Boundary Stabilization of the Rao-Nakra Beam
And
Dual-Phase-Lag One-Dimensional Thermo-Porous-Elasticity with Microtemperatures

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Contents

Chapter 1

Introduction

There is a big gap we wish to bridge before introducing our projects. Therefore, I dedicated a whole chapter to problems that can be viewed as stepping stones to being able to understand our projects. These example problems in Chapter 3 has certain elements that can be applied directly to the systems to be investigated in this thesis. In Chapter 3 we have two sections 3.5 and 3.6 where the damping is internal, similar to what we see in Chapter 4 (Dual-Phase-Lag One-Dimensional Thermo-Porous-Elasticity with Microtemperatures). Also, in chapter 3 we have 3 sections 3.2, 3.3, and 3.4 where the damping occurs on the boundary, these sections are similar to Chapter 5 (Boundary Stabilization of the Rao-Nakra Beam).

This thesis will introduce a few partial differential equation system. We begin by showing well-posedness (uniqueness and existence of the solution) of the problem. Then, given some internal/boundary damping that makes our system dissipative (Our dissipation function is less than or equal to zero for all time) we can possibly achieve a decay rate to the system depending on the damping imposed on the systems.

The biggest struggle we face is to keep this thesis self contained. So we dedicated all of chapter 2 to achieve this goal. In chapter 2 we introduce definitions needed, theorems , and any other mathematically background. Again, Chapter 3 is dedicated to bridge the gap we need to be able to understand Chapter 4 (Dual-Phase-Lag One-Dimensional Thermo-Porous-Elasticity with Microtemperatures) and Chapter 5 (Boundary Stabilization of the Rao-Nakra Beam).

Chapter 2

Mathematical Background

2.1 Definitions/Notations

We first introduce all the definitions we need to keep this thesis self contained

Notations:

f^* : − Complex conjugate of function f
$o(1)$: − Tends to zero
$O(1)$: − Bounded
$\langle , \rangle_{L^2} \equiv \langle , \rangle$

Definition 2.1.0.1 *Sobolev Spaces: Let* $\Omega = [0, L]$, $m \in \mathcal{N}$, $1 \leq p \leq \infty$. $W^{m,p}(\Omega)$ *is the space that contains the collection of functions f in* $L^p(\Omega)$ *with the distribution derivatives of order up to m is also in* $L^p(\Omega)$*. Where the norm is defined by the following*

$$\|f\|_{W^{m,p}(\Omega)} = (\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)})^{1/p}$$

Where $\alpha = \{\alpha_1, ..., \alpha_n\} \in \mathcal{N}^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$*. For when* $p = 2$ *it will be denoted as* $H^m(\Omega)$.

Definition 2.1.0.2 *Norm: From this point on will denote* $\| * \| \equiv \| * \|_{L^2(0,L)}$ *where* $\|f\|_{L^2(0,L)} = \sqrt{\int_0^L f(x) f^*(x) dx} \equiv \sqrt{\int_0^L f^2(x) dx}$

We also have the following properties for it to be a norm

1. $\|f\| \leq 0$
2. $\|f\| = 0$ *iff f=0*
3. $\|\alpha f\| = |\alpha| \|f\|$
4. $\|f + g\| \leq \|f\| + \|g\|$

Note: $\|f\|_{L^\infty(0,L)} = \sup_{x\in[0,L]} |f(x)|$

Definition 2.1.0.3 *L^2 Inner Product: We restrict our discussion of an inner product to the L^2 inner product. The inner product maps two elements vector space (L^2) to the complex field, with the following properties. Let f, g and h be elements of our vector space (V) and α be a scalar that is an element of the complex field. We will denote $\langle f, g\rangle \equiv \int_0^L f(x)g^*(x)dx$* [42]

We also have the four following properties:

1. $\langle f+h, g\rangle = \langle f, g\rangle + \langle h, g\rangle$
2. $\langle g, \alpha f\rangle = \alpha\langle g, f\rangle$
3. $\forall f \in V, \langle f, f\rangle \geq 0 \quad ; \quad \langle f, f\rangle = 0$ *iff* $f = 0$
4. $\langle f, g\rangle = \langle g, f\rangle^*$

We also have,

1. $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$
2. $\langle \alpha g, f\rangle = \alpha^*\langle g, f\rangle$

To go along with our introduction of inner product we have a neat trick that will be used often in the case we have boundary damping, this is why we introduce it here. (since $x \in R$ it can distributed to our dual space and the complex conjugate is just x)

$$\langle f, xf_x\rangle = \langle xf, f_x\rangle \equiv \int_0^L xf^*(x)f_x(x)dx = xf^2(x)|_{x=0,L} - \int_0^L \frac{d}{dx}[xf(x)]^* f(x)dx$$
$$= Lf^2(L) - \int_0^L [f^*(x)f(x) + xf_x^*(x)f(x)]dx \equiv Lf^2(L) - \langle f, f\rangle - \langle xf_x, f\rangle$$

From above we can arrive at

$$\langle f, xf_x\rangle + \langle xf_x, f\rangle = Lf^2(L) - \|f\|^2$$

Taking the reals we see

$$Re[\langle f, xf_x\rangle + \langle xf_x, f\rangle] = 2Re[\langle f, xf_x\rangle] = Lf^2(L) - \|f\|^2$$

Therefore,

$$RE[\langle f, xf_x\rangle] = \frac{L}{2}f^2(L) - \frac{1}{2}\|f\|^2 \tag{2.1.0.1}$$

This trick is used often in the upcoming chapters so now we can save some time when we see $\langle f, xf_x\rangle$.

Definition 2.1.0.4 *C_0-Semigroup: A family $S(t)(0 \leq t < \infty)$ of bounded linear operators in a Banach space H is a C_0-Semigroup if*

1. *$S(t_1 + t_2)$=$S(t_1)S(t_2)$, $\forall t_1, t_2 \geq 0$,*
2. *S(0)=I*
3. *$\forall x \in H$,S(t)x is continuous in t on $[0, \infty)$*

Definition 2.1.0.5 *Let $\mathcal{A}$ be a densely defined linear operator on $\mathcal{H}$ i.e. $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$. We say $\mathcal{A}$ is dissipative if $\forall y \in D(\mathcal{A})$, $Re\langle \mathcal{A}y, y\rangle_{\mathcal{H}} \leq 0$* [25]

Definition 2.1.0.6 *First Order Evolution Equation: We let y be the vector associated with our system then our first order evolution equation which is denoted as $\mathcal{A}$ is the matrix such that $\frac{dy}{dt} = \mathcal{A}y$*

2.2 C_0-Semigroup Generated by a Dissipative Operator

Theorem 2.2.0.1 *Let $\mathcal{A}$ be a linear operator with a dense domain on the Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is dissipative and $0 \in \rho(A)$ then $\mathcal{A}$ is the infinitesimal generator of a C_0-Semigroup of contractions on the Hilbert space $\mathcal{H}$.*

How are we going to show this theorem? We show $0 \in \rho(\mathcal{A})$ by a contradiction.

So if $0 \notin \rho(\mathcal{A})$ then $0 \in \sigma(\mathcal{A})$ aka our eigenvalue $\lambda_n \rightarrow 0$. So we have $\mathcal{A}U = o(1)\ in\ \mathcal{H}$, with normalized eigenfunctions, $\|U\|_{\mathcal{H}} = 1$. If $\mathcal{A}U = o(1)\ in\ \mathcal{H}$ implies $\|U\|_{\mathcal{H}} \neq 1$ we have reached a contradiction and therefore $0 \in \rho(\mathcal{A})$

2.3 Exponential Stability

Definition 2.3.0.1 *$e^{\mathcal{A}t}$ is said to be exponentially stable stable if* [25]

$$\exists \alpha > 0\ and\ M \geq 1 \ni \forall t \geq 0,\ \|e^{\mathcal{A}t}\|_{\mathcal{L}\{\mathcal{H},\mathcal{H}\}} \leq Me^{-\alpha t} \tag{2.3.0.1}$$

Theorem 2.3.0.1 *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-Semigroup of contractions in a Hilbert space. Then S(t) is exponentially stable iff* [42]

$$i\mathcal{R} \subset \rho(\mathcal{A}) \tag{2.3.0.2}$$

$$\overline{\lim}_{|\lambda| \rightarrow \infty} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty \tag{2.3.0.3}$$

How we will show (2.3.0.3) is exact same has how we will show (2.4.0.2) which is in the next section. However, how we will show (2.3.0.2) is proof by contradiction. We have a unit norm and we assume that (2.3.0.2) is false. If (2.3.0.2) is false, then $i\mathcal{R} \not\subset \rho(\mathcal{A})$ Which means $\exists \lambda_n \in \mathcal{R} \ni i\lambda_n$ is an eigenvalue. So this implies $\mathcal{A}U_n = i\lambda_n U_n$ which it can be seen will lead to $\|(i\lambda_n I - \mathcal{A})U_n\|_{\mathcal{H}} = o(1)$. So we can now conclude if we can show $\|(i\lambda_n I - \mathcal{A})U_n\|_{\mathcal{H}} = o(1)$ contradicts that $\|U_n\|_{\mathcal{H}} = 1$ then the proof for (2.3.0.2) will be complete. aka $i\mathcal{R} \subset \rho(\mathcal{A})$

2.4 Polynomial Stability

Theorem 2.4.0.1 *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ generate a C_0-Semigroup of contraction. Assume that* [25]

$$i\mathcal{R} \subset \rho(\mathcal{A}) \tag{2.4.0.1}$$

$$\exists k > 0 \ni \sup_{|\lambda| \to 0} \frac{1}{\lambda^k} \|(i\lambda - \mathcal{A})^{-1}\| < +\infty \tag{2.4.0.2}$$

Then,

$$\exists C > 0 \ni \forall t > 0,\ \|e^{\mathcal{A}t} U_0\|_{\mathcal{H}} \leq C \frac{1}{t^{\frac{1}{k}}} \|U_0\|_{D(\mathcal{A})} \tag{2.4.0.3}$$

For all $U_0 \in D(\mathcal{A})$

How we show (2.4.0.2) is again a proof by contradiction. Assume (2.4.0.2) is false then this implies that $\frac{1}{\lambda^k}\|(i\lambda I - \mathcal{A})^{-1}\|$ is unbounded. Meaning $\exists \|z_n\| = 1 \ni \frac{1}{\lambda^k}(i\lambda I - \mathcal{A})^{-1} z_n = U_n \to \infty$. Since $0 \in \rho(\mathcal{A})$ this implies that $\mathcal{A}$ is invertible. So we can see this implies $z_n = \lambda^k (i\lambda I - \mathcal{A})U_n$. Taking the norm and divide both sides by the norm of $\|U_n\|$ we can see this implies $\frac{\|\lambda^k (i\lambda I - \mathcal{A})U_n\|_{\mathcal{H}}}{\|U_n\|_{\mathcal{H}}} = o(1)$. After denoting the normalized eigenfunctions (U_n) as V_n we can see we will have $\|V\|_{\mathcal{H}} = 1$ and $\|\lambda_n^k (i\lambda_n I - \mathcal{A})V_n\|_{\mathcal{H}} = o(1)$ where λ_n tends to ∞. So if we arrive at a contradiction then we have shown (2.4.0.2)

2.5 Gagliardo-Nirenberg Inequality (GNI)

Theorem 2.5.0.1 *Let j and m be integers satisfying $0 \leq j < m$ and let $1 \leq q, r \leq \infty$, and $p \in \mathcal{R}$, $\frac{j}{m} \geq a \geq 1$ such that*

$$\frac{1}{p} - \frac{j}{n} = a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}$$

Then,

$$\exists C_1, C_2 \ni \|D^j u\| \leq C_1 \|D^m u\|_r^a \|u\|_q^{1-a} + C_2 \|u\|_q \tag{2.5.0.1}$$

(We often drop the $C_2\|u\|_q$ because it always smaller than $C_1\|D^m u\|_r^a \|u\|_q^{1-a}$)

2.6 Poincare Inequality

Theorem 2.6.0.1 *Let Ω be a bounded domain of C^1 in $\mathcal{R}^n$. Then*

$$\exists C \ni \forall u \in H^1(\Omega)\ ,\ \|u\|_{L^2(\Omega)} \leq C(\|u_x\|_{L^2} + |\int_\Omega u dx|) \tag{2.6.0.1}$$

2.7 Holder inequality

Theorem 2.7.0.1 *Let $\frac{1}{p} + \frac{1}{q} = 1$ where $p, q > 1$ then,*

$$\int_a^b |f(x)g(x)|dx \leq [\int_a^b f^p(x)dx]^{1/p}[\int_a^b g^q(x)dx]^{1/q} \tag{2.7.0.1}$$

Chapter 3

Example Problems

3.1 Introduction

We introduce some problems that will build into our projects. We do this to bridge the gap needed to begin to understand our projects. For the Boundary Stabilization of the Rao-Narka Beam, we can think of the system having three elements. Which are the wave equations, an Euler-Beam equation and then the term that couple them together. Therefore in section 3.2 we introduce the boundary stabilization of the wave equation, it has already been done [43]. Then in section 3.3 we wish to address the beam part. This has already been done for the Non-Homogeneous Euler Beam Equation [44]. Finally, how we can get information on the other terms when the feedback controller isn't associated with said variable, we have to get all the information from the coupling term. Therefore, we present the boundary stabilization of the weakly coupled wave equation with one controller turned off and the same wave speed.

Now we introduce of example problems that correlate to our project in Chapter 4. We transition from boundary damping to internal damping we go from a weakly coupled wave equation with boundary damping to a weakly coupled wave equation with internal damping. This problem trivially follows from [45]. Finally, we introduce our last example problem which is exactly our problem we wish to address in Chapter 4, however instead of a 2nd and 1st order Taylor expansion, we just go only to the zero term for the two equations, our problem trivially follows from [24]. [24] is actually the paper we continued off of and explored the critical cases associated with that paper.

3.2 Boundary Stabilization of the Wave Equation

3.2.1 Introduction

Here is the system we wish to first address. The boundary stabilization of the wave equation. Notice that we have no internal damping and the rate of how the solution of the system decays is dictated by the boundary damping. This system has already by addressed in [43]

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \ ; \ x \in (0,L) \ , t \geq 0 \tag{3.2.1.1}$$

$$u(0,t) = 0 \ , \ u_x(L,t) = -k_1 u_t(L,t) \tag{3.2.1.2}$$

3.2.2 Preliminary

First we wish to construct a proper state space so that the energy of our system is dissipative. We begin by figuring out our energy function. To do this we can take the L^2 inner product of (3.2.1.1) with u_t.

$$\langle u_{tt}, u_t \rangle - \langle u_{xx}, u_t \rangle = 0$$

We can integrate by parts on the last term to arrive at the following

$$\langle u_{tt}, u_t \rangle + \langle u_x, u_{xt} \rangle = u_x(L,t)u_t(L,t) - u_x(0,t)u_t(0,t)$$

With (3.2.1.2) we can see the RHS is the following. We also factor out a $\frac{d}{dt}$ to the LHS

$$\frac{1}{2}\frac{d}{dt}[\|u_t\|^2 + \|u_x\|^2] = -k_1 u^2(L,t) \tag{3.2.2.1}$$

From (3.2.2.1) we can see that our energy function is as follows

$$E(t) = \|u_t\|^2 + \|u_x\|^2 \tag{3.2.2.2}$$

We can also deduce our dissipation function is as follows

$$\frac{1}{2}\frac{d}{dt}E(t) = -k_1 u_t^2(L,t) \tag{3.2.2.3}$$

Looking at (3.2.2.2) we can construct a proper energy space

$$(u, u_t) \in \mathcal{H} = H_l^1 \times L^2$$

Where $H_l^1 = \{f \in H^1, f(0) = 0\}$

We denote our $U = (u, u_t) = (u, v)$, then convert our system into a first-order evolution equation on our Hilbert space $\mathcal{H}$.

$$\frac{dU}{dt} = \mathcal{A}U = \begin{pmatrix} v \\ u_{xx} \end{pmatrix} \tag{3.2.2.4}$$

To figure out the domain space we let $\mathcal{A}U = F \equiv (f_1, f_2) \in \mathcal{H}$. Therefore we have the following

$$v = f_1 \in H^1$$
$$u_{xx} = f_2 \in L^2 \to u \in H^2$$

So we will have the following domain space

$$(u, v) \in D(\mathcal{A}) = H^2 \times H^1 \in \mathcal{H}.$$

Theorem 3.2.2.1 *$\mathcal{A}$ is a infinitesimal generator of a C_0-semigroup of contraction on our Hilbert space $\mathcal{H}$*

Proof.

$$Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}\|U\|^2_{\mathcal{H}} = -k_1 u_t^2(L, t) \le 0 \tag{3.2.2.5}$$

So we can see that $\mathcal{A}$ is dissipative and it is easy to see that $D(\mathcal{A})$ is dense in $\mathcal{H}$. Now we just need to show that $0 \in \rho(\mathcal{A})$ so we can use Theorem 2.2.0.1 to complete the proof. We shall show this proof by a contradiction. If $0 \notin \rho(\mathcal{A})$ then since $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ this implies we have an eigenvalue such that $\lambda = 0$. Therefore, if we can show $\mathcal{A}U = 0 \in \mathcal{H}$ implies $\|U\|_{\mathcal{H}} = 0$ we have reached a contradiction . So we assume $\|U\|^2_{\mathcal{H}} = 1$, i.e. $\|u_x\|^2 + \|v\|^2 = 1$. We also assume $\mathcal{A}U = 0 \in \mathcal{H}$,

i.e.

$$v = 0 \in H^1, \tag{3.2.2.6}$$
$$u_{xx} = 0 \in L^2. \tag{3.2.2.7}$$

We can see

$$Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -k_1 u_t^2(L, t) = 0 \tag{3.2.2.8}$$

Also by (3.2.1.2)

$$u_x(L, t) = 0. \tag{3.2.2.9}$$

So if we take the L^2 inner product of (3.2.2.7) with u and integrate by parts, to get

$$\langle u_{xx}, u\rangle = u u_x|_{x=0,L} - \|u_x\|^2 = 0.$$

With (3.2.2.9) and (3.2.1.2) we will get the following.

$$\|u_x\|^2 = 0. \tag{3.2.2.10}$$

Therefore with (3.2.2.10) and (3.2.2.6) we have $\|U\|^2_{\mathcal{H}} = \|u_x\|^2 + \|v\|^2 = o(1)$. Notice this contradicts that we have a unit norm. Therefore $0 \in \rho(\mathcal{A})$. By Theorem 2.2.0.1 the proof is complete.

Now we just have to achieve a decay rate.

Theorem 3.2.2.2 *Let $e^{\mathcal{A}t}$ be the semigroup associated with our system, then this semigroup is exponential stable.*

To finish this proof we use Theorem (2.3.0.1) so we most show (2.3.0.2) and (2.3.0.3). We do this in the following two subsections

3.2.3 Showing $i\mathcal{R} \subset \rho(\mathcal{A})$

Again it is a proof by contradiction we assume $i\mathcal{R} \subset \rho(\mathcal{A})$ is false then there exists $\lambda_n \to \beta$ with a normalized U_n such that (β is a finite value.)

$$\|(i\lambda_n I - \mathcal{A})U_n\|_{\mathcal{H}} = o(1) \tag{3.2.3.1}$$

(We drop the subscript from this point on for convince) i.e.

$$i\lambda u - v = f_1 = o(1) \in H^1 \tag{3.2.3.2}$$

$$i\lambda v - u_{xx} = f_2 = o(1) \in L^2 \tag{3.2.3.3}$$

Taking the $\mathcal{H}$ inner product of $(i\lambda I - \mathcal{A})U$ with U and taking the reals we arrive at the following.

$$RE[\langle (i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}}] = -k_1 v^2(L,t) = o(1) \tag{3.2.3.4}$$

with (3.2.1.2) we also have

$$|u_x(L,t)| = o(1) \tag{3.2.3.5}$$

To begin we plug in (3.2.3.2) into (3.2.3.3) to get the following

$$-\lambda^2 u - u_{xx} = f_2 + i\lambda f_1 \in L^2 \tag{3.2.3.6}$$

Taking (3.2.3.6) with xu_x we get

$$-\lambda^2\langle u, xu_x\rangle - \langle u_{xx}, xu_x\rangle = \langle f_2 + i\lambda f_1, xu_x\rangle$$

When integrating by parts above becomes the following

$$\frac{1}{2}\|\lambda u\|^2 + \frac{1}{2}\|u_x\|^2 - \frac{L}{2}\lambda u^2(L) - \frac{L}{2}u_x^2(L) = Lf_1(L)u(L) - \langle f_1 + xf_{1,x}, \lambda u\rangle$$

We can see $\|v\| = O(1)$ by the norm then by (3.2.3.2) $\|\lambda u\| = O(1)$, since $f_1 = o(1) \in H^1$ this implies $f_{1,x} = o(1)$, this with (3.2.3.5) we can see.

$$\frac{1}{2}\|\lambda u\|^2 + \frac{1}{2}\|u_x\|^2 - \frac{L}{2}\lambda u^2(L) = Lf_1(L)u(L)$$

Now we just have to show the two boundary terms tend to zero. By GNI we have

$$|f_1(L)| \leq \|f_1\|_{L^\infty} \leq K\|f_1\|^{1/2}\|f_{1,x}\|^{1/2} = o(1)$$
$$|i\lambda u(L) - v(L)| \leq \|i\lambda u - v\|_{L^\infty} \leq K\|f_1\|^{1/2}\|f_{1,x}\|^{1/2} = o(1)$$

Since $|i\lambda u(L) - v(L)| = o(1)$ and by (3.2.3.4) $v(L) = o(1)$ this implies $\lambda u(L) = o(1)$. So we can see all our boundary term tends to zero so we arrive at the following

$$\frac{1}{2}\|\lambda u\|^2 + \frac{1}{2}\|u_x\|^2 = o(1) \tag{3.2.3.7}$$

We can see that (3.2.3.7) contradicts that $\|U\|_{\mathcal{H}} = 1$ therefore by proof by contradiction $i\mathcal{R} \subset \rho(\mathcal{A})$.

3.2.4 Further Comments

Now we finish the proof in the next subsection. The proof is actually the exact same as the last subsection for this case however I highlight the search we use to find the proper multipler to show the contradiction. Why we picked xu_x has a sense of ambiguity so we wanted to highlight how we search for these multipliers.

3.2.5 Showing $\overline{\lim}_{|\lambda|\to\infty}\|(i\lambda I - \mathcal{A})^{-1}\| < \infty$

We have $\|U\|_{\mathcal{H}} = 1$, and we complete this proof by contradiction. We assume that $\overline{\lim}_{|\lambda|\to\infty}\|(i\lambda I - \mathcal{A})^{-1}\| < \infty$ is false, which implies $\exists\lambda \to \infty$ such that

$$\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1) \tag{3.2.5.1}$$

i.e.

$$i\lambda u - v = f_1 = o(1) \in H^1 \tag{3.2.5.2}$$

$$i\lambda v - u_{xx} = f_2 = o(1) \in L^2 \tag{3.2.5.3}$$

Taking the inner product of $(i\lambda I - \mathcal{A})U$ with y in $\mathcal{H}$ and taking the reals we arrive at the following

$$RE[\langle(i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}}] = -k_1 v^2(L,t) = o(1) \tag{3.2.5.4}$$

From (3.2.5.2) we can deduce.

$$\begin{aligned}|i\lambda u(L) - v(L)| \leq \|i\lambda u - v\|_{L^\infty} &\leq C_1\|i\lambda u - v\|^{1/2}\|i\lambda u_x - v_x\|^{1/2} + C_2\|i\lambda u - v\| \\ &= C_1\|f_1\|^{1/2}\|f_{1,x}\|^{1/2} + C_2\|f_1\| = o(1)\end{aligned}$$

Hence,

$$i\lambda u(L) - v(L) = o(1)$$

Then by (3.2.5.4) we obtain

$$\lambda u(L) = o(1) \tag{3.2.5.5}$$

We now draw our attention back to (3.2.5.2) if we plug in (3.2.5.2) into (3.2.5.3) we arrive at the following.

$$-\lambda^2 u - u_{xx} = f_2 + i\lambda f_1 \in L^2 \tag{3.2.5.6}$$

Lets begin by taking the L^2 inner product of (3.2.5.6) with $q_1(x)u + q_2(x)u_x$ and we will later restrict these function q_1 and q_2 to guarantee all the expressions our positive definite and tending to zero. (We also take the Reals)

$$\begin{aligned} RE[\langle -\lambda^2 u - u_{xx}, q_1(x)u + q_2(x)u_x\rangle] = RE[\langle f_2 + i\lambda f_1, q_1(x)u + q_2(x)u_x\rangle] = o(1) \\ RE[\langle -\lambda^2 u, q_1(x)u + q_2(x)u_x\rangle] - RE[[\langle u_{xx}, q_1(x)u + q_2(x)u_x\rangle]] = o(1) \\ RE[-\langle \lambda^2 u, q_1(x)u\rangle - \langle \lambda^2 u, q_2(x)u_x\rangle - \langle u_{xx}, q_1(x)u\rangle - \langle u_{xx}, q_2(x)u_x\rangle] = o(1) \end{aligned}$$

With the help of (2.1.0.1) and integration by parts we arrive at the following.

$$\begin{aligned} RE[-\langle \lambda^2 u, q_1(x)u\rangle - \frac{q_2(L)}{2}u^2(L) + \frac{q_2(0)}{2}u^2(0) + \langle u, \lambda^2 \frac{q_2'}{2}u\rangle - u_x(L)q_1(L)u(L) \\ +\langle u_x, q_1'u + q_1 u_x\rangle - \frac{q_2(L)}{2}u_x^2(L) + \frac{q_2(L)}{2}u_x^2(L) + \langle u_x, \frac{q_2'}{2}u_x\rangle] = o(1) \\ RE[-\langle \lambda^2 u, q_1(x)u\rangle - \frac{q_2(L)}{2}u^2(L) + \frac{q_2(0)}{2}u^2(0) + \langle u, \lambda^2 \frac{q_2'}{2}u\rangle - u_x(L)q_1(L)u(L) \\ +\langle u_x, q_1 u_x\rangle + \frac{q_1'(L)}{2}u^2(L) - \langle u, \frac{q_1''}{2}u\rangle - \frac{q_2(L)}{2}u_x^2(L) + \frac{q_2(L)}{2}u_x^2(L) + \langle u_x, \frac{q_2'}{2}u_x\rangle] = o(1) \end{aligned}$$

Combining terms the above expression reduces to the following.

$$\begin{aligned} \langle u, [-\lambda^2 q_1 + \lambda^2 \frac{q_2'}{2} - \frac{q_1''}{2}]u\rangle + \langle u_x, [q_1 + \frac{q_2'}{2}]u_x\rangle - \frac{q_2(L)}{2}u^2(L) + \frac{q_2(0)}{2}u^2(0) \\ -u_x(L)q_1(L)u(L) + \frac{q_1'(L)}{2}u^2(L) - \frac{q_2(L)}{2}u_x^2(L) + \frac{q_2(0)}{2}u_x^2(0) = o(1). \end{aligned} \tag{3.2.5.7}$$

Looking at (3.2.5.7) we need the following restrictions on our q_1 and q_2

$$\begin{aligned} -\lambda^2 q_1 + \lambda^2 \frac{q_2'}{2} - \frac{q_1''}{2} \geq 0 \\ q_1 + \frac{q_2'}{2} \geq 0 \\ q_2(0) \geq 0 \\ q_2(L) \leq 0 \\ q_1(L) = 0 \\ q_1'(L) \geq 0 \end{aligned} \tag{3.2.5.8}$$

If we could come up with a non-trivial solution to (3.2.5.8) we have completed the proof. However, since we have our dissipation function we can use this to make our problem a lot more easy.

