Tournament Matrices with Extremal Spectral Properties[1]

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ABSTRACT

For a tournament matrix M of order n, we define its walk space, W_M, to be

$$\text{Span}\{M^j\mathbf{1} : j = 0, \ldots, n-1\}$$

where $\mathbf{1}$ is the all ones vector. We show that the dimension of W_M equals the number of eigenvalues of M whose real parts are greater than $-1/2$. We then focus on tournament matrices whose walk space has particularly simple structure, and characterize them in terms of their spectra. Specifically, we characterize those tournament matrices such that $M^j\mathbf{1}$ is an eigenvector of M for some $j \geq 0$. We also characterize the tournament matrices M such that $J_n - 2M$ is a skew–Hadamard matrix. Throughout, we illustrate our results with examples.

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1. INTRODUCTION

A tournament of order n is a (loopless) digraph T with vertices 1, 2, ..., n such that exactly one of (i, j) and (j, i) is an arc of T $(1 \leq i < j \leq n)$. A tournament matrix of order n is a $(0, 1)$–matrix $M = [m_{ij}]$ of order n such that

$$M + M^T = J_n - I_n \tag{1}$$

where J_n is the all ones matrix of order n and I_n is the identity matrix of order n. Thus, a tournament matrix M is simply the adjacency matrix of a tournament T. There is an extensive literature on tournaments (see the

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bibliographies in [BR78] and [M68]) and a growing literature on eigenvalues of tournament matrices (see for example, [DGKMP92], [F92], [GKS], [KS90], [K91], [MP90], [S92]). A square matrix A is *reducible* provided there exists a permutation matrix P such that PAP^T has the form

$$\begin{bmatrix} A_1 & O \\ * & A_2 \end{bmatrix}$$

where A_1 and A_2 are square (nonvacuous) matrices. The matrix A is *irreducible* provided it is not reducible. Equivalently, A is irreducible if and only if the digraph associated with A is strongly connected. Since many properties of reducible matrices can be studied in terms of irreducible matrices, we will focus our attention on irreducible tournament matrices.

We denote the complex vector space of n by 1 column vectors by $\mathcal{C}^n$ and the all ones vector by $\mathbf{1}$. Let M be a tournament matrix of order n with corresponding tournament T. The vector $s = M\mathbf{1}$ is called the *score vector* of M. Clearly $\mathbf{1}^T s = \binom{n}{2}$, and the ith entry of s is the outdegree of vertex i in the tournament T. More generally, for any integer $k \geq 2$ the ith entry of $M^k\mathbf{1}$ equals the number of walks in T of length k which start at vertex i. We will call the subspace of $\mathcal{C}^n$ spanned by the vectors

$$\{M^j\mathbf{1} : j = 0, 1, \ldots, n-1\}$$

the *walk space of* M, and denote it by W_M. The *walk polynomial* of M is the unique monic polynomial $p(\lambda)$ of smallest degree such that $p(M)\mathbf{1} = 0$. Since W_M is invariant under multiplication by M, it easy to see that $p(\lambda)$ is the minimum polynomial of the linear transformation obtained by restricting M to W_M.

In Section 2 we investigate basic properties of the walk space and the walk polynomial of tournament matrices. It is known [BG68] that if α is an eigenvalue of a tournament matrix M of order n, then

$$-\frac{1}{2} \leq \mathrm{Re}(\alpha) \leq \frac{n-1}{2}.$$

Our first theorem asserts that the dimension of the walk space of M and the degree of the walk polynomial of M both equal the number of eigenvalues of M with real part not equal to $-1/2$. In addition, we show that the orthogonal complement of W_M in $\mathcal{C}^n$ is the space spanned by the eigenvectors of M corresponding to the eigenvalues whose real parts equal $-1/2$.

In Section 3 we consider matrices whose walk space has special properties. The tournament T is *regular of degree* t, provided the outdegree of each of its vertices is t. The tournament matrix M is a *regular* tournament matrix provided T is regular. If T is a regular tournament of degree t then $n = 2t+1$ and the score vector of M equals $t\mathbf{1}$. Clearly, W_M has dimension 1 if and only if M is a regular tournament matrix. A consequence of the results in Section 2 is that the walk space of M has dimension 2 if and only if there exist constants c and d such that $M^2\mathbf{1} = cs + d\mathbf{1}$, that is, if and only if the number of walks of length two starting at any given vertex can be determined by the outdegree of that vertex. Another consequence is that if W_M has dimension 2, then M has exactly two eigenvalues whose real parts exceed $-1/2$. Tournament matrices with exactly two eigenvalues with real part larger than $-1/2$ arise as the class of tournament matrices for which equality holds in a certain bound on the spectral radius [K91]. We use properties of the walk space to answer a question raised in [K91]. A special type of tournament matrix whose walk space has an interesting structure is one for which there exists an integer k such that the number of walks of length k starting at any given vertex is a constant multiple of the number of walks of length $k-1$ starting at that vertex. Theorem 3 shows that such tournament matrices can be completely characterized in terms of their spectra.

