# AN ERROR ESTIMATE FOR FINITE ELEMENT METHODS FOR SCALAR CONSERVATION LAWS

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# AN ERROR ESTIMATE FOR FINITE ELEMENT METHODS FOR SCALAR CONSERVATION LAWS

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ABSTRACT. In this paper, the Shock-Capturing Streamline Diffusion method and the Shock-Capturing Discontinuous Galerkin method for scalar conservation laws are proven to converge to the entropy solution with a rate of at least $h^{1/8}$ in the $L^\infty(L^1)$-norm. The triangulations are made of general acute simplexes and the approximate solution is taken to be piecewise a polynomial of degree $k$. The result is independent of the dimension of the space.

**1. Introduction.** In this paper, we consider the problem of estimating the difference between the entropy solution of the initial value problem, [7],

$$\partial_t u + \text{ div } f(u) = 0 \qquad \text{in } (0, T_\infty) \times \mathbb{R}^d, \tag{1.1}$$

$$u(0) = u_0 \qquad \text{on } \mathbb{R}^d, \tag{1.2}$$

and the approximate solution given by the so-called Shock-Capturing Streamline Diffusion (SCSD) method, see [3], [9], [10] and the references therein, or given by the so-called Shock-Capturing Discontinuous Galerkin (SCDG) method, see [5] and [4]. More precisely, we prove that the $L^\infty(L^1)$-norm of the above mentioned difference goes to zero at least as $h^{1/8}$ as the discretization parameter $h$ goes to zero. We assume the flux function $f : \mathbb{R} \to \mathbb{R}^d$ to be smooth and take the compactly supported initial data $u_0$ in the space $L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ of bounded functions of bounded variation in $\mathbb{R}^d$.

Convergence of the SCSD method was first obtained by Szepessy [9]. By using the DiPerna theory [2] of measure-valued solutions for (1.1), (1.2), Szepessy proved that the piecewise-linear approximate solution given by the SCSD method converges in $L^p_{loc}((0,T) \times \mathbb{R}^2)$ to the entropy solution of (1.1), (1.2), for any $p \in [1, \infty)$. Later, Szepessy [10] extended this result to the case of a general scalar conservation law in several space dimensions with boundary conditions and an approximate solution which is piecewise polynomial of degree $k$.

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To obtain convergence results in the framework of measure-valued solutions for (1.1), (1.2), [2], the approximate solution must

(i) be bounded in the $L^\infty((0,T_\infty)\times\mathbb{R}^d)$-norm,
(ii) be weakly consistent with all entropy inequalities,
(iii) be strongly consistent with the initial condition.

Recently, Jaffré, Johnson, and Szepessy, [4], proved that the SCDG method converges to the entropy solution of problem (1.1), (1.2) by using an extension of the measure-value convergence theory of DiPerna [2] obtained by Szepessy [11] which allows to replace in (i), the $L^\infty((0,T_\infty)\times\mathbb{R}^d)$-norm by the $L^\infty(0,T_\infty;L^2(\mathbb{R}^d))$-norm. In this paper, we consider the case of piecewise polynomial approximations of degree $k$ and show how to obtain, not only convergence, but error estimates with only a suitable version of (ii); the properties (i) and (iii) do not need to be obtained. The basic idea is to combine the estimates of the entropy dissipation, needed in (ii), with a modification of the Kuznetsov approximation theory [8] obtained by Cockburn, Coquel and LeFloch in [1].

The paper is organized as follows. In §2, we display the SCSD and SCDG methods and state and briefly discuss our main result, Theorem 2.1. The remaining of the paper is devoted to prove Theorem 2.1. In §3, we recall the basic approximation inequality obtained in [1]. This inequality states that, in order to obtain error estimates, only an estimate of the entropy dissipation is required. Such an estimate is obtained in §5 by using an *a priori* estimate and some very simple auxiliary results derived in §4. In §6, we end the proof of Theorem 2.1.

**2. The main result.** In this section, we describe the SCSD and SCDG finite element methods and state and briefly discuss the main result, Theorem 2.1.

The methods we have in mind being essentially implicit, we first decompose our domain into "slabs". More precisely, let $0=t_0<t_1<t_2<\cdots<t_N=T_\infty$ be a sequence of time levels. We set

$$S_n=(t_n,t_{n+1})\times\mathbb{R}^d,\qquad n=0,\dots,N-1,$$
$$\mathbb{R}^d_n=\{t_n\}\times\mathbb{R}^d,\qquad n=0,\dots,N.$$

In each slab $S_n$, we define a triangulation $\mathcal{T}_{h,n}$ of $(d+1)$-simplexes. No compatibility at $t=t_n$ between the meshes of two consecutive slabs $S_n$ and $S_{n+1}$, $n=0,\dots,N-1$, is required. We only assume that the triangulations satisfy the following simple conditions:

$$T \text{ is acute for all } T\in\mathcal{T}_{h,n}, \text{ and } n=0,\cdots,N-1, \tag{2.1a}$$
$$\frac{h_T}{\rho_T}\le\sigma \quad \text{for all } T\in\mathcal{T}_{h,n}, \text{ and } n=0,\cdots,N-1, \tag{2.1b}$$
$$h_T\ge\bar{c}\,h \quad \text{for all } T\in\mathcal{T}_{h,n}, \text{ and } n=0,\cdots,N-1, \tag{2.1c}$$

where $h_T$ is the diameter of $T\in\mathcal{T}_{h,n}$, $\rho_T$ is the diameter of the biggest ball totally included in $T$, $h=\max\{h_T,T\in\mathcal{T}_{h,n},n=0,\dots,N-1\}$, and $\bar{c}$ is a positive constant. The set of $d$-simplexes corresponding to the edges of $\mathcal{T}_{h,n}$ is denoted $\partial\mathcal{T}_{h,n}$.

Taking into account the fact that $u_0$ and (consequently) $u(t,\cdot)$, $t\in[0,T_\infty]$, have compact support, we introduce the following spaces

$$V_{h,n}=\{v;v|_T\in\mathcal{P}_k(T),\forall T\in\mathcal{T}_{h,n},v=0\text{ for }|\,x\,|\geq M\},$$
$$V_h=\{v:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R},v|_{(t_n,t_{n+1})}\in V_{h,n}\},$$

where $M$ is a sufficiently large constant and $\mathcal{P}_k$ stands for the space of polynomials of degree $k$. We emphasize that no continuity requirements are imposed upon the functions in $V_h$. Following [10], we partition each $(d+1)$-simplex $T$ into $k^{d+1}$ congruent $(d+1)$-simplexes, denoted by $T_i$, and introduce the spaces

$$\hat{V}_{h,n}=\{v;v|_{T_i}\in\mathcal{P}_1(T_i),i=1,\cdots,k^{d+1},\forall T\in\mathcal{T}_{h,n},v=0\text{ for }|\,x\,|\geq M\},$$
$$\hat{V}_h=\{v:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R},v|_{(t_n,t_{n+1})}\in\hat{V}_{h,n}\}.$$

For each $v\in V_h$, we define $\hat{v}$ to be the element of $\hat{V}_h$ that coincides with $v$ on each of the vertices of the $(d+1)$-simplexes $T_i$, for $i=1,\cdots,k^{d+1}$ and every $T$ of $\mathcal{T}_{h,n}$, for $n=0,\cdots,N-1$.

