DETERMINING DEGREES OF FREEDOM FOR NONLINEAR DISSIPATIVE EQUATIONS

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DETERMINING DEGREES OF FREEDOM FOR NONLINEAR DISSIPATIVE EQUATIONS

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Abstract. A finite set of linear functionals $\{\ell_i\}_{i=1}^N$ is said to be a set of determining degrees of freedom for a given PDE if when any two solutions of the PDE, u_1 and u_2, are such that $\ell_i(u_1(t) - u_2(t))$ converges to 0 as time t goes to infinity, $1 \leq i \leq N$, then the solutions converge to each other as time goes to infinity. In this paper, we prove the existence of a large class of sets of determining degrees of freedom, with special attention to the standard sets of degrees of freedom used in the finite element method, for the Navier-Stokes equations and provide an estimate on the size of the sets. We then extend and sharpen this result to general nonlinear dissipative evolution equation possessing an inertial manifold.

DEGRES DE LIBERTE DETERMINANTS POUR EQUATIONS NONLINEAIRES DISSIPATIVES

Résumé. On dit qu'un ensemble fini de fonctionnelles linéaires $\{\ell_i\}_{i=1}^N$ est un ensemble déterminant de degrés de liberté pour une équation différentielle aux dérivées partielles si étant données deux solutions de l'équation, u_1 and u_2, telles que $\ell_i(u_1(t) - u_2(t))$ converge vers 0 quand le temps t tend vers l'infini, $1 \leq i \leq N$, alors la différence des solutions converge vers zéro quand le temps tend vers l'infini. Dans cette note, on prouve l'existence d'une très grande classe d'ensembles de degrés de liberté déterminants, qui contient les degrés de liberté classiques utilisés dans la méthode des éléments finis, pour les équations de Navier-Stokes; on donne aussi une estimation du nombre N. Ensuite, on prouve un résultat plus fort pour des équations dissipatives nonlinéaires possédant une variété inertielle.

Version française abrégée. Des arguments de la physique indiquent que pour une classe grande de systèmes de réaction-diffusion de la mécanique des fluides, le comportement asymptotique en temps des solutions peut être décrit par un ensemble fini de degrés de liberté. La première démonstration rigoureuse de ce fait a été donnée par Foias & Prodi (1967), qui ont prouvé que si les N premiers modes de Fourier de deux solutions des équations de Navier-Stokes coïncident quand le temps tend vers l'infini, pour N suffissamment grand, alors les deux solutions coïncident quand le temps tend vers l'infini. La première estimation du nombre N a été obtenue par Foias *et al.* (1983a); elle a été améliorée par Jones & Titi (1993). Foias & Temam (1983b) avaient suggéré que ce résultat devrait rester valable pour des degrés de liberté différents des modes de Fourier considérés par Foias & Prodi (1967). En effet, des résultats similaires ont été prouvés par Foias & Temam (1984), qui ont utilisé comme degrés de liberté les valeurs aux nodes (voir aussi Foias & Titi (1991)), et par Jones & Titi (1992), qui ont utilisé des moyennes sur des volumes finis. Des estimations du nombre de degrés de liberté N ont été obtenues pour les équations de Navier-Stokes en deux dimensions d'espace par Jones & Titi (1993).

Dans cette note, on prouve l'existence d'une très grande classe d'ensembles de degrés de liberté déterminants, qui contient les degrés de liberté classiques utilisés

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dans la méthode des éléments finis, pour les équations de Navier-Stokes dans des domaines bornés en deux dimensions d'espace. On considère le cas d'un carré avec des conditions périodiques pour simplifier notre présentation. Plus précisemment, on considère des degrés de liberté $\{\ell_i\}_{i=1}^N$ avec lesquels ont peut construire des approximations de l'identité, R^h, de la forme (2.3) vérifiant l'inégalité d'approximation (2.4). Des nombreux examples de ces opérateus peuvent être trouvés dans la litérature de la méthode des éléments finis oú l'ont peut voir que R^h est typiquement un opérateur d'interpolation ou une projection sur un espace de dimension fini; voir, par example, Ciarlet (1978)et Girault et Raviart (1979). On démontre alors (voir le Théorème 2.1) que l'ensemble de ces degrés de liberté $\{\ell_i\}_{i=1}^N$ est déterminant pour les équations de Navier-Stokes (2.1) si l'on a $h < h_0 = (\sqrt{2c_2}\,c_1\lambda_1 Gr)^{-1/2}$, oú λ_1 et le nombre de Grashof généralisé Gr sont définis dans (2.2), c_1 est la constante qui apparaît dans l'inégalité (2.4), et c_2 est la constante de l'inégalité de Agmon, voir le Théorème 2.1.