Using (3.2.5.4) and (3.2.1.1) assuming q_1, q_2, q_1' *and* q_2' is bounded (3.2.5.7) is reduced to the following.

$$\langle u, [-\lambda^2 q_1 + \lambda^2 \frac{q_2'}{2} - \frac{q_1''}{2}]u\rangle + \langle u_x, [q_1 + \frac{q_2'}{2}]u_x\rangle + \frac{q_2(0)}{2}u_x^2(0) = o(1) \quad (3.2.5.9)$$

So now we are left with the following restrictions on our functions

$$\begin{aligned} -\lambda^2 q_1 + \lambda^2 \frac{q_2'}{2} - \frac{q_1''}{2} &\geq 0 \\ q_1 + \frac{q_2'}{2} &\geq 0 \\ q_2(0) &\geq 0 \end{aligned} \quad (3.2.5.10)$$

The most common choice for the functions that satisfy (3.2.16) is $q_1 = 0$ and $q_2 = x$

With this choice we can see (3.2.15) reduces to the following

$$\frac{1}{2}\|\lambda u\|^2 + \frac{1}{2}\|u_x\|^2 = o(1) \quad (3.2.5.11)$$

However we can see (3.2.17) contradicts our previous statement of a unit norm therefore the proof is complete.

3.2.6 Further Notes

Notice we could have came up with a wide variety of multipliers to arrive at the contradiction. Another one that works is $2xu_x + u$. If we would of done this multiplier we would end up with.

$$2\|u_x\|^2 = o(1) \quad (3.2.6.1)$$

Then all we would need to do to complete the proof is take the L^2 inner product of (3.2.5.6) with u to eventually arrive at

$$\|\lambda u\| = o(1) \rightarrow \|v\| = o(1) \quad (3.2.6.2)$$

So we can see this path would also contradict that we have a unit norm.

3.3 Boundary Stabilization of the Homogeneous Euler-Beam

3.3.1 Introduction

As stated before this problem follows from [44]. We highlight this problem so that when we introduce the Rao-Narka Beam we know how to deal with the problem given some information on the boundary of our w term.

$$w_{tt} + w_{xxxx} = 0 \ , \tag{3.3.1.1}$$

with the following boundary conditions

$$\begin{aligned} w(0) = w_x(0) = 0 \ , \\ w_{xxx}(L) = k_1 w_t(L) \ , \\ w_{xx}(L) = -k_2 w_{xt}(L) \ . \end{aligned} \tag{3.3.1.2}$$

3.3.2 Preliminary

The first thing we wish to do is to identify the proper Hilbert space such that our energy function is dissipiative. How we do this for the system (3.3.1.1)-(3.3.1.2) is we begin by taking the L^2 inner product of (3.3.1.1) with w_t

$$\langle w_{tt}, w_t \rangle + \langle w_{xxxx}, w_t \rangle = 0.$$

By integration by parts on the last term we can see

$$\langle w_{tt}, w_t \rangle + \langle w_{xx}, w_{xt} \rangle = -w_{xxx}(L) w_t(L) + w_{xx}(L) w_{xt}(L)$$

We can factor out a $\frac{d}{dt}$ to the LHS and we will arrive at our energy function and dissipation function

$$\frac{1}{2}\frac{d}{dt}[\|w_x\|^2 + \|w_t\|^2] = -w_{xxx}(L) w_t(L) + w_{xx}(L) w_{xt}(L). \tag{3.3.2.1}$$

With (3.3.2.1) we arrive at

$$\frac{1}{2}\frac{d}{dt}[\|w_x\|^2 + \|w_t\|^2] = -k_1 w_t^2(L) - k_2 w_{xt}^2(L) \leq 0 \tag{3.3.2.2}$$

From (3.3.2.2) we can deduce that our energy function will be the following

$$E(t) = \|w_{xx}\|^2 + \|w_t\|^2. \tag{3.3.2.3}$$

Denoting $U = (w, w_t) = (w, y)$. From (3.3.1.2) and (3.3.2.3) we can construct a proper energy space.

$$(w, y) \in \mathcal{H} = \{H_l^2 \times L^2\} \tag{3.3.2.4}$$

Where $H_l^2 = \{f \in H^1 | f(0) = f'(0) = 0\}$

Also from (3.3.2.2) we have the following dissipation function

$$\frac{1}{2}\frac{d}{dt}E(t) = -k_1 w_t^2(L) - k_2 w_{xt}^2(L). \tag{3.3.2.5}$$

Looking back at our system we can construct our first order evolution equation of the system

$$\frac{dU}{dt} = \mathcal{A}U \rightarrow \begin{pmatrix} y \\ y_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ D^4 & 0 \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix}. \tag{3.3.2.6}$$

We need to have $\mathcal{A}$ be dense operator and our dissipation function needs to negative definite or zero to be dissipative. Hence, we turn our attention to our domain space. The domain space is constructed in a manner so that $0 \in \rho(\mathcal{A})$ this will be used to show that $\mathcal{A}$ is a infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}$. How our domain space is constructed will be revisited when we show $0 \in \rho(\mathcal{A})$ but for now we just will give our domain space as the following

$$D(\mathcal{A}) = \{H^4 \times H^2\} \in \mathcal{H} \tag{3.3.2.7}$$

From (3.3.2.7) and (3.3.2.4) we have $D(\mathcal{A}) \subset \mathcal{H}$ this implies that $\mathcal{A}$ is a densely defined linear operator. Turing our attention back to our dissipation function we have

$$Re[\langle \mathcal{A}U, U\rangle_{\mathcal{H}}] = -k_1 w_t^2(L) - k_2 w_{xt}^2(L) \leq 0. \tag{3.3.2.8}$$

By Definition (2.1.0.6) we can now say that $\mathcal{A}$ is dissipative.

Theorem 3.3.2.1 *$\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}$.*

Proof: Using Theorem (2.3.0.1) we already can see that $\mathcal{A}$ is dissipative. Now we just have to show $0 \in \rho(\mathcal{A})$. Suppose that $\lambda = 0$ is an eigenvalue and U be the normalized eigenfunction. Then,

$$y = 0 \ in \ H^2, \tag{3.3.2.9}$$

$$w_{xxxx} = 0 \ in \ L^2. \tag{3.3.2.10}$$

Taking the L^2 inner product with w implies

$$\langle w_{xxxx}, w\rangle = \|w_{xx}\|^2 + w_{xxx}(L)w(L) - w_{xxx}(0)w(0) - w_{xx}(L)w_x(L) + w_{xx}(0)w_x(0)$$

Hence with our boundary conditions

$$\|w_{xx}\|^2 = 0 \tag{3.3.2.11}$$

So from (3.3.2.11) and (3.3.2.9) we can see $\|U\|_{\mathcal{H}} = 0$ which contradicts a unit norm. Thus $0 \in \rho(\mathcal{A})$. Therefore $\mathcal{A}$ is is the infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}$.

Theorem 3.3.2.2 *Let $S(t) = e^{\mathcal{A}t}$ be a semigroup associated with our system. Then the semigroup is exponential stable iff*

$$i\mathcal{R} \subset \rho(\mathcal{A}) \tag{3.3.2.12}$$

$$\overline{\lim}_{|\lambda|\to\infty}\|(i\lambda I - \mathcal{A})^{-1}\| < \infty \tag{3.3.2.13}$$

3.3.3 Proof of Theorem 3.3.2

We first wish to show (3.3.2.12). How we do this we assume (3.3.2.12) is false. Then there exists a $\lambda \in \sigma(\mathcal{A})$ and unit norm of U such that

$$\|(i\lambda_n I - \mathcal{A})U_n\|_{\mathcal{H}} = o(1) \tag{3.3.3.1}$$

(We drop the subscript)

i.e.

$$i\lambda w - y = f_1 = o(1) \; in \; H^2 \tag{3.3.3.2}$$

$$i\lambda y + w_{xxxx} = f_2 = o(1) \; in \; L^2 \tag{3.3.3.3}$$

So we have $\|U\|_{\mathcal{H}} = \|y\|^2 + \|w_{xx}\|^2 = 1$ and we want to reach a contradiction to this. So we begin by taking the $\mathcal{H}$ inner product of $(i\lambda I - \mathcal{A})U$ with U and take the reals.

$$Re[\langle(i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}}] = -Re[\langle \mathcal{A}U, U\rangle_{\mathcal{H}}] = -k_1 w_t^2(L) - k_2 w_{xt}^2(L) \tag{3.3.3.4}$$

In the next section the proof for this (3.3.2.12) follows trivially from the proof for (3.3.2.13) so we just show the proof for (3.3.2.13)

We now wish to verify condition (3.3.2.13). Again, it is a proof by contradiction so we assume condition (3.3.2.13) is false and we have a $\|U_n\|_{\mathcal{H}} = 1$. Assuming condition (3.3.2.13) is false implies that there exists $\lambda_n \to \infty$ such that

$$\|(i\lambda_n I - \mathcal{A})U_n\|_{\mathcal{H}} = o(1) \tag{3.3.3.5}$$

(we drop the subscript for convenience)

i.e.

$$i\lambda w - y = f_1 = o(1) \; in \; H^2 \tag{3.3.3.6}$$

$$i\lambda y + w_{xxxx} = f_2 = o(1) \; in \; L^2 \tag{3.3.3.7}$$

So we have $\|U\|_{\mathcal{H}} = \|y\|^2 + \|w_{xx}\|^2 = 1$ and we want to reach a contradiction to this. so we begin by Taking the $\mathcal{H}$ inner product of $(i\lambda I - \mathcal{A})U$ with U and take the reals which will return our dissipation function.

$$Re[\langle (i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}}] = -Re[\langle \mathcal{A}U, U\rangle_{\mathcal{H}}] = -k_1 w_t^2(L) - k_2 w_{xt}^2(L) \tag{3.3.3.8}$$

We begin by plug in (3.3.3.6) into (3.3.3.7) and we are left with

$$-\lambda^2 w + w_{xxxx} = f_2 - i\lambda f_1 = o(1) \tag{3.3.3.9}$$

Now we wish to arrive at a positive definite expression which implies some element of the norm tends to zero to contradict the unit norm. First we begin by taking the L^2 inner product of (3.3.3.9) with xw_x. (We also take the Reals)

$$-\lambda^2\langle w, xw_x\rangle + \langle w_{xxxx}, xw_x\rangle = \langle f_2 - i\lambda f_1, xw_x\rangle$$
$$-\frac{L}{2}\lambda^2 w^2(L) + \frac{1}{2}\|\lambda w\|^2 + Lw_{xxx}(L)w_x(L) - \langle w_{xxx}, w_x + xw_{xx}\rangle = \langle -i[xf_1 + f_{1,x}], \lambda w\rangle$$

We will revisit the above expression. However we first show that the following is bounded. Looking at the norm we can see $\Omega = O(1)$ and $w_{xx} = O(1)$ from (3.3.3.6) since $\Omega = O(1)$ is bounded we can deduce that $\lambda w = O(1)$. We can also show with the interpolation that

$$\|\lambda^{1/2} w_x\| \leq \|\lambda w\|^{1/2}\|w_{xx}\|^{1/2} = O(1) \tag{3.3.3.10}$$

We turn our attention back to the above expression. Since $\lambda w = O(1)$ we can deduce that the RHS tends to zero. So we now we integrate by parts again on the last internal expression of the LHS to arrive at the following.

$$-\frac{L}{2}\lambda^2 w^2(L) + \frac{1}{2}\|\lambda w\|^2 + Lw_{xxx}(L)w_x(L) - \frac{L}{2}w_{xx}^2(L) + \frac{1}{2}\|w_{xx}\|^2$$
$$-w_{xx}(L)w_x(L) + w_{xx}(0)w_x(0) + \|w_{xx}\|^2 = o(1)$$
$$-\frac{L}{2}\lambda^2 w^2(L) + \frac{1}{2}\|\lambda w\|^2 + [Lw_{xxx}(L) - w_{xx}(L)]w_x(L) - \frac{L}{2}w_{xx}^2(L) + \frac{3}{2}\|w_{xx}\|^2 = o(1)$$

We need the following Lemma which will be proved later for the following cases. Case i) $k_1 > 0$ and $k_2 > 0$, Case ii) $k_1 > 0$ and $k_2 = 0$, Case iii) $k_1 = 0$ and $k_2 > 0$. The Lemma will be shown later but as for right now we have.

Lemma 3.3.3.1 $\lambda w(L) = w_x(L) = w_{xx}(L) = w_{xxx}(L) = o(1)$

With Lemma (3.3.3.1) we arrive at the following expression.

$$\frac{1}{2}\|\lambda w\|^2 + \frac{3}{2}\|w_{xx}\|^2 = o(1) \tag{3.3.3.11}$$

We see that (3.3.3.11) Contradicts that $\|U\|_{\mathcal{H}} = 1$ Which completes the proof and we have achieved that the decay rate of the system is exponential. We know have to prove Lemma (3.3.3.1)

3.3.4 Proof of Lemma (3.3.3.1)

We have 3 different cases we wish to show for this lemma.

Case i.) $k_1 > 0$ and $k_2 > 0$

From our dissipation function aka (3.3.3.4) we have.

$$w_t(L) = o(1) \text{ } and \text{ } w_{xt}(L) = o(1) \tag{3.3.4.1}$$

From (3.3.4.1) and GNI we can arrive at the following two expressions.

$$|i\lambda w(L) - \Omega(L)| \leq \|i\lambda w - \Omega\|_{L^\infty} \leq C_1\|f_1\|^{1/2}\|f_{1,x}\|^{1/2} + C_2\|f_1\| = o(1) \tag{3.3.4.2}$$

$$|i\lambda w_x(L) - \Omega_x(L)| \leq \|i\lambda w_x - \Omega_x\|_{L^\infty} \leq C_1\|f_{1,x}\|^{1/2}\|f_{1,xx}\|^{1/2} + C_2\|f_{1,x}\| = o(1) \tag{3.3.4.3}$$

From (3.3.4.1) , (3.3.4.2) and (3.3.4.3) we can the following boundary terms tend to zero.

$$\lambda w(L) = o(1) \text{ } and \text{ } \lambda w_x(L) = o(1) \tag{3.3.4.4}$$

Form (3.3.1.2) and (3.3.4.1) we have the last expressions needed for this case.

$$w_{xx}(L) = o(1) \text{ } and \text{ } w_{xxx}(L) = o(1) \tag{3.3.4.5}$$

For this case the lemma has now been shown.

Case ii.) $k_1 > 0$ and $k_2 = 0$

We have the following from (3.3.2.8) and (3.3.1.2)

$$w_t(L) = w_{xxx}(L) = o(1) \text{ } and \text{ } w_{xx}(L) = 0 \tag{3.3.4.6}$$

From the same logic from the previous case we have

$$\lambda w(L) = o(1) \tag{3.3.4.7}$$

So the only expression we need an estimate on is the boundary term $w_x(L)$. We can get an estimate on this term by using GNI.

$$|w_x(L)| \leq \|w_x\|_{L^\infty} \leq C_1\|w_x\|^{1/2}\|w_{xx}\|^{1/2} + C_2\|w_x\| = o(1) \tag{3.3.4.8}$$

So we have the last expression final, that $w_x(L) = o(1)$ and the Lemma is final complete for case ii).

Case iii.) $k_1 = 0$ and $k_2 > 0$

Its easy to see we have the following by the same logic used case i).

$$w_{xxx}(L) = 0 \; and \; w_{xx}(L) = \lambda w_x(L) = o(1) \tag{3.3.4.9}$$

So we just need the following $\lambda w(L) = o(1)$ to complete the lemma. So how we do this, we take the L^2 inner product of (3.3.3.9) with q(x). We will later put restrictions on q(x) so to that we can get information on $\lambda w(L)$. We go through a long process of integrating by parts below.

$$-\langle \lambda^2 w, q(x)\rangle + \langle w_{xxxx}, q(x)\rangle = \langle f_2 - i\lambda f_1, q(x)\rangle$$

$$-\langle \lambda^2 w, q(x)\rangle + w_{xxx}(L)q(L) - w_{xxx}(0)q(0) - \langle w_{xxx}, q'(x)\rangle = \langle f_2 - i\lambda f_1, q(x)\rangle$$

$$\begin{aligned} -\langle \lambda^2 w, q(x)\rangle + w_{xxx}(L)q(L) - w_{xxx}(0)q(0) - w_{xx}(L)q'(L) + w_{xx}(0)q'(0) + \langle w_{xx}, q''(x)\rangle \\ = \langle f_2 - i\lambda f_1, q(x)\rangle \end{aligned}$$

$$\begin{aligned} -\langle \lambda^2 w, q(x)\rangle + w_{xxx}(L)q(L) - w_{xxx}(0)q(0) - w_{xx}(L)q'(L) + w_{xx}(0)q'(0) \\ +w_x(L)q''(L) - w_x(0)q''(0) - \langle w_x, q'''(x)\rangle = \langle f_2 - i\lambda f_1, q(x)\rangle \end{aligned}$$

$$\begin{aligned} -\langle \lambda^2 w, q(x)\rangle + w_{xxx}(L)q(L) - w_{xxx}(0)q(0) - w_{xx}(L)q'(L) + w_{xx}(0)q'(0) + w_x(L)q''(L) \\ -w_x(0)q''(0) - w(L)q'''(L) + w(0)q'''(0) + \langle w, q''''(x)\rangle = \langle f_2 - i\lambda f_1, q(x)\rangle \end{aligned}$$

$$\begin{aligned} \langle w, -\lambda^2 q(x) + q''''(x)\rangle + w_{xxx}(L)q(L) - w_{xxx}(0)q(0) - w_{xx}(L)q'(L) + w_{xx}(0)q'(0) \\ +w_x(L)q''(L) - w_x(0)q''(0) - w(L)q'''(L) + w(0)q'''(0) = \langle f_2 - i\lambda f_1, q(x)\rangle \end{aligned}$$

From this expression we can start putting restrictions on our q(x). The first thing we want to do is try to get rid of the internal terms. So if we let $-\lambda^2 q + q'''' = 0$. If we do this the above expression reduces to the following. (Also with (3.3.3.2) we can get rid of some boundary terms)

$$\begin{aligned} w_{xxx}(L)q(L) - w_{xxx}(0)q(0) - w_{xx}(L)q'(L) + w_{xx}(0)q'(0) + w_x(L)q''(L) \\ -w(L)q'''(L) = \langle f_2 - i\lambda f_1, q(x)\rangle \end{aligned}$$

From the internal term on the RHS we can see we need to have $\lambda q(x) = O(1)$ or $\lambda \int_a^b |q(x)|dx = O(1)$. Therefore we put those restrictions on our q(x) and we then arrive at

$$w_{xxx}(L)q(L) - w_{xxx}(0)q(0) - w_{xx}(L)q'(L) + w_{xx}(0)q'(0) + w_x(L)q''(L)$$
$$-w(L)q'''(L) = o(1)$$

We need the final restrictions on q to have that $\lambda w(L) = o(1)$. $w_{xxx}(L)q(L) = w_{xxx}(0)q(0) = w_{xx}(L)q'(L) = w_{xx}(0)q'(0) = w_x(L)q''(L) = o(1)$ and $q'''(L) \sim \lambda$. Then we final will arrive at our desired expression

$$\lambda w(L) = o(1) \tag{3.3.4.10}$$

Now we just need to find a function q that satisfies the following properties and the lemma will be complete.

$$-\lambda^2 q + q'''' = 0 \tag{3.3.4.11}$$
$$w_{xxx}(L)q(L) = o(1) \tag{3.3.4.12}$$
$$w_{xxx}(0)q(0) = o(1) = o(1) \tag{3.3.4.13}$$
$$w_{xx}(L)q'(L) = o(1) \tag{3.3.4.14}$$
$$w_{xx}(0)q'(0) = o(1) \tag{3.3.4.15}$$
$$w_x(L)q''(L) = o(1) \tag{3.3.4.16}$$
$$q'''(L) \sim \lambda \tag{3.3.4.17}$$

From (3.3.4.11) and (3.3.4.17) we can see the general form of q(x) will be one of the following.

$$\begin{aligned} q(x) &= \frac{1}{\sqrt{\lambda}}[Asin(\sqrt{\lambda}(x-L)) + Bcos(\sqrt{\lambda}(x-L))] \\ q(x) &= \frac{1}{\sqrt{\lambda}}[Ce^{\sqrt{\lambda}(L-x)} + De^{-\sqrt{\lambda}(L-x)}] \end{aligned} \tag{3.3.4.18}$$

From the GNI we have the following estimates on the two following boundary terms

$$|\frac{w_{xxx}(0)}{\sqrt{\lambda}}| \leq \|\frac{w_{xxx}}{\sqrt{\lambda}}\|_{L^\infty} \leq C_1\|w_{xxx}\|^{1/2}\|\frac{w_{xxxx}}{\lambda}\|^{1/2} \neq O(1) \tag{3.3.4.19}$$

$$|w_{xx}(0)| \leq \|w_{xx}\|_{L^\infty} \leq C_1\|w_{xx}\|^{1/2}\|w_{xxx}\|^{1/2} \neq O(1) \tag{3.3.4.20}$$

These terms are not guaranteed to be bounded. Do to this reason we need $\lambda^{1/4}q(0) = o(1)$ and $\lambda^{1/4}q'(0) = o(1)$ (We can show $\lambda^{-1/2}w_{xxx}(0) \leq O(1)$ that's why we have that power on our λ) so we can see that our function so far needs to be the following.

$$q(x) = \frac{1}{\sqrt{\lambda}}e^{-\sqrt{\lambda}(L-x)} \tag{3.3.4.21}$$

This is the only function that could possible satisfy the restrictions we have made so far. We just need to check that it satisfies (3.3.4.13)-(3.3.4.17). We can see by dissipation function, we can easily show (3.3.4.14) , (3.3.4.16) , (3.3.4.17). Now we just need to check (3.3.4.15) and (3.3.4.13). To this we keep in mind $w_{xx} = O(1)$ and by (3.3.3.3) we can deduce $\frac{w_{xxxx}}{\lambda} = O(1)$ and by the interpolation we have $\|\frac{w_{xxx}}{\sqrt{\lambda}}\| \leq \|w_{xx}\|^{1/2}\|\frac{w_{xxxx}}{\lambda}\|^{1/2}$. With this we can get estimates on the boundary terms using the GNI.

$$|\frac{w_{xxx}(0)}{\lambda^{3/4}}| \leq \|\frac{w_{xxx}}{\lambda^{3/4}}\| \leq C_1\|\frac{w_{xxx}}{\sqrt{\lambda}}\|^{1/2}\|\frac{w_{xxxx}}{\lambda}\|^{1/2} = O(1) \tag{3.3.4.22}$$

It is also easy to see that

$$\lambda^{1/4}e^{-L\sqrt{\lambda}} = o(1) \tag{3.3.4.23}$$

So with (3.3.4.22) and (3.3.4.23) implies (3.3.4.13).

$$|\frac{w_{xx}(0)}{\lambda^{3/4}}| \leq \|\frac{w_{xx}}{\lambda^{3/4}}\| \leq C_1\|\frac{w_{xxx}}{\sqrt{\lambda}}\|^{1/2}\|w_{xx}\|^{1/2} = O(1) \tag{3.3.4.24}$$

So with (3.3.4.23) and (3.3.4.24) implies (3.3.4.15). Therefore q satisfies (3.3.4.11)-(3.3.4.17) which implies (3.3.4.10) so the lemma is complete for the final case.

Conclusion

We can see that the system is exponential stable for the 3 cases that were stated for the homogenous euler-beam equation.