Finally, we define, for all $T\in\mathcal{T}_{h,n}$ and $n=0,\cdots,N-1$,

$$\mathbb{P}_h(v)|_T=\frac{1}{|T|}\int_T v\,dxdt,$$
$$\|\,v\,\|_{\mathbb{P}_h}(t,x)=\frac{1}{|T|^{1/2}}\|\,v\,\|_{L^2(T)},\text{ for }(t,x)\in T,$$

and, for all $e\in\partial\mathcal{T}_{h,n}$ and $n=0,\cdots,N-1$,

$$\mathbb{P}_h(v^T)|_e=\frac{1}{|e|}\int_e v^T\,d\lambda,$$
$$\|\,v^T\,\|_{\mathbb{P}_h}(t,x)=\frac{1}{|e|^{1/2}}\|\,v^T\,\|_{L^2(e)},\text{ for }(t,x)\in e,$$

where for $p=(t,x)\in e\backslash\partial e$, we have used the notation $v^T(p)=\lim_{s\downarrow 0}v(p-sn_{e,T})$, $n_{e,T}$ being the unit outer normal to $T$ along its face $e$.

The approximate solution $u_h$ given by the SCDG method is defined to be the element of $V_h$ such that

$$\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)(v+\delta\tilde{A}(u_h,v))dxdt+\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\Big(f^{LF}_{e,T}(u_h^T,u_h^{T_e})-\tilde{f}(u_h^T)\cdot n_{e,T}\Big)v^T d\lambda$$
$$+\sum_{T\in\mathcal{T}_{h,n}}\int_T\varepsilon_1(u_h)\mathbb{P}_h(\nabla\hat{u}_h\cdot\nabla\hat{v})dxdt+\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\varepsilon_2(u_h^T)\mathbb{P}_h(\nabla_e\hat{u}_h^T\cdot\nabla_e\hat{v}^T)d\lambda=0,$$
$$\forall v\in V_{h,n},\quad n=0,1,\ldots,N-1, \tag{2.2}$$

where

$$A(v)=\partial_t v+\sum_{i=1}^d\partial_i f_i(v),\qquad\tilde{A}(w,v)=\partial_t v+\sum_{i=1}^d f_i'(w)\partial_i v,$$

$$\nabla = (\partial_t, \partial_1, \partial_2, \dots, \partial_d), \qquad \nabla_x = (\partial_1, \partial_2, \dots, \partial_d), \qquad \nabla_e v^T = \nabla(v^T|_e),$$

and where $v^{T_e}(p) = \lim_{s\downarrow 0} v(p + s n_{e,T})$, for $p = (t,x) \in e\backslash\partial e$ and $e \in \partial T$. The initial condition is defined as follows:

$$u_h^{T_e}(0,\cdot) = u_{0h}, \text{ for all } e \in \partial\mathcal{T}_{h,0} \cap \mathbb{R}_0^d, \tag{2.3}$$

$u_{0h}$ being the standard $L^2$-projection of $u_0$ into the space of functions which are polynomials of degree $k$ on $e \in \partial\mathcal{T}_{h,0} \cap \mathbb{R}_0^d$.

We use throughout the paper the so-called Lax-Friedrichs flux

$$f_{e,T}^{LF}(u_h^T, u_h^{T_e}) = \frac{1}{2}(\tilde{f}(u_h^T) + \tilde{f}(u_h^{T_e})) \cdot n_{e,T} + C_e^{LF}(u_h^T - u_h^{T_e}), \tag{2.4a}$$

where $\tilde{f}(u_h) = (u_h, f(u_h))$, $C_e^{LF} = 1/2$ if $n_{e,T} = (\pm 1, 0, \dots, 0)$. We assume that

$$C_e^{LF} - \frac{1}{2}|\tilde{f}'|_{e,\infty} \geq c_\star > 0, \qquad \text{for } e \in \mathcal{E}_n, \tag{2.4b}$$

with $|\tilde{f}'|_{e,\infty} = \max\{|\tilde{f}'(\xi) \cdot n_{e,T}|, \xi \in [\min\{u_h^T, u_h^{T_e}\}, \max\{u_h^T, u_h^{T_e}\}]\}$ and where $\mathcal{E}_n$ denotes the subset of the edges $\partial\mathcal{T}_{h,n}$ whose intersection with $\mathbb{R}_n^d \cup \mathbb{R}_{n+1}^d$ is of measure zero. The condition (2.4b) ensures that the flux $f_{e,T}^{LF}(a,b)$ is a strictly monotone flux, that is, an increasing function of $a$ and a decreasing function of $b$.

Finally, for $T \in \mathcal{T}_{h,n}$, $n = 0, 1, \dots, N-1$, the shock capturing terms $\varepsilon_1$ and $\varepsilon_2$ are defined as follows

$$\varepsilon_1(v(\cdot,\cdot))|_T = \begin{cases} \delta_1 \dfrac{|A(v(\cdot,\cdot))|}{\|\nabla\hat{v}\|_{\mathbb{P}_h}} & \text{if } \|\nabla\hat{v}\|_{\mathbb{P}_h} \neq 0, \\ 0 & \text{if } \|\nabla\hat{v}\|_{\mathbb{P}_h} = 0, \end{cases}$$

and

$$\varepsilon_2(v^T)|_e = \begin{cases} \delta_2 \dfrac{|f_{e,T}^{LM}(v^T, v^{T_e}) - \tilde{f}(v^T) \cdot n_{e,T}|}{\|\nabla_e \hat{v}^T\|_{\mathbb{P}_h}} & \text{if } \|\nabla_e\hat{v}^T\|_{\mathbb{P}_h} \neq 0, \\ 0 & \text{if } \|\nabla_e\hat{v}^T\|_{\mathbb{P}_h} = 0, \end{cases}$$

for any $v \in V_h$. The parameters $\delta$, $\delta_1$, and $\delta_2$ will be defined later.

On the other hand, the SCSD method is the exact analog of SCDG but with *continuous* approximations inside each slab $S_n, n = 0, 1, \dots, N-1$. Namely, instead of $V_{h,n}$, one considers the space $\mathcal{V}_{h,n}$, where

$$\mathcal{V}_{h,n} = \{v \in \mathcal{C}^0(S_n); v|_T \in \mathcal{P}_k(T), \forall T \in \mathcal{T}_{h,n}, v = 0 \text{ for } |x| \geq M\}.$$

The spaces $\mathcal{V}_h$, $\hat{\mathcal{V}}_{h,n}$, $\hat{\mathcal{V}}_h$ are also defined accordingly. Thus, the approximate solution $u_h$ given by the SCSD method is the element of $\mathcal{V}_h$ such that

$$\int_{S_n} A(u_h)(v + \delta\tilde{A}(u_h, v))dxdt + \int_{\mathbb{R}_n^d} (u_{h,+} - u_{h,-})v_+ dx + \int_{S_n} \varepsilon_1(u_h)\mathbb{P}_h(\nabla\hat{u}_h \cdot \nabla\hat{v})dxdt$$
$$+ \int_{\mathbb{R}_n^d} \varepsilon_2^{SD}(u_{h,+})\mathbb{P}_h(\nabla_x\hat{u}_{h,+} \cdot \nabla_x\hat{v}_+)dx = 0, \qquad \forall v \in \mathcal{V}_{h,n}, \quad n = 0, 1, \dots, N-1, \tag{2.5}$$

where

$$\varepsilon_2^{SD}(v(t_n,\cdot))|_K = \begin{cases} \delta_2 \dfrac{|v_+(t_n,\cdot) - v_-(t_n,\cdot)|}{\| \nabla_x \hat{v}_+(t_n) \|_{\mathbb{P}_h}} & \text{if } \| \nabla_x \hat{v}_+(t_n) \|_{\mathbb{P}_h} \neq 0, \\ 0 & \text{if } \| \nabla_x \hat{v}_+(t_n) \|_{\mathbb{P}_h} = 0, \end{cases}$$

for any $v \in \mathcal{V}_h$, and where $u_{h,\pm} = \lim_{s\to\pm 0} v(t+s,x)$ ). Instead of (2.3), the initial condition is now given by

$$u_{h,-}(0,\cdot) = u_{0h}. \tag{2.6}$$

Notice that for both methods, $u_h|_{S_n}$ is not coupled to $u_h|_{S_{n+1}}$, $n = 0,\ldots,N-1$, as a consequence of the "slab structure" and of the upwinding in time used in both numerical schemes.

