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A Mixed Finite Volume Method for Elliptic Problems

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Abstract

We derive a novel finite volume method for the elliptic equation, using the framework of mixed finite element methods to discretize the pressure and velocities on two different grids (covolumes), triangular (tetrahedral) mesh and control volume mesh. The new discretization is defined for tensor diffusion coefficient and well suited for heterogeneous and anisotropic media. When the control volumes are created by connecting the mass center of each triangle to the midpoints of its edges, we show that the discretization is stable and first order accurate for both scalar and vector unknowns.

Key words; finite volume methods, cell-centered finite differences, error estimates

AMS subject classification: 65N06, 65N12, 65N15.

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1 Introduction

Physical diffusion processes are usually modeled by elliptic equations. The heterogeneous diffusion tensor due to the heterogeneous anisotropic media imposes great challenges to the numerical schemes. There are several highly desirable properties of the discretization beyond the classical stability and accuracy, such as local mass conservation, discrete maximum principle, lack of grid orientation effects, etc, that are crucial in capturing the important details of the solutions of complicated (especially nonlinear) problems on relatively coarse grids.

In this paper we derive a method that is *locally mass conservative* and it can be applied to a general finite volume grid constructed from a given triangular or tetrahedral mesh. Typical example is the coupled Voronoi boxes (PEBI mesh) and Delaunay triangles (tetrahedra). When the control volume is formed by mass centers and edge midpoints of the triangles, we show the first order accuracy for the pressure gradient and the velocity. Our discretization is similar to the Finite Volume Element (FVE) method, but it produces direct approximation of the velocity which is not the case for FVE method. This can be very beneficial in some applications. In the proposed method, the velocity unknowns are eliminated and the resulting linear system for the pressure can be solved with the available solvers. Therefore, the estimated computational cost on triangular or tetrahedral mesh will be two to three times less than the cost for solving the same problem with the standard mixed finite element methods.

There have been extensive research on developing numerical schemes for the partial differential equations of interest on a given general grid. The classical finite difference schemes are only applicable to structured grids. The mimetic finite difference schemes proposed by Shashkov [31] are very promising in terms of dealing with highly distorted grids. The convergence and super convergence [6, 7] are also established for smooth problems on smooth meshes by rewriting it into the form of mixed finite element methods. But the computational cost is an issue.

Finite volume methods overcome most of the restrictions of finite difference schemes, and they are usually locally mass conservative. There have been a significant advance in the theory of the finite volume methods applied to diffusion equations with scalar coefficient on unstructured meshes [2, 17, 21, 23, 29]. Several methods for handling tensor coefficient have been proposed [35, 14, 1]. Recently the convergence is established for one of such methods, the multi-point flux approximation, on quadrilateral grids under the assumption of smooth mesh [19]. But a comprehensive theory for general mesh is still not available yet.

Control volume finite element methods, often called finite volume element methods are another worthy alternative [5, 15, 11, 30, 16, 24]. They are applicable on flexible grids and are locally mass conservative, but do not preserve the symmetry of the differential operator, and do not produce direct approximation of the velocity field.

Mixed finite elements, since their introduction [27], have proven to be robust and give very accurate approximations. In some cases it is possible to reduce the system of coupled equations for the pressure and velocity to a system only for the pressure [4, 3], which is much easier to solve. Unfortunately some restrictions apply. There is considerable advance in designing methods for distorted general meshes [22, 10, 33, 34, 12], but they are quite expensive and 3-D case is still under development [25, 20].

The proposed method in considerable extent alleviates the drawbacks of the known methods without significant increase of the computational cost.

The rest of the paper is organized as follows. In Sec. 2 we consider a simple model problem. The discretization is derived in Sec. 3. All necessary formulas for the implementation of the method are provided. Review of the theory for mixed finite element methods is outlined in Sec. 4 and theoretical estimates are proven in Sec. 5. Some numerical results are presented in Sec. 6. Finally we summarize the paper in the conclusions.

2 Model Problem

Consider a model boundary value problem:

$$-div(\mathbf{K}\nabla p(x)) = f(x) \;\text{ in } \Omega, \tag{1a}$$

$$p(x) = 0 \qquad \text{on } \partial\Omega, \tag{1b}$$

where Ω is a domain in $\mathbb{R}^d$, $d = 2$ or 3, and the boundary $\partial\Omega$ is polygonal and convex. The diffusion tensor $\mathbf{K}$ is assumed to be a uniformly symmetric positive definite matrix.

Defining the velocity $\mathbf{u}$ as

$$\mathbf{u} = -\mathbf{K}\nabla p,$$

we can rewrite (1a) as a first order system for the unknowns $\mathbf{u}$ and p

$$\mathbf{u} + \mathbf{K}\nabla p = 0, \tag{2a}$$

$$div(\mathbf{u}) = f. \tag{2b}$$

Note that in some application $\mathbf{K}$ can be discontinuous (heat transfer, reservoir simulation), although the normal component of $\mathbf{u}$ is continuous.

Problems (1) and (2) are equivalent. The weak formulation of problem (1) is:
Find $(\mathbf{u}, p) \in \mathbf{U} \times P$ such that

$$\int_\Omega \mathbf{u}\cdot\mathbf{v} + \int_\Omega \mathbf{K}\nabla p\cdot\mathbf{v} = \quad 0 \quad \forall\,\mathbf{v} \in \mathbf{V}, \tag{3a}$$

$$\int_\Omega div(\mathbf{u})q = \int_\Omega fq \quad \forall\, q \in Q., \tag{3b}$$

Note that the test and trial spaces in (3) can be different. Similar methods have been considered in [33, 34] with particular choice

$$\mathbf{U} = H(div,\Omega), \quad P = H_0^1(\Omega), \quad \mathbf{V} = \left(L^2(\Omega)\right)^d, \quad Q = L^2(\Omega),$$

where

$$H_0^1(\Omega) = \{p \in L^2(\Omega),\, Dp \in L^2(\Omega),\, p = 0 \text{ on } \partial\Omega\},$$

$$H(div,\Omega) = \{\mathbf{v} \in (L^2(\Omega))^d,\, div(\mathbf{v}) \in L^2(\Omega)\}.$$

It is generally difficult to find a stable pair discrete spaces $(\mathbf{U}_h, P_h)$, $\mathbf{U}_h \subset H(div, \Omega)$, $P_h \subset H_0^1(\Omega)$ for unstructured meshes.