Ensuite, on considère des équations dissipatives nonlinéaires (3.1) possédant une variété inertielle qui peut être réprésentée par la graphe de la function $\Phi : P_mH \mapsto Q_m\mathcal{D}(A^\gamma)$, oú $\gamma \geq 0$, $Q_m = Id - P_m$, et P_m est la projection sur l'espace généré par les premiers m vecteurs propres de l'opérateur A. On considère les degrés de liberté avec lesquels on peut construire des approximations de l'identité R^h de la forme (2.3) vérifiant l'inégalité d'approximation (3.2). On démontre alors (voir le Théorème 3.1) que s'il existe t_0 tel que $\ell_i(u(t_0) - v(t_0)) = 0, \quad 1 \leq i \leq N$, pour des solutions u et v de l'équation (3.1) dans la variété inertielle, alors $u(t) = v(t), \quad \forall t \geq 0$, pourvu que $h < (c_4(1+l)\lambda_m^\gamma)^{-1/\beta}$, oú l est la constante de Lipschitz de Φ, λ_m est la m-ième valeur propre de A, et $\beta > 0$ et c_4 sont associées à l'inégalité d'approximation (3.2). De plus, l'ensemble de degrés de liberté $\{\ell_i\}_{i=1}^N$ est déterminant pour l'équation (3.1).

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1. Introduction. For a large class of systems of reaction-diffusion equations arising in fluid mechanics, physical arguments indicate that the long-time behavior of the solutions can be described with only a finite number of degrees of freedom. This was first rigorously proven by Foias & Prodi (1967), who showed that if the first N Fourier modes of any two solutions of the Navier-Stokes equations agree as time goes to infinity, for N sufficiently large, then the two solutions agree everywhere as time goes to infinity. The first estimate of the number N was obtained by Foias *et al.* (1983a); it was later improved by Jones & Titi (1993). It was suggested by Foias & Temam (1983b) that the result obtained by Foias & Prodi (1967) should also hold for degrees of freedom other than the Fourier modes. Indeed, Foias & Temam (1984) proved similar result by taking the values of the solutions at the nodes as degrees of freedom, Foias & Titi (1991), and later Jones & Titi (1992), used averages on finite volumes. Explicit estimates for the number N of these degrees of freedom in the case of the two-dimensional Navier-Stokes equations were obtained by Jones & Titi (1993). In this paper, we further extend the above results and show that the two-dimensional Navier-Stokes equations, in bounded domains, has a large class of determining sets of degrees of freedom which include degrees of freedom used in the finite element method. For simplicity, we will present our results for the Navier-Stokes equations with periodic boundary conditions. We then extend and sharpen this result to nonlinear dissipative evolution equation possessing inertial manifolds. In particular, following the work of Foias & Titi (1991), we show that if a sufficiently large number of degrees of freedom of any two solutions on the inertial manifold agree at one instant, then the solutions agree everywhere and for all non negative time.

The paper is organized as follows. In Section 2, we state and briefly discuss our results concerning determining degrees of freedom for the Navier-Stokes equation on a square with periodic boundary conditions, see Theorem 2.1. In Section 3, we display and discuss our results about determining degrees of freedom for nonlinear dissipative evolution equations, see Theorem 3.1.

