.04
60 yr. old Entrep. Frac.	0.11	0.02

Table 3.7: Numerical Exercise: No Retirement

exercises are available in the on-line appendix to this paper. The lack of significant change when adjusting these parameters indicates the key forces to overcome are the general equilibrium price changes resulting from increased efficiency and the asset decumulation motive present at the end of life in overlapping generations models. As such, it will be important to alter the decision between entrepreneurship and labor by changing other parameters like the skill premium and the return on assets.

For all of my sensitivity analyses, I leave all calibrated parameters, such as the entrepreneurial productivity processes and increased college education cost, the same as above except for the exogenous parameter I am changing. The first exercise I run is an effort to see how important having two periods of retirement is to the result. Table 3.7 shows what happens when I have households work longer but have no extended retirement, meaning they have one extra period of work and one period of retirement, rather than two periods of retirement. The results of this exercise are very similar for education and early cohort entrepreneurship, but the ability to earn in the 70s hurts asset accumulation, and thus hurts entrepreneurship in the 60s.

Table 3.8 presents the results of exogenously eliminating the skill premium for all but those in their 30s and 40s while not extending retirement. In this example, households are allowed to incur student debt, but by eliminating the skill premium, the demand for college has decreased. This result coincides with other results in the literature such as Lochner and Monge-Naranjo (2015), which show that the skill premium sufficiently compensates for the cost of college, but obviously now with exogenously enforced lower returns to college the demand has gone down. Regarding entrepreneurship, this exercise is able to match the data. Entrepreneurship rates are much higher towards the end of life, but there are also fewer educated people in the economy. In the end, this exercise does not match the data as well as my initial results, and, furthermore, the data does not support the lack of a skill premium for older workers.

In the above examples, I have tried to isolate what has the most effect on late-in-life entrepreneurship. Individually, none of them can account for the observed trend in

Moment	Data	Model
College Frac.	0.38	0.30
30 yr. old Entrep. Frac.	0.06	0.06
40 yr. old Entrep. Frac.	0.10	0.13
50 yr. old Entrep. Frac.	0.12	0.14
60 yr. old Entrep. Frac.	0.11	0.07

Table 3.8: Numerical Exercise: No Skill Premium for Older Cohorts

the data for both entrepreneurship and education choices. A common pattern across the individual examples is that as college enrollment declines, the more entrepreneurs there are. I interpret this as further evidence that the best entrepreneurs are constrained not only in terms of capital but also in terms of effective labor. As education increases, the best entrepreneurs are willing to pay more for labor, driving up the general equilibrium wage which decreases the pool of entrepreneurs and increases the pool of workers. This combined with the decumulation pressures at the end of life leads to a sharp decline at the end of life in entrepreneurship in my model. To replicate the data, drastic measures had to be taken by both eliminating the skill premium for all older workers, which is simply not the case in the data. Allowing for an additional period of work gives older households income which leads to decreased asset accumulation throughout the life cycle and further reinforces the decumulation motive at the end of life present in overlapping generations models. There is another possibility that is important to evaluate as well: the distribution of entrepreneurial ability has changed over time and possibly differs based upon age and education now. The next section explores this channel and finds that under this specification the model is also able to replicate the observed data, but implies a large benefit to entrepreneurial ability from age.

3.5.3 Adjusting the Productivity Process

The results from the above robustness checks leads to another possibility to rationalize the observed data. Something must have changed to increase the return to entrepreneurship at older ages. In terms of my model, I will allow the processes for the entrepreneurial productivity shock to have changed from 1983 to today, as well as allow the mean of the process to grow at a constant geometric rate over the course of the life-cycle in an attempt to better match the data. A possible explanation for this is that technology-intensive fields require both higher educational attainment than previous entrepreneurial ventures such as starting a local business like a hardware store, as well as actual experience working in the field prior to fully realizing the potential of the business. Recent papers, such as Azoulay

Moment	Data	Model
College Frac.	0.38	0.33
30 yr. old Entrep. Frac.	0.06	0.06
40 yr. old Entrep. Frac.	0.10	0.11
50 yr. old Entrep. Frac.	0.12	0.07
60 yr. old Entrep. Frac.	0.11	0.16

Table 3.9: Numerical Exercise: Increased Returns to Entrepreneurship

et al. (2017) and Brown et al. (2017), show that the most productive entrepreneurs tend to be both older and have advanced degrees.

Before diving into the values in the table, I will summarize how the calibration has changed. The base is still the case discussed in the compression of morbidity section above: households now work one extra period, have the ability to borrow to finance education, and the costs of an education have slightly more than doubled. The general equilibrium wage has increased relative to 1983's wage by even more than in the base case, now by almost a third as compared to the roughly fifteen percent increase observed above. Total factor productivity has increased only slightly more than in the base case: now by around 70% rather than 67% as above. The productivity processes have roughly the same initial means as before, but the standard deviations have fallen slightly. However, productivity growth across cohorts is significant, growing by around 5.5% per year over the entire working life span of the household, implying a roughly 70% increase in the mean per decade.