We would like to obtain a discrete method that replaces equation (3b) with an equation similar to the one derived from finite volume methods. Suppose the domain Ω is divided into set of control volumes V_i, $i = 1, \ldots, n$, and assume q is smooth enough. With integration by parts, we can rewrite (3b) as

$$\sum_{i=1}^{n} \left[\int_{\partial V_i} \mathbf{u} \cdot \mathbf{n} q - \int_{V_i} \nabla q \cdot \mathbf{u} \right] = \int_{\Omega} f q. \tag{4}$$

Choosing Q_h to be the space of piecewise constants on the control volumes, Eq. (4) becomes

$$\sum_{i=1}^{n} \int_{\partial V_i} \mathbf{u}_h \cdot \mathbf{n}\, q = \int_{\Omega} f\, q, \tag{5}$$

which is a typical equation in finite volume methods.

The equation (5) can be considered as an approximation of Eq. (3b) with non-conforming spaces $\mathbf{U}_h$ and Q_h.

3 Discretization

We introduce two different meshes that are "dual" to each other. The meaning of "dual" will become clear from the following example on which we build the discretization. In particular, the triangular (tetrahedral) mesh, e.g., the Delaunay triangulation $\mathcal{D}_h$, is used for the discretization of pressure p (the triangles on Fig. 1). The velocity $\mathbf{u}$ is approximated on a control volume mesh built from the triangles, for example the Voronoi (PEBI) grid $\mathcal{V}_h$. One Voronoi control volume is depicted on Fig. 1 with dotted line.

We take the finite element methodology to construct the discretization. The first step is to replace the infinitely dimensional spaces $\mathbf{U}$, P,$\mathbf{V}$, and Q in problem (3) with finite subspaces $\mathbf{U}_h$, P_h , $\mathbf{V}_h$, and Q_h. The subindex h is used to emphasize the sequences of spaces for $h \to 0$, i.e., when the meshes become more refined, the dimension of the spaces increases.

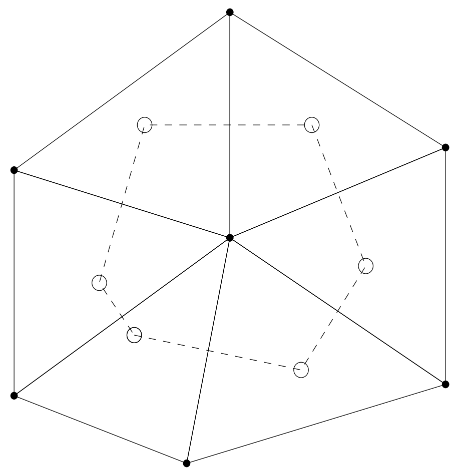

Figure 1: Voronoi box/Delaunay triangles

Denote with $\mathcal{E}_h$ the edges of control volumes in $\mathcal{V}_h$. We approximate the velocity $\mathbf{u}$ with a piecewise constant vector function that have continuous normal components on $\mathcal{E}_h$. More details will be given later on the construction of $\mathbf{U}_h$.

The discrete space P_h of approximated pressure is defined on the triangulation $\mathcal{D}_h$. Every element $p \in P_h$ is a continuous piecewise linear function. We introduce a discrete subspace Q_h of $L^2(\Omega)$ that consists of piecewise constants on control volumes $V \in \mathcal{V}_h$.

Then the discrete problem is defined as follows:

Find $(\mathbf{u}_h, p_h) \in \mathbf{U}_h \times P_h$ such that

$$\int_\Omega \mathbf{u}_h \cdot \mathbf{v}_h + \int_\Omega \mathbf{K}\nabla p_h \cdot \mathbf{v} = \quad 0 \quad \forall\, \mathbf{v}_h \in \mathbf{U}_h, \tag{6a}$$

$$\sum_{V \in \mathcal{V}_h} \int_{\partial V} \mathbf{u}_h \cdot \mathbf{n}\, q_h = \int_\Omega f q_h \quad \forall\, q_h \in Q_h. \tag{6b}$$

Note that $\mathbf{U}_h = \mathbf{V}_h$.

The second step in the finite element discretization is to construct basis for $\mathbf{U}_h$.

3.1 Basis of $\mathbf{U}_h$

We would like to determine the velocity in such a way that the normal components on the edges of the control volumes in any given triangle are continuous. Consider a triangle $D = D_i \cup D_j \cup D_k$ as shown in Fig. 2. In each of the quadrilaterals D_i, D_j and D_k the velocity

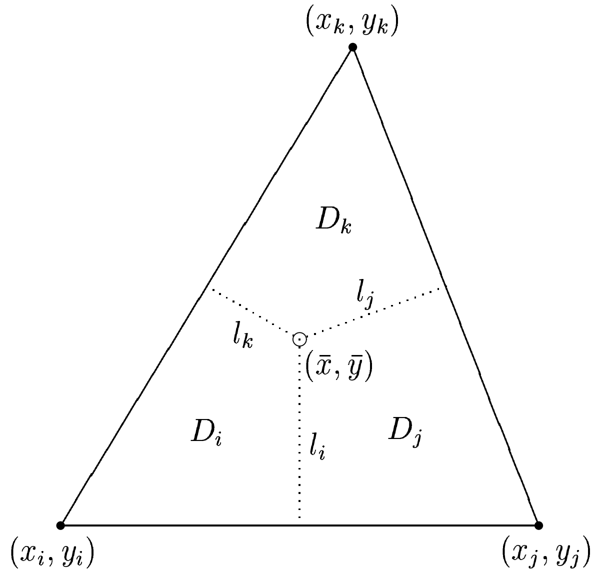

Figure 2: Triangle D

$\mathbf{v}$ is approximated with a constant function (six degrees of freedom). We additionally impose that the normal components on l_i, l_j, l_k are continuous (subtract three degrees of freedom). Therefore, there are three degrees of freedom for the velocity space $\mathbf{U}_h$ in D. We choose them to be the integrals of the normal components on l_i, l_j, l_k, i.e., any vector function $\mathbf{v}_{|D} \in \mathbf{U}_h$ can be uniquely determined by the numbers