Since the only relevant values of the nonlinear flux $f$ are those in the range of the entropy solution $u$, $[a,b]$, we extend each of the components of $f$ smoothly in such a way that the extension is affine linear outside a fixed compact including $[a,b]$. We use that extension, which we still call $f$, to define the above schemes. Existence of a solution to (2.2), (2.3) (or (2.5), (2.6)) is well known and can be obtained by a fixed point argument; see [9], [10] and the references quoted therein.

We can now state our main result.

**Theorem 2.1.** *Let $u$ be the unique weak entropy satisfying solution of* (1.1), *and let $u_h$ be either a solution of* (2.2) *or of* (2.5). *Then, there exists a constant $C$, such that, for $\delta = \mathcal{O}(h)$, $\delta_1 = \mathcal{O}(h^{3/4})$, and $\delta_2 = \mathcal{O}(h^{3/4})$,*

$$\|u_h(\tau,\cdot) - u(\tau,\cdot)\|_{L^1(\mathbb{R}^d)} \le C\Big(\|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)} + h^{1/8}\Big) \qquad \forall \tau \in (0, T_\infty].$$

*The constant $C$ depends on $k, d, \sigma, \bar{c}, c^\star, \| u_0 \|_{L^2(\mathbb{R}^d)}$, and $TV(u_0)$.*

Several versions of the SCSD and SCDG methods (i.e. several definitions of $\varepsilon_1$, $\varepsilon_2$ or $\varepsilon_2^{SD}$) can be found in the literature; see e.g. [3], [6], [9], [10], and [4]). If, for instance, we do not divide by the gradient of the approximate solution in the definition of $\varepsilon_1$, $\varepsilon_2$ or $\varepsilon_2^{SD}$, we obtain different methods which, nevertheless, can be analyzed using the same technique as in this paper. The theoretical order of convergence (with $\delta$, $\delta_1$, and $\delta_2$ of order $h$) is found to be the same as the one in Theorem 2.1.

Since the proofs follow the same lines for both methods (with a few simplifications due to the higher regularity of the approximate solution in the case of SCSD), we will, from now on, exclusively concentrate our attention on the SCDG algorithm. The proof of the above result relies on an approximation inequality obtained in [1] by suitably modifying the Kuznetsov approximation result [8, Lemma 2]. According to this new inequality, the error estimate follows from the estimate of the so-called entropy dissipation form. This approximation result is recalled in the next section.

**3. Kuznetsov approximation theory.** We follow Kuznetsov [8]. We first introduce the notation. Let $w : \mathbb{R} \to \mathbb{R}$ be a smooth nonnegative even function with unit mass and support in $[-1,1]$. We set

$$\varphi = w_{\varepsilon_t}(t-t') \prod_{i=1}^{d} w_{\varepsilon_x}(x_i - x_i'), \qquad (x,t),(x',t') \in \mathbb{R}^d \times \mathbb{R}^+,$$

where $\varepsilon_t$ and $\varepsilon_x$ are two arbitrary positive numbers and $w_\lambda(s) = w(s/\lambda)/\lambda$ for any $s \in \mathbb{R}$, $\lambda = \varepsilon_t,\ \varepsilon_x$.

For $u$ and $v$ right-continuous functions from $(0,T)$ to $L^1(\mathbb{R}^d)$, we define

$$E^{\varepsilon_t,\varepsilon_x}(v,u;\tau) = \int_0^T\int_{\mathbb{R}^d} \Theta_\tau^{\varepsilon_t,\varepsilon_x}(v,u;t,x)\,dxdt$$

with

$$\begin{aligned}\Theta_\tau^{\varepsilon_t,\varepsilon_x}(v,c;t,x) = &-\int_0^\tau\int_{\mathbb{R}^d} U(v(t',x')-c)\partial_{t'}\varphi\,dx'dt' \\ &-\int_0^\tau\int_{\mathbb{R}^d} F(v(t',x'),c)\cdot\nabla_{x'}\varphi\,dx'dt' \\ &-\int_{\mathbb{R}^d} U(v_-(0,x')-c)\,\varphi(t,x,0,x')\,dx' \\ &+\int_{\mathbb{R}^d} U(v(\tau,x')-c)\,\varphi(t,x,\tau,x')\,dx',\end{aligned}$$

where $U$ is an arbitrary even entropy and $F$ its associated flux, i.e., $\partial_u F(u,c) = U'(u)f'(u)$, and $v_-(0,\cdot)$ is the exterior trace of $v$ at $t=0$. We recall that if $u$ is the entropy solution of (1.1), (1.2) and if we set $u_-(0,x) = u_0(x)$, then $E^{\varepsilon_t,\varepsilon_x}(u,v;\tau) \le 0$ for any $v$.

We restrict our attention to one particular family of entropies. Let $G : \mathbb{R} \to \mathbb{R}^+$ be a smooth even function such that

$$G(0) = 0, \quad G'(x) = \begin{cases} 1 & \text{if } x \ge 1 \\ -1 & \text{if } x \le -1 \end{cases}, \quad G'' \ge 0. \tag{3.1}$$

For any $c \in \mathbb{R}$, we define,

$$U_\varepsilon(u) = \varepsilon G(\frac{u}{\varepsilon}) \qquad \text{and} \qquad F_{c,\varepsilon,i}(u) = \int_c^u G'_\varepsilon(\lambda) f'_i(\lambda) d\lambda \qquad i = 1,\cdots,d.$$

Our main result is based on the following basic approximation inequality.

**Lemma 3.1** (Basic approximation inequality, [1]). *Let $u$ be the entropy solution of (1.1). We have*

$$\begin{aligned}\int_{\mathbb{R}^d} U(v(T_\infty,x') - u(T_\infty,x'))\,dx' \le \sqrt{e}\int_{\mathbb{R}^d} U(v_-(0,x') - u_0(x'))dx' \\ + C(T_\infty\varepsilon + \varepsilon_x + \varepsilon_t)TV(u_0) + 2\sqrt{e}\sup_{0\le\tau\le T_\infty} E^{\varepsilon_t,\varepsilon_x}(v,u;\tau),\end{aligned}$$

*where $C$ is a constant independent of $\varepsilon, \varepsilon_x$, and $\varepsilon_t$.*

If we take $v_-(0,x) = u_{0h}$ and set $v$ equal to the right-continuous function that coincides with $u_h$ on each interval of the form $(t_n, t_{n+1})$, we see that to obtain our error estimates, we only have to estimate the entropy dissipation $E^{\varepsilon_t,\varepsilon_x}(v_h,u;\tau)$. From now on, we will not distinguish between $v_h$ and $u_h$.

We devote the remaining part of this section to the rewriting of the entropy dissipation $E^{\varepsilon_t,\varepsilon_x}(u_h,u;\tau)$ as the sum of nine terms, which will then be estimated in §5. For the sake of clarity, we use, from now on, the following simplified notation:

$$\begin{aligned} u_h &= u_h(t',x'), \quad u = u(t,x), \\ u_h^T &= u_h^T(t',x'), \quad u_h^{T_e} = u_h^{T_e}(t',x'), \\ \varphi &= \varphi(t,x,t',x'), \quad \varphi(t_i,p_i) = \varphi(t,x,t_i,p_i). \end{aligned}$$

(We recall that $u_h^T$ and $u_h^{T_e}$ refer to the internal and external traces of $u_h$ on the boundary of the simplex $T$.) Furthermore, we will put a prime as a superscript in an operator when we want to emphasize the fact that such an operator is considered to be acting on the 'primed' variables only.