3.4 Boundary Stabilization of the Weakly Coupled Wave Equation (with $\frac{\rho_1}{C_1} = \frac{\rho_2}{C_2}$)

3.4.1 Introduction

We would like to draw our focus to the boundary stabilization of the weakly coupled wave equation. This problem gives great insight in how we can manipulate the coupling term associated with the Rao-Narka . This problem along with section 3.2 and section 3.3 contains all the elements that we will see in our project in Chapter 5. This problem follows very similar to [46]

$$C_1 u_{tt} - \rho_1 u_{xx} + C\phi = 0, \tag{3.4.1.1}$$

$$C_2 \phi_{tt} - \rho_2 \phi_{xx} + Cu = 0, \tag{3.4.1.2}$$

$$u_t(L) = -ku_x(L)\ \phi(0) = u_x(0) = \phi(L) = 0. \tag{3.4.1.3}$$

3.4.2 Preliminary

Firstly, we need to construct a proper Hilbert space for our energy function. We begin by figure out our norm in $\mathcal{H}$. To do this we take the L^2 inner product of (3.4.1.1) with u_t. Eventually we will add it with the L^2 inner product of (3.4.1.2) with ϕ_t.

$$\langle C_1 u_{tt} - \rho_1 u_{xx} + C\phi, u_t \rangle = 0$$

$$\langle C_1 u_{tt}, u_t \rangle - \rho_1 u_t(L) u_x(L) + \rho_1 u_t(0) u_x(0) + \langle \rho_1 u_x, u_{xt} \rangle + \langle C\phi, u_t \rangle = 0$$

$$\frac{1}{2}\frac{d}{dt}[C_1 \|u_t\|^2 + \rho_1 \|u_x\|^2] + \langle C\phi, u_t \rangle = \rho_1 u_t(L) u_x(L)$$

Now doing it for the ϕ term we have

$$\frac{1}{2}\frac{d}{dt}[C_2 \|\phi_t\|^2 + \rho_2 \|\phi_x\|^2] + C\langle u, \phi_t \rangle = \rho_2 \phi_t(L) \phi_x(L)$$

Adding these together and take the reals we arrive at the following expression.

$$\frac{1}{2}\frac{d}{dt}[C_1 \|u_t\|^2 + \rho_1 \|u_x\|^2 + C_2 \|\phi_t\|^2 + \rho_2 \|\phi_x\|^2 + 2CRe[\langle u, \phi \rangle]] = \rho_2 \phi_t(L) \phi_x(L) + \rho_1 u_t(L) u_x(L) \tag{3.4.2.1}$$

So from (3.4.2.1) we can see our energy function is the following

$$E(t) = C_1 \|u_t\|^2 + \rho_1 \|u_x\|^2 + C_2 \|\phi_t\|^2 + \rho_2 \|\phi_x\|^2 + 2CRe[\langle u, \phi \rangle] \tag{3.4.2.2}$$

With the following dissipation function.

$$\frac{1}{2}\frac{d}{dt}E(t) = \rho_2 \phi_t(L) \phi_x(L) + \rho_1 u_t(L) u_x(L)$$

And with (3.4.1.3) the dissipation function is as follows

$$\frac{1}{2}\frac{d}{dt}E(t) = -k\rho_1 u_x^2(L) \tag{3.4.2.3}$$

Denoting $U = (u, u_t, \phi, \phi_t) = (u, v, \phi, \varphi)$ From (3.4.2.2) and (3.4.1.3) we can construct a proper energy space for our U.

$$U \in \mathcal{H} = \{H_k^1 \times L^2 \times H_0^1 \times L^2\} \tag{3.4.2.4}$$

where $H_0 = \{f \in H^1 | f(0) = f(L) = 0\}$, $H_k = \{f \in H^1 | f_x(0) = 0\}$

Shifting gears looking at our system we construct a first order evolution equation for the system in (3.4.1.1)-(3.4.1.2)

$$\frac{dU}{dt} = \mathcal{A}U \rightarrow \begin{pmatrix} v \\ v_t \\ \varphi \\ \varphi_t \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\rho_1}{C_1}D^2 & 0 & C & 0 \\ 0 & 0 & 0 & 1 \\ C & 0 & -\frac{\rho_2}{C_2}D^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ \phi \\ \varphi \end{pmatrix} \tag{3.4.2.5}$$

We need to have $\mathcal{A}$ be a dense operator and our dissipation function needs to less than or equal to zero to be dissipative. Hence, we turn our attention to our domain space. The domain space is constructed in a manner so that $0 \in \rho(\mathcal{A})$ this will be used to show that $\mathcal{A}$ is a infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}$. It is easy to figure out our domain space is

$$D(\mathcal{A}) = \{H^2 \times H^1 \times H^2 \times H^1\} \in \mathcal{H} \tag{3.4.2.6}$$

From (3.4.2.6) and (3.4.2.4) it is not hard to show $D(\mathcal{A}) \subset \mathcal{H}$ this implies that $\mathcal{A}$ is a densely defined linear operator. Turing our attention back to our dissipation function we have.

$$Re[\langle \mathcal{A}U, U\rangle_{\mathcal{H}}] = -\rho_1 k u_x^2(L) \leq 0 \tag{3.4.2.7}$$

By Definition (2.1.0.5) we can now see that $\mathcal{A}$ is dissipative

Theorem 3.4.2.1 *$\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}$.*

Proof: Using Theorem (2.2.0.1) we already can see that $\mathcal{A}$ is dissipative. Now we just have to show $0 \in \rho(\mathcal{A})$. Suppose that $\lambda = 0$ is an eigenvalue and U be the normalized eigenfunction.

$$v = o(1) \ in \ H^1 \tag{3.4.2.8}$$

$$-\frac{\rho_1}{C_1}u_{xx} + \frac{C}{C_1}\phi = o(1) \ in \ L^2 \tag{3.4.2.9}$$

$$\varphi = o(1) \ in \ H^1 \tag{3.4.2.10}$$

$$-\frac{\rho_2}{C_2}\phi_{xx} + \frac{C}{C_2}u = o(1) \ in \ L^2 \tag{3.4.2.11}$$

So we wish to arrive a contradiction to the unit norm. From our dissipation we have.

$$-k|u_x(L)| = o(1) \tag{3.4.2.12}$$

We now begin by taking the L^2 inner product of (3.4.2.9) with $C_1 u$ to arrive at the following.

$$-\rho_1 u_x(L)u(L) + \rho_1 u_x(L)u(L) + \rho_1\|u_x\|^2 + C\langle\phi, u\rangle = o(1) \tag{3.4.2.13}$$

With our boundary conditions and dissipation we can see

$$\rho_1\|u_x\|^2 + C\langle\phi, u\rangle = o(1) \tag{3.4.2.14}$$

Now we again take the L^2 inner product of (3.4.2.11) with $C_2\phi$ with our boundary conditions we get.

$$\rho_3\|\phi_x\|^2 + C\langle\phi, u\rangle = o(1) \tag{3.4.2.15}$$

We finally add (3.4.2.14) with (3.4.2.15) to arrive at the following

$$\rho_1\|u_x\|^2 + 2C\langle\phi, u\rangle + \rho_3\|\phi_x\|^2 = o(1) \tag{3.4.2.16}$$

We can see by (3.4.2.8) , (3.4.2.10) , and (3.4.2.16) that

$$\|v\|^2 + \|\varphi\|^2\rho_1\|u_x\|^2 + 2C\langle\phi, u\rangle + \rho_3\|\phi_x\|^2 = o(1) \tag{3.4.2.17}$$

From above we can see $\|U\|_{\mathcal{H}} = o(1)$ which contradicts the unit norm. Thus $0 \in \rho(\mathcal{A})$. Therefore $\mathcal{A}$ is is the infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}$.

Theorem 3.4.2.2 *Let $S(t) = e^{\mathcal{A}t}$ be a semigroup associated with our system, the semigroup is polynomial stable of* $1/2$

$$i\mathcal{R} \subset \rho(\mathcal{A}) \tag{3.4.2.18}$$

$$\sup_{|\lambda|\to 0} \frac{1}{\lambda^2}\|(i\lambda - \mathcal{A})^{-1}\| < +\infty \tag{3.4.2.19}$$

3.4.3 Proof of Theorem 3.4.2.2

We first wish to show (3.4.2.18). How we do this we assume (3.4.2.18) is false. Then there is a $\lambda \in \sigma(\mathcal{A})$ and unit norm of U such that

$$\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1) \tag{3.4.3.1}$$

i.e.

$$i\lambda u - v = f_1 = o(1) \; in \; H^1 \tag{3.4.3.2}$$

$$i\lambda v - \frac{\rho_1}{C_1}u_{xx} + \frac{C}{C_1}\phi = f_2 = o(1) \; in \; L^2 \tag{3.4.3.3}$$

$$i\lambda\phi - \varphi = f_3 = o(1) \; in \; H^1 \tag{3.4.3.4}$$

$$i\lambda\varphi - \frac{\rho_3}{C_1}\phi_{xx} + \frac{C}{C_2}u = f_4 = o(1) \; in \; L^2 \tag{3.4.3.5}$$

So we have $\|U\|_{\mathcal{H}} = C_1\|v\|^2 + \rho_1\|u_x\|^2 + C_2\|\varphi\|^2 + \rho_2\|\phi_x\|^2 + 2CRE[\langle\phi, u\rangle] = 1$ and we want to reach a contradiction to this. so we begin by taking the $\mathcal{H}$ inner product of $(i\lambda I - \mathcal{A})U$ with U and taking the reals.

$$Re[\langle(i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}}] = -Re[\langle\mathcal{A}U, U\rangle_{\mathcal{H}}] = -ku_x(L) = o(1) \tag{3.4.3.6}$$

We begin by plug in (3.4.3.2) into (3.4.3.3) and we are left with

$$-\lambda^2 u - \frac{\rho_1}{C_1}u_{xx} + \frac{C}{C_1}\phi = f_2 - i\lambda f_1 \tag{3.4.3.7}$$

We also do the same with (3.4.3.3) into (3.4.3.4)

$$-\lambda^2 \phi - \frac{\rho_3}{C_2}\phi_{xx} + \frac{C}{C_2}u = f_4 - i\lambda f_3 \tag{3.4.3.8}$$

Let's begin by taking the L^2 inner product of (3.4.3.7) with $2xu_x$. We also take the reals. With integration by parts we can arrive at the following and with the help of (3.4.1.3)

$$-C_1|\lambda u(L)|^2 + C_1\|\lambda u\|^2 - \rho_1|u_x(L)|^2 + \rho_1\|u_x\|^2 + 2C\langle\phi, xu_x\rangle = f_2 + i\lambda f_1$$

We can see since λ is finite that the RHS will tend to zero. Looking at (3.4.3.2) with the help of GNI we can get some information on the other boundary term, this is done below.

$$|i\lambda u(L) - v(L)| \leq \|i\lambda u - v\|_{L^\infty} \leq K\|f_1\|^{1/2}\|f_{1,x}\|^{1/2} = o(1)$$

So by our boundary conditions we can see

$$\lambda u(L) = o(1) \tag{3.4.3.9}$$

So revisiting above we can see by (3.4.3.9) and (3.4.3.6) our boundary terms tend to zero therefore

$$C_1\|\lambda u\|^2 + \rho_1\|u_x\|^2 + 2RE[C\langle\phi, xu_x\rangle] = o(1)$$

Using Holder inequality we can see

$$2RE[C\langle\phi, xu_x\rangle] \leq K_1C\|\phi\|\|u_x\| \tag{3.4.3.10}$$

By the norm we can deduce ϕ_x and u_x is bounded and by the poincare inequality we can get u and ϕ are bounded. So now we can see

$$\frac{E_1}{\rho_1}\|u_x\|^2 \leq K_2C\|u_x\|^2$$

Therefore for small enough C this implies $\|u_x\| = o(1)$ by poincare equality we obtain $\|u\| = o(1)$. Looking at (3.4.3.2) we get $\|v\| = o(1)$. We now take the L^2 inner product of (3.4.3.7) with ϕ so we can get an estimate on the ϕ term.

$$-\lambda^2\langle u,\phi\rangle - \frac{\rho_1}{C_1}\langle u_{xx},\phi\rangle + \frac{C}{C_1}\|\phi\|^2 = o(1)$$

We can see the first term tends to zero because $\|u\| = o(1)$ and by integration by parts we arrive at the following.

$$-\frac{\rho_1}{C_1}u_x(L)\phi(L) + \frac{\rho_1}{C_1}u_x(0)\phi(0) + \frac{\rho_1}{C_1}\langle u_x,\phi_x\rangle + \frac{C}{C_1}\|\phi\|^2 = o(1)$$

We can see the boundary terms are of zero, and since $\|u_x\| = o(1)$ we then can see.

$$\|\phi\| = o(1) \tag{3.4.3.11}$$

Then if we take the L^2 inner product of (3.4.3.8) it is easy to conclude $\|\phi_x\| = o(1)$. Finally with (3.4.3.11), looking at (3.4.3.4) we obtain $\|\varphi\| = o(1)$. With all this we can see it implies $\|U\|_{\mathcal{H}} = o(1)$, which is a contradiction to the unit norm. Hence, (3.4.2.18) is satisfied. Now it can remains to show (3.4.2.19)

We now wish to verify condition (3.4.2.19). Again, it is a proof by contradiction so we assume condition (3.4.2.19) is false and we have a $\|U\|_{\mathcal{H}} = 1$. Assuming condition (3.4.2.19) is false implies that there exists $\lambda \to \infty$ such that

$$\|\lambda^p(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1) \tag{3.4.3.12}$$

i.e.

$$\lambda^p[i\lambda u - v] = f_1 = o(1)\ in\ H^1 \tag{3.4.3.13}$$

$$\lambda^p[iC_1\lambda v - \rho_1 u_{xx} + C\phi] = f_2 = o(1)\ in\ L^2 \tag{3.4.3.14}$$

$$\lambda^p[i\lambda\phi - \varphi] = f_3 = o(1)\ in\ H^1 \tag{3.4.3.15}$$

$$\lambda^p[i\lambda C_2\varphi - \rho_2\phi_{xx} + Cu] = f_4 = o(1)\ in\ L^2 \tag{3.4.3.16}$$

So we have $\|U\|_{\mathcal{H}} = C_1\|v\|^2 + \rho_1\|u_x\|^2 + C_2\|\varphi\|^2 + \rho_2\|\phi_x\|^2 + 2CRE[\langle\phi,u\rangle] = 1$ and we want to reach a contradiction to this. so we begin by Taking the $\mathcal{H}$ inner product of $\lambda^p(i\lambda I - \mathcal{A})U$ with U and taking the reals.

$$Re[\langle\lambda^p(i\lambda I - \mathcal{A})U,U\rangle_{\mathcal{H}}] = -Re[\langle\mathcal{A}U,U\rangle_{\mathcal{H}}] = -k\lambda^{p/2}u_x(L) = o(1) \tag{3.4.3.17}$$

It is easy to see with (3.4.3.13) we also have

$$\lambda^{p/2+1}u(L) = o(1) \tag{3.4.3.18}$$

We begin plugging in (3.4.3.13) into (3.4.3.14) and (3.4.3.15) into (3.4.3.16). Doing this we will arrive at the following 2 equations.

$$-\lambda^{p+2}C_1u - \lambda^p\rho_1u_{xx} + C\lambda^p\phi = i\lambda f_1 + f_2 \in L^2 \tag{3.4.3.19}$$

$$-\lambda^{p+2}C_2\phi - \lambda^p\rho_2\phi_{xx} + C\lambda^p u = i\lambda f_3 + f_4 \in L^2 \tag{3.4.3.20}$$

It is easy to obtain from our norm (3.4.3.13) and (3.4.3.15) the following

$$\begin{aligned} \|\lambda u\| = O(1) \; and \; \|u_x\| = O(1) \\ \|\lambda \phi\| = O(1) \; and \; \|\phi_x\| = O(1) \end{aligned} \tag{3.4.3.21}$$

If we take the L^2 inner product of (3.4.3.19) with $2xu_x$ and then take the reals we will get the following with the use of (3.4.3.17) and (3.4.3.18)

$$C_1\|\lambda^2 u\|^2 + \rho_1\|\lambda u_x\|^2 + 2CRe[\langle \lambda\phi, x\lambda u_x\rangle] = \frac{Re[\langle i\lambda f_1 + f_2, xu_x\rangle]}{\lambda^{p-2}} \tag{3.4.3.22}$$

Since $f_1 \in H_1$ with integration by parts it is easy to see that the RHS will tend to zero. Since $\|\lambda\phi\| = O(1)$ we can deduce.

$$C_1\|\lambda^2 u\|^2 + \rho_1\|\lambda u_x\|^2 = \frac{O(1)}{\lambda^{p-2}} \tag{3.4.3.23}$$

We again take the L^2 inner product of (3.4.3.19) with ϕ to arrive at the following.

$$-\lambda^{p+2}\langle u, \phi\rangle - \lambda^p \frac{\rho_1}{C_1} u_x\phi|_{x=0,L} + \lambda^p \frac{\rho_1}{C_1}\langle u_x, \phi_x\rangle + \frac{C}{C_1}\lambda^p\|\phi\|^2 = \langle i\lambda f_1 + f_2, \phi\rangle = o(1) \tag{3.4.3.24}$$

Now we take the L^2 inner product of (3.4.3.20) with -u

$$\lambda^{p+2}\langle \phi, u\rangle + \lambda^p \frac{\rho_2}{C_2} u\phi_x|_{x=0,L} - \lambda^p \frac{\rho_2}{C_2}\langle \phi_x, u_x\rangle = o(1) \tag{3.4.3.25}$$

If we add (3.4.3.24) with (3.4.3.25) then take the reals since we have the same wave speed we can see it reduces to the following

$$-\lambda^p \frac{\rho_1}{C_1} u_x\phi|_{x=0,L} + \lambda^p \frac{\rho_2}{C_2} u\phi_x|_{x=0,L} + \frac{C}{C_1}\lambda^p\|\phi\|^2 = o(1)$$

With our boundary conditions we can see that

$$\|\lambda^{p/2}\phi\| = o(1) \tag{3.4.3.26}$$

Now taking the inner product of (3.4.3.20) with ϕ we can deduce

$$\|\lambda^{p/2-1}\phi\| = o(1) \tag{3.4.3.27}$$

so when our $p \geq 2$ we can see that it implies $\|U\|_{\mathcal{H}} = o(1)$ but however this contradicts that $\|U\|_{\mathcal{H}} = 1$. Therefore the proof is complete by contradiction.

3.5 Weakly Coupled Wave Equation with Viscous Damping Term

What better way to transition into internal damping than to introduce that exact same problem we did in the previous section but this time instead of boundary damping, we have internal damping. This is why we introduce the weakly coupled wave equation with viscous damping term so we can begin to introduce our other project in Chapter 4. The problem introduced here follows from [45]

3.5.1 Introduction

$$u_{tt} - k_1 u_{xx} + k_2 \phi = 0, \tag{3.5.1.1}$$

$$\phi_{tt} - k_3 \phi_{xx} + k_2 u + k_4 \phi_t = 0, \tag{3.5.1.2}$$

We will impose the following initial and boundary conditions

$$u(0) = u(L) = \phi(0) = \phi(L) = 0. \tag{3.5.1.3}$$

3.5.2 Preliminary

First, we begin by construction a proper state space so the energy function of the system is dissipative. To do this we start by figuring out our energy function for the system to do this we take the L^2 inner product of (3.5.1.1) with u_t, (3.5.1.2) with ϕ_t and then add them together. Doing so with our boundary conditions we will arrive at the following.

$$\frac{1}{2}\frac{d}{dt}[\|u_t\|^2 + k_1\|u_x\|^2 + \|\phi_t\|^2 + k_3\|\phi_x\|^2 + 2k_2 Re\langle u, \phi\rangle] = -k_4\|\phi_t\|^2 \tag{3.5.2.1}$$

Looking at (3.5.2.1) we can see our energy function is as follows

$$E(t) = \|u_t\|^2 + k_1\|u_x\|^2 + \|\phi_t\|^2 + k_3\|\phi_x\|^2 + 2k_2 Re\langle u, \phi\rangle \tag{3.5.2.2}$$

With the following dissipation function

$$\frac{1}{2}\frac{d}{dt}E(t) = -k_4\|\phi_t\|^2 \tag{3.5.2.3}$$

Looking at our energy function we can deduce that the norm is in the following Hilbert space.

$$\mathcal{H} = H_0^1 \times L^2 \times H_0^1 \times L^2$$

Denoting $U = (u, v, \phi, \varphi)$, we then convert our system into a first order evolution equation on $\mathcal{H}$

$$\frac{dU}{dt} = \mathcal{A}U = \begin{pmatrix} v \\ k_1 u_{xx} - k_2 \phi \\ \varphi \\ k_3 \phi_{xx} - k_2 u - k_4 \varphi \end{pmatrix} \tag{3.5.2.4}$$

With the following domain space.

$$D(\mathcal{A}) = H^2 \times H^1 \times H^2 \times H^1 \in \mathcal{H}$$

It is easy to show that $D(\mathcal{A})$ is dense in $\mathcal{H}$. It is also easy to see that.

$$Re\langle AU, U\rangle_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}E(t) = -k_4\|\phi_t\|^2 \leq 0 \quad (3.5.2.5)$$

Therefore we can conclude that our system is dissipative.

Theorem 3.5.2.1 *$\mathcal{A}$ is a infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}$.*

Proof.

Since $D(\mathcal{A})$ is dense in $\mathcal{H}$ and $\mathcal{A}$ is dissipative. We now just have to show that $0 \in \rho(\mathcal{A})$, we do this proof by a proof by contradiction we assume that $0 \notin \rho(\mathcal{A})$ and $\|U\|_{\mathcal{H}} = 1$. If $0 \notin \rho(\mathcal{A})$ then $\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$ where $\lambda = 0$

i.e.

$$v = o(1) \in H^1 \quad (3.5.2.6)$$

$$k_1 u_{xx} - k_2\phi = o(1) \in L^2 \quad (3.5.2.7)$$

$$\varphi = o(1) \in H^1 \quad (3.5.2.8)$$

$$k_3\phi_{xx} - k_2 u - k_4\varphi = o(1) \in L^2 \quad (3.5.2.9)$$

From our dissipation function we have

$$\|\varphi\| = o(1) \quad (3.5.2.10)$$

We begin by taking the L^2 inner product of (3.5.2.7) with -u

$$k_1\|u_x\|^2 + k_2\langle\phi, u\rangle = o(1) \quad (3.5.2.11)$$

Then we take the inner product L^2 inner product of (3.5.2.9) with $-\phi$

$$k_3\|\phi_x\|^2 + k_2\langle u, \phi\rangle = o(1) \quad (3.5.2.12)$$

Adding the two equations together and taking the reals we arrive at

$$k_3\|\phi_x\|^2 + k_1\|u_x\|^2 + 2k_2\langle\phi, u\rangle = o(1) \quad (3.5.2.13)$$

Then by (3.5.2.6) , (3.5.2.8), and (3.5.2.13) we see that $\|U\|_{\mathcal{H}} = o(1)$ which is a contradiction to that $\|U\|_{\mathcal{H}} = 1$ then by Lumer-Phillips Theorem the proof is complete.

Theorem 3.5.2.2 *Let $S(t) = e^{\mathcal{A}t}$ be a semigroup associated with our system, the semigroup is polynomial stable of* $1/4$

To show this proof we use theorem (2.4.0.1) as we can see we have two parts to this proof which we do in the following subsections

3.5.3 Showing $i\mathcal{R} \subset \rho(\mathcal{A})$

We first wish to show $i\mathcal{R} \subset \rho(\mathcal{A})$. How we do this we assume it is false. Then there is a $\lambda \in \sigma(\mathcal{A})$ and unit norm of U such that

$$\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1) \tag{3.5.3.1}$$

i.e.

$$i\lambda u - v = o(1) \in H^1 \tag{3.5.3.2}$$

$$i\lambda v - k_1 u_{xx} - k_2 \phi = o(1) \in L^2 \tag{3.5.3.3}$$

$$i\lambda \phi - \varphi = o(1) \in H^1 \tag{3.5.3.4}$$

$$i\lambda \varphi - k_3 \phi_{xx} - k_2 u - k_4 \varphi = o(1) \in L^2 \tag{3.5.3.5}$$

We have

$$Re\langle(i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}} = -Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -k_4\|\varphi\|^2 = o(1) \tag{3.5.3.6}$$

Looking at (3.5.3.4) with (3.5.3.6) we have

$$\|\phi\| = o(1) \tag{3.5.3.7}$$

Taking the L^2 inner product of (3.5.4.4) with ϕ we can see

$$\|\phi_x\| = o(1) \tag{3.5.3.8}$$

Now taking the L^2 inner product of u with (3.5.4.4) with integration by parts and since our boundary terms vanish we have.

$$\|u\| = o(1) \tag{3.5.3.9}$$

Looking at (3.5.3.2) with (3.5.3.9) we have

$$\|v\| = o(1) \tag{3.5.3.10}$$

Now if we take the L^2 inner product of (3.5.3.3) with u we finally have

$$\|u_x\| = o(1) \tag{3.5.3.11}$$

We can now see we have reached the promised contradiction of having a unit norm since from everything above we have $\|U\|_{\mathcal{H}} = o(1)$. Therefore we can conclude that $i\mathcal{R} \subset \rho(\mathcal{A})$. Now we just have to show $\sup_{|\lambda|\to 0} \frac{1}{\lambda^4}\|(i\lambda - \mathcal{A})^{-1}\| < +\infty$ to complete the proof which is done in the next subsection

3.5.4 Showing $\sup_{|\lambda|\to 0} \frac{1}{\lambda^4}\|(i\lambda - \mathcal{A})^{-1}\| < +\infty$

We do the proof by a contradiction we assume that $\sup_{|\lambda|\to 0} \frac{1}{\lambda^4}\|(i\lambda - \mathcal{A})^{-1}\| < +\infty$ is false. this implies that $\exists \lambda \to \infty \ni \lambda^4\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$. This is written out below, we also have a unit norm aka $\|U\|_{\mathcal{H}} = 1$. We now express $\lambda^4\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$.

i.e.

$$\lambda^4[i\lambda u - v] = o(1) \in H^1 \tag{3.5.4.1}$$

$$\lambda^4[i\lambda v - k_1 u_{xx} - k_2\phi] = o(1) \in L^2 \tag{3.5.4.2}$$

$$\lambda^4[i\lambda\phi - \varphi] = o(1) \in H^1 \tag{3.5.4.3}$$

$$\lambda^4[i\lambda\varphi - k_3\phi_{xx} - k_2 u - k_4\varphi] = o(1) \in L^2 \tag{3.5.4.4}$$

We have

$$\lambda^4 Re\langle(i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}} = -\lambda^4 Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -k_4\|\lambda^2\varphi\|^2 = o(1) \tag{3.5.4.5}$$

We can deduce by the $\mathcal{H}$ norm the following are bounded.

$$\|v\| = O(1)\ ,\ \|u_x\| = O(1)\ ,\ \|\phi_x\| = O(1) \tag{3.5.4.6}$$

From (3.5.4.1) , (3.5.4.3) , (3.5.4.4) and (3.5.4.6) we can see

$$\|\frac{v_x}{\lambda}\| = O(1)\ ,\ \|\frac{\varphi_x}{\lambda}\| = O(1)\ ,\ \|\lambda u\| = O(1)\ ,\ \|\lambda^3\phi\| = o(1) \tag{3.5.4.7}$$

We begin by taking the L^2 inner product of (3.5.4.4) with ϕ

$$\langle i\lambda^2\varphi, \lambda^3\phi\rangle + k_3\lambda^4\langle\phi_{xx}, \phi\rangle - k_2\langle\lambda u, \lambda^3\phi\rangle - k_4\langle\lambda\varphi, \lambda^3\phi\rangle = o(1) \tag{3.5.4.8}$$

We can easily see

$$k_3\lambda^4\langle\phi_{xx}, \phi\rangle = o(1) \tag{3.5.4.9}$$

Finally with integration by parts this will imply

$$\|\lambda^2\phi_x\| = o(1) \tag{3.5.4.10}$$

If we turn attention back to (3.5.4.4), Let us take the L^2 inner product with u.

$$\langle i\lambda^2\varphi, \lambda u\rangle + k_3\lambda^2\langle\phi_{xx}, u\rangle - k_2\|\lambda u\|^2 - k_4\langle\lambda\varphi, \lambda\phi\rangle = o(1)$$

It is easy to see we will be left with the following.

$$+k_3\lambda^2 u\phi_x|_{x=0,L} - k_3\lambda^2\langle\phi_x, u_x\rangle - k_2\|\lambda u\|^2 = o(1)$$

With GNI we can show that the boundary term tends to zero no matter what boundary condition we choice for our u and ϕ, this is done below.

$$|\lambda^{1/2}u|_{x=0orL}| \leq \|\lambda^{1/2}u\|_{L^\infty} \leq K\|\lambda u\|^{1/2}\|u_x\|^{1/2} = O(1)$$

Also looking at (3.5.4.4) we can deduce $\|\lambda\phi_{xx}\| = O(1)$

$$|\lambda^{3/2}\phi_x|_{x=0orL}| \leq \|\lambda^{3/2}\phi_x\|_{L^\infty} \leq K\|\lambda^2\phi_x\|^{1/2}\|\lambda\phi_{xx}\|^{1/2} = o(1)$$

So we can see the boundary term tends to zero with (3.5.4.10) and (3.5.4.6) we can see we will arrive at the following

$$\|\lambda u\| = o(1) \tag{3.5.4.11}$$

by (3.5.4.1) we can now see with (3.5.4.11)

$$\|v\| = o(1) \tag{3.5.4.12}$$

Finally taking the L^2 inner product of (3.5.4.2) with u it is easy to see we will arrive at

$$\|u_x\| = o(1) \tag{3.5.4.13}$$

Therefore we can now see that $\|U\|_{\mathcal{H}} = o(1)$, but this contradicts that $\|U\|_{\mathcal{H}} = 1$ hence we arrive at the contradiction and the proof is complete.