The tournament T is *doubly regular* of degree t provided any two vertices of T jointly dominate precisely t vertices. It is easy to see that if T is doubly regular, then T is regular with degree $2t+1$. Thus, T is doubly regular of degree t if and only if M satisfies $MM^T = tJ_n + (t+1)I_n$ where $n = 4t+3$. In [RB72] the existence of a doubly regular tournament with degree t is shown to be equivalent to the existence of a skew–Hadamard matrix of order $4t+4$ (that is, a $(1,-1)$–matrix Q of order $4t+4$ such that $Q - I_n$ is a skew–symmetric matrix and $Q^TQ = (4t+4)I_{4t+4}$). In Theorem 4 we show that if M is an irreducible singular tournament matrix of order $n \geq 4$ with 0 as a simple eigenvalue, then M has at least 4 distinct eigenvalues with equality if and only if $J_n - 2M$ is a skew–Hadamard matrix.

2. THE WALK SPACE AND THE WALK POLYNOMIAL

We begin by proving a few basic properties of eigenvalues of tournament matrices. The assertions (i), (ii), (iii) and (iv) in Lemma 1 below have been proven in [BG68], [BG68], [MP90] and [DGKMP92], respectively. However, because of their simplicity, we give complete proofs of these assertions.

LEMMA 1. *Let M be a tournament matrix of order n and let z be an eigenvector of M with corresponding eigenvalue α. Then*

(i) $\mathrm{Re}(\alpha) \geq -1/2$ *with equality if and only if* $z^*\mathbf{1} = 0$.

(ii) $\text{Re}(\alpha) \leq (n-1)/2$ *with equality only if* M *is a regular tournament matrix.*
(iii) *If* $\text{Re}(\alpha) \neq -1/2$, *then* α *has geometric multiplicity* 1.
(iv) *If* $\text{Re}(\alpha) = -1/2$, *then the geometric and algebraic multiplicity of* α *coincide.*
(v) *If* $\text{Re}(\alpha) = -1/2$, *then* $z^*v = 0$ *for every vector* $v \in W_M$.

Proof. By pre– and post–multiplying (1) by z^* and z, respectively, we see that

$$(2\text{Re}(\alpha)+1)z^*z = |z^*\mathbf{1}|^2. \tag{2}$$

Assertion (i) is now an immediate consequence of (2), and assertion (ii) follows from (2) and the Cauchy–Schwartz inequality.

Consider eigenvectors x and y corresponding to the eigenvalue α. Then applying (2) to the vector $z = (x^*\mathbf{1})y - (y^*\mathbf{1})x$, we see that either $z = 0$ or $\text{Re}(\alpha) = -1/2$. Thus if $\text{Re}(\alpha) \neq -1/2$, then x and y are multiples of each other, and hence it follows that (iii) holds.

To prove (iv) assume that $\text{Re}(\alpha) = -1/2$, and suppose to the contrary that the geometric multiplicity of α is less than its algebraic multiplicity. Then there exist nonzero vectors v and w such that

$$Mw = \alpha w, \quad Mv = \alpha v + w \quad \text{and} \quad w^*v = 0.$$

Pre– and post–multiplying (1) by w^* and v, respectively, and then simplifying we obtain

$$\alpha(w^*v) + w^*w + \bar{\alpha}w^*v = (w^*\mathbf{1})(\mathbf{1}^Tv) - w^*v.$$

Since $w^*v = 0$ and since (i) implies that $w^*\mathbf{1} = 0$, we have $w^*w = 0$, a contradiction. Therefore, (iv) holds.