Using the definitions of the form $\Theta_\tau^{\varepsilon_t,\varepsilon_x}$ and of the function $F$, we obtain, after a simple integration by parts,

$$\begin{aligned} \Theta_\tau^{\varepsilon_t,\varepsilon_x}(u_h,u;t,x) = &\sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T A'(u_h)U'(u_h-u)\,\varphi\,dx'dt' \\ &- \sum_{n=0}^{N_\tau-1} \sum_{e\in\partial\mathcal{T}_{h,n}} \int_e \Big(\tilde F(u_h^T,u) - \tilde F(u_h^{T_e},u)\Big)\cdot n_{e,T}\,\varphi\,d\lambda', \end{aligned}$$

where $\tau, 0 < \tau \le T_\infty$, is supposed to be such that there exists $N_\tau, 0 < N_\tau \le N$, with $t_{N_\tau} = \tau$, and where $\tilde F(u_h,u) = (U(u_h-u), F(u_h,u))$ is the "extended" entropy flux. Following [9], we introduce the classical nodal interpolation operator $\Pi_h'$ (which acts only on the variables $t'$ and $x'$) and rewrite $\Theta_\tau^{\varepsilon_t,\varepsilon_x}$ as follows:

$$\begin{aligned} \Theta_\tau^{\varepsilon_t,\varepsilon_x}(u_h,u;t,x) = &\sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T A'(u_h)\,\Pi_h'(U'(u_h-u)\,\varphi)\,dx'dt' \\ &+ \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T A'(u_h)\Big(U'(u_h-u)\,\varphi - \Pi_h'(U'(u_h-u)\,\varphi)\Big)\,dx'dt' \\ &- \sum_{n=0}^{N_\tau-1} \sum_{e\in\partial\mathcal{T}_{h,n}} \int_e \Big(\tilde F(u_h^T,u) - \tilde F(u_h^{T_e},u)\Big)\cdot n_{e,T}\,\varphi\,d\lambda'. \end{aligned}$$

By inserting the definition of $u_h$ and setting $v = \Pi_h'(U'(u_h-u)\,\varphi)$ in (2.2), we get

$$\begin{aligned} &\Theta^{\varepsilon_t,\varepsilon_x}(u_h,u;t,x) = \\ &\quad -\delta \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T A'(u_h)\,\tilde A'\Big(u_h, \Pi_h'[U'(u_h-u)\,\varphi]\Big)\,dx'dt' \\ &\quad - \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e \Big(f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde f(u_h^T)\cdot n_{e,T}\Big)\Pi_h'[U'(u_h^T-u)\,\varphi]d\lambda' \end{aligned}$$

$$- \sum_{n=0}^{N_\tau -1} \sum_{T\in\mathcal{T}_{h,n}} \int_T \varepsilon_1(u_h)\mathbb{P}'_h(\nabla'\hat{u}_h \cdot \nabla'\hat{\Pi}'_h[U'(u_h-u)\,\varphi])\,dx'dt'$$

$$- \sum_{n=0}^{N_\tau -1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e \varepsilon_2(u_h^T)\mathbb{P}'_h(\nabla'_e\hat{u}_h^T \cdot \nabla'_e\hat{\Pi}'_h[U'(u_h^T-u)\,\varphi])\,d\lambda'$$

$$+ \sum_{n=0}^{N_\tau -1} \sum_{T\in\mathcal{T}_{h,n}} \int_T A'(u_h)\Big(U'(u_h-u)\,\varphi - \Pi'_h[U'(u_h-u)\,\varphi]\Big)\,dx'dt'$$

$$- \sum_{n=0}^{N_\tau -1} \sum_{e\in\partial\mathcal{T}_{h,n}} \int_e \Big(\tilde{F}(u_h^T,u) - \tilde{F}(u_h^{T_e},u)\Big)\cdot n_{e,T}\,\varphi\,d\lambda'.$$

Noting that, by (2.4a),

$$\sum_{n=0}^{N_\tau -1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e \Big(f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}\Big)U'(u_h^T-u)\,\varphi d\lambda'\,d\lambda'$$

$$= \sum_{n=0}^{N_\tau -1} \sum_{e\in\mathcal{E}_n} \int_e \Big(-f_{e,T}^{LF}(u_h^T,u_h^{T_e})(U'(u_h^T-u) - U'(u_h^{T_e}-u))$$

$$+ \tilde{f}(u_h^T)\cdot n_{e,T}U'(u_h^T-u) - \tilde{f}(u_h^{T_e})\cdot n_{e,T}U'(u_h^{T_e}-u)\Big)\,\varphi d\lambda'$$

$$+ \sum_{n=0}^{N_\tau -1} \int_{\mathbb{R}^d_n} (u_h^T - u_h^{T_e})U'(u_h^T-u)\,\varphi dx',$$

and that, by definition of $\tilde{F}(u_h,u)$,

$$\sum_{n=0}^{N_\tau -1} \sum_{e\in\partial\mathcal{T}_{h,n}} \int_e \Big(\tilde{F}(u_h^T,u) - \tilde{F}(u_h^{T_e},u)\Big)\cdot n_{e,T}\,\varphi\,d\lambda'$$

$$= \sum_{n=0}^{N_\tau -1} \sum_{e\in\mathcal{E}_n} \int_e (\tilde{F}(u_h^T,u) - \tilde{F}(u_h^{T_e},u))\cdot n_{e,T}\,\varphi d\lambda'$$

$$+ \sum_{n=0}^{N_\tau -1} \int_{\mathbb{R}^d_n} (U(u_h^T-u) - U(u_h^{T_e}-u))\,\varphi dx',$$

we obtain, after a few manipulations and rearrangements,

$$\Theta^{\varepsilon_t,\varepsilon_x}(u_h,u;t,x) =$$

$$\sum_{n=0}^{N_\tau -1} \sum_{e\in\mathcal{E}_n} \int_e \Big(-f_{e,T}^{LF}(u_h^T,u_h^{T_e})(U'(u_h^T-u) - U'(u_h^{T_e}-u)) + \tilde{f}(u_h^T)\cdot n_{e,T}U'(u_h^T-u)$$

$$- \tilde{f}(u_h^{T_e})\cdot n_{e,T}U'(u_h^{T_e}-u) - (\tilde{F}(u_h^T,u) - \tilde{F}(u_h^{T_e},u))\cdot n_{e,T}\Big)\,\varphi d\lambda'$$

$$
\begin{aligned}
&+ \sum_{n=0}^{N_\tau-1} \int_{\mathbb{R}^d_n} \Big( -(u_h^T - u_h^{T_e}) U'(u_h^T - u) + U(u_h^T - u) - U(u_h^{T_e} - u) \Big) \varphi \, dx' \\
&- \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T \varepsilon_1(u_h) \mathbb{P}'_h(\nabla' \hat{u}_h \cdot \nabla' \hat{\Pi}'_h[U'(u_h - u)\,\mathbb{P}'_h \varphi])\, dx'\, dt' \\
&- \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e \varepsilon_2(u_h^T) \mathbb{P}'_h(\nabla'_e \hat{u}_h^T \cdot \nabla'_e \hat{\Pi}'_h[U'(u_h^T - u)\,\mathbb{P}'_h \varphi])\, d\lambda' \\
&- \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T \varepsilon_1(u_h) \mathbb{P}'_h(\nabla' \hat{u}_h \cdot \nabla' \hat{\Pi}'_h[U'(u_h - u)(\varphi - \mathbb{P}'_h \varphi)])\, dx'\, dt' \\
&- \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e \varepsilon_2(u_h^T) \mathbb{P}'_h(\nabla'_e \hat{u}_h^T \cdot \nabla'_e \hat{\Pi}'_h[U'(u_h^T - u)(\varphi - \mathbb{P}'_h \varphi)])\, d\lambda' \\
&+ \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T A'(u_h) \Big( U'(u_h - u)\,\varphi - \Pi'_h[U'(u_h - u)\,\varphi] \Big)\, dx'\, dt' \\
&- \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e \Big( f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T} \Big) \Big( U'(u_h^T - u)\,\varphi - \Pi'_h[U'(u_h^T - u)\,\varphi] \Big)\, d\lambda' \\
&- \delta \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T A'(u_h)\, \tilde{A}' \Big( u_h, \Pi'_h[U'(u_h - u)\,\varphi] \Big)\, dx'\, dt' \\
&= \sum_{i=1}^{9} \Theta_i^{\varepsilon_t,\varepsilon_x}(u_h, u; t, x).
\end{aligned}
$$