3.6 Fourier One-Dimensional Thermo-Porous-Elasticity with Microtemperatures

3.6.1 Introduction

We consider the case where $\mu\xi > 4\mu_0^2$ with this restriction we can obtain that 0 will be an element of the resolvent set that's why we make those restrictions. The system that we will introduce is very similar to the problem we will address in chapter 4. In fact the paper that preceded our project [24] is almost exactly the same with one exception. The system in [24] is derived from a 1st order and a 2nd order taylor expansion. We introduce the easier case where we only keep the zeroth order term in our expansion. We as of now take the system as is but in our project we show where it actually comes from. Our system is introduced below

$$\rho u_{tt} = \mu u_{xx} + \mu_0\phi_x - \beta_0\theta_x, \tag{3.6.1.1}$$

$$J\phi_{tt} = a_0\phi_{xx} - \mu_0 u_x - \mu_2 T_x + \beta_1\theta - \xi\phi, \tag{3.6.1.2}$$

$$a\theta_t = -\beta_0 u_{xt} - \beta_1\phi_t + k\theta_{xx} + k_1 T_x, \tag{3.6.1.3}$$

$$bT_t = -\mu_2\phi_{xt} + k_4 T_{xx} + -k_2 T - k_1\theta_x, \tag{3.6.1.4}$$

3.6.2 Preliminary

First, we wish to construct a proper Hilbert space for our energy function. We begin by figure out our norm in $\mathcal{H}$. The first step we make is we take the L^2 inner product of (3.6.1.1) with u_t, We can conclude the following

$$\frac{1}{2}\frac{d}{dt}[\rho\|u_t\|^2 + \mu\|u_x\|^2] + \langle -\mu_0\phi_x + \beta_0\theta_x, u_t\rangle = 0 \tag{3.6.2.1}$$

Now we take the inner product of (3.6.1.2) with ϕ_t

$$\frac{1}{2}\frac{d}{dt}[J\|\phi_t\|^2 + a_0\|\phi_x\|^2 + \xi\|\phi\|^2] + \langle \mu_0 u_x + \mu_2 T_x - \beta_1\theta, \phi_t\rangle = 0 \tag{3.6.2.2}$$

Taking the reals of (3.6.2.1) and (3.6.2.2). Then adding the two together we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[J\|\phi_t\|^2 + a_0\|\phi_x\|^2 + \xi\|\phi\|^2 + \rho\|u_t\|^2 + \mu\|u_x\|^2] \\ &\quad + \langle -\mu_0\phi_x + \beta_0\theta_x, u_t\rangle + \langle \mu_2 T_x - \beta_1\theta, \phi_t\rangle = 0 \end{aligned} \tag{3.6.2.3}$$

Taking the inner product of (3.6.1.3) with θ leads to the following

$$\frac{1}{2}\frac{d}{dt}[a\|\theta\|^2] + \langle \beta_0 u_{xt} + \beta_1\phi_t - k_1 T_x, \theta\rangle = -k\|\theta_x\|^2 \tag{3.6.2.4}$$

Taking the inner product of (3.6.1.4) with T leads to the following

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[b\|T\|^2] + \langle \mu_2\phi_{xt} + k_1\theta_x, T\rangle \\ &\quad = -k_4\|T_x\|^2 - k_2\|T\|^2 \end{aligned} \tag{3.6.2.5}$$

Adding (3.6.2.4) and (3.6.2.5). Then taking the reals combine to the following.

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[a\|\theta\|^2 + b\|T\|^2] \\ &+ \langle \beta_0 u_{xt} + \beta_1\phi_t, \theta\rangle + \mu_2\langle \phi_{xt}, T\rangle = 2k_1 Re\langle T, \theta_x\rangle - k_4\|T_x\|^2 - k_2\|T\|^2 - k\|\theta_x\|^2 \end{aligned} \tag{3.6.2.6}$$

We finally combine (3.6.2.3) with (3.6.2.6) and we will arrive at the following

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[a\|\theta\|^2 + b\|T\|^2 \\ J\|\phi_t\|^2 + a_0\|\phi_x\|^2 + \xi\|\phi\|^2 + \rho\|u_t\|^2 + \mu\|u_x\|^2] &= 2k_1 Re\langle T, \theta_x\rangle \\ &\quad -k_4\|T_x\|^2 - k_2\|T\|^2 - k\|\theta_x\|^2 \end{aligned} \tag{3.6.2.7}$$

To guarantee this system will be dissipative we need to make the restriction that $kk_2 > 2k_1^2$. From (3.6.2.7) we can see our energy function is the following

$$E(t) = a\|\theta\|^2 + b\|T\|^2 + k_4\tau_2\|T_x\|^2$$
$$J\|\phi_t\|^2 + a_0\|\phi_x\|^2 + \xi\|\phi\|^2 + \rho\|u_t\|^2 + \mu\|u_x\|^2 \tag{3.6.2.8}$$

Also from (3.6.2.7) we can see our dissipation function is as follows

$$\frac{1}{2}\frac{d}{dt}E(t) = 2k_1 Re\langle T, \theta_x\rangle - k_4\|T_x\|^2 - k_2\|T\|^2 - k\|\theta_x\|^2 \leq 0 \tag{3.6.2.9}$$

With our restrictions on the constants we can see our dissipation function is dissipative. From looking at our energy function we can construct a proper a Hilbert space.

$$\mathcal{H} = H^1 \times L^2 \times H^1 \times L^2 \times H^1 \times H^1$$

Such that $\mathcal{H}$ contains (3.6.1.5)

Denoting $U = (u, v, \phi, \varphi, \theta, T)$, we then convert our system into a first order evolution equation on Hilbert space $\mathcal{H}$

$$\frac{dU}{dt} = \mathcal{A}U = \begin{pmatrix} v \\ \frac{\mu}{\rho}u_{xx} + \frac{\mu_0}{\rho}\phi_x - \frac{\beta_0}{\rho}\theta_x \\ \varphi \\ \frac{a_0}{J}\phi_{xx} - \frac{\mu_0}{J}u_x - \frac{\mu_2}{J}T_x + \frac{\beta_1}{J}\theta - \frac{\xi}{J}\phi \\ -\frac{\beta_0}{a}u_{xt} - \frac{\beta_1}{a}\phi_t + \frac{k}{a}\theta_{xx} + \frac{k_1}{a}T_x \\ -\frac{\mu_2}{b}\phi_{xt} + \frac{k_4}{b}T_{xx} - \frac{k_2}{b}T - \frac{k_1}{b}\theta_x \end{pmatrix}$$

and

$$D(\mathcal{A}) = H^2 \times H^1 \times H^2 \times H^1 \times H^2 \times H^2 \in \mathcal{H}$$

Such that $\mathcal{H}$ contains (3.6.1.5)

Theorem 3.6.2.1 *$\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on the Hilbert space $\mathcal{H}$*

Proof.

$$Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}\|U\|^2_{\mathcal{H}} = 2k_1 Re\langle T, \theta_x\rangle - k_4\|T_x\|^2 - k_2\|T\|^2 - k\|\theta_x\|^2 \leq 0 \tag{3.6.2.10}$$

We can see that $\mathcal{A}$ is dissipative. It is easy to show $D(\mathcal{A})$ is dense in $\mathcal{H}$. Let our eigenvalue $\lambda = 0$ and our eigenfunctions U be normalized. So for the proof we Assume $\|U\|_{\mathcal{H}} = 1$ and we wish to show that this is contradicted for $\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$ where $\lambda = 0$.

i.e.

$$v = o(1) \in H^1 \tag{3.6.2.11}$$

$$\frac{\mu}{\rho}u_{xx} + \frac{\mu_0}{\rho}\phi_x - \frac{\beta_0}{\rho}\theta_x \in L^2 \tag{3.6.2.12}$$

$$\varphi = o(1) \in H^1 \tag{3.6.2.13}$$

$$\frac{a_0}{J}\phi_{xx} - \frac{\mu_0}{J}u_x - \frac{\mu_2}{J}T_x + \frac{\beta_1}{J}\theta - \frac{\xi}{J}\phi = o(1) \in L^2 \tag{3.6.2.14}$$

$$-\frac{\beta_0}{a}v_x - \frac{\beta_1}{a}\varphi + \frac{k}{a}\theta_{xx} + \frac{k_1}{a}T_x = o(1) \in L^2 \tag{3.6.2.15}$$

$$-\frac{\mu_2}{b}\varphi_x + \frac{k_4}{b}T_{xx} - \frac{k_2}{b}T - \frac{k_1}{b}\theta_x = o(1) \in L^2 \tag{3.6.2.16}$$

From the dissipation

$$\|T_x\| = \|T\| = \|\theta_x\| = o(1) \tag{3.6.2.17}$$

We can see from (3.6.2.10) and Poincare's inequality

$$\|\theta\| = o(1) \tag{3.6.2.18}$$

We begin by taking the L^2 inner product of (3.6.2.14) with $-\phi$. We can see with (3.6.2.17) and (3.6.2.18) that we can achieve the following.

$$a_0\|\phi_x\|^2 + \mu_0\langle u_x, \phi\rangle + \xi\|\phi\|^2 = o(1) \tag{3.6.2.19}$$

Now we take the L^2 inner product of (3.6.2.12) with $-u$ with (3.6.2.17) and integration by parts we get.

$$\mu\|u_x\|^2 + \mu_0\langle \phi, u_x\rangle - \mu_0\phi(x)u(x)|_{x=0,L} = o(1) \tag{3.6.2.20}$$

Adding (3.6.2.19) with (3.6.2.20), taking the reals and with our boundary conditions we get.

$$\mu\|u_x\|^2 + 2\mu_0 Re\langle \phi, u_x\rangle + a_0\|\phi_x\|^2 + \xi\|\phi\|^2 = o(1) \tag{3.6.2.21}$$

With the restriction we made early that $\mu\xi > 4\mu_0^2$ we can see the following

$$\|u_x\| = \|\phi_x\| = \|\phi\| = o(1) \tag{3.6.2.22}$$

and with poncare inequality and (3.6.2.22) this implies

$$\|u\| = o(1) \tag{3.6.2.23}$$

Now with (3.6.2.11) , (3.6.2.13), (3.6.2.17) , (3.6.2.22) and (3.6.2.23) we can see this implies $\|U\|_{\mathcal{H}} = o(1)$ which is a contradiction that we have a unit norm. Therefore, the proof is complete so $0 \in \rho(\mathcal{A})$ and since $\mathcal{A}$ is dissipative and $D(\mathcal{A})$ is dense in $\mathcal{H}$. We can see now that the proof is complete and $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on the Hilbert space $\mathcal{H}$ by Theorem 2.2.0.1.

We now turn our attention to the problem of achieve a decay rate for the system.

Theorem 3.6.2.2 *Let $e^{\mathcal{A}t}$ be the semigroup associated with our system then the semigroup $e^{\mathcal{A}t}$ is exponential stable.*

We complete this proof by using theorem (2.3.0.1). I decided to break this proof into two subsection shown below.

3.6.3 Proof of Theorem 3.6.2.2 ($i\mathcal{R} \subset \rho(\mathcal{A})$)

We begin by showing $i\mathcal{R} \subset \rho(\mathcal{A})$. To do this we assume that it is false. Then there exist sequence λ_n that approaches a finite value such that $\|(i\lambda I - \mathcal{A})U\| = o(1)$,

i.e.

$$i\lambda u - v = o(1) \in H^1 \tag{3.6.3.1}$$

$$i\lambda\rho v - \mu u_{xx} - \mu_0\phi_x + \beta_0\theta_x = o(1) \in L^2 \tag{3.6.3.2}$$

$$i\lambda\phi - \varphi = o(1) \in H^1 \tag{3.6.3.3}$$

$$i\lambda J\varphi - a_0\phi_{xx} + \mu_0 u_x + \mu_2 T_x - \beta_1\theta + \xi\phi = o(1) \in L^2 \tag{3.6.3.4}$$

$$i\lambda a\theta + \beta_0 v_x + \beta_1\varphi - k\theta_{xx} - k_1 T_x = o(1) \in L^2 \tag{3.6.3.5}$$

$$i\lambda bT + \mu_2\varphi_x - k_4 T_{xx} + k_2 T + k_1\theta_x = o(1) \in L^2 \tag{3.6.3.6}$$

From the dissipation

$$\|T_x\| = \|T\| = \|\theta_x\| = o(1) \tag{3.6.3.7}$$

We can see from (3.6.2.10) and Poincare's inequality

$$\|\theta\| = o(1) \tag{3.6.3.8}$$

Looking at (3.6.3.6) we can see after using (3.6.3.7) and (3.6.3.8)

$$\mu_2\varphi_x - k_4 T_{xx} = o(1) \in L^2$$

We can plug in (3.6.3.3) into above to achieve the following

$$\mu_2 i\lambda\phi_x - k_4 T_{xx} = o(1)$$

Now if we take the L^2 inner product of above with ϕ_x integrating by parts and with our boundary conditions we get

$$\|\phi_x\| = o(1) \tag{3.6.3.9}$$

Then from (3.6.3.3) and poincare inequality we have the three following expressions

$$\|\phi\| = \|\varphi_x\| = \|\varphi\| = o(1) \tag{3.6.3.10}$$

Taking the L^2 inner product of u_x it is not hard to see we can get

$$\|u_x\| = o(1) \tag{3.6.3.11}$$

With poincare inequality and (3.6.3.1) we have

$$\|u\| = \|v\| = o(1) \tag{3.6.3.12}$$

We can now see we have reached the promised contradiction of having a unit norm since from everything above we have $\|U\|_{\mathcal{H}} = o(1)$. Therefore we can conclude that $i\mathcal{R} \subset \rho(\mathcal{A})$. Now we just have to show $\overline{\lim}_{|\lambda|\to\infty}\|(i\lambda I - \mathcal{A})^{-1}\| < \infty$ to complete the proof which is done in the next subsection

3.6.4 Proof of Theorem 3.6.2.2 ($\overline{\lim}_{|\lambda|\to\infty}\|(i\lambda I - \mathcal{A})^{-1}\| < \infty$)

This proof is a proof by contraction so we assume $\overline{\lim}_{|\lambda|\to\infty}\|(i\lambda I - \mathcal{A})^{-1}\| < \infty$ is false which implies $\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$ we let our eigenfunctions be normalized so this implies $\|U\|_{\mathcal{H}} = 1$ so we eventually wish to reach a contradiction to the unit norm. So to reiterate we have $\overline{\lim}_{|\lambda|\to\infty}\|(i\lambda I - \mathcal{A})^{-1}\| < \infty$

i.e.

$$i\lambda u - v = o(1) \in H^1 \tag{3.6.4.1}$$

$$i\lambda\rho v - \mu u_{xx} - \mu_0\phi_x + \beta_0\theta_x = o(1) \in L^2 \tag{3.6.4.2}$$

$$i\lambda\phi - \varphi = o(1) \in H^1 \tag{3.6.4.3}$$

$$i\lambda J\varphi - a_0\phi_{xx} + \mu_0 u_x + \mu_2 T_x - \beta_1\theta + \xi\phi = o(1) \in L^2 \tag{3.6.4.4}$$

$$i\lambda a\theta + \beta_0 v_x + \beta_1\varphi - k\theta_{xx} - k_1 T_x = o(1) \in L^2 \tag{3.6.4.5}$$

$$i\lambda bT + \mu_2\varphi_x - k_4 T_{xx} + k_2 T + k_1\theta_x = o(1) \in L^2 \tag{3.6.4.6}$$

From the dissipation

$$\|T_x\| = \|T\| = \|\theta_x\| = o(1) \tag{3.6.4.7}$$

We can see from (3.6.2.10) and Poincare's inequality

$$\|\theta\| = o(1) \tag{3.6.4.8}$$

we can deduce from the norm

$$\|v\| = O(1)\ ,\ \|u_x\| = O(1)\ ,\ \|\varphi\| = O(1)\ ,\ \|\phi_x\| = O(1) \tag{3.6.4.9}$$

from (3.6.4.9) we can see with (3.6.4.1) and (3.6.4.3)

$$\|\frac{\varphi_x}{\lambda}\| = O(1)\ ,\ \|\lambda\phi\| = O(1)\ ,\|\frac{v_x}{\lambda}\| = O(1)\ ,\ \|\lambda u\| = O(1) \tag{3.6.4.10}$$

We can also deduce with (3.6.4.9) if we turn our attention to (3.6.4.1) and (3.6.4.3)

$$\|\frac{u_{xx}}{\lambda}\| = O(1)\ ,\ \|\frac{\phi_{xx}}{\lambda}\| = O(1) \tag{3.6.4.11}$$

We turn our attention to (3.6.4.6) in the view of (3.6.4.3) and using (3.6.4.7) we will arrive at the following.

$$i\lambda bT + \mu_2 i\lambda\phi_x - k_4 T_{xx} = o(1) \in L^2$$

We now take the L^2 inner product with ϕ_x. We also divide by λ to conclude

$$\langle ibT, \phi_x\rangle + \mu_2 i\|\phi_x\|^2 - \frac{k_4}{\lambda}\langle T_{xx}, \phi_x\rangle = o(1)$$

By (3.6.4.7) and (3.6.4.9) we can see the first term tends to zero. To deal with the last term we integrate by parts. Doing so we arrive at

$$\mu_2 i\|\phi_x\|^2 - \frac{k_4}{\lambda}\phi_x T_x|_{x=0,L} + k_4\langle T_x, \frac{\phi_{xx}}{\lambda}\rangle = o(1)$$

If we have the approiate boundary conditions the boundary term will be zero but if we were to have dirchlet condtions on our ϕ and T we need GNI to show that these terms tend to zero. This is which I do below.

$$|\lambda^{-1/2}\phi_x||_{x=0\ or\ L} \leq \|\lambda^{-1/2}\phi_x\|_{L^\infty} \leq K\|\phi_x\|^{1/2}\|\frac{\phi_{xx}}{\lambda}\|^{1/2} = O(1)$$

$$|\lambda^{-1/2}T_x||_{x=0\ or\ L} \leq \|\lambda^{-1/2}T_x\|_{L^\infty} \leq K\|T_x\|^{1/2}\|\frac{T_{xx}}{\lambda}\|^{1/2} = o(1)$$

Therefore we can conclude $\frac{1}{\lambda}\phi_x T_x|_{x=0,L} = o(1)$ so no matter if our boundary conditions are mixed, neuman or dirchlet our boundary terms are either zero or tend to zero. Now looking at our last term by (3.6.4.7) and (3.6.4.11) we can see it also will tend to zero. So we finally arrive at

$$\|\phi_x\| = o(1) \tag{3.6.4.12}$$

With poincare's inequality we also have

$$\|\phi\| = o(1) \tag{3.6.4.13}$$

Now we turn our attention to (3.6.4.5) if you divide by λ using (3.6.4.7) and (3.6.4.9) we get

$$\frac{\beta_0}{\lambda}v_x - \frac{k}{\lambda}\theta_{xx} = o(1) \in L^2$$

In the view of (3.6.4.1) we have

$$i\beta_0 u_x - \frac{k}{\lambda}\theta_{xx} = o(1) \in L^2$$

Now we take the inner product with u_x and with integration by parts we can see the last term will tend to zero. Therefore we arrive at

$$\|u_x\| = o(1) \tag{3.6.4.14}$$

With the poincare inequlity we have

$$\|u\| = o(1) \tag{3.6.4.15}$$

Now we turn our attention to (3.6.4.2) we can see we now we will have the following

$$i\rho v - \frac{\mu}{\lambda} u_{xx} = o(1) \tag{3.6.4.16}$$

So if we take the inner product with v and by integration by parts on the last term with (3.6.4.10) and (3.6.4.14) we can conclude

$$\|v\| = o(1) \tag{3.6.4.17}$$

Doing the same thing with φ and (3.6.4.2) we can arrive at the last piece

$$\|\varphi\| = o(1) \tag{3.6.4.18}$$

We can now see we have reached a contradiction that we have a unit norm therefore by proof by contradiction $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty$ and with that we can conclude that the semigroup $e^{\mathcal{A}t}$ is exponential stable

3.6.5 Further Notes

We can see that there is a problem with our proof when $\mu_2 = 0$ or when $\beta_0 = 0$. We have the same problem when looking at the system introduce in Chapter 4 the paper [24] explored the case where $\mu_2 \neq 0$ and $\beta_0 \neq 0$ so we choice to explore this critical case in Chapter 4 when they are zero and how we can still achieve a decay rate for that system. We explore one of these critical cases right now to highlight that the proof changes. Subsection 3.6.4 , is the only non-trival part when we consider $\beta_0 \neq 0$ so that is the only part we choice to revisit.

3.6.6 Proof of polynomial stable of $1/4$ when $\beta_0 = 0$

It again is a proof by contradiction we assume that $\sup_{|\lambda|\to 0} \frac{1}{\lambda^4} \|(i\lambda - \mathcal{A})^{-1}\| < +\infty$ is false then

$\exists \lambda \to \infty \ni \lambda^4 \|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$. we let our eignefunctions be normalized so this implies $\|U\|_{\mathcal{H}} = 1$ so we eventually wish to reach a contradiction to the unit norm. So to reiterate we have $\lambda \to \infty \ni \lambda^4 \|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$

i.e.

$$\lambda^4 [i\lambda u - v] = o(1) \in H^1 \tag{3.6.6.1}$$

$$\lambda^4 [i\lambda \rho v - \mu u_{xx} - \mu_0 \phi_x + \beta_0 \theta_x] = o(1) \in L^2 \tag{3.6.6.2}$$

$$\lambda^4 [i\lambda \phi - \varphi] = o(1) \in H^1 \tag{3.6.6.3}$$

$$\lambda^4 [i\lambda J \varphi - a_0 \phi_{xx} + \mu_0 u_x + \mu_2 T_x - \beta_1 \theta + \xi \phi] = o(1) \in L^2 \tag{3.6.6.4}$$

$$\lambda^4[i\lambda a\theta + \beta_1\varphi - k\theta_{xx} - k_1 T_x] = o(1) \in L^2 \tag{3.6.6.5}$$

$$\lambda^4[i\lambda bT + \mu_2\varphi_x - k_4 T_{xx} + k_2 T + k_1\theta_x] = o(1) \in L^2 \tag{3.6.6.6}$$

From the dissipation

$$\|\lambda^2 T_x\| = \|\lambda^2 T\| = \|\lambda^2\theta_x\| = o(1) \tag{3.6.6.7}$$

We can see from (3.6.2.10) and Poincare's inequality

$$\|\lambda^2\theta\| = o(1) \tag{3.6.6.8}$$

we can deduce from the norm

$$\|v\| = O(1)\ ,\ \|u_x\| = O(1)\ ,\ \|\varphi\| = O(1)\ ,\ \|\phi_x\| = O(1) \tag{3.6.6.9}$$

from (3.6.6.9) we can see with (3.6.6.1) and (3.6.6.3)

$$\|\frac{\varphi_x}{\lambda}\| = O(1)\ ,\ \|\lambda\phi\| = O(1)\ ,\|\frac{v_x}{\lambda}\| = O(1)\ ,\ \|\lambda u\| = O(1) \tag{3.6.6.10}$$

We can also deduce with (3.6.6.9) if we turn our attention to (3.6.6.1) and (3.6.6.3)

$$\|\frac{u_{xx}}{\lambda}\| = O(1)\ ,\ \|\frac{\phi_{xx}}{\lambda}\| = O(1) \tag{3.6.6.11}$$

Taking the L^2 inner product of (3.6.6.5) with φ we can see

$$\beta_1\lambda^2\|\varphi\|^2 - k\lambda^2\langle\theta_{xx},\varphi\rangle = o(1)$$

With integration by parts we can see if we have neumann conditions on ϕ we will arrive at the following.

$$\|\lambda\varphi\| = o(1) \tag{3.6.6.12}$$

Now we take the L^2 inner product of (3.6.6.6) with ϕ_x we can see

$$\lambda^2\mu_2\langle\varphi_x,\phi_x\rangle - k_4\lambda^2\langle T_{xx},\phi_x\rangle = o(1)$$

With (3.6.6.3) and integration by parts we can see

$$\lambda^3\mu_2\|\phi_x\|^2 - k_4\lambda^2 T_x\phi_x|_{x=0,L} + k_4\lambda^3\langle T_x, \frac{\phi_{xx}}{\lambda}\rangle = o(1)$$

With neumann conditions our u we can see the boundary terms vanish and if we divide by λ using (3.6.6.11) and (3.6.6.7) it is clear the internal term tends to zero. Therefore we arrive at

$$\|\lambda\phi_x\| = o(1) \tag{3.6.6.13}$$

Looking at (3.6.6.4) it is clear that

$$-a_0\phi_{xx} + \mu_0 u_x = o(1) \in L^2$$

Therefore if take the L^2 inner product with the expression above and u_x we get after integrate by parts the following

$$-a_0\phi_x u_x|_{x=0,L} + a_0\langle \lambda\phi_x, \frac{u_{xx}}{\lambda}\rangle + \mu_0\|u_x\|^2 = o(1)$$

We can now see that

$$\|u_x\| = o(1) \tag{3.6.6.14}$$

Finally taking the L^2 product of (3.6.6.2) with v it is easy to deduce

$$i\rho\|v\|^2 - \mu\langle \frac{u_{xx}}{\lambda}, v\rangle = o(1)$$

With integration by parts

$$i\rho\|v\|^2 - \mu\langle u_x, \frac{v_x}{\lambda}\rangle = o(1)$$

Therefore

$$\|v\| = o(1) \tag{3.6.6.15}$$

so we finally reach the promised contradiction because $\|U\|_{\mathcal{H}} = o(1)$ but the contradicts that $\|U\|_{\mathcal{H}} = 1$. Therefore, the proof is complete

Chapter 4

Dual-Phase-Lag One-Dimensional Thermo-Porous-Elasticity with Microtemperatures

4.1 Introduction

At the begin of the last century Cosserat brothers (6) proposed the study of the micropolar elastic solids. That is elastic materials such that the material points can rotate and therefore microstructure is taken into account. In the second part of the last century a big deal was launched concerning the study of elastic materials with microstructure based in the axioms of thermomechanics. In the work of Goodman and Cowin (13) the authors proposed the foundations of a continuum theory for granular materials with interstitial voids. Their basic idea consists in writing the bulk density as the product of the density matrix by the volume fraction. Based on this point of view Cowin and Nunziato (7; 8; 32) set down the theory of elastic solids with voids. The intention was to model the deformations of solids with pores or small voids distributed within them. Thermal effects were also included (16; 17; 18). It is worth recalling that this theory has received a lots of attention in the past years (10; 12; 20; 26; 27; 30; 31; 33; 40) to understand the relevance of the microstructural component in the whole material. In fact the microstructure has deserved much attention and one of its possible components is the microtemperature (1; 5; 14; 15; 19; 28; 34; 38; 39; 11).

Heat conduction is usually based on the Fourier law. In this case the heat flux vector is expressed as a linear form of the gradient of temperature. But this assumption bring us to a paradox because the thermal waves propagates instantaneously and therefore the causality principle is violated. It has been natural to look for an alternative law for the heat flux vector. Cattaneo and

Maxwell proposed the introduction of a relaxation parameter which brings to an hyperbolic damped equation. Other authors have proposed alternative laws as Green and Naghdi. In this paper we recover the one proposed by Tzou (41). The introduction of two delay parameters are considered and the Cattaneo and Maxwell law can be seen as a particular case. It is worth recalling that the first contribution concerning the decay of solutions for the dual-phase-lag thermoelasticity was presented by Quintanilla and Racke (36). Many works dealing this thermoelatic theory have been obtained later. Recently, Liu *et al.* (23) proposed the study of the dual-phase-lag heat conduction with microtemperatures and gave sufficient conditions to guarantee the stability of the problem (3; 22; 21; 29).

In this short note, we want to focus our attention to porous-thermo-elastic materials with microtemperatures in the context of the dual-phase-lag theory. That is to consider the elastic materials with voids where the heat conduction and the microheat conduction are determined by the dual-phase-lag theory. In this sense, the present paper develops three main objectives: the first one is to propose the one dimensional thermo-porous-elasticity with microtemperatures in the context of the dual-phase-lag heat conduction. The second one is to provide of a family of conditions on the parameters of the system guaranteeing the well-posedness of the problem in a suitable Hilbert space. The third one is to prove an exponential stability result for the solutions. We also obtain the polynomial stability of the solutions when the relaxation parameters satisfy a certain relation (limit case).