Table 3.9 shows the final results of this calibration. This is significantly closer than the exercises above, though still problematic. Again, roughly 33% of the population gets a post-secondary degree. Now, I match both entrepreneurs in their 30s and 40s. There is a dip in the rate of entrepreneurship for fifty-somethings, followed by a sharp increase in the rate of entrepreneurship for sixty-somethings. This is likely an interaction between the skill premium and geometric growth rate. Skill premia follow a parabolic path, peaking in the 40s and 50s, so that combined with the increased general equilibrium wage likely explains the dip in the fifties. What is most promising is the increase in sixty-something entrepreneurs. Instead of continuing working with a slightly lower wage, older households take advantage of their increased skill to either remain or enter entrepreneurship. These results indicate that including an idea of experience or learning through working or entrepreneurship through the course of one's working life might be of value in future extensions of this initial model.

3.6 Conclusion

My paper links changes in education cost to the decline in entrepreneurship in the United States over the past 30 years. Specifically, as college costs have increased, the net asset position of the youngest generation is diminished, making it harder to finance entrepreneurial ventures. I document changes that are consistent with this in the data, and propose a reasonable initial model to rationalize these trends. In the model, the increase in the skill premium over time makes college more appealing even with its higher cost, but, in my framework, it also leads to an important counterfactual. In the data, the rates of entrepreneurship for older cohorts have been increasing steadily since the middle of the 1990s and entrepreneurs are better educated on average than the rest of the population. Unfortunately, neither of these can be the case if the skill premium alone increased. Fortunately, the data on the compression of morbidity provides insight into another important trend that counters these pressures. In an initial attempt to rationalize the increased prevalence of older entrepreneurs, I incorporate into the model the compression of morbidity by extending retirement, which changes asset accumulation throughout the life cycle. In this specification, the model is able to successful explain both the increase in observed educational attainment, as well as the cohort-specific rates of entrepreneurship.

To explore possible alternative solutions to diminished entrepreneurship among older cohorts in the model, I ran sensitivity checks for both exogenous parameters such as the rate of depreciation, risk aversion, and the discount rate, to the eliminating the skill premia for 50 and 60 year olds. While other specifications are able to brute force replicate the results, the data does not support their usage. Most importantly, however, the results from these robustness checks have two key implications. First, increased efficiency is associated with increased college attendance. The best entrepreneurs are not only constrained by a lack of capital but also by a lack of effective labor. Increased college attendance loosens this constraint, and the increased demand for labor from the best entrepreneurs raises the general equilibrium wage, causing former marginal entrepreneurs to drop out and become workers. Importantly, both the general equilibrium wage and total factor productivity have increased significantly since 1983. These improvements hold under all the sensitivity analyses.

The second key implication of the sensitivity analyses is that none of the proposed alternative solutions alone can significantly alter the decisions of sixty-something entrepreneurs while still being true to the data without altering the returns to entrepreneurship. Recent work has shown that the best entrepreneurs tend to be older and well-educated, as well as finding previous experience is fundamental to entrepreneurial success. Given this, I ran a new calibration that the allowed returns to entrepreneurship to have changed both since 1983 as well as over the lifespan of a household. In this new calibration, while the initial means stayed roughly the same as the mean value of the process from the 1983 calibration, I found that a combination of decreased standard deviations of productivity shocks and an annualized growth rate of roughly 5.5% in the mean entrepreneurial ability can qualitatively match most of the observed data. This points towards a possible avenue for improving the model: including an idea of learning by doing over the course of the working life span of households.

This paper leads to multiple avenues for potential research. A key extension to my model would be to endogenize the retirement decision of the older households, rather than enforce a fixed retirement age of 70. Another important addition would be an idea of human capital that would allow for a more precise understanding of the improvements to the entrepreneurial return process.

To briefly summarize, the observed decline in entrepreneurship is accompanied by a concurrent change in the age composition of entrepreneurs. The change in education costs can explain the decline in younger entrepreneurs, but matching the increase in older entrepreneurs requires understanding the importance of the compression of morbidity observed over the same time frame. Another implication of the model is that the decline in entrepreneurship in the context of an increasingly educated populace is not necessarily a bad outcome. In fact, the decline of entrepreneurs is a sign of increased efficiency.