$$v_i = \int_{l_i} \mathbf{v} \cdot \mathbf{n}_i \,, \quad v_j = \int_{l_j} \mathbf{v} \cdot \mathbf{n}_j \,, \quad v_k = \int_{l_k} \mathbf{v} \cdot \mathbf{n}_k .$$

Note that the values of the velocity in one triangle are not directly connected with the values in the neighboring triangles.

Consider vector functions $\mathbf{e}_i$, $\mathbf{e}_j$, $\mathbf{e}_k$ defined with the relations

$$\int_{l_i} \mathbf{e}_i \cdot \mathbf{n}_i = 1, \quad \int_{l_j} \mathbf{e}_i \cdot \mathbf{n}_j = 0, \quad \int_{l_k} \mathbf{e}_i \cdot \mathbf{n}_k = 0, \tag{7a}$$

$$\int_{l_i} \mathbf{e}_j \cdot \mathbf{n}_i = 0, \quad \int_{l_j} \mathbf{e}_j \cdot \mathbf{n}_j = 1, \quad \int_{l_k} \mathbf{e}_j \cdot \mathbf{n}_k = 0, \tag{7b}$$

$$\int_{l_i} \mathbf{e}_k \cdot \mathbf{n}_i = 0, \quad \int_{l_j} \mathbf{e}_k \cdot \mathbf{n}_j = 0, \quad \int_{l_k} \mathbf{e}_k \cdot \mathbf{n}_k = 1. \tag{7c}$$

It is easy to see that $\mathbf{e}_i$, $\mathbf{e}_j$, $\mathbf{e}_k$ are linearly independent and therefore form a basis. So $\mathbf{v}$ restricted to the triangle D is equal to

$$\mathbf{v}_{|D} = v_i \mathbf{e}_i + v_j \mathbf{e}_j + v_k \mathbf{e}_k .$$

Each of the vectors $\mathbf{e}_i, \mathbf{e}_j$ and $\mathbf{e}_k$, is defined on the whole triangle D, being piecewise constant on each of the quadrilaterals D_i, D_j and D_k. One can compute the basis vectors once the normal vectors $\mathbf{n}_i$, $\mathbf{n}_j$, $\mathbf{n}_k$, and the lengths of the intervals l_i, l_j, l_k are given. For example, $\mathbf{e}_i$ can be represented as

$$\mathbf{e}_i = \left((e^{(i)}_{1|D_i}, e^{(i)}_{2|D_i}),\ (e^{(i)}_{1|D_j}, e^{(i)}_{2|D_j}),\ (e^{(i)}_{1|D_k}, e^{(i)}_{2|D_k}) \right).$$

According to the conditions (7a), in the quadrilateral D_i we have the identities $\int_{l_i} \mathbf{e}_i \cdot \mathbf{n}_i = 1$ and $\int_{l_k} \mathbf{e}_i \cdot \mathbf{n}_k = 0$ that lead to the linear system for the unknown components

$$\begin{aligned} e^{(i)}_{1|D_i} n^{(i)}_1 + e^{(i)}_{2|D_i} n^{(i)}_2 &= 1/|l_i|, \\ e^{(i)}_{1|D_i} n^{(k)}_1 + e^{(i)}_{2|D_i} n^{(k)}_2 &= 0. \end{aligned}$$

Similarly we compute $e^{(i)}_{1|D_j}$ and $e^{(i)}_{1|D_k}$. Permuting appropriately the indexes i, j and k in the expressions above we can get the formulas for $\mathbf{e}_j$ and $\mathbf{e}_k$.

3.2 Assembling

The last step is to derive the actual finite volume scheme, in finite element terminology, to assemble the matrix. Consider Eq. (6a) with $\mathbf{v}_h = \mathbf{e}_i$, $\mathbf{e}_j$ and $\mathbf{e}_k$, i.e.,

$$\begin{aligned} u_1 \int_D \mathbf{e}_i \cdot \mathbf{e}_i + u_2 \int_D \mathbf{e}_j \cdot \mathbf{e}_i + u_3 \int_D \mathbf{e}_k \cdot \mathbf{e}_i + \int_D \mathbf{K}\nabla p \cdot \mathbf{e}_i = 0, \\ u_1 \int_D \mathbf{e}_i \cdot \mathbf{e}_j + u_2 \int_D \mathbf{e}_j \cdot \mathbf{e}_j + u_3 \int_D \mathbf{e}_k \cdot \mathbf{e}_j + \int_D \mathbf{K}\nabla p \cdot \mathbf{e}_j = 0, \\ u_1 \int_D \mathbf{e}_i \cdot \mathbf{e}_k + u_2 \int_D \mathbf{e}_j \cdot \mathbf{e}_k + u_3 \int_D \mathbf{e}_k \cdot \mathbf{e}_k + \int_D \mathbf{K}\nabla p \cdot \mathbf{e}_k = 0. \end{aligned}$$