4.2 Basic equations

In this section we recall the basic equations for the one-dimensional problem of the thermo-porous-elasticity with microtemperatures for isotropic and homogeneous materials in the context of the dual-phase-lag theory. We assume that our rod occupies the interval of length π. The evolution equations are:

$$\begin{aligned} &\rho\ddot{u} = t_x, \quad J\ddot{\phi} = h_x + g, \\ &\rho T_0\dot{\eta} = q_x, \quad \rho\dot{\epsilon} = P_x + q - Q, \end{aligned}$$

and the constitutive equations:

$$\begin{aligned} &t = \mu u_x + \mu_0\phi - \beta_0\theta, \quad h = a_0\phi_x - \mu_2 T, \\ &g = -\mu_0 u_x - \xi\phi + \beta_1\theta, \quad \rho\eta = \beta_0 u_x + \beta_1\phi + a\theta, \\ &\rho\epsilon = -\mu_2\phi_x - bT \\ &q + \tau_1\dot{q} + \frac{\tau_1^2}{2}\ddot{q} = (k\theta_x + k_1 T) + \tau_2(k\dot{\theta}_x + k_1\dot{T}), \\ &P + \tau_1\dot{P} + \frac{\tau_1^2}{2}\ddot{P} = -k_4(S_x + \tau_2\dot{S}_x), \\ &Q + \tau_1\dot{Q} + \frac{\tau_1^2}{2}\ddot{Q} = (k - k_1)\theta_x + (k_1 - k_2)T + \tau_2((k - k_1)\dot{\theta}_x + (k_1 - k_2)\dot{T}). \end{aligned}$$

In the above system of equations ρ is the mass density, u is the displacement, t is the stress, h is the equilibrated stress, g is the equilibrated body force, η is the entropy, q is the heat flux, J is the equilibrated inertia, T_0 is the reference temperature at the equilibrium state (that we will assume equal to one), ϵ is the first moment of the energy, Q is the microheat flux average, P is the first heat flux moment, ϕ is the volume fraction, θ is the temperature, T is the microtemperature, τ_1 and τ_2 are the relaxation parameters. The constitutive parameters, $\mu, \mu_0, \beta_0, \beta_1, a_0, \mu_2, \xi, k$ and k_i define the characteristics of the material and in particular they define the couplings.

We will assume that

$$\mu > 0, \quad \mu\xi > \mu_0^2, \quad a_0 > 0, \quad k > 0, \quad k_4 > 0, \quad \rho > 0, \tag{4.2.0.1}$$

$$J > 0, \quad a > 0, \quad b > 0, \quad kk_2 > k_1^2, \quad 2\tau_2 \geq \tau_1 > 0. \tag{4.2.0.2}$$

These assumptions are natural in the context of the theory. They imply that the internal energy and the dissipation are positive definite forms. These properties are related to the elastic stability. We also mention that the last condition on the relaxation parameters implies that the heat conduction is stable and dissipative (23) (see also (2)). When the relaxation parameters do not satisfy this condition the instability of solutions hold (35).

Substitution of the constitutive equations into the evolution equations, bring us to the following linear system:

$$\begin{aligned}
&\rho\ddot{u} = \mu u_{xx} + \mu_0\phi_x - \beta_0\theta_x,\\
&J\ddot{\phi} = a_0\phi_{xx} - \mu_0 u_x - \mu_2 T_x + \beta_1\theta - \xi\phi,\\
&a(\dot{\theta} + \tau_1\ddot{\theta} + \frac{\tau_1^2}{2}\dddot{\theta}) = -\beta_0(\dot{u}_x + \tau_1\ddot{u}_x + \frac{\tau_1^2}{2}\dddot{u}_x)\\
&\qquad -\beta_1(\dot{\phi} + \tau_1\ddot{\phi} + \frac{\tau_1^2}{2}\dddot{\phi}) + k(\theta_{xx} + \tau_2\dot{\theta}_{xx}) + k_1(T_x + \tau_2\dot{T}_x),\\
&b(\dot{T} + \tau_1\ddot{T} + \frac{\tau_1^2}{2}\dddot{T}) = -\mu_2(\dot{\phi}_x + \tau_1\ddot{\phi}_x + \frac{\tau_1^2}{2}\dddot{\phi}_x) + k_4(T_{xx} + \tau_2\dot{T}_{xx})\\
&\qquad -k_2(T + \tau_2\dot{T}) - k_1(\theta_x + \tau_2\dot{\theta}_x).
\end{aligned}$$

If we denote by $\hat{f} = f + \tau_1\dot{f} + \frac{\tau_1^2}{2}\ddot{f}$ we can write our system as

$$\rho\ddot{\hat{u}} = \mu\hat{u}_{xx} + \mu_0\hat{\phi}_x - \beta_0(\theta_x + \tau_1\dot{\theta}_x + \frac{\tau_1^2}{2}\ddot{\theta}_x), \tag{4.2.0.3}$$

$$J\ddot{\hat{\phi}} = a_0\hat{\phi}_{xx} - \mu_0\hat{u}_x - \mu_2(T_x + \tau_1\dot{T}_x + \frac{\tau_1^2}{2}\ddot{T}_x) + \beta_1(\theta + \tau_1\dot{\theta} + \frac{\tau_1^2}{2}\ddot{\theta}) - \xi\hat{\phi}, \tag{4.2.0.4}$$

$$a(\dot{\theta} + \tau_1\ddot{\theta} + \frac{\tau_1^2}{2}\dddot{\theta}) = -\beta_0\dot{\hat{u}}_x - \beta_1\dot{\hat{\phi}} + k(\theta_{xx} + \tau_2\dot{\theta}_{xx}) + k_1(T_x + \tau_2\dot{T}_x), \tag{4.2.0.5}$$

$$b(\dot{T} + \tau_1\ddot{T} + \frac{\tau_1^2}{2}\dddot{T}) = -\mu_2\dot{\hat{\phi}}_x + k_4(T_{xx} + \tau_2\dot{T}_{xx}) - k_2(T + \tau_2\dot{T}) - k_1(\theta_x + \tau_2\dot{\theta}_x). \tag{4.2.0.6}$$

From now on, we will omit the hats on the mechanical variables to simplify the notation.

To propose the well posed problem we will need to impose the initial conditions

$$\begin{array}{llll} u(x,0)=u^0(x), & \dot{u}(x,0)=v^0(x), & \phi(x,0)=\phi^0(x), & \dot{\phi}(x,0)=\varphi^0(x),\\ \theta(x,0)=\theta^0(x), & T(x,0)=T^0(x), & \dot{\theta}(x,0)=\vartheta(x), & \dot{T}(x,0)=S^0(x)\\ \ddot{\theta}(x,0)=\zeta^0(x), & \ddot{T}(x,0)=R^0(x) & & \end{array}$$

where u^0, v^0, ϕ^0, φ^0, θ^0, ϑ^0, ζ^0, T^0, S^0 and R^0 are given functions.

We assume homogeneous Dirichlet boundary conditions for u and T and homogeneous Neumann conditions fot ϕ and θ. That is

$$u(x,t)=\phi_x(x,t)=\theta_x(x,t)=T(x,t)=0, \quad t\in[0,\infty), \quad x=0,\pi.$$

The energy of the system is:

$$E(t)=\frac{1}{2}\int_0^\pi(\rho|\dot{u}|^2+J|\dot{\phi}|^2+\mu|u_x|^2+2\mu_0u_x\phi+\xi|\phi|^2+a|\phi_x|^2)dx$$

$$+\frac{1}{2}\int_0^\pi(a|\hat{\theta}|^2+b|\hat{T}|^2+k(\tau_1+\tau_2)|\theta_x|^2+\frac{k\tau_1^2\tau_2}{2}|\dot{\theta}_x|^2+k_2(\tau_1+\tau_2)|T|^2+\frac{k_2\tau_1^2\tau_2}{2}|\dot{T}|^2+k\tau_1^2\theta_x\dot{\theta}_x)dx$$

$$+\frac{1}{2}\int_0^\pi(k_2T\dot{T}+2(\tau_1+\tau_2)k_1\theta_xT+k_1\tau_1^2(\theta_x\dot{T}+\dot{\theta}_xT)+k_1\tau_1^2\tau_2\dot{\theta}_x\dot{T}+k_4(\tau_1+\tau_2)|T_x|^2+\frac{\tau_1^2}{2}k_4|\dot{T}_x|^2+\tau_1^2k_4T_x\dot{T}_x)dx.$$

The dissipation is:

$$D(t)=\int_0^\pi\left[k|\theta_x|^2+k(\tau_1\tau_2-\frac{\tau_1^2}{2})|\dot{\theta}_x|^2+k_2|T|^2+k_2(\tau_1\tau_2-\frac{\tau_1^2}{2})|\dot{T}|^2+2k_1\theta_xT+2k_1(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{\theta}_x\dot{T}\right]dx$$

$$+\int_0^\pi\left[k_4|T_x|^2+(\tau_1\tau_2-\frac{\tau_1^2}{2})|\dot{T}_x|^2\right]dx$$

In view of the assumptions we imposed previously the energy and the dissipation are positive definite functions. In fact, we have:

$$E(t)+\int_0^t D(\xi)d\xi=E(0).$$

Therefore we can expect the stability of the solutions of the problem. In fact we will show the polynomial stability for two degenerate cases.

4.3 Main Results

This section is devoted to the well-posedness of the problem and to the statement of the main results. Note that the well-posedness of the problem with Dirichlet boundary conditions was given in ((24)). To transform our problem into a Cauchy problem in a suitable Hilbert space, we consider:

$$\mathcal{H}=H_0^1\times L^2\times H_*^1\times L_*^2\times H_*^1\times H_*^1\times L_*^2\times H_0^1\times H_0^1\times L^2.$$

Where

$$H^1_*(0,\pi)=\{f(x)\in H^1\Big|\int_0^\pi f(x)dx=0\},\quad L^2_*(0,\pi)=\{f(x)\in L^2\Big|\int_0^\pi f(x)dx=0\}.$$

It is worth noting that now we consider that the elements take values in the complex field.

An element in this space will be denoted by $U=(u,v,\phi,\varphi,\theta,\vartheta,\zeta,T,S,R)$.

Defining an operator $\mathcal{A}:\mathcal{H}\to\mathcal{H}$ by

$$\mathcal{A}U=\begin{cases} v\\ \frac{1}{\rho}[\mu u_x+\mu_0\phi-\beta_0(\theta+\tau_1\vartheta+\frac{\tau_1^2}{2}\zeta)]_x\\ \varphi\\ \frac{1}{J}[(a_0\phi_x-\mu_2(T+\tau_1 S+\frac{\tau_1^2}{2}R))_x-\mu_0u_x-\xi\phi+\beta_1(\theta+\tau_1\vartheta+\frac{\tau_1^2}{2}\zeta)]\\ \vartheta\\ \zeta\\ \frac{2}{a\tau_1^2}[-\beta_0v_x-\beta_1\varphi+k(\theta_x+\tau_2\vartheta_x)_x+k_1(T_x+\tau_2S_x)-a\vartheta-a\tau_1\zeta]\\ S\\ R\\ \frac{2}{b\tau_1^2}[-\mu_2\varphi_x+k_4(T_x+\tau_2S_x)_x-k_2(T+\tau_2S)-k_1(\theta+\tau_2\vartheta_x)-aS-a\tau_1R] \end{cases} \tag{4.3.0.1}$$

with the domain

$$\mathcal{D}(\mathcal{A})=\{U\in\mathcal{H}\mid v,S,R,\phi_x,\theta_x,\ \vartheta_x\in H_0^1,\varphi,\vartheta,\zeta\in H_*^1,u_x,T_x,S_x\in H^1\}.$$

We then write our problem as

$$\frac{dU}{dt}=\mathcal{A}U,\quad U(0)=U^0, \tag{4.3.0.2}$$

where $U^0=(u^0,v^0,\phi^0,\varphi^0,\theta^0,\vartheta^0,\zeta^0,T^0,S^0,R^0)$.

Given $U=(u,v,\phi,\varphi,\theta,\vartheta,\zeta,T,S,R)$ and $U^*=(u^*,v^*,\phi^*,\varphi^*,\theta^*,\vartheta^*,\zeta^*,T^*,S^*,R^*)$, we define the inner product

$$\langle U,U^*\rangle_{\mathcal{H}}=\frac{1}{2}\int_0^\pi(\rho v\bar{v}^*+J\varphi\bar{\varphi}^*+\mu u_x\bar{u}_x^*+\mu_0(u_x\bar{\phi}^*+\bar{u}_x^*\phi)+\xi\phi\bar{\phi}^*+a_0\phi_x\bar{\phi}_x^*+c\theta\bar{\hat{\theta}}^*+bT\bar{\hat{T}}^*)dx$$

$$+\frac{1}{2}\int_0^\pi(k(\tau_1+\tau_2)\theta_x\bar{\theta}_x^*+\frac{k\tau_1^2\tau_2}{2}\vartheta_x\bar{\vartheta}_x^*+k_2(\tau_1+\tau_2)T\bar{T}^*+\frac{k_2\tau_1^2\tau_2}{2}T\bar{T}^*+\frac{k\tau_1^2}{2}(\theta_x\bar{\vartheta}_x^*+\bar{\theta}_x^*\vartheta_x))dx$$

$$+\frac{1}{2}\int_0^\pi(\frac{k_2}{2}(T\bar{S}^*+\bar{T}^*S)+k_1(\tau_1+\tau_2)(\theta_x\bar{T}^*+\bar{\theta}_x^*T)+\frac{k_1\tau_1^2\tau_2}{2}(\vartheta_x\bar{S}^*+\bar{\vartheta}_x^*S))dx$$

$$+\frac{1}{2}\int_0^\pi(\frac{k_1\tau_1^2}{2}(\theta_x\bar{S}^*+\bar{\theta}_x^*S)+k_4(\tau_1+\tau_2)T_x\bar{T}_x^*+\frac{k_4\tau_1^2}{2}S_x\bar{S}_x^*+\frac{k_4\tau_1^2}{2}(T_x\bar{S}_x^*+\bar{T}_x^*S_x))dx$$

$$+\frac{1}{2}\int_0^\pi\frac{k_1\tau_1^2}{2}(\vartheta_x\bar{T}^*+\bar{\vartheta}_x^*T)dx.$$

As usual, from now on, the bar denotes the conjugated complex. It is clear that this inner product is equivalent to the usual one in the Hilbert space $\mathcal{H}$. It was proved that the domain of the operator is dense, the real part of the product of $\langle \mathcal{A}U, U\rangle_{\mathcal{H}}$ is less or equal to zero for every U at the domain of the operator and that the zero belongs to the resolvent of the operator. Therefore:

Theorem 4.3.0.1 *Let us assume that the conditions (4.2.0.1)-(4.2.0.2) hold. Then, operator $\mathcal{A}$ generates a C_0 semigroup of contractions.*

Proof.
Case.) $\mu_2 \neq 0$ and $\beta_1 = \beta_0 = 0$ and $2\tau_2 > \tau_1$

All the other cases fall out trivially from one case so we just highlight this one case.

It is easy to see $D(\mathcal{A})$ is dense in $\mathcal{H}$ If we have $kk_2 > k_1^2$ we can see

$$Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -k\|\theta_x\|^2 - k(\tau_1\tau_2 - \frac{\tau_1^2}{2})\|\vartheta_x\|^2 - k_2\|T\|^2 - 2k_1 Re\langle\theta_x, T\rangle$$
$$-2k_1(\tau_1\tau_2 - \frac{\tau_1^2}{2})Re\langle\vartheta_x, S\rangle - k_4\|T_x\|^2 - (\tau_1\tau_2 - \frac{\tau_1^2}{2})\|S_x\|^2 \leq 0 \tag{4.3.0.3}$$

Therefore our system is dissipative, now we just have to show that $0 \in \rho(\mathcal{A})$ to use the following theorem to prove this theorem. To do this we let our eigenvalue $\lambda = 0$. We then have

$$v_n = o(1) \text{ in } H^1, \tag{4.3.0.4}$$
$$-\mu u_{n,xx} - \mu_0\phi_{n,x} = o(1) \text{ in } L^2, \tag{4.3.0.5}$$
$$\varphi_n = o(1) \text{ in } H^1, \tag{4.3.0.6}$$
$$-a_0\phi_{n,xx} + \mu_0 u_{n,x} + \xi\phi_n$$
$$\mu_2(T_{n,x} + \tau_1 S_{n,x} + \frac{\tau_1^2}{2}R_{n,x})) = o(1) \text{ in } L^2, \tag{4.3.0.7}$$
$$\vartheta_n = o(1) \text{ in } H^1, \tag{4.3.0.8}$$
$$\zeta_n = o(1) \text{ in } H^1, \tag{4.3.0.9}$$
$$-k(\theta_{n,xx} + \tau_2\vartheta_{n,xx})$$
$$-k_1(T_{n,x} + \tau_2 S_{n,x}) + a\vartheta_n + a\tau_1\zeta_n) = o(1) \text{ in } L^2, \tag{4.3.0.10}$$
$$S_n = o(1) \text{ in } H^1, \tag{4.3.0.11}$$
$$R_n = o(1) \text{ in } H^1, \tag{4.3.0.12}$$
$$\mu_2\varphi_{n,x} - k_4(T_{n,xx} + \tau_2 S_{n,xx}) + k_2(T_n + \tau_2 S_n)$$
$$+k_1(\theta_{n,x} + \tau_2\vartheta_{n,x}) + aS_n + a\tau_1 R_n) = o(1) \text{ in } L^2. \tag{4.3.0.13}$$

We can see $Re\langle \mathcal{A}U, U\rangle$ implies

$$\|\theta_x\| = \|\vartheta_x\| = \|T\| = \|T_x\| = \|S_x\| = o(1) \tag{4.3.0.14}$$

With poincare inequality we can see

$$\|\theta\| = \|\vartheta\| = \|T\| = \|S\| = o(1) \tag{4.3.0.15}$$

Looking at (4.3.0.7) with (4.3.0.12) with (4.3.0.14) we can reduce it to

$$-a_0\phi_{n,xx} + \mu_0 u_{n,x} + \xi\phi_n = o(1) \in L^2 \tag{4.3.0.16}$$

Taking the L^2 inner product of above with ϕ we can see

$$a_0\|\phi_{n,x}\|^2 + \mu_0\langle u_{n,x}, \phi_n\rangle + \xi\|\phi_n\|^2 = o(1) \tag{4.3.0.17}$$

Taking the L^2 inner product of (4.3.0.5) with u_n we get the following after integrating by parts since our boundary term vanishes by our boundary conditions

$$\mu\|u_{n,x}\|^2 + \mu_0\langle \phi_n, u_{n,x}\rangle = o(1) \tag{4.3.0.18}$$

Adding (4.3.0.17) with (4.3.0.18) and taking the reals we have

$$a_0\|\phi_{n,x}\|^2 + 2\mu_0 Re\langle u_{n,x}, \phi_n\rangle + \xi\|\phi_n\|^2 + \mu\|u_{n,x}\|^2 = o(1) \tag{4.3.0.19}$$

We can see when $a_0\xi > 4\mu_0^2$ we can see this will imply

$$\|\phi_{n,x}\| = \|\phi_n\| = \|u_{n,x}\| = o(1) \tag{4.3.0.20}$$

Then by poincare inequality we have

$$\|u_n\| = o(1) \tag{4.3.0.21}$$

We can see we have reached a contradicition to our unit norm. So this implies $0 \in \rho(\mathcal{A})$ and by Lumer-Phillips theroem the proof is complete.

The proof for the other case are done the same way so we just do the one case.

The next theorem follows from Theorem 4.3.0.1.

Theorem 4.3.0.2 *If we assume that conditions (4.2.0.1)-(4.2.0.2) hold, then, for every $U^0 \in \mathcal{D}(\mathcal{A})$, there exists a unique solution to problem (4.3.0.2).*

Since the operator generates a contractive semigroup, our problem is stable and well posed in the sense of Hadamard. Furthermore, in case that we impose supply terms with suitable regularity conditions the solutions will depend continuously on the supply terms.

The asymptotic stability of the system (4.3.0.2) was investigated in (24). It was proved that the system is exponentially stable if $2\tau_2 > \tau_1$, and is polynomially stable of order 2 if $2\tau_2 > \tau_1$. In this paper, we are interested in the two degenerate cases, either $\beta_0 = \beta_1 = 0$ or $\mu_2 = 0$, i.e., one of the heat equation is decoupled from the elastic equations. Then, the thermal damping is weaker. It is natural to expect a slower solution decay rate.

The main results of this paper are the following two theorems.

Theorem 4.3.0.3 *Assume that (4.2.0.1)-(4.2.0.2) hold, and that* $\beta_0 = \beta_1 = 0$*,* $\mu_2 \neq 0$*. Then the semigroup generated by the operator* $\mathcal{A}$ *is polynomially stable of order* k*. More precisely, there exist positive constants* C *which is independent of the initial data such that*

$$\|U(t)\|_{\mathcal{H}} \leq Ct^{-1/p}\|U(0)\|_{\mathcal{D}(\mathcal{A})}$$

for every $U(0) \in \mathcal{D}(\mathcal{A})$*, where*

$$p = \begin{cases} 4, & \text{if } 2\tau_2 > \tau_1 \\ 6, & \text{if } 2\tau_2 = \tau_1. \end{cases}$$

Theorem 4.3.0.4 *Assume that (4.2.0.1)-(4.2.0.2) hold, and that* $\beta_0, \beta_1 \neq 0$*,* $\mu_2 = 0$*. In addition, assume that* $\mu_0\beta_0 - \mu\beta_1 \geq 0$*. Then the semigroup generated by the operator* $\mathcal{A}$ *is polynomially stable of order* k*. More precisely, there exist positive constants* C *which is independent of the initial data such that*

$$\|U(t)\|_{\mathcal{H}} \leq Ct^{-1/p}\|U(0)\|_{\mathcal{D}(\mathcal{A})}$$

for every $U(0) \in \mathcal{D}(\mathcal{A})$*, where*

$$p = \begin{cases} 4, & \text{if } 2\tau_2 > \tau_1 \\ 6, & \text{if } 2\tau_2 = \tau_1. \end{cases}$$

Theorem 4.3.0.5 *Assume that (4.2.0.1)-(4.2.0.2) hold, and that* $\beta_0 \neq 0$ *,* $\beta_1 = 0$*,* $\mu_2 = 0$*. Then the semigroup generated by the operator* $\mathcal{A}$ *is polynomially stable of order* k*. More precisely, there exist positive constants* C *which is independent of the initial data such that*

$$\|U(t)\|_{\mathcal{H}} \leq Ct^{-1/p}\|U(0)\|_{\mathcal{D}(\mathcal{A})}$$

for every $U(0) \in \mathcal{D}(\mathcal{A})$*, where*

$$p = \begin{cases} 4, & \text{if } 2\tau_2 > \tau_1 \\ 6, & \text{if } 2\tau_2 = \tau_1. \end{cases}$$

Theorem 4.3.0.6 *Assume that (4.2.0.1)-(4.2.0.2) hold, and that* $\beta_0 = 0$ $\beta_1 \neq 0$*,* $\mu_2 \neq 0$*. Then the semigroup generated by the operator* $\mathcal{A}$ *is polynomially stable of order* k*. More precisely, there exist positive constants* C *which is independent of the initial data such that*

$$\|U(t)\|_{\mathcal{H}} \leq Ct^{-1/p}\|U(0)\|_{\mathcal{D}(\mathcal{A})}$$

for every $U(0) \in \mathcal{D}(\mathcal{A})$*, where*

$$p = \begin{cases} 4, & \text{if } 2\tau_2 > \tau_1 \\ 6, & \text{if } 2\tau_2 = \tau_1. \end{cases}$$

Theorem 4.3.0.7 *Assume that (4.2.0.1)-(4.2.0.2) hold, and that* $\beta_0 = 0$ $\beta_1 \neq 0$, $\mu_2 = 0$. *Then the semigroup generated by the operator* $\mathcal{A}$ *is polynomially stable of order* k. *More precisely, there exist positive constants* C *which is independent of the initial data such that*

$$\|U(t)\|_{\mathcal{H}} \leq Ct^{-1/p}\|U(0)\|_{\mathcal{D}(\mathcal{A})}$$

for every $U(0) \in \mathcal{D}(\mathcal{A})$, *where*

$$p = \begin{cases} 8, & \text{if } 2\tau_2 > \tau_1 \\ 10, & \text{if } 2\tau_2 = \tau_1. \end{cases}$$

4.4 Proof of Theorem 4.3.0.3

The aim of this section is to prove that the solutions of our problem decay polynomially to the equilibrium solution when $\beta_0 = \beta_1 = 0$ and $\mu_2 \neq 0$. Note that the first heat equation about θ is now decoupled from the two elastic equations. To prove this result we will use the characterization of a polynomially stable semigroups given by Borichev and Tomilov in (4), which states that the semigroup $e^{\mathcal{A}t}$ is polynomially stable of order p, i.e.,

$$||U(t)||_{\mathcal{H}} \leq Ct^{-1/p}||U(0)||_{\mathcal{D}(\mathcal{A})}$$

if and only if

$$iR \subset \rho(\mathcal{A}) \tag{4.4.0.1}$$

$$\overline{\lim_{\lambda\in R, |\lambda|\to\infty}} |\lambda|^{-p}\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{4.4.0.2}$$

We first check condition 4.4.0.2. If it is false, then there will exist a sequence of real numbers λ_n with $|\lambda_n| \to \infty$ and a sequence of vectors $U_n = (u_n, v_n, \phi_n, \varphi_n, \theta_n, \vartheta_n, \zeta_n, T_n, S_n, R_n)$ in $\mathcal{D}(\mathcal{A})$ with unit norm, such that

$$\lim_{n\to\infty} |\lambda_n|^p \|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\|_{\mathcal{H}} = 0. \tag{4.4.0.3}$$

Without losing generality, we assume that $\lambda_n > 0$.
Case 1: $2\tau_2 > \tau_1$.

(4.4.0.3) can be written as

$$\lambda_n^p(i\lambda_n u_n - v_n) = o(1) \text{ in } H^1, \tag{4.4.0.4}$$

$$\lambda_n^p(i\rho\lambda_n v_n - (\mu u_{n,xx} + \mu_0\phi_{n,x})) = o(1) \text{ in } L^2, \tag{4.4.0.5}$$

$$\lambda_n^p(i\lambda_n \phi_n - \varphi_n) = o(1) \text{ in } H^1, \tag{4.4.0.6}$$

$$\lambda_n^p(iJ\lambda_n\varphi_n - a_0\phi_{n,xx} + \mu_0 u_{n,x} + \xi\phi_n \\ \mu_2(T_{n,x} + \tau_1 S_{n,x} + \frac{\tau_1^2}{2}R_{n,x})) = o(1) \text{ in } L^2, \tag{4.4.0.7}$$

$$\lambda_n^p(i\lambda_n\theta_n - \vartheta_n) = o(1) \text{ in } H^1, \tag{4.4.0.8}$$

$$\lambda_n^p(i\lambda_n\vartheta_n - \zeta_n) = o(1) \text{ in } H^1, \tag{4.4.0.9}$$

$$\lambda_n^p(\frac{ia\tau_1^2\lambda_n}{2}\zeta_n - k(\theta_{n,xx} + \tau_2\vartheta_{n,xx}) \\ -k_1(T_{n,x} + \tau_2 S_{n,x}) + a\vartheta_n + a\tau_1\zeta_n) = o(1) \text{ in } L^2, \tag{4.4.0.10}$$

$$\lambda_n^p(i\lambda_n T_n - S_n) = o(1) \text{ in } H^1, \tag{4.4.0.11}$$

$$\lambda_n^p(i\lambda_n S_n - R_n) = o(1) \text{ in } H^1, \tag{4.4.0.12}$$

$$\lambda_n^p(\frac{ib\tau_1^2\lambda_n}{2}R_n + \mu_2\varphi_{n,x} - k_4(T_{n,xx} + \tau_2 S_{n,xx}) + k_2(T_n + \tau_2 S_n) \\ +k_1(\theta_{n,x} + \tau_2\vartheta_{n,x}) + aS_n + a\tau_1 R_n) = o(1) \text{ in } L^2. \tag{4.4.0.13}$$

Hereafter, we use the notation $\|\cdot\|$ and $\langle\cdot,\ \cdot\rangle$ to denote the L^2 norm and inner product, respectively. Our goal is to show $\|U_n\|_{\mathcal{H}} = o(1)$ which is a contraction.