Finally, by definition of $E^{\varepsilon_t,\varepsilon_x}(u_h, u; \tau)$, we can write,

$$
E^{\varepsilon_t,\varepsilon_x}(u_h, u; \tau) = \sum_{i=1}^{9} E_i^{\varepsilon_t,\varepsilon_x}(u_h, u; \tau),
$$

where, for $i = 1, \dots, 9$,

$$
E_i^{\varepsilon_t,\varepsilon_x}(u_h, u; \tau) = \int_0^\tau \int_{\mathbb{R}^d} \Theta_i^{\varepsilon_t,\varepsilon_x}(u_h, u; t, x)\, dx\, dt.
$$

This is the desired expression of the entropy dissipation form.

**4. Regularity of the approximate solution and auxiliary results.** To be able to estimate the entropy dissipation form, we need (i) *a priori* estimates for the approximate solution $u_h$, see Lemma 4.1, which could be considered as regularity results for $u_h$, (ii) suitable approximation results, see Lemma 4.4, that allow us to use the *a priori* estimates in the estimation of the entropy dissipation form, and (iii) 'nonnegativity' results, see Lemma 4.5, which reflect the nature of the so-called shock-capturing terms used in the definition of the scheme. The *a priori* estimates are essentially an $L^2$-stability result that follows easily from the variational formulation. The approximation result is a standard finite element result in which the regularity of the triangulation, condition (2.1b), is essential, as expected. The 'nonnegativity' result holds because our simplexes are supposed to be acute, see condition (2.1a)

**Lemma 4.1** (A priori estimate). *For any $N_\tau$, $0 \le N_\tau \le N$, we have*

$$\frac{1}{2}\int_{\mathbb{R}^d_{N_\tau}} {u_h^T}^2 dx + \frac{1}{2}\sum_{n=0}^{N_\tau-1}\int_{\mathbb{R}^d_n}(u_h^T - u_h^{T_e})^2 dx$$
$$+\frac{1}{2}\sum_{n=0}^{N_\tau-1}\sum_{e\in\mathcal{E}_n}\int_e\int_{u_h^{T_e}}^{u_h^T}\Big(f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde f(s)\cdot n_{e,T}\Big)ds\,d\lambda$$
$$+\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T \varepsilon_1(u_h)\|\,\nabla\hat u_h\,\|^2_{\mathbb{P}_h}\,dx\,dt + \sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e \varepsilon_2(u_h^T)\|\,\nabla_e\hat u_h^T\,\|^2_{\mathbb{P}_h}\,d\lambda$$
$$+\,\delta\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)^2\,dx\,dt \le \frac{1}{2}\|\,u_0\,\|^2_{L^2(\mathbb{R}^d)}.$$

*Proof.* We consider only the non trivially positive terms in (2.2) when $v$ is chosen as $u_h$, namely,

$$\Psi^n = \sum_{T\in\mathcal{T}_{h,n}}\int_T\Big(\frac{1}{2}\partial_t u_h^2 + \sum_{i=1}^d f_i'(u_h)\,u_h\,\partial_i u_h\Big)dx\,dt$$
$$+\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\Big(f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde f(u_h^T)\cdot n_{e,T}\Big)u_h^T d\lambda.$$

By introducing the auxiliary pair entropy-entropy flux $\tilde F_a(u) = (U_a(u), F_a(u))$, where

$$U_a(u) = \frac{1}{2}u^2 \quad\text{and}\quad F_{a,i}'(u) = f_i'(u)U_a'(u) = f_i'(u)u \quad i = 1,\dots,d,$$

the term $\Psi^n$ may be expressed as

$$\Psi^n = \sum_{T\in\mathcal{T}_{h,n}}\int_T \nabla\tilde F_a(u_h)dx\,dt + \sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\Big(f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde f(u_h^T)\cdot n_{e,T}\Big)u_h^T d\lambda$$
$$= \sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\Big(\tilde F_a(u_h^T)\cdot n_{e,T} + (f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde f(u_h^T)\cdot n_{e,T})u_h^T\Big)d\lambda.$$

After reordering, we get

$$\Psi^n = -\frac{1}{2}\int_{\mathbb{R}^d_n} {u_h^{T_e}}^2 dx + \frac{1}{2}\int_{\mathbb{R}^d_n}(u_h^T - u_h^{T_e})^2 dx + \frac{1}{2}\int_{\mathbb{R}^d_{n+1}} {u_h^T}^2 dx$$
$$+\sum_{e\in\mathcal{E}_n}\int_e\Big((\tilde F_a(u_h^T) - \tilde F_a(u_h^{T_e}))\cdot n_{e,T} + f_{e,T}^{LF}(u_h^T,u_h^{T_e})(u_h^T - u_h^{T_e})$$
$$-(\tilde f(u_h^T)u_h^T - \tilde f(u_h^{T_e})u_h^{T_e})\cdot n_{e,T}\Big)d\lambda.$$

Since

$$\tilde{F}_a(u_h^T) - \tilde{F}_a(u_h^{T_e}) = \int_{u_h^{T_e}}^{u_h^T} \tilde{F}_a'(s)ds = \int_{u_h^{T_e}}^{u_h^T} \tilde{f}'(s)s\,ds$$
$$= -\int_{u_h^{T_e}}^{u_h^T} \tilde{f}(s)ds + \tilde{f}(u_h^T)u_h^T - \tilde{f}(u_h^{T_e})u_h^{T_e},$$

we get

$$\Psi^n = -\frac{1}{2}\int_{\mathbb{R}_n^d} {u_h^{T_e}}^2 dx + \frac{1}{2}\int_{\mathbb{R}_n^d} (u_h^T - u_h^{T_e})^2 dx + \frac{1}{2}\int_{\mathbb{R}_{n+1}^d} {u_h^T}^2 dx$$
$$+ \sum_{e\in\mathcal{E}_n} \int_e \int_{u_h^{T_e}}^{u_h^T} \left( f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(s)\cdot n_{e,T} \right) ds d\lambda.$$

Summing over $n$ from 0 to $N_\tau - 1$ completes the proof. □

For any monotone numerical flux, the terms appearing in the previous lemma are all nonnegative. In the particular case of the Lax-Friedrichs flux, we can be more precise. If the parameter $C_e^{LF}$ is suitably chosen, the above result also gives an $L^2$-estimate of the jump of $u_h$ across the inner edges $\mathcal{E}_n$.

**Corollary 4.2.** *If $C_e^{LF}$ satisfies condition* (2.4b), *then*

$$\sum_{n=0}^{N_\tau-1} \sum_{e\in\mathcal{E}_n} \int_e (u_h^T - u_h^{T_e})^2 d\lambda \le \frac{\|u_0\|^2_{L^2(\mathbb{R}^d)}}{c_\star}.$$

*Proof.* By the definition of the Lax-Friedrichs flux, (2.4a), we have

$$\int_e \int_{u_h^{T_e}}^{u_h^T} \left( f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(s)\cdot n_{e,T} \right) ds d\lambda = \int_e C_e^{LF}(u_h^T - u_h^{T_e})^2\, d\lambda$$
$$+ \int_e \int_{u_h^{T_e}}^{u_h^T} \left( \frac{1}{2}(\tilde{f}(u_h^T) + \tilde{f}(u_h^{T_e}))\cdot n_{e,T} - \tilde{f}(s)\cdot n_{e,T} ds \right) d\lambda$$
$$\ge \int_e C_e^{LF}(u_h^T - u_h^{T_e})^2\, d\lambda - \int_e \frac{1}{2}|\tilde{f}'|_{e,\infty}(u_h^T - u_h^{T_e})^2 d\lambda$$
$$\ge c_\star \int_e (u_h^T - u_h^{T_e})^2\, d\lambda,$$

by the condition (2.4b). The result follows easily from Lemma 4.1. □

The following result is a simple consequence of the fact that all norms are equivalent in finite dimensional spaces.