2. Sets of determining degrees of freedom for the two-dimensional Navier-Stokes equations. In this section we consider the problem of finding sets of determining degrees of freedom for the two-dimensional Navier-Stokes equations for a viscous incompressible fluid on the square $\Omega = (0, L) \times (0, L)$ with periodic boundary conditions. It is well-known that such a system of equations can be rewritten as the following evolutionary equation for the velocity

$$\frac{du}{dt} + \nu A u + B(u, u) = \Phi, \tag{2.1}$$

where A is the so-called Stokes operator, defined by $Au = -P\Delta u, \quad \forall u \in \mathcal{D}(A) = V \cap (H^2(\Omega) \times H^2(\Omega))$, where P is the L^2-projection onto H, and $B(u, v) = P((u \cdot \nabla)v), \quad \forall u, v \in \mathcal{D}(A)$ (see, for instance, Lions (1969), Temam (1983) and Constantin & Foias (1988)). The spaces H and V are the closure in $(L^2(\Omega))^2$ and in $(H^1(\Omega))^2$, respectively, of the space of divergence-free functions which are zero-mean trigonometric polynomials with period L in each component.

We assume that $\Phi = P\Phi$ and that Φ belongs $L^\infty((0, \infty); H)$, that is, $\sup_{t \geq 0} |\Phi(t)| < \infty$, where $|\cdot|$ denotes the L^2-norm. This hypothesis allows us to introduce the so-called generalized Grashof number Gr, see Foias *et al.* (1983a), defined as follows:

$$Gr = \frac{F}{\lambda_1 \nu^2} = \frac{L^2 F}{4\pi^2 \nu^2}, \quad F = \limsup_{t \to \infty} |\Phi(t)|. \tag{2.2}$$

Note that if Φ is time independent, then Gr is the Grashof number $G = \frac{L^2 |\Phi|}{4\pi^2 \nu^2}$.

Next, we focus our attention on the type of degrees of freedom we want to deal with. Since our goal is to compare the asymptotic behavior as time t of two solutions of the Navier-Stokes equations (2.1) in terms of the long time behavior of their degrees of freedom, it is reasonable to consider degrees of freedom with which it is possible to reconstruct a good approximation of the original functions. More precisely, given an arbitrary set of degrees of freedom $\{\ell_i\}_{i=1}^N$, we consider 'reconstruction' operators R^h of the form

$$R^h(u) = \sum_{i=1}^N \ell_i(u)\, \phi_i, \tag{2.3}$$

where $\phi_i \in (L^2(\Omega))^2$, such that

$$|\, u - R^h(u)\,| \leq c_1 h^2 \,|\, Au\,|. \tag{2.4}$$

Note that although we do not require the functions ϕ_i to belong to the space H, we do require the operator R^h to be a 'good' approximation of the identity on $\mathcal{D}(A)$. In fact, the only property of the operator R^h that is used in our analysis is the approximation inequality (2.4). This means that we should not talk about determining degrees of freedom but of determining operators R^h; we kept the term degrees of freedom for historical reasons.

Typically, the operator R^h is an interpolation operator or a projection operator. Classical examples are constructed as follows. Let T^h be a triangulation, made of triangles, of the domain Ω and let $\{x_i\}_{i=1}^N$ be the set of all the vertices of the triangles

$T \in \mathrm{T}^h$. Let V^h be the set of continuous functions with values in $\mathbb{R}^2$ whose restrictions to each triangle $T \in \mathrm{T}^h$ is affine in each component. Then, take ϕ_i to be the element of V^h such that $\phi_i(x_j) = \delta_{ij}$. We can define R^h as an interpolation operator if we take $\ell_i(u) = u(x_i)$. (Note that the degrees of freedom ℓ_i are well defined for functions $u \in \mathcal{D}(A)$ since, by Sobolev's inequality, $\mathcal{D}(A) \subset (\mathrm{C}^0(\Omega))^2$.) We can also define R^h as the L^2-projection of u into V^h, that is, $R^h(u)$ is the only element in V^h such that $\int_\Omega R^h(u) \cdot v_h = \int_\Omega u \cdot v_h, \quad \forall v_h \in V^h$. Both of the operators R^h defined above satisfy (2.4) with h equal to the maximum of the diameters of the triangles $T \in \mathrm{T}^h$. In general, if $\mathrm{span}\{\phi_i, 1 \leq i \leq N\}$ includes V^h and if $R^h(v_h) = v_h$ for every $v_h \in V^h$, then the inequality (2.4) holds. See, for example, Ciarlet (1978) and Girault and Raviart (1979).

section. We use the following standard notation $\|v\|^2 = \sum_{i,j=1}^2 \int_\Omega \left|\frac{\partial v_i}{\partial x_j}\right|^2 dx$.