To prove (v) we assume that $\text{Re}(\alpha) = -1/2$. We show by induction on k that $z^*M^k\mathbf{1} = 0$ for $k = 0, \ldots, n-1$. We have already seen that $z^*\mathbf{1} = 0$. Suppose $k \geq 1$ and that $z^*M^{k-1}\mathbf{1} = 0$. Then

$$\begin{aligned} z^*M^k\mathbf{1} &= z^*(J_n - I_n - M^T)M^{k-1}\mathbf{1} \\ &= z^*\mathbf{1}\mathbf{1}^TM^{k-1}\mathbf{1} - z^*M^{k-1}\mathbf{1} - \bar{\alpha}z^*M^{k-1}\mathbf{1}. \end{aligned}$$

Hence $z^*M^k\mathbf{1} = 0$, and it follows that (v) holds. ■

THEOREM 2. *Let* M *be a tournament matrix of order* n *and let* l *be the number of eigenvalues, counting multiplicity, of* M *with real part equal to* $-1/2$. *Then*

(i) $n = \dim W_M + l$,

(ii) *the orthogonal complement of* W_M *is the space spanned by the eigenvectors of* M *corresponding to eigenvalues with real part equal to* $-1/2$, *and*

(iii) *the walk polynomial of* M *is*

$$p(\lambda) = \prod(\lambda - \beta_j)^{k_j}$$

where the product is over all eigenvalues β_j *of* M *with* $\mathrm{Re}(\beta_j) \neq -1/2$ *and where* k_j *is the algebraic multiplicity of* β_j.

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_c$ be the distinct eigenvalues of M with real part equal to $-1/2$, and let $\beta_1, \beta_2, \ldots, \beta_d$ be the distinct eigenvalues of M with with real part not equal to $-1/2$. Also, let $k_1, k_2, \ldots, k_d$ be the algebraic multiplicity of the eigenvalues $\beta_1, \beta_2, \ldots, \beta_d$, respectively. It follows from (iii) and (iv) of Lemma 1 that the minimum polynomial of M on $\mathcal{C}^n$ is

$$m(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)\cdots(\lambda - \alpha_c)(\lambda - \beta_1)^{k_1}(\lambda - \beta_2)^{k_2}\cdots(\lambda - \beta_d)^{k_d}.$$

For each integer j with $1 \leq j \leq d$, let

$$r_j(\lambda) = \frac{m(\lambda)}{(\lambda - \beta_j)}.$$

Then $r_j(M) \neq O$, and the rows of $r(M)$ are either zero vectors or left eigenvectors of M corresponding to the eigenvalue β_j. By (i) of Lemma 1 (applied to M^T) we have $r(M)\mathbf{1} \neq O$, and hence $r_j(M)\mathbf{1}$ is a right eigenvector of M corresponding to the eigenvalue β_j. It now follows from (iii) of Lemma 1 that

$$\{(M - \beta_j I_n)^m q_j(M)\mathbf{1} : 0 \leq m \leq k_j - 1\}$$

is a basis for the generalized eigenspace E_{β_j} of M corresponding to β_j, where

$$q_j(\lambda) = \frac{m(\lambda)}{(\lambda - \beta_j)^{k_j}}.$$

Therefore,

$$\oplus_{j=1}^{d} E_{\beta_j} \subseteq W_M.$$

Since $\mathcal{C}^n$ is the direct sum of the generalized eigenspaces of M, and since the algebraic and geometric multiplicities of the eigenvalues with real part equal to $-1/2$ coincide, the dimension of $\oplus_{j=1}^{d} E_{\beta_j}$ equals $n-l$. It follows from (iv) and (v) of Lemma 1, that $\dim W_M \le n-l$. Therefore

$$\oplus_{j=1}^{d} E_{\mu_j} = W_M,$$

and (i) and (ii) hold. Clearly, the minimum polynomial of M restricted to W_M is

$$(\lambda-\beta_1)^{k_1}(\lambda-\beta_2)^{k_2}\cdots(\lambda-\beta_d)^{k_d},$$

and hence (iii) holds. ■

Since the characteristic equation of any tournament matrix has integer coefficients, it follows that $-1/2$ is not an eigenvalue of a tournament matrix. Further a tournament matrix has real entries, so its nonreal eigenvalues occur in complex conjugate pairs. As a result, the number of eigenvalues of a tournament matrix M having real part equal to $-1/2$ is even. In particular, Theorem 2 implies that $\dim W_M$ and n have the same parity.

We now describe an iterative method for finding the walk polynomial of a tournament matrix whose validity follows from Theorem 2.5 and the discussion in Section 3 of [F92]. Let M be a tournament matrix of order n with score vector s, and suppose that the dimension of W_M is k. Let A be the skew–symmetric matrix $(\frac{1}{2})(M-M^T) = M - \frac{1}{2}J_n + \frac{1}{2}I_n$. It is not difficult to see that W_M is spanned by the vectors $\mathbf{1}$, $A\mathbf{1}$, $A^2\mathbf{1}$, …, $A^{k-1}\mathbf{1}$. We construct an orthonormal basis e^1, e^2, …, e^k of W_M by defining $\{e^1, e^2, \ldots, e^k\}$ to be the orthonormal set of vectors obtained by applying the Gram–Schmidt process to the vectors $\mathbf{1}$, $A\mathbf{1}$, $A^2\mathbf{1}$, …, $A^{k-1}\mathbf{1}$. Using the skew–symmetry of A, it is easy to verify that