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Appendix A

Analytical Derivations

A.1 Chapter 2: Equivalent Economy

Our economy with mitigated shocks is equivalent to an economy that features state-contingent claims and an enforcement constraint on how much income above the lowest possible income state can be hidden. First, calculate the expected value of bond holdings:

$$\bar{b_t} = \sum_{s_{t+1}} b(s_{t+1})g(s_t, s_{t+1})$$

Then rewrite individual bond holdings as:

$$b(s_{t+1}) = \bar{b_t} + x(s_{t+1})$$

Where:

$$\sum_{s_{t+1}} x(s_{t+1})g(s_t, s_{t+1}) = 0$$

Then, the evolution of assets is able to be rewritten as:

$$a(s_{t+1}) = y_{t+1} + \sum_{l=1}^{N} \left[k_{l,t} + z_{l,t+1}k_{l,t}^{\nu}\right] + \bar{b_t} + x(s_{t+1})$$

Next, rewrite the realized shock value j in relation to the lowest shock state 1:

$$a(s_j) = a(s_1) + (1 - \phi^i)\left[w(s_j) - w(s_1) + \sum_{l=1}^{N}(z_l(s_j) - z_l(s_1))k_l^{\nu}\right]$$

Now, substitute in the law of motion to get:

$$x(s_j) - x(s_1) = -\phi^i\left[w(s_j) - w(s_1) + \sum_{l=1}^{N}(z_l(s_j) - z_l(s_1))k_l^{\nu}\right]$$

The above holds $\forall j \in \{2, ..., J\}$, where J is the number of possible shock states. Recall that $\sum_j x(s^j)g(s_t, s^J) = 0$. Then:

$$x(s_j) = -\phi^i W^j(s_t) - \phi^i \sum_{l=1}^{N} Z_l^j(s_t)k_l^{\nu}$$

Where:

$$W^j(s_t) = w^j - \sum_i g(s_t, s_i)w^i$$

And:

$$Z_l^j(s_t) = z_l^j - \sum_i g(s_t, s_i)z_l^i$$

Define:

$$\tilde{w}^j(s_t) = w^j - \phi W^j(s_t)$$

$$\tilde{z}_l^j(s_t) = z_l^j - \phi Z_l^j(s_t)$$

Then, the formulation used in the model is obtained.

$$a(s^j) = \tilde{w}^j(s_t) + \sum_{l=1}^{N}\left[k_{l,t} + \tilde{z}_l^j(s_t)k_{l,t}^{\nu}\right] + \bar{b_t}$$

A.2 Chapter 2: Characterization Algebra

To characterize the solution to this problem, first take the first order conditions with respect to each choice variable.

$$c: \qquad \lambda = u'(c)$$

$$k_1': \qquad \beta E\left[\frac{\partial V'}{\partial k_1'}\right] - \lambda + E\left[\mu(s')\frac{\partial a(s')}{\partial k_1'}\right] = 0$$

$$k_2': \qquad \beta E\left[\frac{\partial V'}{\partial k_2'}\right] - \lambda + E\left[\mu(s')\frac{\partial a(s')}{\partial k_2'}\right] = 0$$

$$b': \qquad \beta E\left[\frac{\partial V'}{\partial b'}\right] - \lambda + E\left[\mu(s')\frac{\partial a(s')}{\partial b'}\right] = 0$$

After calculating the first order conditions, we need the envelope conditions for each choice variable. Note that, for all choice variables $\{k_1', k_2', b'\} \in x$, we must use the chain rule: $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial a}\frac{\partial a}{\partial x}$. Taking these envelope conditions yields:

$$\frac{\partial V}{\partial a} = \lambda$$

$$\frac{\partial a(s)}{\partial k_1'} = (\alpha \tilde{z}_1 k_1^{\alpha-1} + 1)$$

$$\frac{\partial a(s)}{\partial k_2'} = (\alpha \tilde{z}_1 k_1^{\alpha-1} + 1)$$

$$\frac{\partial a(s)}{\partial b'} = (1 + R(A_1^-, A_2^-))$$

$$\frac{\partial V}{\partial k_1} = \lambda(\alpha \tilde{z}_1 k_1^{\alpha-1} + 1)$$

$$\frac{\partial V}{\partial k_2} = \lambda(\alpha \tilde{z}_1 k_1^{\alpha-1} + 1)$$

$$\frac{\partial V}{\partial b} = \lambda(1 + R(A_1^-, A_2^-)$$

Now that we have the components, it simply remains to plug the envelope conditions into the first order conditions to generate the Euler equations:

$$\beta E\left[u'(c')(\alpha\tilde{z}'_1 k_1'^{\alpha-1}+1)\right]+E\left[\mu(s')\alpha(\tilde{z}'_1 k_1'^{\alpha-1}+1)\right]=u'(c)$$

The Euler equation for capital in country 2 is:

$$\beta E\left[u'(c')(\alpha\tilde{z}'_2 k_2'^{\alpha-1}+1)\right]+E\left[\mu(s')\alpha(\tilde{z}'_2 k_2'^{\alpha-1}+1)\right]=u'(c)$$

The Euler equation for bonds is:

$$\beta E\left[u'(c')(1+R(A_1,A_2))\right]+E\left[\mu(s')(1+R(A_1,A_2)\right]=u'(c)$$