Here we have taken into account that the restriction of $\mathbf{u}_h$ to the triangle D is $\mathbf{u}_{h|D} = u_i \mathbf{e}_i + u_j \mathbf{e}_j + u_k \mathbf{e}_k$. Note that the matrix $\{\mathbf{e}_m \cdot \mathbf{e}_n\}_{m,n=i,j,k}$ is symmetric and positive definite. Therefore, we can solve for the fluxes u_i, u_j and u_k and substitute them into Eq. (6b). Let

$$t_{mn} = \int_D \mathbf{e}_m \cdot \mathbf{e}_n,$$

and denote the matrix

$$T^{-1} = \begin{bmatrix} t_{ii} & t_{ji} & t_{ki} \\ t_{ij} & t_{jj} & t_{kj} \\ t_{ik} & t_{jk} & t_{kk} \end{bmatrix}.$$

On the triangle D, we have

$$p = p_i q_i + p_j q_j + p_k q_k,$$

where q_i, q_j and q_k are the basis “hat” functions. Denote

$$m_{mn} = \int_D \mathbf{K} \nabla q_m \cdot \mathbf{e}_n$$

and

$$M = \begin{bmatrix} m_{ii} & m_{ji} & m_{ki} \\ m_{ij} & m_{jj} & m_{kj} \\ m_{ik} & m_{jk} & m_{kk} \end{bmatrix}.$$

We call M the mass matrix. Then

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = - \begin{bmatrix} t_{ii} & t_{ji} & t_{ki} \\ t_{ij} & t_{jj} & t_{kj} \\ t_{ik} & t_{jk} & t_{kk} \end{bmatrix}^{-1} \begin{bmatrix} m_{ii} & m_{ji} & m_{ki} \\ m_{ij} & m_{jj} & m_{kj} \\ m_{ik} & m_{jk} & m_{kk} \end{bmatrix} \begin{bmatrix} p_i \\ p_j \\ p_k \end{bmatrix} = -TM\mathbf{p}.$$

Having expressed $\mathbf{u}_h$ through p_h, we obtain the final equation for each p_i by substituting $\mathbf{u}_h$ into (6b) and testing with the basis of Q_h. Let D_1, $D_2, \ldots,$ D_m are the triangles in the support of q_i. Then the equation for p_i is

$$\sum_{V \in \mathcal{V}_h} \int_{\partial V} -T_i M_i \mathbf{p}_i \cdot \mathbf{n} \, q_h = \int_\Omega f q_h.$$

Here the subscript means T_i, M_i and $\mathbf{p}_i$ are defined on triangle D_i.

Remark 1 *The proposed method belongs to the class of "$\mathbf{K}$ methods" [18]. One can obtain a "$\mathbf{K}^{-1}$ method" by first rewriting Eq.* (2a) *as*

$$\mathbf{K}^{-1}\mathbf{u} + \nabla p = 0,$$

and then following the above procedure.

4 Theoretical Framework

In this section we state a theorem which is an easy generalization of the results by Nicolaides [26], Brezzi and Fortin [8] and Thomas and Trujillo [33, 34].

Take an abstract problem

$$\text{Find } (\mathbf{u},p) \in \quad \mathbf{U}\times P$$
$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = < g,\mathbf{v} > \quad \forall\, \mathbf{v}\in\mathbf{V}, \tag{8a}$$
$$c(\mathbf{u},q) = < f,q > \quad \forall\, q \in Q, \tag{8b}$$

where $(\mathbf{U},\|.\|_{\mathbf{U}})$, $(P,\|.\|_P)$, $(\mathbf{V},\|.\|_{\mathbf{V}})$ and $(Q,\|.\|_Q)$ are four Hilbert spaces, $a(.,.)$, $b(.,.)$ and $c(.,.)$ are bilinear forms defined respectively on $\mathbf{U}\times\mathbf{V}$, $P\times\mathbf{V}$ and $\mathbf{U}\times Q$. The right hand sides are defined for $g\in\mathbf{V}'$, $f\in Q'$, where $\mathbf{V}'$ and Q' are the dual spaces of $\mathbf{V}$ and Q correspondingly.

Consider the discrete problem:

$$\text{Find } (\mathbf{u}_h,p_h) \in \quad \mathbf{U}_h\times P_h$$
$$a_h(\mathbf{u}_h,\mathbf{v_h}) + b_h(\mathbf{v_h},p_h) = < g,\mathbf{v}_h >_h \quad \forall\, \mathbf{v}_h\in\mathbf{V}_h, \tag{9a}$$
$$c_h(\mathbf{u}_h,q_h) = < f,q_h >_h \quad \forall\, q_h \in Q_h, \tag{9b}$$

where $(\mathbf{U}_h,\|.\|_{\mathbf{U}_h})$, $(P_h,\|.\|_{P_h})$, $(\mathbf{V}_h,\|.\|_{\mathbf{V}_h})$ and $(Q_h,\|.\|_{Q_h})$ are four Hilbert spaces, $a_h(.,.)$, $b_h(.,.)$ and $c_h(.,.)$ are bilinear forms defined respectively on $\mathbf{U}_h\times\mathbf{V}_h$, $P_h\times\mathbf{V}_h$ and $\mathbf{U}_h\times Q_h$. Suppose that $\mathbf{U}_h\not\subset\mathbf{U}$ and $\mathbf{V}_h\not\subset\mathbf{V}$, i.e., discretization (9) is non-conforming.