(i) $e^1 = \frac{1}{\sqrt{n}}\mathbf{1}$,

(ii) if $k \ge 2$ then

$$e^2 = \frac{Ae^1}{||Ae^1||} = \frac{s - (\frac{n-1}{2})\mathbf{1}}{\sqrt{s^T s - \frac{n(n-1)^2}{4}}},$$

and

(iii) if $k \ge 3$ then

$$e^j = \frac{Ae^{j-1} - ((e^{j-2})^T Ae^j)e^{j-2}}{||Ae^{j-1} - ((e^{j-2})^T Ae^j)e^{j-2}||} \qquad (j = 3, \ldots, k). \tag{3}$$

Let $\alpha_1 = ||Ae^1||$ and $\alpha_j = ||Ae^j + \alpha_{j-1}e^{j-1}||$ for $j = 2, \ldots, k-1$. Simple calculations show that $\alpha_1 = \sqrt{\frac{4s^T s - n(n-1)^2}{n}}$, and that $\alpha_j = -(e^{j-1})^T Ae^{j+1}$ for $j = 2, \ldots, k-1$. Moreover, with respect to the basis $e^1, \ldots, e^k$, the linear transformation corresponding to M restricted to W_M is given by tridiagonal matrix

$$\widehat{M} = \begin{bmatrix} \frac{n-1}{2} & \alpha_1 & 0 & 0 & \cdots & 0 \\ -\alpha_1 & -\frac{1}{2} & \alpha_2 & 0 & \cdots & 0 \\ 0 & -\alpha_2 & -\frac{1}{2} & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\alpha_{k-2} & -\frac{1}{2} & \alpha_{k-1} \\ 0 & 0 & \cdots & 0 & -\alpha_{k-1} & -\frac{1}{2} \end{bmatrix}.$$

The characteristic polynomial of $\widehat{M} + \frac{1}{2}I_n$ can be computed recursively by setting $q_0(\lambda) = 1$, $q_1(\lambda) = \lambda - \frac{n}{2}$, and $q_{j+1} = \lambda q_j \lambda + \alpha_j^2 q_{j-1}(\lambda)$ for $j = 2, \ldots k$, and the walk polynomial of M equals $q_k(\lambda + 1/2)$.

In applying the above method, it was not necessary to assume that the the dimension of W_M was known beforehand. Using the fact that the vectors $e^1, \ldots, e^k$ constructed above are an orthonormal basis of W_M, it is not difficult to show that

$$0 = Ae^k - (e^{k-1}Ae^k)e^{k-1},$$

and hence $k+1$ is the first positive integer j for which the numerator of the quantity on the right hand side of (3) is the zero vector.

To illustrate this method for calculating the walk polynomial, consider

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Then $s = (1, 2, 2, 2, 3)^T$. According to the above algorithm, $e^1 = 1/\sqrt{5}(1, 1, 1, 1, 1)^T$ and $e^2 = 1/\sqrt{2}(-1, 0, 0, 0, 1)^T$. Since

$$A = \frac{1}{2}\begin{bmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

we have $Ae^2 = \frac{1}{2\sqrt{2}}(1, -2, -2, -2, 1)^T$, and hence $(e^1)^T Ae^2 = -\frac{2}{\sqrt{10}}$. It follows that $e^3 = \frac{1}{\sqrt{30}}(3, -2, -2, -2, 3)^T$. Further, $Ae^3 = \frac{1}{2\sqrt{30}}(9, 0, 0, 0, -9)^T$, from which we find that

$$Ae^3 - (e^2Ae^3)e^2 = 0,$$

and hence W_M has dimension 3. Now $\alpha_1 = \frac{\sqrt{2}}{\sqrt{5}}$, and $\alpha_2 = (e^3)^T A e^2 = \frac{9}{\sqrt{60}}$. Also, we see that $q_0(\lambda) = 1$, $q_1(\lambda) = \lambda - 5/2$ and

$$q_3 = \lambda(\lambda^2 - \frac{5}{2}\lambda + \frac{2}{5}) + \frac{27}{20}(\lambda - \frac{5}{2}) = \lambda^3 - \frac{5}{2}\lambda^2 + \frac{7}{4}\lambda - \frac{27}{8}.$$

Finally, a few simplifications yield that the walk polynomial of M is

$$q_3(\lambda + 1/2) = \lambda^3 - \lambda^2 - 3.$$

Note that the eigenvalues of M are (approximately) 1.8637, $-.4319 \pm 1.1.930i$, and $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, and that as expected the first three of these are the roots of $\lambda^3 - \lambda^2 - 3$.