**Lemma 4.3.** *There is a constant $C$ that depends only on $k$ and $d$ such that*

$$\| \nabla' u_h \|_{L^\infty(T)} \le C \| \nabla' \hat{u}_h \|_{\mathbb{P}_h} \quad \text{on } T, \forall T \in \mathcal{T}_{h,n}, n = 0, \cdots, N-1,$$
$$\| \nabla'_e u_h^T \|_{L^\infty(e)} \le C \| \nabla'_e \hat{u}_h^T \|_{\mathbb{P}_h} \quad \text{on } e, \forall e \in \partial\mathcal{T}_{h,n}, n = 0, \cdots, N-1.$$

**Lemma 4.4** (Approximation properties of the operator $\Pi_h$). *Let $s \in L^1((0,T_\infty)\times \mathbb{R}^d)$ be a piecewise continuous function. We have*

$$\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T s(t',x')\int_0^T\int_{\mathbb{R}^d}\Big(U'(u_h-u)\,\varphi-\Pi'_h[U'(u_h-u)\,\varphi]\Big)\,dx\,dt\,dx'\,dt'$$
$$\leq \frac{C}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |s(t',x')|\,\|\,\nabla'\hat{u}_h\,\|_{\mathbb{P}_h}\,h_T\,dx'\,dt'$$
$$+\,C\frac{h}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |s(t',x')|\,dx'\,dt',$$

$$\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e s^T\int_0^T\int_{\mathbb{R}^d}\Big(U'(u_h^T-u)\,\varphi-\Pi'_h[U'(u_h^T-u)\,\varphi]\Big)\,dx\,dt\,d\lambda'$$
$$\leq \frac{C}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |s^T|\,\|\,\nabla_e\hat{u}_h^T\|_{\mathbb{P}_h}\,h_e\,d\lambda'$$
$$+\,C\frac{h}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |s^T|\,d\lambda',$$

*where $h_e$ is the diameter of $e$ and where $C$ depends on $k, d$, and $\sigma$ only.*

*Proof.* Since the proofs of these inequalities are almost identical, we prove only the first one; let us denote by $L$ its left hand side. Let $\{(t_j,p_j)\}_{j=1}^{N_{d+1,k}}$ be the nodal points associated with the canonical basis $\{\Psi_j^T\}_{j=1}^{N_{d+1,k}}$ of the space $\mathcal{P}^k(T)$ where, of course, $N_{d+1,k}=\binom{d+k+1}{k}$. With this notation, we can express $L$ as follows:

$$L=\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T s(t',x')\int_0^T\int_{\mathbb{R}^d}\chi\,dx\,dt\,dt'\,dx',$$

where $\chi=\chi(t,x,t',x')$ is defined by

$$\chi = U'(u_h-u)\,\varphi-\sum_{j=1}^{N_{d+1,k}}[U'(u_{h,j}-u)\,\varphi_j]\Psi_j^T$$
$$=\sum_{j=1}^{N_{d+1,k}}\Big(U'(u_h-u)\,\varphi-U'(u_{h,j}-u)\,\varphi_j\Big)\Psi_j^T,$$

(since $\sum_{j=1}^{N_{d+1,k}}\Psi_j^T\equiv 1$ on $T$) where $u_{h,j}=u_h(t_j,p_j)$ and $\varphi_j=\varphi(t_j,p_j)$. Moreover, since

$$(U'(u_h-u)\,\varphi-U'(u_{h,j}-u)\,\varphi_j=(U'(u_h-u)-U'(u_{h,j}-u))\,\varphi$$
$$-\,U'(u_{h,j}-u)(\varphi_j-\varphi),$$

we obtain, after simple manipulations,

$$L \leq \frac{C}{\varepsilon} \sum_{n=0}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \int_T |s(t',x')| \sum_{j=1}^{N_{d+1,k}} |u_h - u_{h,j}| |\Psi_j^T(t',x')| dx'$$
$$+ C \sum_{n=0}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \int_T |s(t',x')| \sum_{j=1}^{N_{d+1,k}} \int_0^T \int_{\mathbb{R}^d} |\varphi - \varphi_j| |\Psi_i^T(t',x')| dt dx dt' dx',$$

where we have used the fact that $U' \leq 1$ and $U'' \leq 1/\varepsilon$ by (3.1), and that $\int_0^T \int_{\mathbb{R}^d} \varphi \leq 1$. From this inequality, we immediately get

$$L \leq \frac{C}{\varepsilon} \sum_{n=0}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \int_T |s(t',x')| \| \nabla' u_h \|_{L^\infty(T)} h_T \, dt' dx'$$
$$+ \frac{C\,h}{\varepsilon_x + \varepsilon_t} \sum_{n=0}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \int_T |s(t',x')| \, dt' dx',$$

The result follows from Lemma 4.3. This completes the proof. □

**Lemma 4.5** (Nonnegativity of the shock-capturing terms). *Suppose that the condition* (2.1a) *on the triangulations is satisfied. Then*

$$\nabla' \hat{u}_h \cdot \nabla' \hat{\Pi}_h' [U'(u_h - u)] \geq 0,$$
$$\nabla_e' \hat{u}_h^T \cdot \nabla_e' \hat{\Pi}_h' [U'(u_h^T - u)] \geq 0.$$

*Proof.* We only prove the first inequality since the proof for the second is similar. By definition, we have that, on each $(d+1)$-simplex $T_\ell$,

$$\hat{u}_h(t',x') = \sum_{i=1}^{d+2} u_{h,\ell,i} \Psi_i^{T_\ell}(t',x'),$$

where $\{ \Psi_i^{T_\ell} \}_{i=1}^{d+2}$ is the canonical basis of the space $\mathcal{P}_1(T_\ell)$, $\{(t_{\ell,i}, p_{\ell,i})\}_{i=1}^{d+2}$ are the vertices of $T_\ell$, and $u_{h,\ell,i} = u_h(t_{\ell,i}, p_{\ell,i})$.

Thus, for $(t',x') \in T_\ell$,

$$\theta = \nabla' \hat{u}_h \cdot \nabla' (\hat{\Pi}_h' [U'(u_h - u)])$$
$$= \sum_{i,j=1}^{d+2} u_{h,\ell,i} U'(u_{h,\ell,j} - u) \, \Lambda_{i,j},$$

where $\Lambda_{i,j} = \nabla' \Psi_i^{T\ell}(t',x') \cdot \nabla' \Psi_j^{T_\ell}(t',x')$. Since $\sum_{i=1}^{d+2} \Psi_i^{T\ell}(t',x') \equiv 1$ on $T_\ell$, we have

$$\theta = \sum_{i,j=1}^{d+2} (-u_{h,\ell,j} + u_{h,\ell,i}) U'(u_{h,\ell,j} - u) \Lambda_{i,j}.$$

Interchanging $i$ and $j$ in the previous relation and averaging the two expressions for $\theta$, we get

$$\theta = \frac{1}{2} \sum_{i,j=1}^{d+2} \big(-u_{h,\ell,j} + u_{h,\ell,i}\big)\big(-U'(u_{h,\ell,j} - u) + U'(u_{h,\ell,i} - u)\big)\,\big(-\Lambda_{i,j}\big).$$

Since by (2.1a) the $(d+1)$-simplex $T_\ell$ is acute, we have $-\Lambda_{i,j} \geq 0$ on $T_\ell$. Hence by convexity of $U$, $\theta \geq 0$ on $T_\ell$. This completes the proof. □

**5. Estimating the entropy dissipation form.** In this section, we estimate each of the terms $E_i^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)$.