Let $\mathbf{U}_{0h}$ and $\mathbf{V}_{1h}$ be the spaces defined by

$$\mathbf{U}_{0h} = \{\mathbf{u}_h\in\mathbf{U}_h, \quad \forall\, q_h\in Q_h, \quad c_h(\mathbf{u}_h,q_h)=0\},$$
$$\mathbf{V}_{1h} = \{\mathbf{v}_h\in\mathbf{V}_h, \quad \forall\, \mathbf{u}_h\in\mathbf{U}_{0h}, \quad a_h(\mathbf{u}_h,\mathbf{v}_h)=0\}.$$

Assume that there exists three constants A, B and C independent of h such that

$$a_h(\mathbf{u}_h,\mathbf{v}_h) \le A\|\mathbf{u}_h\|_{\mathbf{U}_h}\|\mathbf{v}_h\|_{\mathbf{V}_h}, \tag{10a}$$
$$b_h(\mathbf{v}_h,p_h) \le B\|\mathbf{v}_h\|_{\mathbf{V}_h}\|p_h\|_{P_h}. \tag{10b}$$
$$c_h(\mathbf{u}_h,q_h) \le C\|\mathbf{u}_h\|_{\mathbf{U_h}}\|q_h\|_{Q_h}. \tag{10c}$$

Then we have the following result:

Theorem 1 *Assume that the next three Babuška-Brezzi conditions are satisfied:*

$$i) \quad \inf_{\mathbf{u}_h \in \mathbf{U}_{0h}} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{a_h(\mathbf{u}_h, \mathbf{v}_h)}{\|\mathbf{u}_h\|_{\mathbf{U}_h} \|\mathbf{v}_h\|_{\mathbf{V}_h}} \geq \alpha, \tag{11a}$$

$$ii) \quad \inf_{p_h \in P_h} \sup_{\mathbf{v}_h \in \mathbf{V}_{1h}} \frac{b_h(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{\mathbf{V}_h} \|p_h\|_{P_h}} \geq \beta, \tag{11b}$$

$$iii) \quad \inf_{q_h \in Q_h} \sup_{\mathbf{u}_h \in \mathbf{U}_h} \frac{c_h(\mathbf{u}_h, q_h)}{\|\mathbf{u}_h\|_{\mathbf{U}_h} \|q_h\|_{Q_h}} \geq \gamma. \tag{11c}$$

and that

$$\dim(\mathbf{U}_h) + \dim(P_h) = \dim(\mathbf{V}_h) + \dim(Q_h).$$

Then problem (9) *has a unique solution* $(\mathbf{u}_h, p_h)$. *Moreover, if* α, β *and* γ *are independent of* h, *there exists positive constant* C *independent of* h *such that*

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{U}_h} + \|p - p_h\|_{P_h} \\ &\quad \leq C \left(\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{V}_h} + \inf_{p_h \in P_h} \|p - p_h\|_{P_h} + M_{1h} + M_{2h} + M_{3h} + M_{4h} \right), \end{aligned} \tag{12}$$

where

$$\begin{aligned} M_{1h} &= \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p) - < q, \mathbf{v}_h >}{\|\mathbf{v}_h\|_{\mathbf{V}_h}}, \\ M_{2h} &= \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{< g, \mathbf{v}_h > - < g, \mathbf{v}_h >_h}{\|\mathbf{v}_h\|_{\mathbf{V}_h}}, \\ M_{3h} &= \sup_{q_h \in q_h} \frac{c_h(\mathbf{u}, q_h) - < f, q_h >}{\|q_h\|_{Q_h}}, \\ M_{4h} &= \sup_{q_h \in Q_h} \frac{< f, q_h > - < f, q_h >_h}{\|q_h\|_{Q_h}}. \end{aligned}$$

5 Stability and Error Estimates

We start with several auxiliary results. The primary triangular mesh is assumed to be quasi-regular as in the analysis of conforming finite element methods. The dual mesh, i.e., the control volumes, are formed by connecting the mass centers of the triangles to their edge midpoints. Some proofs are omitted for simplicity.

Lemma 1

$$\|\mathbf{u}_h\|^2_{0.\Omega} \sim \sum_{E_{ij} \in \mathcal{E}_h} \left[\int_{E_{ij}} \mathbf{u}_h \cdot \mathbf{n} \right]^2, \quad \forall\, \mathbf{u}_h \in \mathbf{U}_h$$

(The $\sim$ is use to indicate the equivalence of the norms.)

Proof: The proof follows from the definition of the space $\mathbf{U}_h$. $\square$

The error estimate for the constant interpolant in $\mathbf{V}_h$ is provided below.

Lemma 2

$$\|\mathbf{v} - I_h^c \mathbf{v}\|_{0,\Omega} \leq Ch\|\mathbf{v}\|_{1,\Omega}.$$

Proof: We follow the methodology developed by Ciarlet [13, Theorems 3.1.3–3.1.6, pp. 120–124]. Consider the reference element $\hat{D}$ and the constant interpolant $\hat{I}^c$ defined by

$$\hat{I}^c \hat{\mathbf{v}} = \left[\int_{l_i} \hat{\mathbf{v}} \cdot \mathbf{n}_i \right] \mathbf{e}_i + \left[\int_{l_j} \hat{\mathbf{v}} \cdot \mathbf{n}_j \right] \mathbf{e}_j + \left[\int_{l_k} \hat{\mathbf{v}} \cdot \mathbf{n}_k \right] \mathbf{e}_k.$$

We have

$$\begin{aligned} \|\hat{I}^c \hat{\mathbf{v}}\|_{0,\hat{D}} &\leq C(\hat{D}) \left(\left| \int_{l_i} \hat{\mathbf{v}} \cdot \mathbf{n}_i \right| |\mathbf{e}_i| + \left| \int_{l_j} \hat{\mathbf{v}} \cdot \mathbf{n}_j \right| |\mathbf{e}_j| + \left| \int_{l_k} \hat{\mathbf{v}} \cdot \mathbf{n}_k \right| |\mathbf{e}_k| \right) \\ &\leq C_1(\hat{D})(\|\hat{\mathbf{v}}\|_{0,l_i} + \|\hat{\mathbf{v}}\|_{0,l_j} + \|\hat{\mathbf{v}}\|_{0,l_k}) \\ &\leq C_2 \|\hat{\mathbf{v}}\|_{1,\hat{D}}. \end{aligned}$$

$\square$

In order to use **Theorem** 1 one has to specify the spaces and norms mentioned in it. We define the norms as

$$\|p_h\|_{P_h} = \|p_h\|_{1,\Omega}, \quad \|q_h\|_{Q_h} = \left(\sum_{x_i \in \omega} \sum_{j \in \Pi(i)} (q_i - q_j)^2 \right)^{1/2},$$

$$\|\mathbf{u}_h\|_{\mathbf{U}_h} = \|\mathbf{u}_h\|_{\mathbf{V}_h} = \|\mathbf{u}_h\|_{0,\Omega}.$$

Recall a well known result for the equivalence of the norm on P_h with H_0^1-seminorm [9]:

Lemma 3

$$\|q_h\| \sim |q_h|_{1,\Omega} \quad \forall\, q_h \in Q_h.$$

The following theorem is the error estimate for the method proposed in this paper.