Theorem 2 helps to answer a question posed in [K91]. In that paper, it is shown that if $s^T s < (n^3 + 4n(n-1)^2)/16$, then A has a real positive eigenvalue ρ such that

$$\rho \geq \frac{n - 2 + \sqrt{n^2 + 4(n-1)^2 - 16 s^T s/n}}{4} \tag{4}$$

and each of its remaining eigenvalues α satisfy

$$\mathrm{Re}(\alpha) \leq \frac{n - 2 - \sqrt{n^2 + 4(n-1)^2 - 16 s^T s/n}}{4}. \tag{5}$$

In addition, M has an eigenvalue ρ and another eigenvalue α for which equality holds in (4) and (5) if and only if M has at least $n-2$ eigenvalues with real part equal to $-1/2$, that is, if and only if the $\dim W_M \leq 2$. The question asks whether or not there exist tournament matrices for which equality holds in exactly one of (4) and (5). The following argument shows that there do not exist such tournament matrices.

Assume that M is a tournament matrix of order n, and that β is an eigenvalue of M with corresponding eigenvector z, for which equality holds in either (4) or (5). Let s be the score vector of M, and assume that $s^T s > n(n-1)^2/4$ (otherwise, M is regular, and the result follows easily). Let

$$v = z - \left((\frac{\mathbf{1}^T z}{n})\mathbf{1} \right) - \left(\frac{s^T z - (\frac{n-1}{2})\mathbf{1}^T}{s^T s - \frac{n(n-1)^2}{4}} \right) \left(s - (\frac{n-1}{2})\mathbf{1} \right).$$

From (1), we find that

$$s^T z = (n - 1 - \beta)\mathbf{1}^T z. \tag{6}$$

Using (2) and (6), a number of algebraic manipulations yield

$$v^* v = \frac{z^* z(\frac{n-1}{2} - \mathrm{Re}\beta)}{s^T s - \frac{n(n-1)^2}{4}} \left[2(\mathrm{Re}(\beta))^2 - (n-2)\mathrm{Re}(\beta) - \frac{n(n-1)}{2} + \frac{2s^T s}{n} \right] - \frac{z^* z(2\mathrm{Re}(\beta)+1)(\mathrm{Im}(\beta))^2}{s^T s - \frac{n(n-1)^2}{4}}. \tag{7}$$

Since equality holds in either (4) or (5), the first term in (7) is zero, and it follows that β must be a real eigenvalue. Thus $v = 0$, and since z is a linear combination of $\mathbf{1}$ and s, the dimension of W_M equals 2. Hence, by Theorem 2, M has $n-2$ eigenvalues with real part equal to $-1/2$. Consequently, from Theorem 2 of [K91], M must have real eigenvalues ρ, and α (one of which is β) such that ρ yields equality in (3), and α yields equality in (4).

3. EXTREMAL TOURNAMENTS

In this section we study tournament matrices whose walk spaces have simple structure. We begin by discussing how tournament matrices with walk space of dimension 2 arise in the study of the relationship between the score vector and the Perron–vector of tournament matrices.

Let M be an irreducible tournament matrix of order n with spectral radius ρ and score vector s. It is known [S92] that if M is singular, then $s^T s \geq n^2(n-1)/4$, and $\rho \leq (n-2)/2$. In addition, if $s^T s = n^2(n-1)/4$, then M is singular if and only if $Ms = ((n-2)/2)s$. Thus, the singular tournament matrices whose score vector has smallest possible length each have a walk space of dimension 2, and for such tournament matrices the score vector is an eigenvector. Motivated by this relationship between singular tournament matrices and tournament matrices whose score vector is a Perron–vector, we consider tournament matrices M for which the Perron–vector is a vector of the form $M^k\mathbf{1}$. The following theorem shows that such tournament matrices are completely characterized by their spectra.

Theorem 3. *Let M be a tournament matrix of order $n \geq 2$, and suppose that $k \geq 1$. Then the following are equivalent:*

(i) *$M^k\mathbf{1}$ is an eigenvector of M, but $M^{k-1}\mathbf{1}$ is not.*
(ii) *There is a constant ρ such that $M^k\mathbf{1} - \rho M^{k-1}\mathbf{1}$ is a nonzero vector in the nullspace of M.*
(iii) *The eigenvalues of M are $(n-1-k)/2$, 0 with multiplicity k, and $n-k-1$ eigenvalues with real part equal to $-1/2$.*

(iv) *The walk polynomial of M is $p(\lambda) = \lambda^k(\lambda - (n-k-1)/2)$.*

Proof. First assume (i). Then $M^{k+1}\mathbf{1} = \rho M^k\mathbf{1}$ for some number ρ. This implies that $M(M^k\mathbf{1} - \rho M^{k-1}\mathbf{1}) = 0$. Since $M^{k-1}\mathbf{1}$ is not an eigenvector of M, $M^k\mathbf{1} - \rho M^{k-1}\mathbf{1}$ is nonzero, and thus (i) implies (ii).