**Lemma 5.1.** *Suppose that $C_e^{LF}$ satisfies the condition* (2.4b). *Then we have*

$$E_1^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau) \leq 0.$$

*Proof.* We have

$$\begin{aligned}\tilde{F}(u_h^T, u) - \tilde{F}(u_h^{T_e}, u) &= \int_{u_h^{T_e}}^{u_h^T} \frac{d}{ds}\tilde{F}(s,u)ds = \int_{u_h^{T_e}}^{u_h^T} \tilde{f}'(s)U'(s-u)\,ds \\ &= -\int_{u_h^{T_e}}^{u_h^T} \tilde{f}'(s)U''(s-u)ds + \tilde{f}(u_h^T)U'(u_h^T - u) - \tilde{f}(u_h^{T_e})U'(u_h^{T_e} - u),\end{aligned}$$

and thus

$$\begin{aligned}E_1^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau) &= \int_0^\tau \int_{\mathbb{R}^d} \sum_{n=0}^{N_\tau - 1} \sum_{e \in \mathcal{E}_n} \int_e \Big( -f_{e,T}^{LF}(u_h^T, u_h^{T_e})(U'(u_h^T - u) - U'(u_h^{T_e} - u)) \\ &\qquad + \int_{u_h^{T_e}}^{u_h^T} \tilde{f}'(s)U''(s-u)ds \Big) dx dt d\lambda' \\ &= \int_0^\tau \int_{\mathbb{R}^d} \sum_{n=0}^{N_\tau - 1} \sum_{e \in \mathcal{E}_n} \int_e \int_{u_h^{T_e}}^{u_h^T} \Big(\tilde{f}'(s)\cdot n_{e,T} - f_{e,T}^{LF}(u_h^T, u_h^{T_e})\Big) U''(s-u)\, dx dt ds d\lambda'.\end{aligned}$$

Since $U$ is convex, and the numerical flux is monotone, by (2.4b), the above quantity is nonnegative. This proves the lemma. □

**Lemma 5.2.** *We have*

$$E_2^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau) \leq 0.$$

*Proof.* The result immediately follows from the nonnegativity of the function $\varphi$ and the convexity of the entropy $U$. □

**Lemma 5.3.** *Suppose that the condition* (2.1a) *on the triangulations is satisfied. Then we have*

$$E_3^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau) \leq 0.$$

*Proof.* The result directly follows from Lemma 4.5 and the fact that $\mathbb{P}_h'\varphi$ is a nonnegative piecewise constant function. □

**Lemma 5.4.** *Suppose that the condition (2.1a) on the triangulations is satisfied. Then we have*

$$E_4^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)\le 0.$$

The proof of this result is similar to that of the preceding lemma.

**Lemma 5.5.** *Suppose that the condition (2.1b) on the triangulations is satisfied. Then we have*

$$E_5^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)\le C\frac{\delta_1}{(\varepsilon_x+\varepsilon_t)\delta^{1/2}},$$

*where the constant $C$ depends on $k,d,\sigma$, and $\| u_0\|_{L^2(\mathbb{R}^d)}$ only.*

*Proof.* We have, by definition,

$$\begin{aligned}
&E_5^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)\\
&\quad=-\int_0^T\!\!\int_{\mathbb{R}^d}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T \varepsilon_1(u_h)\mathbb{P}'_h(\nabla'\hat{u}_h\cdot\nabla'\hat{\Pi}'_h[U'(u_h-u)(\varphi-\mathbb{P}'_h\varphi)])\,dt'dx'dtdx\\
&\quad\le\int_0^T\!\!\int_{\mathbb{R}^d}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T \varepsilon_1(u_h)\|\nabla'\hat{u}_h\|_{\mathbb{P}_h}\|\Gamma\|_{L^\infty(T)}\,dt'dx'dtdx,
\end{aligned}$$

where $\Gamma=\nabla'\hat{\Pi}'_h[U'(u_h-u)(\varphi-\mathbb{P}'_h\varphi)]$. Since, on each $(d+1)$-simplex $T_\ell$,

$$\begin{aligned}
\Gamma&=\nabla'\big(\hat{\Pi}'_h[U'(u_h-u)(\varphi-\mathbb{P}'_h\varphi)]\big)\\
&=\sum_{i=1}^{d+2}U'(u_{h,\ell,i}-u)(\varphi_{\ell,i}-\mathbb{P}'_h\varphi)\nabla'\Psi_i^{T_\ell}(t',x'),
\end{aligned}$$

and since $|\nabla'\Psi_i^{T_\ell}|\le c_0/\rho_T$ on $T$ and $|U'|\le 1$, by (3.1), we have

$$|\Gamma|\le\frac{c_0}{\rho_T}\sum_{i=1}^{d+2}|\varphi_{\ell,i}-\mathbb{P}'_h\varphi|,$$

which yields by (2.1b)

$$\int_0^T\!\!\int_{\mathbb{R}^d}|\Gamma|\le c_0\frac{h_T}{\rho_T}\frac{1}{\varepsilon_x+\varepsilon_t}\le c_0\frac{\sigma}{\varepsilon_x+\varepsilon_t}.$$

Finally, using the definition of $\varepsilon_1$, we get

$$\begin{aligned}
E_5^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)&\le\frac{c_0\sigma}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T\varepsilon_1(u_h)\|\nabla'\hat{u}_h\|_{\mathbb{P}_h}\,dx'dt'\\
&=\frac{c_0\sigma}{\varepsilon_x+\varepsilon_t}\delta_1\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T|A'(u_h)|dx'dt'.
\end{aligned}$$

The result follows easily from the *a priori* estimate of Lemma 4.1. This completes the proof. □

**Lemma 5.6.** *Suppose that the conditions* (2.1b), (2.1c) *on the triangulations and* (2.4b) *on the coefficients* $C_e^{LF}$ *are satisfied. Then, we have*

$$E_6^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) \le C\frac{\delta_2}{(\varepsilon_x+\varepsilon_t)h^{1/2}},$$

*where the constant* $C$ *depends on* $k, d, \sigma, \bar{c}, c_\star$, *and* $\| u_0 \|_{L^2(\mathbb{R}^d)}$ *only.*

*Proof.* Proceeding as in the preceding lemma, we obtain

$$\begin{aligned} E_6^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) &\le \frac{C}{\varepsilon_x+\varepsilon_t}\delta_2 \sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_{\partial T}|f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}|d\lambda' \\ &\le \frac{C}{\varepsilon_x+\varepsilon_t}\delta_2 \sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |C_e^{LF}(u_h^T-u_h^{T_e}) - \frac{1}{2}(\tilde{f}(u_h^T)-\tilde{f}(u_h^{T_e}))\cdot n_{e,T}|d\lambda'. \end{aligned}$$

By (2.4b), we have

$$E_6^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) \le \frac{C}{\varepsilon_x+\varepsilon_t}\delta_2\, c_\star \sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |u_h^T-u_h^{T_e}|d\lambda'.$$

The result then follows from the Cauchy-Schwarz inequality, Corollary 4.2, and condition (2.1c). □

**Lemma 5.7.** *We have*

$$E_7^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) \le C\Big(\frac{h}{\varepsilon\delta_1} + \frac{h}{(\varepsilon_x+\varepsilon_t)\delta^{1/2}}\Big),$$