From (4.4.0.3) and the dissipation inequality we see that

$$\lambda_n^{p/2}\|\theta_{n,x}\|,\ \lambda_n^{p/2}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|S_{n,x}\| = o(1). \tag{4.4.0.14}$$

This, combined with (4.4.0.9) and (4.4.0.12), implies that

$$\lambda_n^{p/2-1}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-1}\|R_{n,x}\| = o(1), \tag{4.4.0.15}$$

which further leads to, by the Poincàre inequality,

$$\lambda_n^{p/2-1}\|\zeta_n\|,\ \lambda_n^{p/2-1}\|R_n\| = o(1). \tag{4.4.0.16}$$

Now, let's turn our attention to the four elastic components of U_n. Notice that since $\beta_0 = \beta_1 = 0$, the coupling between the heat and elastic equations are only through φ and φ_x. It is natural to estimate these two terms first.

From (4.4.0.14), (4.4.0.6), and (4.4.0.16), (4.4.0.13) is simplified into

$$i\frac{b\tau_1^2}{2}\lambda_n^{p/2}R_n + \lambda_n^{p/2}\mu_2\phi_{n,x} - \lambda_n^{p/2-1}k_4(T_{n,xx} + \tau_2 S_{n,xx}) = o(1) \text{ in } L^2. \tag{4.4.0.17}$$

Next, we take the L^2 inner product of (4.4.0.17) with $\phi_{n,x}$, then integrate by parts to obtain that

$$-i\frac{b\tau_1^2}{2}\langle\lambda_n^{p/2-1}R_{n,x}, \lambda_n\phi_n\rangle + \lambda_n^{p/2}\mu_2\|\phi_{n,x}\|^2 + \lambda_n^{p/2}k_4\langle(T_{n,x} + \tau_2 S_{n,x}), \lambda_n^{-1}\phi_{n,xx}\rangle = o(1). \tag{4.4.0.18}$$

Since $\|\lambda_n \phi\|$ and $\|\lambda_n^{-1}\phi_{n,xx}\|$ are uniformly bounded in n which can be seen from (4.4.0.6) and (4.4.0.7), the first and third term in (4.4.0.18) are of $o(1)$. We now obtain from (4.4.0.18) that

$$\lambda_n^{p/4}\|\phi_{n,x}\| = o(1), \tag{4.4.0.19}$$

which further leads to, thanks to 4.4.0.6,

$$\lambda_n^{p/4-1}\|\varphi_{n,x}\| = o(1), \quad \lambda_n^{p/4-1}\|\varphi_n\| = o(1). \tag{4.4.0.20}$$

Take $p = 4$. In summary, we have already obtained

$$\begin{cases} \lambda_n^2\|\theta_{n,x}\|,\ \lambda_n^2\|\vartheta_{n,x}\|,\ \lambda_n\|\zeta_{n,x}\|,\ \lambda_n^2\|T_{n,x}\|,\ \lambda_n^2\|S_{n,x}\|,\ \lambda_n\|R_{n,x}\| = o(1), \\ \lambda_n\|\phi_{n,x}\|,\ \|\varphi_n\| = o(1). \end{cases}$$

Next, we take the L^2 inner product of (4.4.0.7) with $\lambda_n^{-4}u_{n,x}$ and integrate by parts to get

$$-iJ\langle\varphi_{n,x}, \lambda_n u_n\rangle + \langle\lambda_n\phi_{n,x}, \lambda_n^{-1}u_{n,xx}\rangle + \mu_0\|u_{n,x}\|^2 + \frac{\mu_2\tau_1^2}{2}\langle\lambda_n R_n, \lambda_n^{-1}u_{n,xx}\rangle + \xi\langle\phi_{n,x}, u_n\rangle = o(1), \tag{4.4.0.21}$$

which implies that

$$\|u_{n,x}\| = o(1) \tag{4.4.0.22}$$

since $\|\lambda_n^{-1}u_{n,xx}\|$ and $\|\lambda_n u_n\|$ are bounded by (4.4.0.5) and (4.4.0.4). Finally, taking the L^2 inner product of (4.4.0.5) with $\lambda_n^{-5}v_n$ and using (4.4.0.22) yields

$$\|v_n\| = o(1). \tag{4.4.0.23}$$

We have reached the promised contradiction $\|U_n\|_{\mathcal{H}} = o(1)$.

Case 2: $2\tau_2 = \tau_1$

In this case, the energy dissipation is weaker. This time, from (4.4.0.3) and dissipation inequality we only have

$$\lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|\theta_{n,x}\| = o(1), \tag{4.4.0.24}$$

which further leads to

$$\lambda_n^{p/2-1}\|S_{n,x}\|,\ \lambda_n^{p/2-1}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2-2}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-2}\|R_{n,x}\| = o(1). \tag{4.4.0.25}$$

Thus, we can simplify (4.4.0.13) into

$$i\frac{b\tau_1^2}{2}\lambda_n^{p/2-1}R_n + \lambda_n^{p/2-2}\mu_2\varphi_{n,x} - \lambda_n^{p/2-2}k_4(T_{n,xx} + \tau_2 S_{n,xx}) = o(1) \text{ in } L^2. \tag{4.4.0.26}$$

Taking the L^2 inner product of (4.4.0.26) with $\phi_{n,x}$ and using (4.4.0.6) we obtain

$$i\frac{b\tau_1^2}{2}\langle\lambda_n^{p/2-2}R_{n,x}, \lambda\phi_n\rangle - i\mu_2\lambda_n^{p/2-1}\|\phi_{n,x}\|^2 + \langle\lambda_n^{p/2-1}k_4(T_{n,x}+\tau_2 S_{n,x}), \lambda_n^{-1}\phi_{n,xx}\rangle = o(1) \tag{4.4.0.27}$$

where the first and last term are from integrating by parts.

Since $\|\lambda_n\phi_n\|$ and $\|\lambda_n^{-1}\phi_{n,xx}\|$ are uniformly bounded in n, the first and last term in (4.4.0.27) are of $o(1)$. Therefore, (4.4.0.27) leads to

$$\|\lambda_n^{p/4-1/2}\phi_{n,x}\| = o(1), \quad \|\lambda_n^{p/4-3/2}\varphi_{n,x}\| = o(1).$$

Take $p = 6$. In summary, we have already obtained

$$\begin{cases} \lambda_n^3\|\theta_{n,x}\|,\ \lambda_n^2\|\vartheta_{n,x}\|,\ \lambda_n\|\zeta_{n,x}\|,\ \lambda_n^3\|T_{n,x}\|,\ \lambda_n^2\|S_{n,x}\|,\ \lambda_n\|R_{n,x}\| = o(1), \\ \lambda_n\|\phi_{n,x}\|,\ \|\varphi_{n,x}\| = o(1). \end{cases}$$

Next, we take the L^2 inner product of (4.4.0.7) with $\lambda_n^{-6}u_{n,x}$. The rest of the proof can be done by repeating the argument from (4.4.0.21) to (4.4.0.23).

4.5 Proof of Theorem 4.3.0.4

When $\beta_0, \beta_1 \neq 0$ and $\mu_2 = 0$, the second heat equation about T is now decoupled from the two elastic equations. However, the first heat equation is coupled to the two wave equations via v_x and ϕ.

Similar to the proof of Theorem 3, if condition (4.4.0.2) is false, we have

$$\lambda_n^p(i\lambda_n u_n - v_n) = f_n = o(1) \text{ in } H^1, \tag{4.5.0.1}$$

$$\lambda_n^p(i\rho\lambda_n v_n - (\mu u_{n,xx} + \mu_0\phi_{n,x})) + \beta_0(\theta_n + \tau_1\vartheta_n + \frac{\tau_1^2}{2}\zeta_n)_x = o(1) \text{ in } L^2, \tag{4.5.0.2}$$

$$\lambda_n^p(i\lambda_n\phi_n - \varphi_n) = o(1) \text{ in } H^1, \tag{4.5.0.3}$$

$$\lambda_n^p(iJ\lambda_n\varphi_n - a_0\phi_{n,xx} + \mu_0 u_{n,x} + \xi\phi_n - \beta_1(\theta_n + \tau_1\vartheta_n + \frac{\tau_1^2}{2}\zeta_n) = o(1) \text{ in } L^2, \tag{4.5.0.4}$$

$$\lambda_n^p(i\lambda_n\theta_n - \vartheta_n) = o(1) \text{ in } H^1, \tag{4.5.0.5}$$

$$\lambda_n^p(i\lambda_n\vartheta_n - \zeta_n) = o(1) \text{ in } H^1, \tag{4.5.0.6}$$

$$\lambda_n^p(\frac{ia\tau_1^2\lambda_n}{2}\zeta_n + \beta_0 v_{n,x} + \beta_1\varphi_n - k(\theta_{n,xx} + \tau_2\vartheta_{n,xx}) - k_1(T_{n,x} + \tau_2 S_{n,x}) + a\vartheta_n + a\tau_1\zeta_n) = o(1) \text{ in } L^2, \tag{4.5.0.7}$$

$$\lambda_n^p(i\lambda_n T_n - S_n) = o(1) \text{ in } H^1, \tag{4.5.0.8}$$

$$\lambda_n^p(i\lambda_n S_n - R_n) = o(1) \text{ in } H^1, \tag{4.5.0.9}$$

$$\lambda_n^p(\frac{ib\tau_1^2\lambda_n}{2}R_n - k_4(T_{n,xx} + \tau_2 S_{n,xx}) + k_2(T_n + \tau_2 S_n) + k_1(\theta_{n,x} + \tau_2\vartheta_{n,x}) + aS_n + a\tau_1 R_n) = o(1) \text{ in } L^2. \tag{4.5.0.10}$$

Case 1: $2\tau_2 > \tau_1$.

From (4.4.0.3) and the dissipation inequality we again have

$$\begin{cases} \lambda_n^{p/2}\|\theta_{n,x}\|,\ \lambda_n^{p/2}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|S_{n,x}\| = o(1) \\ \lambda_n^{p/2-1}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-1}\|R_{n,x}\|,\ \lambda_n^{p/2-1}\|\zeta_n\|,\ \lambda_n^{p/2-1}\|R_n\| = o(1). \end{cases} \tag{4.5.0.11}$$

Since the heat equation (4.5.0.7) contains both $v_{n,x}$ and φ_n, we are not able to obtain $\|u_{n,x}\|, \|\varphi_n\| = o(1)$ from it as we did in last section for $\|\phi_{n,x}\| = o(1)$ from equation (4.4.0.13). Here, we will take advantage of the additional condition $\mu_0\beta_0 - \mu\beta_1 \geq 0$.

First, we substitute (4.5.0.1) into (4.5.0.2) to get

$$\lambda_n^p[-\rho\lambda_n^2 u_n - \mu u_{n,xx} - \mu_0\phi_{n,x} + \beta_0(\theta_{n,x} + \tau_1\vartheta_{n,x} + \frac{\tau_1^2}{2}\zeta_{n,x})] = i\lambda_n f_n + o(1) \ in\ L^2 \tag{4.5.0.12}$$

Since

$$\langle\lambda_n^{p/2}(\theta_{n,x}+\tau_1\vartheta_{n,x}),\ u_n\rangle = o(1), \quad \langle\lambda_n^{p/2-1}\zeta_{n,x},\ \lambda_n u_n\rangle = o(1), \quad \langle f_n,\ \lambda_n u_n\rangle = o(1),$$

the L^2 inner product of (4.5.0.12) with $\lambda_n^{-p/2}\beta_1 u_n$ yields

$$\lambda^{p/2}[-\beta_1\rho\|\lambda_n u_n\|^2 + \mu\beta_1\|u_{n,x}\|^2 + \mu_0\beta_1\langle\phi_n, u_{n,x}\rangle] = o(1). \tag{4.5.0.13}$$

Next, we substitute (4.5.0.1) and (4.5.0.3) into (4.5.0.7), then divide by $\lambda_n^{p/2}$ to get

$$i\lambda_n^{p/2}[\frac{a\tau_1^2}{2}\zeta_n + \beta_0 u_{n,x} + \beta_1\phi_n] - \lambda_n^{p/2-1}(k\theta_{n,xx} + k\tau_2\vartheta_{n,xx}) = o(1) \text{ in } L^2. \tag{4.5.0.14}$$

Since

$$\langle\lambda_n^{p/2}(k\theta_{n,x} + k\tau_2\vartheta_{n,x}),\ \lambda_n^{-1}u_{n,xx}\rangle = o(1), \quad \langle\lambda_n^{p/2-1}\zeta_{n,x},\ \lambda_n u_n\rangle = o(1),$$

the L^2 inner product of (4.5.0.14) with $i\mu_0 u_{n,x}$ yields

$$\lambda_n^{p/2}[-\mu_0\beta_0\|u_{n,x}\|^2 - \mu\beta_1\langle\phi_n, u_{n,x}\rangle] = o(1). \tag{4.5.0.15}$$

Thus, the sum of (4.5.0.13) and (4.5.0.15) gives

$$\lambda_n^{p/2}[-\beta_1\rho\|\lambda_n u_n\|^2 - (\mu_0\beta_0 - \mu\beta_1)\|u_{n,x}\|^2 = o(1). \tag{4.5.0.16}$$

Take $p = 4$. By the condition $\mu_0\beta_0 - \mu\beta_1 \geq 0$, we have arrived

$$\lambda_n^2\|u_n\| = o(1),$$

which further leads to

$$\lambda_n\|v_n\| = o(1). \tag{4.5.0.17}$$

If $\mu_0\beta_0 - \mu\beta_1$ is strictly positive, we also have $\|\lambda_n u_{n,x}\| = o(1)$. Otherwise, it follows from the L^2 inner product of (4.5.0.2) with $\lambda_n^{-4}u_n$.

In summary, we have already obtained

$$\begin{cases} \lambda_n^2\|\theta_{n,x}\|,\ \lambda_n^2\|\vartheta_{n,x}\|,\ \lambda_n^2\|T_{n,x}\|,\ \lambda_n^2\|S_{n,x}\| = o(1) \\ \lambda_n\|\zeta_{n,x}\|,\ \lambda_n\|R_{n,x}\|,\ \lambda_n\|\zeta_n\|,\ \lambda_n\|R_n\| = o(1) \\ \lambda_n\|v_n\|,\ \lambda_n\|u_{n,x}\| = o(1). \end{cases} \tag{4.5.0.18}$$

With these results at hand, we can simplify (4.5.0.2) into

$$-\mu u_{n,xx} - \mu_0 \phi_x = o(1) \text{ in } L^2. \tag{4.5.0.19}$$

Take the L^2 inner product of (4.5.0.19) with $\phi_{n,x}$ to get

$$\|\phi_{n,x}\| = o(1) \tag{4.5.0.20}$$

because

$$\langle u_{n,xx},\ \phi_{n,x}\rangle = -\langle \lambda_n u_{n,x},\ \lambda_n^{-1}\phi_{n,xx}\rangle = o(1)$$

Finally, taking the L^2 inner product of (4.5.0.4) with $\lambda_n^{-5}\varphi_n$ yields

$$\|\varphi_n\| = o(1). \tag{4.5.0.21}$$

We have reached the promised contradiction $\|U_n\|_{\mathcal{H}} = o(1)$.

Case 2: $2\tau_2 = \tau_1$

In this case, the energy dissipation is weaker. This time, from (4.4.0.3) and dissipation inequality we only have

$$\lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|\theta_{n,x}\| = o(1), \tag{4.5.0.22}$$

which further leads to

$$\lambda_n^{p/2-1}\|S_{n,x}\|,\ \lambda_n^{p/2-1}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2-2}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-2}\|R_{n,x}\| = o(1). \tag{4.5.0.23}$$

Since

$$\langle \lambda_n^{p/2}(\theta_{n,x}+\tau_1\vartheta_{n,x}),\ u_n\rangle = o(1), \quad \langle \lambda_n^{p/2-1}\zeta_{n,x},\ u_n\rangle = o(1), \quad \langle f_n,\ \lambda_n u_n\rangle = o(1),$$

the L^2 inner product of (4.5.0.12) with $\lambda_n^{-p/2-1}\beta_1 u_n$ yields

$$\lambda^{p/2-1}[-\beta_1\rho\|\lambda_n u_n\|^2 + \mu\beta_1\|u_{n,x}\|^2 + \mu_0\beta_1\langle \phi_n, u_{n,x}\rangle] = o(1). \tag{4.5.0.24}$$

On the other hand, we also have

$$\langle \lambda_n^{p/2-1}(k\theta_{n,x} + k\tau_2\vartheta_{n,x}),\ \lambda_n^{-1}u_{n,xx}\rangle = o(1), \quad \langle \lambda_n^{p/2-2}\zeta_{n,x},\ \lambda_n u_n\rangle = o(1).$$

Then the L^2 inner product of (4.5.0.14) with $i\lambda_n^{-1}\mu_0 u_{n,x}$ yields

$$\lambda_n^{p/2-1}[-\mu_0\beta_0\|u_{n,x}\|^2 - \mu\beta_1\langle \phi_n, u_{n,x}\rangle] = o(1). \tag{4.5.0.25}$$

It follows the sum of (4.5.0.24) and (4.5.0.25) that

$$\lambda_n^{p/2-1}[-\beta_1\rho\|\lambda_n u_n\|^2 - (\mu_0\beta_0 - \mu\beta_1)\|u_{n,x}\|^2 = o(1). \tag{4.5.0.26}$$

Take $p = 6$. By the condition $\mu_0\beta_0 - \mu\beta_1 \geq 0$, we have arrived

$$\lambda_n^2\|u_n\| = o(1),$$

which further leads to

$$\lambda_n\|v_n\| = o(1). \tag{4.5.0.27}$$

If $\mu_0\beta_0 - \mu\beta_1$ is strictly positive, we also have $\|\lambda_n u_{n,x}\| = o(1)$. Otherwise, it follows from the L^2 inner product of (4.5.0.2) with $\lambda_n^{-6}u_n$.

In summary, we have already obtained

$$\begin{cases} \lambda_n^3\|\theta_{n,x}\|,\ \lambda_n^3\|T_{n,x}\| = o(1), \\ \lambda_n^2\|\vartheta_{n,x}\|,\ \ \lambda_n^2\|S_{n,x}\| = o(1) \\ \lambda_n\|\zeta_{n,x}\|,\ \lambda_n\|R_{n,x}\|,\ \lambda_n\|\zeta_n\|,\ \lambda_n\|R_n\| = o(1) \\ \lambda_n\|v_n\|,\ \lambda_n\|u_{n,x}\| = o(1). \end{cases} \tag{4.5.0.28}$$

Repeating the arguments from (4.5.0.19)-(4.5.0.21), we again have $\|\phi_x\|, \|\varphi\| = o(1)$. Thus, the proof is completed.

4.6 Proof of Theorem 4.3.0.5

Similar to the proof of Theorem 3, if condition (4.4.0.2) is false, we have

$$\lambda_n^p(i\lambda_n u_n - v_n) = f_n = o(1) \text{ in } H^1, \tag{4.6.0.1}$$

$$\lambda_n^p(i\rho\lambda_n v_n - (\mu u_{n,xx} + \mu_0\phi_{n,x})) + \beta_0(\theta_n + \tau_1\vartheta_n + \frac{\tau_1^2}{2}\zeta_n)_x = o(1) \text{ in } L^2, \tag{4.6.0.2}$$

$$\lambda_n^p(i\lambda_n\phi_n - \varphi_n) = o(1) \text{ in } H^1, \tag{4.6.0.3}$$

$$\lambda_n^p(iJ\lambda_n\varphi_n - a_0\phi_{n,xx} + \mu_0 u_{n,x} + \xi\phi_n = o(1) \text{ in } L^2, \tag{4.6.0.4}$$

$$\lambda_n^p(i\lambda_n\theta_n - \vartheta_n) = o(1) \text{ in } H^1, \tag{4.6.0.5}$$

$$\lambda_n^p(i\lambda_n\vartheta_n - \zeta_n) = o(1) \text{ in } H^1, \tag{4.6.0.6}$$

$$\lambda_n^p(\frac{ia\tau_1^2\lambda_n}{2}\zeta_n + \beta_0 v_{n,x} - k(\theta_{n,xx} + \tau_2\vartheta_{n,xx}) - k_1(T_{n,x} + \tau_2 S_{n,x}) + a\vartheta_n + a\tau_1\zeta_n) = o(1) \text{ in } L^2, \tag{4.6.0.7}$$

$$\lambda_n^p(i\lambda_n T_n - S_n) = o(1) \text{ in } H^1, \tag{4.6.0.8}$$

$$\lambda_n^p(i\lambda_n S_n - R_n) = o(1) \text{ in } H^1, \tag{4.6.0.9}$$

$$\lambda_n^p(\frac{ib\tau_1^2\lambda_n}{2}R_n - k_4(T_{n,xx} + \tau_2 S_{n,xx}) + k_2(T_n + \tau_2 S_n) + k_1(\theta_{n,x} + \tau_2\vartheta_{n,x}) + aS_n + a\tau_1 R_n) = o(1) \text{ in } L^2. \tag{4.6.0.10}$$

Case 1: $2\tau_2 > \tau_1$.

From (4.4.0.3) and the dissipation inequality we again have

$$\begin{cases} \lambda_n^{p/2}\|\theta_{n,x}\|,\ \lambda_n^{p/2}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|S_{n,x}\| = o(1) \\ \lambda_n^{p/2-1}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-1}\|R_{n,x}\|,\ \lambda_n^{p/2-1}\|\zeta_n\|,\ \lambda_n^{p/2-1}\|R_n\| = o(1). \end{cases} \tag{4.6.0.11}$$

Taking the L^2 inner product of (4.6.0.7) with u_x we arrive at the following with the help of (4.6.0.11) with our boundary conditions after integrating by parts we see.

$$\frac{ia\tau_1^2}{2}\langle\lambda_n^{p/2-1}\zeta_{n,x},\lambda_n u_n\rangle+\lambda_n^{p/2-1}\beta_0\langle v_{n,x},u_{n,x}\rangle+\lambda_n^{p/2}\langle k(\theta_{n,x}+\tau_2\vartheta_{n,x}),\frac{u_{n,xx}}{\lambda_n}\rangle=o(1)$$

We then plug in (4.6.0.7) to our equation, after using (4.6.0.11) we can see since $\|\frac{u_{n,xx}}{\lambda_n}\|=O(1)$

$$\|\lambda_n^{p/4}u_{n,x}\|=o(1) \tag{4.6.0.12}$$

We next take the L^2 inner product of (4.6.0.2) with u_n to get the following after using (4.6.0.11)

$$i\rho\lambda_n^{p/2+1}\langle v_n,u_n\rangle-\mu\|\lambda_n^{p/4}u_{n,x}\|^2+\mu_0\langle\lambda_n\phi_n,\lambda_n^{p/2-1}u_{n,x}\rangle=o(1)$$

Its easy to see when $p\leq 4$ the last term tends to zero by (4.6.0.12) likewise with the second term. At this point we will let $p=4$ after plugging in (4.6.0.2) we arrive at

$$\|\lambda_n v\|=o(1) \tag{4.6.0.13}$$

Taking the L^2 inner product of (4.6.0.2) with $\phi_{n,x}$ it is easy to see.

$$-\mu\langle u_{n,xx},\phi_{n,x}\rangle-\mu_0\|\phi_{n,x}\|^2=o(1)$$

After integrating by parts with our boundary conditions we can deduce

$$\|\phi_{n,x}\|=o(1) \tag{4.6.0.14}$$

Finally, it is easy to see taking the L^2 inner product of (4.6.0.4) with φ_n that

$$\|\varphi_n\|=o(1) \tag{4.6.0.15}$$

We can see we have finally reached a contradiction to the unit norm therefore the proof is complete.

Case 2: $2\tau_2=\tau_1$.

In this case, the energy dissipation is weaker. This time, from (4.4.0.3) and dissipation inequality we only have

$$\lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|\theta_{n,x}\|=o(1), \tag{4.6.0.16}$$

which further leads to

$$\lambda_n^{p/2-1}\|S_{n,x}\|,\ \lambda_n^{p/2-1}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2-2}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-2}\|R_{n,x}\|=o(1). \tag{4.6.0.17}$$

Taking the L^2 inner product of (4.6.0.7) with u_x we arrive at the following with the help of (4.6.0.16), (4.6.0.17) and with our boundary conditions after integrating by parts we see.

$$\frac{ia\tau_1^2}{2}\langle\lambda_n^{p/2-2}\zeta_{n,x},\lambda_n u_n\rangle+\lambda_n^{p/2-2}\beta_0\langle v_{n,x},u_{n,x}\rangle+\lambda_n^{p/2-1}\langle k(\theta_{n,x}+\tau_2\vartheta_{n,x}),\frac{u_{n,xx}}{\lambda_n}\rangle=o(1)$$

We then plug in (4.6.0.7) to our equation, after using (4.6.0.16) and (4.6.0.17) we can see since $\|\frac{u_{n,xx}}{\lambda_n}\| = O(1)$

$$\|\lambda_n^{p/4-1/2} u_{n,x}\| = o(1) \tag{4.6.0.18}$$

We next take the L^2 inner product of (4.6.0.2) with u_n to get the following after using after using (4.6.0.16) and (4.6.0.17)

$$i\rho\lambda_n^{p/2}\langle v_n, u_n\rangle - \mu\|\lambda_n^{p/4-1/2} u_{n,x}\|^2 + \mu_0\langle\lambda_n\phi_n, \lambda_n^{p/2-2} u_{n,x}\rangle = o(1)$$

Its easy to see when $p \leq 6$ the last term tends to zero by (4.6.0.12) likewise with the second term. At this point we will let $p = 6$ after plugging in (4.6.0.2) we arrive at

$$\|\lambda_n v\| = o(1) \tag{4.6.0.19}$$

Taking the L^2 inner product of (4.6.0.2) with $\phi_{n,x}$ it is easy to see.

$$-\mu\langle u_{n,xx}, \phi_{n,x}\rangle - \mu_0\|\phi_{n,x}\|^2 = o(1)$$

After integrating by parts with our boundary conditions we can deduce since $\|\frac{\phi_{n,xx}}{\lambda_n}\| = O(1)$ by (4.6.0.18)

$$\|\phi_{n,x}\| = o(1) \tag{4.6.0.20}$$

Finally, it is easy to see taking the L^2 inner product of (4.6.0.4) with φ_n that

$$\|\varphi_n\| = o(1) \tag{4.6.0.21}$$

We can see we have finally reached a contradiction to the unit norm therefore the proof is complete.