Now assume (ii). Then the walk polynomial $p(\lambda)$ of M divides $\lambda^{k+1} - \rho\lambda^k = \lambda^k(\lambda - \rho)$. Hence by Theorem 2, β is an eigenvalue of M with $\mathrm{Re}(\beta) \neq -1/2$ only if $\beta = 0$ or $\beta = \rho$. Since $M^k\mathbf{1} - \rho M^{k-1}\mathbf{1} \neq 0$, we conclude that 0 is an eigenvalue of M with algebraic multiplicity k, and since $M^k\mathbf{1} \neq 0$ we conclude that ρ is an eigenvalue of M with algebraic multiplicity 1. Because M is a real matrix, the assumptions imply that ρ is real. That (iii) holds is now a consequence of the facts that the real part of each eigenvalue of M is at least $-1/2$ and that the trace of M, which is 0, is the sum of the eigenvalues of M.

That (iii) implies (iv) is an immediate consequence of Theorem 1.

Finally, assume (iv) holds. Then $M(M^k - \frac{n-k-1}{2}M^{k-1})\mathbf{1} = 0$, and $M^k\mathbf{1}$ is an eigenvector of M. Suppose that $M^{k-1}\mathbf{1}$ is an eigenvector of M. Then by what we have already shown M would have an eigenvalue of the form $(n-1-l)/2$ for some $l < k$. Since such an eigenvalue is not a root of the walk polynomial, we have obtained a contradiction. Hence (iv) implies (i). ∎

It is not difficult to construct examples of tournament matrices of the type characterized in Theorem 3. Assume that n and k are positive integers of opposite parity, and let R be a regular tournament matrix of order $n-k$. Let S be the tournament matrix of order k having 0's on and above the main diagonal, and 1's below the main diagonal, and let $J_{n-k,k}$ be the $n-k$ by k matrix of all 1's. Then

$$M = \left[\begin{array}{c|c} S & O \\ \hline J_{n-k,k} & R \end{array}\right]$$

is a tournament matrix of order n such that $(n-k-1)/2$ is a simple eigenvalue, 0 is an eigenvalue of multiplicity k and the remaining $n-k-1$ eigenvalues each have real part equal to $-1/2$. Thus M is of the type described in Theorem 3. However, M is a reducible matrix, and it would be more interesting to construct examples which were irreducible. While we are unable to construct such examples for general values of k, we now discuss a class of irreducible examples for the case $k = 1$.

Let p and q be nonnegative integers with $q \neq 0$, such that $p+1$ divides $q(q+1)$. Let $a = 2p+1$ and $b = 2q+1$, and let $m = \frac{ab^2+1}{a+1}$. Then m is an odd integer. Let R_1 be a regular tournament matrix of order m, R_2 a regular tournament matrix of order am, and let $B_1, B_2, \ldots, B_a$, be (0,1)–matrices of order m each of whose row and column sums equal $(m-b)/2$.

(Note that $m > b$ since $q \neq 0$, and that $m - b$ is even since both m and b are odd). The matrix

$$M = \left[\begin{array}{c|ccc} \multicolumn{1}{c|}{R_1} & B_1 & \ldots & B_a \\ \hline J_m - B_1^T & & & \\ \vdots & & R_2 & \\ J_m - B_a^T & & & \end{array}\right]$$

is a tournament matrix of order $(a+1)m$, and since both R_1 and R_2 are irreducible and since B_1 is neither O nor J_m, it follows that M is irreducible. It is easy to see that the first m rows of M each contain $\frac{ab(b-1)}{2}$ ones, and each of the remaining rows contain exactly $\frac{b(ab+1)}{2}$ ones. Further, one verifies that $M^2\mathbf{1} = (\frac{ab^2-1}{2})M\mathbf{1}$. Thus $\mathbf{1}$ is not an eigenvector of M, but $M\mathbf{1}$ is. It follows from Theorem 3 that $\frac{ab^2-1}{2}$ and 0 are eigenvalues of M of algebraic multiplicitly 1, and that the remaining $(a+1)m - 2$ eigenvalues each have real part equal to $-1/2$. We note that the construction for the special case $p = 0$ (and hence $a = 1$) appears in [MP90] and [S92].