Theorem 2 *Let* $(\mathbf{u}, p)$ *be the solution of problem* (3) *and* $(\mathbf{u}_h, p_h)$ *be the solution of* (6). *Then there exists a positive constant* C *such that*

$$\|p - p_h\|_{1,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \le Ch(|p|_{2,\Omega} + \|\mathbf{u}\|_{1,\Omega}).$$

Proof: It is easy to verify (10a) and (11a) for the bilinear form $a_h(.,.)$.

We check the conditions (10b) and (11b) for the bilinear form $b_h(.,.)$ as follows. Since $\mathbf{K}$ is a uniformly positive definite matrix, there exists positive constants k_1 and k_2 such that

$$k_1 \xi^t \xi \le \xi^t \mathbf{K} \xi \le k_2 \xi^t \xi, \qquad \forall\, \xi \in \mathbf{R}^d.$$

So,

$$b_h(\mathbf{v}_h, p_h) = \int_\Omega \mathbf{v}_h \cdot \mathbf{K}\nabla p_h \le \left(\int_\Omega \mathbf{K}\mathbf{v}_h \cdot \mathbf{v}_h\right)^{1/2} \left(\int_\Omega \mathbf{K}\nabla p_h \cdot \nabla p_h\right)^{1/2} \le k_2 \|\mathbf{v}_h\|_{0,\Omega} \|p_h\|_{1,\Omega}.$$

Choosing $\mathbf{v}_h(p_h) = \nabla p_h$. Then

$$\begin{aligned} b_h(\mathbf{v}_h(p_h), p_h) = \int_\Omega \mathbf{v}_h(p_h) \cdot \mathbf{K}\nabla p_h &= \left(\int_\Omega \mathbf{K}\mathbf{v}_h(p_h) \cdot \mathbf{v}_h(p_h)\right)^{1/2} \left(\int_\Omega \mathbf{K}\nabla p_h \cdot \nabla p_h\right)^{1/2} \\ &\ge k_1 \|\mathbf{v}_h(p_h)\|_{0,\Omega}\, |p_h|_{1,\Omega} \ge C \|\mathbf{v}_h(p_h)\|_{0,\Omega}\, \|p_h\|_{1,\Omega}. \end{aligned}$$

Hence, in order to verify that (11b) is true, it suffices to show that $\nabla p_h \in \mathbf{V}_{1h}, \forall p_h \in P_h$, i.e., we only need to show

$$a_h(\mathbf{u}_h, \nabla p_h) = \sum_{D \in \mathcal{D}_h} \int_D \mathbf{u}_h \cdot \nabla p_h = 0, \qquad \forall \mathbf{u}_h \in \mathbf{U}_{0h}, p_h \in P_h. \tag{13}$$

Consider a triangle D as shown in Fig. 2. Denote the normal vectors for l_i, l_j, l_k as $\tilde{\mathbf{n}}_i, \tilde{\mathbf{n}}_j, \tilde{\mathbf{n}}_k$ with the direction $D_i \to D_j \to D_k \to D_i$. Note that $\tilde{\mathbf{n}}_i, \tilde{\mathbf{n}}_j, \tilde{\mathbf{n}}_k$ have length $|l_i|, |l_j|, |l_k|$ respectively, different from those normal vectors defined in Sec. 3.1. Let s be the area of triangle D. Take $\mathbf{N}_i, \mathbf{N}_j$ and $\mathbf{N}_k$ to be the outwards normal vectors of triangle D. The usual convention applies, i.e., $\mathbf{N}_i$ is the normal vector of the edge $(x_j, y_j) \to (x_k, y_k)$, and the norm of $\mathbf{N}_i$ is equal to the length of the corresponding edge. Since the edge midpoints and mass center are used to form the control volumes, one can show

$$|D_i| = |D_j| = |D_k| = \frac{s}{3}, \qquad \tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_j + \tilde{\mathbf{n}}_k = 0,$$
$$\mathbf{N}_i = 2(\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k), \qquad \mathbf{N}_j = 2(\tilde{\mathbf{n}}_j - \tilde{\mathbf{n}}_i), \qquad \mathbf{N}_k = 2(\tilde{\mathbf{n}}_k - \tilde{\mathbf{n}}_j).$$

Since p_h is a linear function on D, we have

$$\nabla p_h = -\frac{1}{2s}\left(p_i\mathbf{N}_i + p_j\mathbf{N}_j + p_k\mathbf{N}_k\right).$$

Therefore,

$$\begin{aligned}\int_D \mathbf{u}_h \cdot \nabla p_h &= \int_{D_i+D_j+D_k} \mathbf{u}_h \cdot \nabla p_h \\ &= -\frac{1}{3}\left(\mathbf{u}_h|_{D_i} + \mathbf{u}_h|_{D_j} + \mathbf{u}_h|_{D_k}\right) \cdot \left(p_i(\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k) + p_j(\tilde{\mathbf{n}}_j - \tilde{\mathbf{n}}_i) + p_k(\tilde{\mathbf{n}}_k - \tilde{\mathbf{n}}_j)\right),\end{aligned}$$