4.7 Proof of Theorem 4.3.0.6

Similar to the proof of Theorem 3, if condition (4.4.0.2) is false, we have

$$\lambda_n^p(i\lambda_n u_n - v_n) = f_n = o(1) \text{ in } H^1, \tag{4.7.0.1}$$

$$\lambda_n^p(i\rho\lambda_n v_n - (\mu u_{n,xx} + \mu_0\phi_{n,x})) = o(1) \text{ in } L^2, \tag{4.7.0.2}$$

$$\lambda_n^p(i\lambda_n\phi_n - \varphi_n) = o(1) \text{ in } H^1, \tag{4.7.0.3}$$

$$\lambda_n^p(iJ\lambda_n\varphi_n - a_0\phi_{n,xx} + \mu_0 u_{n,x} + \xi\phi_n + \mu_2(T_{n,x} + \tau_1 S_{n,x} + \frac{\tau_1^2}{2}R_{n,x})) = o(1) \text{ in } L^2, \tag{4.7.0.4}$$

$$\lambda_n^p(i\lambda_n\theta_n - \vartheta_n) = o(1) \text{ in } H^1, \tag{4.7.0.5}$$

$$\lambda_n^p(i\lambda_n\vartheta_n - \zeta_n) = o(1) \text{ in } H^1, \tag{4.7.0.6}$$

$$\lambda_n^p(\frac{ia\tau_1^2\lambda_n}{2}\zeta_n + \beta_1\varphi_n - k(\theta_{n,xx} + \tau_2\vartheta_{n,xx}) - k_1(T_{n,x} + \tau_2 S_{n,x}) + a\vartheta_n + a\tau_1\zeta_n) = o(1) \text{ in } L^2, \tag{4.7.0.7}$$

$$\lambda_n^p(i\lambda_n T_n - S_n) = o(1) \text{ in } H^1, \tag{4.7.0.8}$$

$$\lambda_n^p(i\lambda_n S_n - R_n) = o(1) \text{ in } H^1, \tag{4.7.0.9}$$

$$\lambda_n^p(\frac{ib\tau_1^2\lambda_n}{2}R_n + \mu_2\varphi_{n,x} - k_4(T_{n,xx} + \tau_2 S_{n,xx}) + k_2(T_n + \tau_2 S_n) + k_1(\theta_{n,x} + \tau_2\vartheta_{n,x}) + aS_n + a\tau_1 R_n) = o(1) \text{ in } L^2. \tag{4.7.0.10}$$

Case 1: $2\tau_2 > \tau_1$.

From (4.4.0.3) and the dissipation inequality we again have

$$\begin{cases} \lambda_n^{p/2}\|\theta_{n,x}\|,\ \lambda_n^{p/2}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|S_{n,x}\| = o(1) \\ \lambda_n^{p/2-1}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-1}\|R_{n,x}\|,\ \lambda_n^{p/2-1}\|\zeta_n\|,\ \lambda_n^{p/2-1}\|R_n\| = o(1). \end{cases} \tag{4.7.0.11}$$

We begin by taking the L^2 inner product of (4.7.0.10) with ϕ_x. Doing so with (4.7.0.11) and integrating by parts since our boundary terms vanish we can see.

$$\frac{ib\tau_1^2\lambda_n^{p/2+1}}{2}\langle\lambda_n^{p/2-1}R_{n,x}, \lambda_n\phi_n\rangle + \mu_2\lambda_n^{p/2-1}\langle\varphi_{n,x}, \phi_{n,x}\rangle - k_4\lambda_n^{p/2}\langle(T_{n,x}+\tau_2 S_{n,x}), \frac{\phi_{n,xx}}{\lambda_n}\rangle = o(1)$$

We can see with (4.7.0.11) if we substitute (4.7.0.3) into the expression this will imply

$$\|\lambda_n^{p/4}\phi_{n,x}\| = o(1) \tag{4.7.0.12}$$

At this point we will let $p = 4$. We take the L^2 inner product of (4.7.0.4) with ϕ_n with (4.7.0.11) and integration by parts we can see. Also by poincare inequality we can see (4.7.0.12) implies $\|\lambda_n\phi_n\| = o(1)$

$$iJ\lambda_n^3\langle\varphi_n, \phi_n\rangle + a_0\|\lambda_n\phi_{n,x}\|^2 + \mu_0\langle\lambda_n u_n, \lambda_n\phi_{n,x}\rangle = o(1)$$

since $\|\lambda_n u\| = O(1)$ and with (4.7.0.12) we can see all the terms besides the first one tend to zero. The last step to this part is we plug in (4.7.0.3) and we can see this implies

$$\|\lambda_n\varphi_n\| = o(1) \tag{4.7.0.13}$$

We can see since $\frac{u_{n,xx}}{\lambda_n} = O(1)$ that if we take the L^2 inner product of $u_{n,x}$ with (4.7.0.4) we can see we will arrive at the following

$$\|u_{n,x}\| = o(1) \tag{4.7.0.14}$$

Finally, taking the L^2 inner product of (4.7.0.2) with v_n since $\|\frac{v_{n,x}}{\lambda_n}\| = O(1)$ we can deduce that.

$$\|v_n\| = o(1) \tag{4.7.0.15}$$

Therefore we reach a contradiction when $p = 4$ to use having a unit norm therefore the proof is complete.

Case 2: $2\tau_2 = \tau_1$. In this case, the energy dissipation is weaker. This time, from (4.4.0.3) and dissipation inequality we only have

$$\lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|\theta_{n,x}\| = o(1), \tag{4.7.0.16}$$

which further leads to

$$\lambda_n^{p/2-1}\|S_{n,x}\|,\ \lambda_n^{p/2-1}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2-2}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-2}\|R_{n,x}\| = o(1). \tag{4.7.0.17}$$

We begin by taking the L^2 inner product of (4.7.0.10) with ϕ_x. Doing so with (4.7.0.16), (4.7.0.17) and integrating by parts since our boundary terms vanish we can see.

$$\frac{ib\tau_1^2}{2}\langle\lambda_n^{p/2-2}R_{n,x},\lambda_n\phi_n\rangle+\mu_2\lambda_n^{p/2-2}\langle\varphi_{n,x},\phi_{n,x}\rangle-k_4\lambda_n^{p/2-1}\langle(T_{n,x}+\tau_2 S_{n,x}),\frac{\phi_{n,xx}}{\lambda_n}\rangle = o(1)$$

We can see with (4.7.0.16), (4.7.0.17) and if we substitute (4.7.0.3) into the expression this will imply

$$\|\lambda_n^{p/4-1/2}\phi_{n,x}\| = o(1) \tag{4.7.0.18}$$

At this point we will let $p = 6$. We take the L^2 inner product of (4.7.0.4) with ϕ_n with (4.7.0.16), (4.7.0.17) and integration by parts we can see. Also by poincare inequality we can see (4.7.0.18) implies $\|\lambda_n\phi_n\| = o(1)$

$$iJ\lambda_n^3\langle\varphi_n,\phi_n\rangle + a_0\|\lambda_n\phi_{n,x}\|^2 + \mu_0\langle\lambda_n u_n,\lambda_n\phi_{n,x}\rangle = o(1)$$

since $\|\lambda_n u\| = O(1)$ and with (4.7.0.18) we can see all the terms besides the first one tend to zero. The last step to this part is we plug in (4.7.0.3) and we can see this implies

$$\|\lambda_n\varphi_n\| = o(1) \tag{4.7.0.19}$$

We can see since $\frac{u_{n,xx}}{\lambda_n} = O(1)$ that if we take the L^2 inner product of $u_{n,x}$ with (4.7.0.4) we can see we will arrive at the following

$$\|u_{n,x}\| = o(1) \tag{4.7.0.20}$$

Finally, taking the L^2 inner product of (4.7.0.2) with v_n since $\|\frac{v_{n,x}}{\lambda_n}\| = O(1)$ we can deduce that.

$$\|v_n\| = o(1) \tag{4.7.0.21}$$

Therefore we reach a contradiction when $p = 6$ to use having a unit norm therefore the proof is complete.

4.8 Proof of Theorem 4.3.0.7

Similar to the proof of Theorem 3, if condition (4.4.0.2) is false, we have

$$\lambda_n^p(i\lambda_n u_n - v_n) = f_n = o(1) \text{ in } H^1, \tag{4.8.0.1}$$

$$\lambda_n^p(i\rho\lambda_n v_n - (\mu u_{n,xx} + \mu_0\phi_{n,x})) = o(1) \text{ in } L^2, \tag{4.8.0.2}$$

$$\lambda_n^p(i\lambda_n\phi_n - \varphi_n) = o(1) \text{ in } H^1, \tag{4.8.0.3}$$

$$\begin{aligned}\lambda_n^p(&iJ\lambda_n\varphi_n - a_0\phi_{n,xx} + \mu_0 u_{n,x} + \xi\phi_n\\ &-\beta_1(\theta_n + \tau_1\vartheta_n + \frac{\tau_1^2}{2}\zeta_n) = o(1) \text{ in } L^2,\end{aligned} \tag{4.8.0.4}$$

$$\lambda_n^p(i\lambda_n\theta_n - \vartheta_n) = o(1) \text{ in } H^1, \tag{4.8.0.5}$$

$$\lambda_n^p(i\lambda_n\vartheta_n - \zeta_n) = o(1) \text{ in } H^1, \tag{4.8.0.6}$$

$$\begin{aligned}\lambda_n^p(&\frac{ia\tau_1^2\lambda_n}{2}\zeta_n + \beta_1\varphi_n - k(\theta_{n,xx} + \tau_2\vartheta_{n,xx})\\ &-k_1(T_{n,x} + \tau_2 S_{n,x}) + a\vartheta_n + a\tau_1\zeta_n) = o(1) \text{ in } L^2,\end{aligned} \tag{4.8.0.7}$$

$$\lambda_n^p(i\lambda_n T_n - S_n) = o(1) \text{ in } H^1, \tag{4.8.0.8}$$

$$\lambda_n^p(i\lambda_n S_n - R_n) = o(1) \text{ in } H^1, \tag{4.8.0.9}$$

$$\begin{aligned}\lambda_n^p(&\frac{ib\tau_1^2\lambda_n}{2}R_n - k_4(T_{n,xx} + \tau_2 S_{n,xx}) + k_2(T_n + \tau_2 S_n)\\ &+k_1(\theta_{n,x} + \tau_2\vartheta_{n,x}) + aS_n + a\tau_1 R_n) = o(1) \text{ in } L^2.\end{aligned} \tag{4.8.0.10}$$

Case 1: $2\tau_2 > \tau_1$.

From (4.4.0.3) and the dissipation inequality we again have

$$\begin{cases} \lambda_n^{p/2}\|\theta_{n,x}\|,\ \lambda_n^{p/2}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|S_{n,x}\| = o(1)\\ \lambda_n^{p/2-1}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-1}\|R_{n,x}\|,\ \lambda_n^{p/2-1}\|\zeta_n\|,\ \lambda_n^{p/2-1}\|R_n\| = o(1). \end{cases} \tag{4.8.0.11}$$

We begin by taking the L^2 inner product of (4.8.0.8) with φ with (4.8.0.11) and integration by parts we can deduce

$$\lambda_n^{p/2-1}\frac{ia\tau_1^2}{2}\langle\zeta_n, \varphi_n\rangle + \beta_1\|\lambda_n^{p/4-1}\varphi_n\|^2 - \langle k\lambda_n^{p/2-1}(\theta_{n,x} + \tau_2\vartheta_{n,x}), \frac{\varphi_{n,x}}{\lambda_n}\rangle = o(1)$$

We can see with (4.8.0.11) and since $\|\frac{\varphi_n}{\lambda_n}\| = O(1)$ we can see it implies

$$\|\lambda_n^{p/4-1}\varphi_n\| = o(1) \tag{4.8.0.12}$$

At this point we will let $p = 8$ from (4.8.0.3) and (4.8.0.12) we can see

$$\|\lambda_n^2\phi_n\| = o(1) \tag{4.8.0.13}$$

Taking the L^2 inner product of (4.8.0.3) with ϕ_n we can easily see

$$\langle iJ\lambda_n\varphi_n, \lambda_n^2\phi\rangle - a_0\|\lambda_n\phi_{n,x}\| + \mu_0\langle u_{n,x}, \lambda_n^2\phi_n\rangle = o(1)$$

From (4.8.0.12) and (4.8.0.13) we can arrive at

$$\|\lambda_n \phi_{n,x}\| = o(1) \tag{4.8.0.14}$$

Now we take the L^2 inner product of (4.8.0.4) with $u_{n,x}$ it is easy to see

$$-a_0\langle \lambda_n \phi_{n,x}, \frac{u_{n,xx}}{\lambda_n}\rangle + \mu_0\|u_{n,x}\|^2 = o(1)$$

Since $\|\frac{u_{n,xx}}{\lambda_n}\| = O(1)$ by (4.8.0.14) we have

$$\|u_{n,x}\| = o(1) \tag{4.8.0.15}$$

Since $\|\frac{v_{n,x}}{\lambda_n}\| = O(1)$ we can see taking the L^2 inner product of (4.8.0.2) we will eventually arrive at

$$\|v_n\| = o(1) \tag{4.8.0.16}$$

We have finally reached the promised contradiction when $p = 8$. Therefore the proof is now complete.

Case 2: $2\tau_2 = \tau_1$.

In this case, the energy dissipation is weaker. This time, from (4.4.0.3) and our dissipation we only have

$$\lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|\theta_{n,x}\| = o(1), \tag{4.8.0.17}$$

which further leads to

$$\lambda_n^{p/2-1}\|S_{n,x}\|,\ \lambda_n^{p/2-1}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2-2}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-2}\|R_{n,x}\| = o(1). \tag{4.8.0.18}$$

We begin by taking the L^2 inner product of (4.8.0.8) with φ with (4.8.0.17) and (4.8.0.18) and integration by parts we can deduce

$$\lambda_n^{p/2-2}\frac{ia\tau_1^2}{2}\langle \zeta_n, \varphi_n\rangle + \beta_1\|\lambda_n^{p/4-3/2}\varphi_n\|^2 - \langle k\lambda_n^{p/2-2}(\theta_{n,x} + \tau_2\vartheta_{n,x}), \frac{\varphi_{n,x}}{\lambda_n}\rangle = o(1)$$

We can see with (4.8.0.17),(4.8.0.18) and since $\|\frac{\varphi_{n,x}}{\lambda_n}\| = O(1)$ we can see it implies

$$\|\lambda_n^{p/4-3/2}\varphi_n\| = o(1) \tag{4.8.0.19}$$

At this point we will let $p = 10$ from (4.8.0.3) and (4.8.0.19) we can see

$$\|\lambda_n^2\phi_n\| = o(1) \tag{4.8.0.20}$$

Taking the L^2 inner product of (4.8.0.3) with ϕ_n we can easily see

$$\langle iJ\lambda_n\varphi_n, \lambda_n^2\phi\rangle - a_0\|\lambda_n\phi_{n,x}\| + \mu_0\langle u_{n,x}, \lambda_n^2\phi_n\rangle = o(1)$$

From (4.8.0.19) and (4.8.0.20) we can arrive at

$$\|\lambda_n\phi_{n,x}\| = o(1) \tag{4.8.0.21}$$

Now we take the L^2 inner product of (4.8.0.4) with $u_{n,x}$ it is easy to see

$$-a_0\langle \lambda_n \phi_{n,x}, \frac{u_{n,xx}}{\lambda_n}\rangle + \mu_0\|u_{n,x}\|^2 = o(1)$$

Since $\|\frac{u_{n,xx}}{\lambda_n}\| = O(1)$ by (4.8.0.21) we have

$$\|u_{n,x}\| = o(1) \tag{4.8.0.22}$$

Since $\|\frac{v_{n,x}}{\lambda_n}\| = O(1)$ we can see taking the L^2 inner product of (4.8.0.2) we will eventually arrive at

$$\|v_n\| = o(1) \tag{4.8.0.23}$$

We have finally reached the promised contradiction when $p = 10$. Therefore the proof is now complete.

Case 2: $2\tau_2 > \tau_1$. when $\theta(0) = \theta(L) = \vartheta(0) = \vartheta(L) = \phi_x(0) = \phi_x(L) = u(0) = u(L) = 0$

If we consider the following boundary terms instead we can no longer conclude that the boundary terms vanish. So we wanted to explore how our p value changes with these new boundary conditions. We have already conclude we have the following.

From (4.4.0.3) and the dissipation inequality we again have

$$\begin{cases} \lambda_n^{p/2}\|\theta_{n,x}\|,\ \lambda_n^{p/2}\|\vartheta_{n,x}\|,\ \lambda_n^{p/2}\|T_{n,x}\|,\ \lambda_n^{p/2}\|S_{n,x}\| = o(1) \\ \lambda_n^{p/2-1}\|\zeta_{n,x}\|,\ \lambda_n^{p/2-1}\|R_{n,x}\|,\ \lambda_n^{p/2-1}\|\zeta_n\|,\ \lambda_n^{p/2-1}\|R_n\| = o(1). \end{cases} \tag{4.8.0.24}$$

We begin by taking the L^2 inner product of (4.8.0.8) with φ with (4.8.0.11) and integration by parts we can deduce

$$\lambda_n^{p/2-1}\frac{ia\tau_1^2}{2}\langle \zeta_n, \varphi_n\rangle + \beta_1\|\lambda_n^{p/4-1}\varphi_n\|^2 + k\lambda_n^{p/2-2}(\theta_{n,x} + \tau_2\vartheta_{n,x})\varphi_n|_{x=0,L}$$
$$-\langle k\lambda_n^{p/2-1}(\theta_{n,x} + \tau_2\vartheta_{n,x}), \frac{\varphi_{n,x}}{\lambda_n}\rangle = o(1)$$

We can see with (4.8.0.24) and since $\|\frac{\varphi_n}{\lambda_n}\| = O(1)$ we can see it implies

$$\beta_1\|\lambda_n^{p/4-1}\varphi_n\|^2 = k\lambda_n^{p/2-2}(\theta_{n,x} + \tau_2\vartheta_{n,x})\varphi_n|_{x=0,L} + o(1)$$

We need to show that the boundary term tends to zero. How we do this we use GNI to get an estimate on the boundary terms

$$|\lambda_n^{p/4-1/2}(\theta_{n,x} + \tau_2\vartheta_{n,x})||_{x=0,L} \le \|\lambda_n^{p/4-1/2}(\theta_{n,x} + \tau_2\vartheta_{n,x})\|_{L^\infty} \le$$
$$K\|\lambda_n^{p/2}(\theta_{n,x} + \tau_2\vartheta_{n,x})\|^{1/2}\|\frac{\theta_{n,xx} + \tau_2\vartheta_{n,xx}}{\lambda_n}\|^{1/2} = o(1)$$

$$|\lambda_n^{p/4-3/2}\varphi_n|_{x=0,L} \le \|\lambda_n^{p/4-3/2}\varphi_n\|_{L^\infty} \le K\|\lambda_n^{p/2-2}\varphi_n\|^{1/2}\|\frac{\varphi_{n,x}}{\lambda_n}\|^{1/2}$$

So we can see

$$\beta_1\|\lambda_n^{p/4-1}\varphi_n\|^2 \le o(1)K\|\lambda_n^{p/2-2}\varphi_n\|^{1/2} + o(1)$$

At this point we restrict $p \ge 4$, if this is true we can divide by $\lambda^{k/6-2/3}$ to arrive at

$$\beta_1\|\lambda_n^{p/6-2/3}\varphi_n\|^2 \le o(1)K\|\lambda_n^{p/6-2/3}\varphi_n\|^{1/2} + o(1)$$

Therefore

$$\|\lambda_n^{p/6-2/3}\varphi_n\| = o(1) \quad (4.8.0.25)$$

We can see by (4.8.0.3) and (4.8.0.25)

$$\|\lambda_n^{p/6+1/3}\phi_n\| = o(1) \quad (4.8.0.26)$$

Taking the L^2 inner product of (4.8.0.3) with ϕ_n we can easily see

$$\langle iJ\lambda_n^{p/6-2/3}\varphi_n, \lambda_n^{p/6+1/3}\phi\rangle - a_0\|\lambda_n^{p/6-2/3}\phi_{n,x}\| + \mu_0\langle u_{n,x}, \lambda_n^{p/3-4/3}\phi_n\rangle = o(1)$$

When $p \le 10$ we can see from (4.8.0.25) and (4.8.0.27) we can arrive at

$$\|\lambda_n^{p/6-2/3}\phi_{n,x}\| = o(1) \quad (4.8.0.27)$$

At this point we let $p = 10$.Then, we take the L^2 inner product of (4.8.0.4) with $u_{n,x}$ it is easy to see

$$+a_0\lambda_n u_{n,x} - a_0\langle\lambda_n\phi_{n,x}, \frac{u_{n,xx}}{\lambda_n}\rangle + \mu_0\|u_{n,x}\|^2 = o(1)$$

Since $\|\frac{u_{n,xx}}{\lambda_n}\| = O(1)$ by (4.8.0.27) we have

$$\|u_{n,x}\| = o(1) \quad (4.8.0.28)$$

Since $\|\frac{v_{n,x}}{\lambda_n}\| = O(1)$ we can see taking the L^2 inner product of (4.8.0.2) we will eventually arrive at

$$\|v_n\| = o(1) \quad (4.8.0.29)$$

We have finally reached the promised contradiction when $p = 10$. Therefore the proof is now complete.

4.9 Conclusion

It is easy to show $i\mathcal{R} \subset \rho(\mathcal{A})$ for all the case. So in summary we were able to achieve a decay rate for all the critical cases of our system.

Chapter 5

Boundary Stabilization of the Rao-Nakra Beam

5.1 Introduction

5.1.1 Introduction to the Model

The Generalized Rao-Nakra beam is a three-layer laminated beam that was first modeled in [47]. The assumption made in deriving the system in [47] are as follows; the total thickness of the beam is small compared to any other characteristic of length, the stress and displacement fields do not vary violently across that thickness , and the displacement and stress are continuous across the interfaces of the beam. In these assumption they were able to derive the Generalized Rao-Nakra beam model

$$\rho_1 h_1 u^{(1)}_{tt} - E_1 h_1 u^{(1)}_{xx} - \phi = 0 \tag{5.1.1.1}$$

$$\rho_1 I_1 y^{(1)}_{tt} - \frac{h_1}{2}\phi + G_1 h_1 (w_x + y^{(1)}) = 0 \tag{5.1.1.2}$$

$$\rho h w_{tt} + EI w_{xxxx} - G_1 h_1 (w_x + y^{(1)})_x - G_3 h_3 (w_x + y^{(3)})_x - h_2 \phi_x = 0 \tag{5.1.1.3}$$

$$\rho_1 h_1 u^{(3)}_{tt} - E_1 h_1 u^{(3)}_{xx} + \phi = 0 \tag{5.1.1.4}$$

$$\rho_3 I_3 y^{(3)}_{tt} - E_3 I_3 y^{(3)}_{xx} - \frac{h_3}{2}\phi + G_3 h_3 (w_x + y^{(3)}) = 0 \tag{5.1.1.5}$$

Where $u^{(i)}$ is the longitudinal displacement of the i th layer, w is the transverse displacement of the beam, $y^{(i)}$ is the shear angle of the i th layer, and the parameters E_i, I_i, h_i, ρ_i and G_i are the Young's modulus, moment of inertia, thickness, density and shear modulus of the ith layer respectively, and ϕ is the shear stress in the core layer which is the following

$$\phi = -u^{(1)} - \frac{h_1}{2} y^{(1)} + h_2 w_x + u^{(3)} - \frac{h_3}{2} y^{(3)}$$

To arrive at the Rao-Nakra model two more assumptions are made. i) Rotatory inertia and transverse shear of the bottom and top layer are neglected. This implies that (5.1.1.2) and (5.1.1.5) reduce to $y^{(1)} - w_x = y^{(3)} - w_x = 0$. ii.) Finally, if the core material is to be linearly elastic i.e. $\phi = \frac{G_2}{h_2}(-u^{(1)}+u^{(3)}+\alpha w_x)$. The Rao-Nakra model [50] is now able to be introduced.

$$\rho_1 h_1 u^{(1)}_{tt} - E_1 h_1 u^{(1)}_{xx} - G_2 \tau = 0 \tag{5.1.1.6}$$

$$\rho_3 h_3 u^{(3)}_{tt} - E_3 h_3 u^{(3)}_{xx} + G_2 \tau = 0 \tag{5.1.1.7}$$

$$\rho h w_{tt} + EI w_{xxxx} - G_2 \alpha \tau_x = 0 \tag{5.1.1.8}$$

where

$$\tau = \frac{1}{h_2}(-u + \phi + \alpha w_x) \tag{5.1.1.9}$$

A brief overview of the assumption and how to arrive at the model are given for the system in this subsection, the derivation of (5.1.1.1)-(5.1.1.5) is in [47]. Also a good reference for this subsection is [49]. We will introduce our specific system in the following subsection.

5.1.2 Introduction to the Project

Our final project is on the Rao-Nakra beam. The Rao-Nakra beam is a 3-layer laminated beam, the two top plates our linear elastic and the core is made of a material were the shear stress can be related to the shear strain. [47]. The varibles h_1, h_3 our the thickness of the top and bottom layer respectfully. h_2 is the thickness of the middle layer, and h is the thickness of the whole beam. ρ_i, E_i , *and* I_i are the density , Young's modulus and moments of inertia of the respected beam. G_2 is the shear modulus of the middle beam. A figure of the system was provided in [47]

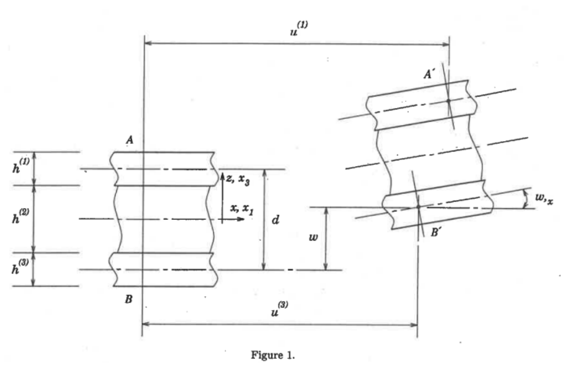

Figure 5.1: This picture was provided from [47]. The $u^{(1)} \equiv u$ and $u^{(3)} \equiv \phi$. We can see by the figure our u correlates to the longitudinal displacement of the top beam and ϕ the bottom beam. w is the transverse displacement of the beam.

The internal damping case for this system has already be solved by [48]. Therefore, we drew our attention to the cases where the damping is on the boundary. We have this boundary damping by a series of feedback controller located on each layer of on end of the beam. Looking at the dissipation function of the system we can see that the feedback controllers will have to correlate to the following. (dissipation function of the system will be introduced later.)

$$\begin{aligned} u_x(L) = -k_1 u_t(L)\ ,\ -EIw_{xxx}(L) + G_2\alpha\tau(L) = -k_2 w_t(L) \\ w_{xx}(L) = k_3 w_{xt}(L)\ ,\ \phi_x(L) = -k_4\phi_t(L) \end{aligned} \tag{5.1.2.1}$$

As far as I can find it appears only 4 cases have been solved so far. Case i) $k_1 > 0\ ,\ k_2 > 0\ ,\ k_3 > 0\ ,\ k_4 > 0$ []. Case ii) $k_1 > 0\ ,\ k_2 > 0\ ,\ k_3 = 0\ ,\ k_4 > 0$ []. Case iii) $k_1 > 0\ ,\ k_2 = 0\ ,\ k_3 > 0\ ,\ k_4 > 0$ []. Case iV) $k_1 > 0\ ,\ k_2 > 0\ ,\ k_3 = 0\ ,\ k_4 = 0$ When $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ This case is exactly the same one as $k_1 = 0\ ,\ k_2 > 0\ ,\ k_3 = 0\ ,\ k_4 > 0$ When $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$[46].