We conclude this section by considering tournament matrices with few distinct eigenvalues. As mentioned in the introduction, it is shown in [DGKMP] that every tournament matrix of order $n \geq 3$ has at least 3 distinct eigenvalues, and there is a correspondence between skew–Hadamard matrices of order n and tournament matrices of order $n+1$ with exactly 3 distinct eigenvalues. The following theorem shows that there is also a correspondence between skew–Hadamard matrices of order n and tournament matrices of order n which have 0 as a simple eigenvalue and have exactly 3 other distinct eigenvalues.

Theorem 4. *Let M be an irreducible tournament matrix of order $n \geq 2$ such that 0 is an eigenvalue with algebraic multiplicity 1. Then M has at least 4 distinct eigenvalues, and M has exactly 4 distinct eigenvalues if and only if $J_n - 2M$ is a skew–Hadamard matrix.*

Proof. Because M is a tournament matrix each of the main diagonal entries of M^2 is 0. Since the trace of M^2 is the sum of the squares of the eigenvalues of M and since the spectral radius $\rho > 0$ of M is an eigenvalue of M, it follows that M has at least one nonreal eigenvalue μ. Because M is a real matrix, $\bar{\mu}$ is also an eigenvalue of M. Hence M has at least four distinct eigenvalues.

Now assume that M has exactly 4 distinct eigenvalues, say, 0, $\mu = a+bi$, $\bar{\mu}$, and ρ. Since μ is not real, we may assume that $b > 0$. Then μ and $\bar{\mu}$

both have algebraic multiplicity $(n-2)/2$. In particular, n is even. The trace of M equals zero and is the sum of the eigenvalues of M, and thus

$$0 = \rho + (n-2)a. \tag{8}$$

Similarly, since the trace of M^2 equals zero, we have

$$0 = \rho^2 + (n-2)(a^2 - b^2).$$

These equations imply that

$$a = -\frac{\rho}{n-2} \quad \text{and} \quad b = \frac{\rho\sqrt{n-1}}{n-2}. \tag{9}$$

Since M is a $(0,1)$–matrix, the minimum polynomial $m(\lambda)$ of M has integer coefficients and has the form

$$m(\lambda) = \lambda(\lambda-\rho)(\lambda-\mu)^k(\lambda-\bar{\mu})^k$$

for some integer k. Since an irreducible polynomial over the rational numbers has distinct roots, the minimum polynomial of ρ over the rationals is either $(\lambda-\rho)$ or $(\lambda-\rho)(\lambda-\mu)(\lambda-\bar{\mu})$. It follows that ρ is rational (in the latter case we see that the coefficient of λ^2 is $\rho - 2a = \rho(n-4)/n-2)$, and hence that ρ is rational). Since M is a $(0,1)$–matrix, its eigenvalues are algebraic integers, and we conclude that ρ is an integer. Since μ is an algebraic integer, so is $2a = \mu + \overline{\mu}$, and it follows from (9) that $(n-2)/2$ divides ρ. Since $a \geq -1/2$, (8) now implies that $\rho = (n-2)/2$, $a = -1/2$, and $b = \sqrt{n-1}/2$.

Let $B = J_n - 2M$. Then clearly, B is a $(1,-1)$–matrix, and $B - I_n$ is a skew–symmetric matrix. By Theorem 2, $\mathcal{C}^n$ is a direct sum of the M–invariant subspaces W_M and W_1 where W_1 is the space spanned by the eigenvectors of M corresponding to μ and $\bar{\mu}$. It follows from (i) of Lemma 1 that W_1 is invariant under multiplication by B, and that the eigenvalues of B on W_1 are -2μ and $-2\bar{\mu}$. It follows from (1) and (i) of Lemma 1 that if x is an eigenvector of M corresponding to μ, respectively $\bar{\mu}$, then x is an eigvenvector of M^T corresponding to the eigenvalue $\bar{\mu}$, respectively μ. This implies that there is a basis of W_1 which consists of common eigenvectors of B^T and B. We conclude that the action of B^TB of W_1 is multiplication by $4||\mu||^2 = n$. Clearly, W_M is invariant under multiplication by B, and under multiplication by $B^T = 2M + I_n - J_n$. It follows from (i) of Theorem 2 that the walk space of M has dimension 2 and is spanned by

$\mathbf{1}$ and s. Further, by Theorem 3, we have $Ms = \frac{n-2}{2}s$. Consequently, on the invariant subspace W_M, B acts as follows

$$(B)\mathbf{1} = n\mathbf{1} - 2s \quad \text{and} \quad (B)s = \frac{n(n-1)}{2}\mathbf{1} - (n-2)s.$$

Since $B^T = 2M - J_n + 2I_n$, we have

$$B^T\mathbf{1} = -(n-2)\mathbf{1} + 2s \quad \text{and} \quad B^Ts = -\frac{n(n-1)}{2}\mathbf{1} + (n)s.$$

A simple computation now shows that the action of B^TB on W_M is multiplication by n. Therefore, $B^TB = nI_n$, and $J_n - 2M$ is a skew–Hadamard matrix.