THEOREM 3.1. *Let u and v be the solutions of the Navier-Stokes equations* (2.1) *with $\Phi = f$, $u(0) = u_0$ and $\Phi = g$, $v(0) = v_0$, respectively. Suppose that f and g are such that*

$$\lim_{t\to\infty} |f(t) - g(t)| = 0$$

and that

$$\lim_{t\to\infty} \ell_i(u(t) - v(t)) = 0, \quad 1 \leq i \leq N.$$

Then the set of degrees of freedom $\{\ell_i\}_{i=1}^N$ is determining for the Navier-Stokes equations (2.1), *that is,*

$$\lim_{t\to\infty} \| u(t) - v(t) \| = 0,$$

provided

$$h < h_0 = (\sqrt{2\, c_2}\, c_1 \lambda_1 Gr)^{-1/2},$$

where the smallest eigenvalue of A, λ_1, and the generalized Grashof number Gr are defined in (2.2), *c_1 is the constant in the approximation inequality* (2.4), *and c_2 is the constant of Agmon's inequality, namely, $\|u\|_{L^\infty(\Omega)} \leq c_2 |u|^{\frac{1}{2}} |Au|^{\frac{1}{2}}, \quad \forall u \in \mathcal{D}(A)$.*

COROLLARY. *Suppose that $N = c_3 L^2 h^{-2}$. Then the set of degrees of freedom $\{\ell_i\}_{i=1}^N$ is determining for the equations* (2.1) *provided $N > 4\pi^2 \sqrt{2\, c_2}\, c_1\, c_3\, Gr$.*

We recall that the best known upper bound for the fractal dimension of the attractor given in Constantin, Foias, Temam (1988) is of the order $G^{2/3}(1 + \log(G))^{1/3}$. This estimate agrees up tc the logarithmic term with the number of degrees of freedom predicted by physical arguments. A rigorous lower bound for the Hausdorff dimension of the attractor was obtained by Babin & Vishik (1983) (see also Liu (1993) for this case). The estimate in our corollary is consistent with the bound derived for the upper bound for the number of determining nodes, determining finite volume elements, and determining modes which are all of the order Gr; see Jones and Titi (1993). Alternatively, as discussed in Constantin, Foias, Temam (1988), the Kraichnan dissipative micro scale is of the order $L\,G^{-1/3}$, whereas Theorem 2.1 shows that the polynomials need to be defined on a mesh with $h \leq CL/\sqrt{Gr}$ in order to be determining.

3. Sets of determining degrees of freedom for PDEs with Inertial Manifolds. In this section, we consider sets of determining degrees of freedom for nonlinear evolution equations a Hilbert space H (with inner product $(\cdot,\cdot)$ and norm $|\cdot|$) of the form

$$\frac{du}{dt} + Au + R(u) = f, \tag{3.1}$$

that possess inertial manifolds; the operator A is assumed to be a self-adjoint positive definite operator with compact inverse. Examples are the Kuramoto-Sivashinsky equation, the complex Ginzburg-Landau equation, the Cahn-Hilliard equation and certain reaction-diffusion equations; see, for example, Foias, Sell & Temam (1988), Temam (1988), Constantin *et al.* (1989) and the references therein.