where the coefficient of $(-\frac{1}{3}p_i)$ is equal to

$$\begin{aligned}&\left(\mathbf{u}_h|_{D_i} + \mathbf{u}_h|_{D_j} + \mathbf{u}_h|_{D_k}\right) \cdot (\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k) \\ &\quad= \mathbf{u}_h|_{D_i} \cdot (\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k) + \mathbf{u}_h|_{D_j} \cdot (\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k) + \mathbf{u}_h|_{D_k} \cdot (\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k) \\ &\quad= \mathbf{u}_h|_{D_i} \cdot (\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k) + \mathbf{u}_h|_{D_j} \cdot (2\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_j) + \mathbf{u}_h|_{D_k} \cdot (\tilde{\mathbf{n}}_j - 2\tilde{\mathbf{n}}_k) \\ &\quad= \mathbf{u}_h|_{D_i} \cdot (\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k) + 2\mathbf{u}_h|_{D_j} \cdot \tilde{\mathbf{n}}_i - \mathbf{u}_h|_{D_j} \cdot \tilde{\mathbf{n}}_j + \mathbf{u}_h|_{D_k} \cdot \tilde{\mathbf{n}}_j - 2\mathbf{u}_h|_{D_k} \cdot \tilde{\mathbf{n}}_k \\ &\quad= \mathbf{u}_h|_{D_i} \cdot (\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k) + 2\mathbf{u}_h|_{D_i} \cdot \tilde{\mathbf{n}}_i - \mathbf{u}_h|_{D_j} \cdot \tilde{\mathbf{n}}_j + \mathbf{u}_h|_{D_j} \cdot \tilde{\mathbf{n}}_j - 2\mathbf{u}_h|_{D_i} \cdot \tilde{\mathbf{n}}_k \\ &\quad= 3\mathbf{u}_h|_{D_i} \cdot (\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k).\end{aligned}$$

In the above equalities, we use the fact that the normal component of u_h is continuous across the edges l_i, l_j, l_k. So,

$$\int_D \mathbf{u}_h \cdot \nabla p_h = -\left(p_i\mathbf{u}_h|_{D_i} \cdot (\tilde{\mathbf{n}}_i - \tilde{\mathbf{n}}_k) + p_j\mathbf{u}_h|_{D_j} \cdot (\tilde{\mathbf{n}}_j - \tilde{\mathbf{n}}_i) + p_k\mathbf{u}_h|_{D_k} \cdot (\tilde{\mathbf{n}}_k - \tilde{\mathbf{n}}_j)\right).$$

Substitute it into Eq. (13) and sum all the coefficients of p_i. Then,

$$a_h(\mathbf{u}_h, \nabla p_h) = \sum_{V \in \mathcal{V}_h} \int_{\partial V} \mathbf{u}_h \cdot \mathbf{n}\, p_V = 0,$$

where p_V denotes the value of p_h at the 'center' of control volume V and the last equality follows from the fact $\mathbf{u}_h \in \mathbf{U}_{0h}$.

Now we check the conditions (10c) and (11c). The condition (10c) can be easily verified. Given $q_h \in Q_h$, construct $\mathbf{u}(q_h)$ such that $\mathbf{u}(q_h) \cdot \mathbf{n}_{|E_{ij}} |E_{ij}| = q_i - q_j$. Then

$$\begin{aligned} c_h(\mathbf{u}(q_h), q_h) &= \sum_{V \in \mathcal{V}_h} \int_{\partial V} \mathbf{u}(q_h) \cdot \mathbf{n}\, q_h = \sum_{E_{ij} \in \mathcal{E}_h} \mathbf{u}(q_h) \cdot \mathbf{n}_{|E_{ij}} |E_{ij}|\; (q_i - q_j) \\ &= \left(\sum_{E_{ij} \in \mathcal{E}_h} \left[\int_{E_{ij}} \mathbf{u}(q_h) \cdot \mathbf{n} \right]^2 \right)^{1/2} \left(\sum_{E_{ij} \in \mathcal{E}_h} (q_i - q_j)^2 \right)^{1/2} \\ &\geq C \|\mathbf{u}(q_h)\|_{0,\Omega} \|q_h\|_{1,\Omega}. \end{aligned}$$

Therefore,

$$\sup_{\mathbf{u} \in \mathbf{U}_h} \frac{c_h(\mathbf{u}, q_h)}{\|\mathbf{u}\|_{0,\Omega}} \geq \frac{c_h(\mathbf{u}(q_h), q_h)}{\|\mathbf{u}(q_h)\|_{0,\Omega}} \geq C \|q_h\|_{1,\Omega}.$$

To finish the proof of the theorem we note that M_{1h}, M_{2h}, M_{3h} and M_{4h} are equal to zero for the corresponding bilinear forms and **Lemma** 2 provides a bound for the interpolation of $\mathbf{u}$ in (12). □

6 Numerical Results

In this section we present numerical experiments with three model problems on a domain Ω - a pentagon with a square hole inside. The coordinates of the pentagon vertexes are $((-1,0), (1,0), (1.6,2), (0,4), (-1.6,2))$ and the coordinates of the square hole are $((-0.7,0.9), (0.7,0.9), (0.7,2.3), (-0.7,2.3))$. The domain Ω is shown on Fig. 3 with 66 mesh points and 92 triangles. We choose such a domain to illustrate the flexibility of the finite volume method.

The Delaunay triangulations are generated using the software product **triangle** developed by J. R. Shewchuk [32]. **Triangle** is a Delaunay triangulator and it provides the control

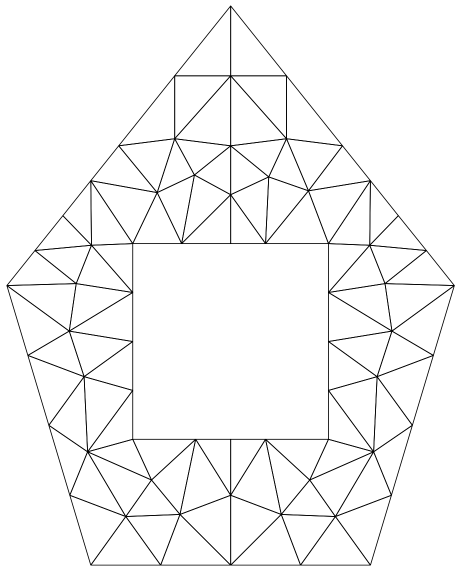

Figure 3: Domain Ω and mesh points

on the maximum area of the triangles. We used this option to generate five Delaunay triangulations. The maximum area of the triangles decreases four times on every successive level. The number of nodes increase roughly by four. The control volumes are created by connecting the center of gravity of each triangle to the center of its edges.