Now suppose that $B = J_n - 2M$ is a skew–Hadamard matrix. Then

$$\begin{aligned} nI_n &= BB^T \\ &= nJ_n - 2MJ_n - 2J_nM^T + 4MM^T. \end{aligned} \tag{10}$$

Since M is a tournament matrix $MM^T = MJ_n - M - M^2$. Substituting this identity into (10) and simplifying we obtain

$$\begin{aligned} (4M^2 + 4M + nI) &= nJ_n + 2MJ_n - 2J_nM^T \\ &= n(\mathbf{1}\mathbf{1}^T) + 2(s\mathbf{1}^T) - 2(\mathbf{1}s^T). \end{aligned} \tag{11}$$

Post–multiplying both sides of (11) by $\mathbf{1}$ we have

$$4Ms + 4s + n\mathbf{1} = (n^2)\mathbf{1} + (2n)s - (n(n-1))\mathbf{1},$$

and hence $Ms = ((n-2)/2)s$. In particular, this implies that M is singular. Post–multiplying (11) by M and $Ms = ((n-2)/2)s$, we have

$$M(4M^2 + 4M + nI_n) = 2(n-1)s\mathbf{1}^T - 2ss^T.$$

Similarly,

$$M^2(4M^2 + 4M + nI_n) = \frac{n-2}{2}(2(n-1)s\mathbf{1}^T - 2ss^T).$$

It follows that

$$M(M-(\frac{n-2}{2})I_n)(4M^2+4M+nI_n)=O,$$

and hence that M has at most 4 distinct eigenvalues. Since a singular tournament matrix of order at $n \geq 4$ has at least 4 distinct eigenvalues, M has exactly 4 distinct eigenvalues. ■

We can construct examples of tournament matrices of the type described in Theorem 4 as follows. Suppose H is a Hadamard tournament matrix of order $n-1$ and $n \geq 8$, and let

$$M = \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 1 & & & \\ \vdots & & H & \\ 1 & & & \end{array}\right].$$

It can be verified directly that $J_n - 2M$ is a skew–Hadamard matrix, so that

$$(M^T - M + I_n)(M - M^T + I_n) = nI_n.$$

Note that M is a reducible tournament matrix.

To construct an irreducible tournament matrix with the desired properties, fix $2 \leq i \leq n$, and let M_i be the matrix obtained from M by "switching" its ith row and column, that is, M_i is the same as M except the ith row of M_i equals the ith column of M, and the ith column of M_i equals the ith row of M. By considering score vectors it is easy to show that any principal submatrix of H of order $n-2$ is irreducible. It now follows that M_i is also irreducible. Further, $M_i^T - M_i = D(M^T - M)D$, where D is the diagonal matrix with a -1 in its ith diagonal entry and 1's in each of its remaining diagonal entries. Thus we have

$$\begin{aligned}(J_n - 2M_i)(J_n - 2M_i^T) &= (M_i^T - M_i + I_n)(M_i - M_I^T + I_n) \\ &= D(M^T - M + I_n)DD(M - M^T + I_n)D \\ &= nD^2 \\ &= nI_n.\end{aligned}$$

Hence M_i is an irreducible tournament matrix and such that $J - 2M_i$ is a skew–Hadamard matrix. Evidently, any further "switches" performed on M_i would also maintain the desired skew–Hadamard property, although they may not maintain irreducibility.

For example, let

$$H = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and note that H is a Hadamard tournament matrix of order 7. Then

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Now M_2 is irreducible, as is the matrix

$$\widehat{M_2} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

obtained from M_2 by switching its third row and column. Thus both $J_8 - 2\widehat{M_2}$ and $J_8 - 2M_2$ are skew–Hadamard matrices, but since their score vectors are different, $\widehat{M_2}$ and M_2 are not permutationally equivalent. However, a straightforward exercise reveals that $\widehat{M_2}$ is permutationally equivalent to M_2^T. Thus, while the tournaments $\widehat{T_2}$ and T_2 corresponding to $\widehat{M_2}$ and M_2, respectively, are not isomorphic, $\widehat{T_2}$ is isomorphic to the complement of T_2 in the complete graph K_8.

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