We assume that (3.1) has an inertial manifold $\mathcal{M}$ representable as the graph of a Lipschitz function $\Phi : P_m H \mapsto Q_m \mathcal{D}(A^\gamma)$, for some fixed $\gamma \geq 0$ and for some $m > 0$, where $Q_m = I - P_m$ and P_m the projection onto the span of the first m eigenfunctions the operator A. Once more, we consider operators R^h of the form given by (2.3) where this time we require ϕ_i to belong to H. As in the preceding section, we restrict ourselves to those operators R^h which are a 'good' approximation of the identity, but this time we ask this property to hold on $\mathcal{D}(A^\gamma)$, not on $\mathcal{D}(A)$ as in the case of the Navier-Stokes equations (2.1). More precisely, we require the following approximation inequality to hold:

$$|u - R^h u| \leq c_4 h^\beta |A^\gamma u|, \quad \forall u \in \mathcal{D}(A^\gamma), \tag{3.2}$$

for some positive number β. We are now ready to state the main results of this section which strengthen the results of Theorem 2.1.

THEOREM 3.1. *Suppose that* (3.1) *has an inertial manifold* $\mathcal{M}$ *representable as the graph of a function* $\Phi : P_m H \mapsto Q_m \mathcal{D}(A^\gamma)$. *Suppose that the approximation inequality* (3.2) *is satisfied. Take* $u(0), v(0) \in Graph(\Phi)$, *and assume that for some* $t_0 \geq 0$ *we have*

$$\ell_i(u(t_0) - v(t_0)) = 0, \quad 1 \leq i \leq N.$$

Then

$$u(t) = v(t), \quad \forall t \geq 0,$$

provided

$$h < (c_4(1 + l)\lambda_m^\gamma)^{-1/\beta},$$

where l is the Lipschitz constant of Φ and λ_m is the m-th eigenvalue of A considering that the eigenvalues are ordered in a nondecreasing order, including their multiplicities.

Moreover, under the above condition on h, the set of degrees of freedom $\{\ell_i\}_{i=1}^N$ is determining for (3.1).

This result states that the intepolate or projection operators R^h are injective on the inertial manifold. In particular, these operators are injective on the global attractor, which is contained in the inertial manifold. Whether this property holds for dissipative PDEs that are not known to have an inertial manifold, such as the Navier-Stokes equations, remains an open question.

To illustrate the above result, we consider the Kuramoto-Sivashinsky equation $\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0$, restricted to the invariant subspace of periodic odd functions. We take the Hilbert space H to be the space $H = \{u \in L^2((0,L)) | u(x) = u(x + L), u(x) = -u(L-x), x \in I\!R\}$. Here $A = \frac{\partial^4}{\partial x^4}$ with domain $\mathcal{D}(A) = H^4_{per}((0,L)) \cap H$, where $H^m_{per}(0,L)$ denote the Sobolev space of functions that are periodic with period L along with their $m-1$ derivatives and have zero average. The eigenvalues of A are $\lambda_j = (2\pi j/l)^4$, for $j = 1, 2, \cdots$. The space V is given by $V = \mathcal{D}(A^{1/2}) = H^2_{per}(0,L) \cap H$ and $\gamma = 1/2$.

We construct the operator R^h as follows. First, we divide the domain $(0, L]$ into N elements of width $h = L/M$ and set $x_j = j\,h$ for $1 \leq j \leq M$. Then we take $V^h = \text{span}\{\phi_i, 1 \leq i \leq N\} \equiv \text{span}\{\phi_{1,j}, \phi_{2,j}, 1 \leq j \leq M\}$, where the functions $\phi_{1,i}$

and $\phi_{2,i}$ are piecewise-cubic C^1 functions such that $\phi_{1,i}(x_j) = \delta_{ij}$, $\phi_{2,i}(x_j) = 0$, and $\frac{\partial \phi_{1,i}}{\partial x}(x_j) = 0$, $\frac{\partial \phi_{2,i}}{\partial x}(x_j) = \delta_{ij}$. These functions are the so-called Hermite cubics. Now, we define $R^h(u)$ to be the L^2-projection of u into V^h. The operator R^h thus constructed does satisfy the approximation inequality (3.2) with $\beta = 2$. For the existence of an inertial manifold for the KS equation the reader is referred to Foias *et al.* (1988a,b, 1989), Constantin *et al.* (1989) and the references therein. Note that Theorem 3.1 requires that $M \sim \lambda_m^{\gamma/2} \sim m$. This is the same order as the dimension of the inertial manifold which is of course has dimension m.

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