Problem 1 (Laplacian) *Take*

$$\mathbf{K}(\mathrm{x}) = I$$

and the exact solution

$$p(x, y) = \sin(\pi x)\sin(\pi y).$$

Problem 2 (Smooth full tensor (see Shashkov [31])) *Take*

$$\mathbf{K}(x, y) = R\ D(x, y)\ R^T,$$

where

$$R = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}, \quad D(x, y) = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

and

$$\phi = \frac{\pi}{4},$$
$$d_1 = 1 + 2x^2 + y^2 + y^5,$$
$$d_2 = 1 + x^2 + 2y^2 + x^3,$$

and the exact solution is

$$p(x,y) = \sin(\pi x)\sin(\pi y).$$

Problem 3 (Discontinuous full tensor) *Let*

$$\mathbf{K}(x,y) = \begin{cases} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \textit{for } x \le 0, \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \textit{for } x > 0. \end{cases}$$

The exact solution is given as

$$p(x,y) = \begin{cases} (1+x)y, & \textit{for } x \le 0, \\ (1+x)y + x(1+y), & \textit{for } x > 0. \end{cases}$$

We compute the L^2 error of p_h and $\mathbf{u}_h$, the H^1 error of p_h, and the $H(div)_h$ error for $\mathbf{u}_h$. In particular, the $H(div)_h$ error takes the form

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(div),h} = \left(\sum_{K_i \in \mathcal{F}_h} \int_{\partial K_i} (\mathbf{u}_h \cdot \mathbf{n} + \mathbf{K}\nabla p \cdot \mathbf{n})^2 \, dl \right)^{1/2}.$$

The numerical results reported in Tab. 1-3 confirm the first order error estimates for p_h and $\mathbf{u}_h$ proven in **Theorem** 2. We also observe that there is second order convergence of the pressure in L^2-norm and a half order convergence for the discrete divergence of the velocity.

Remark 2 *Similar results are obtained for all the three test problems with the "*$\mathbf{K}^{-1}$*" method discussed in* **Remark 1**.

Table 1: Discrete norms of the error for Problem 1

N	$\|p - p_h\|_{0,\Omega}$	$\|p - p_h\|_{1,\Omega}$	$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u} - \mathbf{u}_h\|_{H(div),h}$
66	$3.127.10^{-1}$	$2.067.10^{-0}$	$2.603.10^{-0}$	$2.891.10^{-0}$
233	$7.319.10^{-2}$	$9.070.10^{-1}$	$1.210.10^{-0}$	$1.700.10^{-0}$
878	$1.815.10^{-2}$	$4.440.10^{-1}$	$5.973.10^{-1}$	$1.112.10^{-0}$
3353	$4.466.10^{-3}$	$2.160.10^{-1}$	$2.928.10^{-1}$	$7.609.10^{-1}$
13105	$1.124.10^{-3}$	$1.070.10^{-1}$	$1.453.10^{-1}$	$5.303.10^{-1}$
Order	2.121	1.110	1.085	0.633

Table 2: Discrete norms of the error for Problem 2

N	$\|p - p_h\|_{0,\Omega}$	$\|p - p_h\|_{1,\Omega}$	$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u} - \mathbf{u}_h\|_{H(div),h}$
66	$3.847.10^{-1}$	$2.493.10^{-0}$	$2.776.10^{+2}$	$3.614.10^{+2}$
233	$9.474.10^{-2}$	$9.802.10^{-1}$	$1.575.10^{+2}$	$2.621.10^{+2}$
878	$2.513.10^{-2}$	$4.797.10^{-1}$	$7.894.10^{+1}$	$1.631.10^{+2}$
3353	$5.892.10^{-3}$	$2.266.10^{-1}$	$4.001.10^{+1}$	$1.146.10^{+2}$
13105	$1.698.10^{-3}$	$1.120.10^{-1}$	$1.785.10^{+1}$	$7.764.10^{+1}$
Order	2.056	1.157	1.036	0.589

Table 3: Discrete norms of the error for Problem 3

N	$\|p - p_h\|_{0,\Omega}$	$\|p - p_h\|_{1,\Omega}$	$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u} - \mathbf{u}_h\|_{H(div),h}$
66	$4.862.10^{-2}$	$5.756.10^{-1}$	$8.721.10^{-1}$	$9.048.10^{-1}$
233	$9.450.10^{-3}$	$2.577.10^{-1}$	$4.193.10^{-1}$	$5.906.10^{-1}$
878	$2.609.10^{-3}$	$1.298.10^{-1}$	$2.073.10^{-1}$	$4.075.10^{-1}$
3353	$5.818.10^{-4}$	$6.339.10^{-2}$	$1.028.10^{-1}$	$2.806.10^{-1}$
13105	$1.603.10^{-5}$	$3.155.10^{-2}$	$5.103.10^{-2}$	$1.959.10^{-1}$
Order	2.145	1.088	1.069	0.574

7 Conclusion

We have developed a locally conservative and flux-continuous "**K** method" for the elliptic diffusion equation with anisotropic and heterogeneous diffusion tensor. The method is well defined on very general meshes and is directly applicable to 3-D problems. Stability and error estimate are also established when the control volumes are formed by connecting the mass center of the triangles to their edge midpoints. Numerical experiments confirm our theoretical results. Similar numerical results were obtained for its corresponding "$\mathbf{K}^{-1}$ method".

There are still several theoretical and practical issues left unresolved like the convergence of the pressure in L^2-norm and convergence of the velocity in the discrete $H(div)$-norm.

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