*where* $C$ *depends on* $k, d$, *and* $\| u_0 \|_{L^2(\mathbb{R}^d)}$ *only.*

*Proof.* A simple application of Lemma 4.4 gives us

$$\begin{aligned} E_7^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) \le& C\frac{h}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A'(u_h)|\| \nabla'\hat{u}_h \|_{\mathbb{P}_h} dx'dt' \\ &+ \frac{Ch}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A'(u_h)|dx'dt'. \end{aligned}$$

The proof then follows from the Cauchy-Schwarz inequality and the *a priori* estimates of Lemma 4.1. □

**Lemma 5.8.** *Suppose that the condition* (2.4b) *on the coefficients* $C_e^{LF}$ *and the condition* (2.1c) *on the triangulations are satisfied. Then we have*

$$E_8^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) \le C\Big(\frac{h}{\varepsilon\delta_2} + \frac{h^{1/2}}{\varepsilon_x}\Big),$$

*where $C$ depends on $k, d, c^\star, \bar{c}$ and $\| u_0 \|_{L^2(\mathbb{R}^d)}$ only.*

*Proof.* The proof is similar to the proof of the previous result. After a straightforward application of Lemma 4.4, we get

$$E_8^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) \leq C\frac{h}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}| \, \| \nabla_e \hat{u}_h^T \|_{\mathbb{P}_h} d\lambda'$$

$$+C\frac{h}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}|d\lambda'.$$

The first term is estimated using directly the definition of $\varepsilon_2$ and the *a priori* estimates of Lemma 4.1. The second term has already been studied in Lemma 5.6. □

**Lemma 5.9.** *Assume that the condition* (2.1b) *on the triangulations is satisfied. Then we have*

$$E_9^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) \leq C(\frac{\delta}{\varepsilon\delta_1} + \frac{\delta^{1/2}}{(\varepsilon_x+\varepsilon_t)}),$$

*where $C$ depends on $d, k, \sigma$, and $\| u_0 \|_{\mathbb{R}^d}$ only.*

*Proof.* By definition, we have

$$E_9^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) = -\delta\int_0^\tau\int_{\mathbb{R}^d}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A'(u_h)\,\tilde{A}'\Big(u_h,\Pi_h'[U'(u_h-u)\,\varphi]\Big)\,dx'dt'dtdx$$

$$\leq \delta\int_0^\tau\int_{\mathbb{R}^d}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |\,A'(u_h)\,|\,|\,\tilde{A}'\Big(u_h,\Pi_h'[U'(u_h-u)\,\varphi]\Big)|\,dx'dt'dtdx.$$

Taking into account the linearity of $\tilde{A}'(u_h,\cdot)$ and the definition of $\Pi_h'$, we get

$$\begin{aligned}\Gamma &= \tilde{A}'(u_h,\Pi_h'[U'(u_h-u)\,\varphi]) \\ &= \sum_{i=1}^{N_{d+1,k}} U'(u_{h,i}-u)\,\varphi_i\tilde{A}'(u_h,\Psi_i^T(t',x')) \\ &= \sum_{i=1}^{N_{d+1,k}} (U'(u_{h,i}-u)\,\varphi_i - U'(u_h-u)\,\varphi)\tilde{A}'(u_h,\Psi_i^T(t',x'))\end{aligned}$$

since $\sum_{i=1}^{N_{d+1,k}}\Psi_i^T(t',x')) \equiv 1$ on $T$. Moreover, since $|\,\tilde{A}'(u_h,\Psi_{i,n}^T)\,| \leq C/\rho_T$ on $T$, we obtain

$$\begin{aligned}|\,\Gamma\,| &\leq \frac{C}{\rho_T}\sum_{i=1}^{N_{d+1,k}} |\,U'(u_{h,i}-u)\,\varphi_i - U'(u_h-u)\,\varphi\,| \\ &\leq \frac{C}{\rho_T}\sum_{i=1}^{N_{d+1,k}} |\,U'(u_{h,i}-u) - U'(u_h-u)\,|\,\varphi + \frac{C}{\rho_T}\sum_{i=1}^{N_{d+1,k}} |\,\varphi_i - \varphi\,|,\end{aligned}$$

where we have used the fact that $|U'| \leq 1$ by (3.1). Thus,

$$\int_0^\tau \int_{\mathbb{R}^d} |\Gamma| \leq C\frac{h_T}{\rho_T}\left(\frac{1}{\varepsilon}\,\|\nabla'\hat{u}_h\|_{\mathbb{P}_h} + \frac{1}{\varepsilon_x+\varepsilon_t}\right) \leq C\sigma\left(\frac{1}{\varepsilon}\,\|\nabla'\hat{u}_h\|_{\mathbb{P}_h} + \frac{1}{\varepsilon_x+\varepsilon_t}\right),$$

by condition (2.1b). Now, we can write

$$\begin{aligned} E_9^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) &\leq C\frac{\delta}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|\,\|\nabla'\hat{u}_h\|_{\mathbb{P}_h}\,dx'dt' \\ &+ C\frac{\delta}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|\,dx'dt'. \end{aligned}$$

The result follows by using the *a priori* estimates of Lemma 4.1. □

**6. Proof of Theorem 2.1.** We are now ready to prove our main result. Inserting into Lemma 3.1 all the estimates of the lemmas obtained in §5, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} U(u_h(\tau,x)-u(\tau,x))dx &\leq \sqrt{e}\int_{\mathbb{R}^d} U(u_{0h}(x)-u_0(x))dx \\ &+ C\left(\varepsilon + \frac{1}{\varepsilon}\left(\frac{h}{\delta_1}+\frac{h}{\delta_2}+\frac{\delta}{\delta_1}\right)\right) \\ &+ C\left(\varepsilon_x+\varepsilon_t+\frac{1}{\varepsilon_x+\varepsilon_t}\left(\frac{\delta_1}{\delta^{1/2}}+\frac{\delta_2}{h^{1/2}}+\frac{h}{\delta^{1/2}}+h^{1/2}+\delta^{1/2}\right)\right). \end{aligned}$$

From this inequality, it is clear that we must take $\delta = h$ and $\delta_1 = \delta_2 = \delta^\star$.

Now, consider the function $B(w) = |w| - U(w)$. By definition (3.1) of $G$, $B(w) \geq 0$ and

$$B(w) = \varepsilon\,(|w/\varepsilon| - G(w/\varepsilon)) \leq \varepsilon\,\sup_{v\in\mathbb{R}}\big|\,|v| - G(v)\big| \leq \varepsilon\sup_{|v|\leq 1}\big|\,|v|-G(v)\big| = c_0\,\varepsilon.$$

This implies that $|w| - c_0\,\varepsilon \leq U(w) \leq |w|$, and so

$$\begin{aligned} \|u_h(\tau,\cdot)-u(\tau,\cdot)\|_{L^1(\mathbb{R}^d)} &\leq \sqrt{e}\|u_{0h}-u_0\|_{L^1(\mathbb{R}^d)} + C\left(\varepsilon + \frac{1}{\varepsilon}\frac{h}{\delta^\star}\right) \\ &+ C\left(\varepsilon_x+\varepsilon_t+\frac{1}{\varepsilon_x+\varepsilon_t}\left(\frac{\delta^\star}{h^{1/2}}+h^{1/2}\right)\right). \end{aligned}$$

Minimizing over $\varepsilon$ and over $\varepsilon_x+\varepsilon_t$, we obtain

$$\|u_h(\tau,\cdot)-u(\tau,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \sqrt{e}\|u_{0h}-u_0\|_{L^1(\mathbb{R}^d)} + C\left(\frac{h^{1/2}}{(\delta^\star)^{1/2}}+\frac{(\delta^\star)^{1/2}}{h^{1/4}}+h^{1/4}\right).$$

The result of Theorem 2.1 is obtained by minimizing over $\delta^\star$.

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