In our project we have achieved a decay rate for 2 new cases so far. Those cases are Section 5.2 ($k_1 > 0\ ,\ k_2 = 0\ ,\ k_3 = 0\ ,\ k_4 > 0$) and Section 5.3 ($k_1 > 0\ ,\ k_2 = 0\ ,\ k_3 > 0\ ,\ k_4 = 0$ When $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ and this case is the exact same as $k_1 = 0\ ,\ k_2 = 0\ ,\ k_3 > 0\ ,\ k_4 > 0$ When $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$).

The system for the Rao-Narka beam is the following

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - G_2\tau = 0 \tag{5.1.2.2}$$

$$\rho_3 h_3 \phi_{tt} - E_3 h_3 \phi_{xx} + G_2\tau = 0 \tag{5.1.2.3}$$

$$\rho h w_{tt} + EIw_{xxxx} - G_2\alpha\tau_x = 0 \tag{5.1.2.4}$$

where

$$\tau = -u + \phi + \alpha w_x \tag{5.1.2.5}$$

τ is commonly refered to as shear angle.

5.2 Case i) $k_1 > 0\ ,\ k_2 = 0\ ,\ k_3 = 0\ and\ k_4 > 0$

5.2.1 Prelimanry and Main Results

We introduce before the Rao-Narka Beam system:

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - G_2\tau = 0 \tag{5.2.1.1}$$

$$\rho_3 h_3 \phi_{tt} - E_3 h_3 \phi_{xx} + G_2\tau = 0 \tag{5.2.1.2}$$

$$\rho h w_{tt} + EIw_{xxxx} - G_2\alpha\tau_x = 0 \tag{5.2.1.3}$$

where

$$\tau = -u + \phi + \alpha w_x \tag{5.2.1.4}$$

We consider the following boundary conditions

$$\begin{aligned} u_x(L) = -k_1 u_t(L)\ ,\ \phi_x(L) = -k_4 \phi_t(L)\ , \\ w_x(0) = w(0) = w_{xx}(L) = 0\ ,\ w_{xxx}(L) - \tau(L) = 0 \\ u(0) = \phi_x(0) = 0 \end{aligned} \tag{5.2.1.5}$$

First it is important to figure out our proper Hilbert space so that our energy function is dissipative. To begin this step we can take the L^2 inner product of (5.2.1.1) with u_t, (5.2.1.2) with ϕ_t, (5.2.1.3) with w_t and we will add all these together.

$$\rho_1 h_1 \langle u_{tt}, u_t \rangle - E_1 h_1 \langle u_{xx}, u_t \rangle - G_2 \langle \tau, u_t \rangle = 0$$

With integration by parts we can arrive at the following expression.

$$\rho_1 h_1 \langle u_{tt}, u_t \rangle + E_1 h_1 \langle u_x, u_{xt} \rangle - G_2 \langle \tau, u_t \rangle = E_1 h_1 u_t(L) u_x(L) - E_1 h_1 u_t(0) u_x(0)$$

We can factor a derivative with respect to time and with our boundary conditions proposed in (5.2.1.5) we will arrive at the following.

$$\frac{1}{2}\frac{d}{dt}[\rho_1 h_1 \|u_t\|^2 + E_1 h_1 \|u_x\|^2] - G_2 \langle \tau, u_t \rangle = E_1 h_1 u_t(L) u_x(L) \tag{5.2.1.6}$$

Doing the same with ϕ it is easy to see.

$$\frac{1}{2}\frac{d}{dt}[\rho_3 h_3 \|\phi_t\|^2 + E_3 h_3 \|\phi_x\|^2] + G_2 \langle \tau, \phi_t \rangle = E_3 h_3 \phi_t(L) \phi_x(L) \tag{5.2.1.7}$$

With the last part we have

$$\rho h \langle w_{tt}, w_t \rangle + EI \langle w_{xxxx}, w_t \rangle - G_2 \alpha \langle \tau_x, w_t \rangle = 0$$

Integrating by parts we have

$$\begin{aligned} \rho h \langle w_{tt}, w_t \rangle + EI \langle w_{xx}, w_{xxt} \rangle + G_2 \alpha \langle \tau, w_{xt} \rangle = [-EI w_{xxx}(L) + G_2 \alpha \tau(L)] w_t(L) \\ -[-EI w_{xxx}(0) + G_2 \alpha \tau(0)] w_t(0) + EI w_{xx}(L) w_{xt}(L) - EI w_{xx}(0) w_{xt}(0) \end{aligned}$$

We can factor a derivative with respect to time on a few of our terms and with our boundary conditions we arrive at the following.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}[\rho h \|w_t\|^2 + EI \|w_{xx}\|^2] + G_2 \alpha \langle \tau, w_{xt} \rangle = [-EI w_{xxx}(L) + G_2 \alpha \tau(L)] w_t(L) \\ + EI w_{xx}(L) w_{xt}(L) \end{aligned} \tag{5.2.1.8}$$

Now adding (5.2.1.6)-(5.2.1.8) we will arrive at the following

$$\frac{1}{2}\frac{d}{dt}[[\rho_1 h_1\|u_t\|^2 + E_1 h_1\|u_x\|^2 + \rho_3 h_3\|\phi_t\|^2 + E_3 h_3\|\phi_x\|^2$$
$$\rho h\|w_t\|^2 + EI\|w_{xx}\|^2 + G_2\|\tau\|^2] = [-EIw_{xxx}(L) + G_2\alpha\tau(L)]w_t(L)$$
$$+EIw_{xx}(L)w_{xt}(L) + E_3h_3\phi_t(L)\phi_x(L) + E_1h_1u_t(L)u_x(L)$$

From (5.2.1.5) we can see the RHS will be the following.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[[\rho_1 h_1\|u_t\|^2 + E_1 h_1\|u_x\|^2 + \rho_3 h_3\|\phi_t\|^2 + E_3 h_3\|\phi_x\|^2 \\ \rho h\|w_t\|^2 + EI\|w_{xx}\|^2 + G_2\|\tau\|^2] = -k_4E_3h_3\phi_t^2(L) - k_1E_1h_1u_t^2(L)\end{aligned} \quad (5.2.1.9)$$

We can see from (5.2.1.9) our energy function of the system will be the following

$$\begin{aligned}E(t) = \rho_1 h_1\|u_t\|^2 + E_1 h_1\|u_x\|^2 + \rho_3 h_3\|\phi_t\|^2 + E_3 h_3\|\phi_x\|^2 \\ +\rho h\|w_t\|^2 + EI\|w_{xx}\|^2 + G_2\|\tau\|^2\end{aligned} \quad (5.2.1.10)$$

With the following dissipation function

$$\frac{1}{2}\frac{dE(t)}{dt} = -k_4E_3h_3\phi_t^2(L) - k_1E_1h_1u_t^2(L) \quad (5.2.1.11)$$

With our energy function we can start to construct a proper state space. Let

$$\mathcal{H} = H_l^1 \times L^2 \times H_k^1 \times L^2 \times H_p^2 \times L^2$$

Where $H_l^1 = \{f \in H^1 | f(0) = 0\}$, $H_k^1 = \{f \in H^1 | f_x(0) = 0\}$, and $H_p^2 = \{w \in H^2 | w_x(0) = w(0) = w_{xx}(L) = w_{xxx}(L) - \tau(L) = 0\}$

Denoting $U = (u, u_t, \phi, \phi_t, w, w_t) = (u, v, \phi, \varphi, w, \Omega)$. We then convert our system to a first-order evolution equation on our Hilbert space $\mathcal{H}$

$$\frac{dU}{dt} = \mathcal{A}U = \begin{pmatrix} v \\ \frac{E_1}{\rho_1}u_{xx} + \frac{G_2}{\rho_1 h_1}\tau \\ \varphi \\ \frac{E_3}{\rho_3}\phi_{xx} - \frac{G_2}{\rho_3 h_3}\tau \\ \Omega \\ -\frac{EI}{\rho h}w_{xxxx} + \frac{G_2\alpha}{\rho_1 h_1}\tau_x \end{pmatrix} \quad (5.2.1.12)$$

With

$$D(\mathcal{A}) = H^2 \times H^1 \times H^2 \times H^1 \times H^4 \times H^2 \in \mathcal{H}$$

Where $D(\mathcal{A})$ contains (5.2.1.5) as well

Theorem 5.2.1.1 *$\mathcal{A}$ is a infinitesimal generator of a C_0-semigroup of contractions on the Hilbert space $\mathcal{H}$*

Proof.

$$Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}\|U\|^2 = -k_4E_3h_3\phi_t^2(L) - k_1E_1h_1u_t^2(L) \leq 0 \qquad (5.2.1.13)$$

Thus $\mathcal{A}$ is dissipative. It also easy to see that $D(\mathcal{A})$ is dense in $\mathcal{H}$. To begin we assume our eigenvalue $\lambda = 0$ and we have a unit norm aka $\|U\|_{\mathcal{H}} = 1$ and we wish to show this is contradict for $\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$

i.e.

$$v = o(1) \in H^1 \qquad (5.2.1.14)$$

$$-\frac{E_1}{\rho_1}u_{xx} - \frac{G_2}{\rho_1 h_1}\tau = o(1) \in L^2 \qquad (5.2.1.15)$$

$$\varphi = o(1) \in H^1 \qquad (5.2.1.16)$$

$$-\frac{E_3}{\rho_3}\phi_{xx} + \frac{G_2}{\rho_3 h_3}\tau = o(1) \in L^2 \qquad (5.2.1.17)$$

$$\Omega = o(1) \in H^2 \qquad (5.2.1.18)$$

$$\frac{EI}{\rho h}w_{xxxx} - \frac{G_2\alpha}{\rho h}\tau_x = o(1) \in L^2 \qquad (5.2.1.19)$$

From our dissipation function we have

$$k_2E_3h_3\phi_t^2(L) = k_1E_1h_1u_t^2(L) = o(1) \qquad (5.2.1.20)$$

Taking the L^2 inner product of (5.2.1.15) with ρ_1h_1u and Taking the L^2 inner product of (5.2.1.17) with $\rho_3h_3\phi$. We then add them together we get the following after integration by parts and using (5.2.1.20).

$$E_1h_1\|u_x\|^2 + E_3h_3\|\phi_x\|^2 + G_2\langle \tau, -u + \phi\rangle = o(1) \qquad (5.2.1.21)$$

Now we take the L^2 inner product of (5.2.1.19) with ρhw. With integration by parts and with our boundary conditions we can arrive at the following.

$$EI\|w_{xx}\|^2 + G_2\langle \tau, \alpha w_x\rangle = o(1) \qquad (5.2.1.22)$$

Adding (5.2.1.21) with (5.2.1.22) we get the following

$$E_1h_1\|u_x\|^2 + E_3h_3\|\phi_x\|^2 EI\|w_{xx}\|^2 + G_2\|\tau\|^2 = o(1) \qquad (5.2.1.23)$$

We can see with (5.2.1.14) , (5.2.1.16) , (5.2.1.18) and (5.2.1.23) implies $\|U\|_{\mathcal{H}} = o(1)$. However, this a contradiction to $\|U\|_{\mathcal{H}} = 1$. Therefore $0 \in \rho(\mathcal{A})$. With Theorem 2.2.0.1 we can conclude that the proof is complete.

Theorem 5.2.1.2 *Let $e^{\mathcal{A}t}$ be the semigroup associated with our system then it is polynomially stable of order 1/3. When $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ or when $\frac{\rho_3 E_1}{\rho_1 E_3} \notin \mathcal{Q}$.*

This proof will be done in section 5.2.4. We can see the following theorem is include in this theorem. However, I believe theorem 5.2.1.3 is critical in regards to trying to achieve a decay rate for 1-feedback controller that's why it was left in this thesis.

Theorem 5.2.1.3 *Let $e^{\mathcal{A}t}$ be the semigroup associated with our system then it is polynomially stable of order 1/3. When $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ and $\rho_1 h_1 = \rho_3 h_3$ (This theorem is also covered in theorem 5.2.1.2, I only kept this proof because I think it will be useful to eventually being able to achieve a decay rate for one feedback controller.)*

This Theorem will be shown in the following section. We use Theorem 2.4.0.1 to do so.

5.2.2 Proof of Theorem 5.2.1.3 $i\mathcal{R} \subset \rho(\mathcal{A})$

We begin by showing $i\mathcal{R} \subset \rho(\mathcal{A})$. To do this we assume that it is false. Then there exist sequence λ_n that approaches a finite value such that $\|(i\lambda_n I - \mathcal{A})U_n\|_{\mathcal{H}} = o(1)$, (we will omit the n for convenience)

i.e.

$$i\lambda u - v = o(1) \in H^1 \tag{5.2.2.1}$$

$$i\lambda v - \frac{E_1}{\rho_1} u_{xx} - \frac{G_2}{\rho_1 h_1}\tau = o(1) \in L^2 \tag{5.2.2.2}$$

$$i\lambda\phi - \varphi = o(1) \in H^1 \tag{5.2.2.3}$$

$$i\lambda\varphi - \frac{E_3}{\rho_3}\phi_{xx} + \frac{G_2}{\rho_3 h_3}\tau = o(1) \in L^2 \tag{5.2.2.4}$$

$$i\lambda w - \Omega = o(1) \in H^2 \tag{5.2.2.5}$$

$$i\lambda\Omega + \frac{EI}{\rho h} w_{xxxx} - \frac{G_2\alpha}{\rho h}\tau_x = o(1) \in L^2 \tag{5.2.2.6}$$

The proof for this section follows very similar to [46]

We can see

$$Re\langle (i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}} = -k_1 u_t^2(L) - k_4\phi_t^2(L) = o(1) \tag{5.2.2.7}$$

From (5.2.2.7) we can easily deduce

$$v(L) = \varphi(L) = u_x(L) = \phi_x(L) = o(1) \tag{5.2.2.8}$$

We begin by plugging in (5.2.2.1) into (5.2.2.2) , (5.2.2.3) into (5.2.2.4) and (5.2.2.5) into (5.2.2.6) we then arrive at the 3 following expressions.

$$-\lambda^2 u_1 - \frac{E_1}{\rho_1} u_{1,xx} - \frac{G_2}{\rho_1 h_1}\tau = \frac{g_2 + i\lambda g_1}{\lambda^k} \tag{5.2.2.9}$$

$$-\lambda^2\phi_1 - \frac{E_3}{\rho_3}\phi_{1,xx} + \frac{G_2}{\rho_3 h_3}\tau = \frac{g_4 + i\lambda g_3}{\lambda^k} \quad (5.2.2.10)$$

$$-\lambda^2 w_1 + \frac{EI}{h\rho}w_{1,xxxx} - \frac{G_2\alpha}{\rho h}\tau_x = \frac{g_6 + i\lambda g_5}{\lambda^k} \quad (5.2.2.11)$$

If we take the L^2 inner product of (5.2.2.9) with xu_x we can see after taking the reals and integrating by parts

$$\|\lambda u\|^2 + \frac{E_1}{\rho_3}\|u_x\|^2 = 2Re\langle\frac{G_2}{\rho_1 h_1}\tau, xu_x\rangle + o(1) \quad (5.2.2.12)$$

By the Holder inequality we will have

$$\frac{G_2}{\rho_1 h_1}\tau, xu_x\rangle \leq C(\|u\|\|u_x\| + \|\phi\|\|u_x\| + \|w\|\|u_x\|) \quad (5.2.2.13)$$

Which implies since everything is bounded

$$\frac{E_1}{\rho_1}\|u_x\|^2 \leq \frac{G_2}{\rho_1 h_1}K\|u_x\|^2 \quad (5.2.2.14)$$

So for small enough $\frac{G_2}{\rho_1 h_1}$ we can see $\|u_x\| = o(1)$ and by poincare inequality we have $\|u\| = o(1) \to \|v\| = o(1)$. We can repeat almost the same exact process for the ϕ to get $\|\phi_x\| = o(1)$, $\|\phi\| = o(1)$, $\|\varphi\| = o(1)$. Now we can take the L^2 inner product of (5.2.2.9) with w_x to get $\|w_x\| = o(1)$ and with poincare inequality we get $\|w\| = o(1) \to \|\Omega\| = o(1)$. We finally take the L^2 inner product of (5.2.2.11) to get. (its easy to see $w_{xx} = O(1)$)

$$[+\frac{EI}{h\rho}w_{xxx}(x) - \frac{G_2\alpha}{\rho h}\tau(x)]w(x)|_{x=0,L} = o(1)$$
$$-\frac{EI}{h\rho}w_{xx}(x)w_x(x)|_{x=0,L} + \frac{EI}{h\rho}\|w_{xx}\|^2 = o(1)$$

Which implies

$$\|w_{xx}\| = o(1) \quad (5.2.2.15)$$

Therefore, we have reached the promised contradiction.

5.2.3 Proof of Theorem 5.2.1.2 $\sup_{|\lambda|\to 0}\frac{1}{\lambda^3}\|(i\lambda - \mathcal{A})^{-1}\| < +\infty$ with $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ and $\rho_1 h_1 = \rho_3 h_3$

We do the proof by a contradiction we assume that $\sup_{|\lambda|\to 0}\frac{1}{\lambda^3}\|(i\lambda - \mathcal{A})^{-1}\| < +\infty$ is false. this implies that $\exists\lambda \to \infty \ni \lambda^3\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$. This is written out below, we also have a unit norm aka $\|U\|_{\mathcal{H}} = 1$. We now express $\lambda^3\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$.

i.e.

$$\lambda^k[i\lambda u - v] = f_1 = o(1) \in H^1 \tag{5.2.3.1}$$

$$\lambda^k[i\lambda v - \frac{E_1}{\rho_1}u_{xx} - \frac{G_2}{\rho_1 h_1}\tau] = f_2 = o(1) \in L^2 \tag{5.2.3.2}$$

$$\lambda^k[i\lambda\phi - \varphi] = f_3 = o(1) \in H^1 \tag{5.2.3.3}$$

$$\lambda^k[i\lambda\varphi - \frac{E_3}{\rho_3}\phi_{xx} + \frac{G_2}{\rho_3 h_3}\tau] = f_4 = o(1) \in L^2 \tag{5.2.3.4}$$

$$\lambda^k[i\lambda w - \Omega] = f_5 = o(1) \in H^2 \tag{5.2.3.5}$$

$$\lambda^k[i\lambda\Omega + \frac{EI}{\rho h}w_{xxxx} - \frac{G_2\alpha}{\rho h}\tau_x] = f_6 = o(1) \in L^2 \tag{5.2.3.6}$$

We begin by plugging in (5.2.3.1) into (5.2.3.2), (5.2.3.3) into (5.2.3.4), and (5.2.3.5) into (5.2.3.6) to arrive at the three expressions below

$$-\lambda^2 u_1 - \frac{E_1}{\rho_1}u_{1,xx} - \frac{G_2}{\rho_1 h_1}\tau = \frac{g_2 + i\lambda g_1}{\lambda^k} \tag{5.2.3.7}$$

$$-\lambda^2 \phi_1 - \frac{E_3}{\rho_3}\phi_{1,xx} + \frac{G_2}{\rho_3 h_3}\tau = \frac{g_4 + i\lambda g_3}{\lambda^k} \tag{5.2.3.8}$$

$$-\lambda^2 w_1 + \frac{EI}{h\rho}w_{1,xxxx} - \frac{G_2\alpha}{\rho h}\tau_x = \frac{g_6 + i\lambda g_5}{\lambda^k} \tag{5.2.3.9}$$

From our dissipation we have

$$\lambda^{k/2}v(L) = \lambda^{k/2}u_x(L) = \lambda^{k/2}\varphi(L) = \lambda^{k/2}\phi_x(L) = o(1) \tag{5.2.3.10}$$

With (5.2.3.1) using GNI we have

$$|i\lambda^{k/2+1}u(L) - \lambda^{k/2}v(L)| \le \|i\lambda^{k/2+1}u - \lambda^{k/2}v\|_{L^\infty} \le K\|f_1\|^{1/2}\|f_{1,x}\|^{1/2} = o(1)$$

Now by (5.2.3.10) we have

$$\lambda^{k/2+1}u(L) = o(1) \tag{5.2.3.11}$$

Likewise for the ϕ

$$\lambda^{k/2+1}\phi(L) = o(1) \tag{5.2.3.12}$$

From the norm it is clear

$$\|\lambda u\| = O(1) \; and \; \|\lambda\phi\| = O(1) \tag{5.2.3.13}$$

$$\|w_{xx}\| = O(1) \; \|\lambda^{1/2}w_x\| = O(1) \; \|\lambda w\| = O(1) \tag{5.2.3.14}$$

We can see (5.2.3.13) and (5.2.3.14) implies

$$\|\lambda^{1/2}\tau\| = O(1) \tag{5.2.3.15}$$

Taking the L^2 inner product of xu_x with (5.2.3.7) and then taking the reals, we will arrive at the following with the help of (5.2.3.15). Also we use (5.2.3.10) and (5.2.3.11) to get rid of the boundary terms

$$\|\lambda^{3/2}u\|^2 + \frac{E_1}{\rho_1}\|\lambda^{1/2}u_x\|^2 = \frac{G_2\alpha}{\rho_1 h_1}\langle\lambda^{1/2}\tau, x\lambda^{1/2}u_x\rangle = O(1) \tag{5.2.3.16}$$

Doing the same with (5.2.3.8) and $x\phi_X$ implies

$$\|\lambda^{1/2}\phi_x\| = O(1)\ \|\lambda^{3/2}\phi\| = O(1) \tag{5.2.3.17}$$

To finish the proof we need to show $\|\lambda w\|$ and $\|w_{xx}\|$ tend to zero. We start by taking the inner product of (5.2.3.9) with xw_x

$$\|\lambda w\|^2 + \frac{3EI}{2h\rho}\|w_{xx}\|^2 = L|\lambda w(L)|^2 + \frac{o(1)}{\lambda^k} \tag{5.2.3.18}$$

We can now conclude that once we show the boundary term tends to zero the proof is complete. We start to switch gears and get an estimate on the following boundary terms to use later in our proof. With the restrictions on our constants. We can see if we add (5.2.3.7) and (5.2.3.8) we are left with.

$$-\lambda^{k+2}[u+\phi] - \frac{E_1}{\rho_1}\lambda^k[u_{xx}+\phi_{xx}] = o(1) \in L^2 \tag{5.2.3.19}$$

Now lets take the inner product of (5.2.3.19) with $u_x + \phi_x$ we can see

$$\begin{aligned} &-\lambda^{k+2}[\langle u, u_x\rangle + \langle \phi, u_x\rangle + \langle u, \phi_x\rangle + \langle \phi, \phi_x\rangle] \\ -\frac{E_1}{\rho_1}\lambda^k[&\langle u_{xx}, u_x\rangle + \langle \phi_{xx}, u_x\rangle + \langle u_{xx}, \phi_x\rangle + \langle \phi_{xx}, \phi_x\rangle] = o(1) \end{aligned}$$

With integration by parts we have

$$\begin{aligned} -\lambda^{k+2}[\langle u, u_x\rangle - \langle \phi_x, u\rangle + \langle u, \phi_x\rangle + \langle \phi, \phi_x\rangle] - \lambda^{k+2}u(L)\phi(L) + \lambda^{k+2}u(0)\phi(0) \\ -\frac{E_1}{\rho_1}\lambda^k[\langle u_{xx}, u_x\rangle + \langle \phi_{xx}, u_x\rangle - \langle u_x, \phi_{xx}\rangle + \langle \phi_{xx}, \phi_x\rangle] \\ -\frac{E_1}{\rho_1}\lambda^k u_x(L)\phi_x(L) + \frac{E_1}{\rho_1}\lambda^k u_x(0)\phi_x(0) = o(1) \end{aligned}$$

We can see with our boundary conditions that our boundary terms will go away. We can also see that the cross terms cancel out (If we take the reals).

$$-\lambda^{k+2}Re[\langle u, u_x\rangle + \langle \phi, \phi_x\rangle] - \frac{E_1}{\rho_1}\lambda^k Re[\langle u_{xx}, u_x\rangle + \langle \phi_{xx}, \phi_x\rangle] = o(1)$$

Now we integrate by parts again and we arrive at the following

$$-\lambda^{k+2}u^2(L) - \lambda^{k+2}\phi^2(L) - \frac{E_1}{\rho_1}\lambda^k u_x^2(L) - \frac{E_1}{\rho_1}\lambda^k \phi_x^2(L)$$
$$+\lambda^{k+2}u^2(0) + \lambda^{k+2}\phi^2(0) + \frac{E_1}{\rho_1}\lambda^k u_x^2(0) + \frac{E_1}{\rho_1}\lambda^k \phi_x^2(0) = o(1)$$

With (5.2.3.10)-(5.2.3.12) implies the above expression reduces to the following

$$\lambda^{k+2}\phi^2(0) + \frac{E_1}{\rho_1}\lambda^k u_x^2(0) = o(1) \tag{5.2.3.20}$$

So (5.2.3.21) implies the following

$$|\lambda^{k/2+1}\phi(0)| = o(1) \; and \; |\lambda^{k/2}u_x(0)| = o(1) \tag{5.2.3.21}$$

We will use (5.2.3.22) quite frequently.

We now turn our attention to the main part of our the proof. We first take the L^2 inner product of (5.2.3.7) with u_x after taking the reals

$$-\lambda^{k+2}u^2(L) - \lambda^k u_x^2(L) + \frac{E_1}{\rho_1}\lambda^{k+2}u^2(0) + \frac{E_1}{\rho_1}\lambda^k u_x^2(0) - \frac{G_2}{\rho_1 h_1}\lambda^k Re\langle[-u+\phi+\alpha w_x], u_x\rangle = o(1)$$

With our boundary conditions, (5.2.3.21) , (5.2.3.11) , and (5.2.3.10)

$$\alpha\lambda^k Re\langle w_x, u_x\rangle = -\lambda^k\langle\phi, u_x\rangle + o(1) \tag{5.2.3.22}$$

Now we take the inner product of (5.2.3.7) with αw_{xx} and we get the following.

$$-\alpha\lambda^{k+2}\langle u, w_{xx}\rangle - \frac{E_1\alpha}{\rho_1}\lambda^k\langle u_{xx}, w_{xx}\rangle - \alpha\frac{G_2}{\rho_1 h_1}\lambda^k\langle[-u+\phi+\alpha w_x], w_{xx}\rangle = o(1)$$

Now we integrate by parts and divide by λ^2 with taking the reals

$$-\alpha\lambda^k u(L)w_x(L) + \alpha\lambda^k u(0)w_x(0) + \alpha\lambda^k\langle u_x, w_x\rangle - \frac{E_1\alpha}{\rho_1}\lambda^{k-1}\langle u, \frac{w_{xxxx}}{\lambda}\rangle$$
$$-\frac{E_1\alpha}{\rho_1}\lambda^{k-2}u_x(L)w_{xx}(L) + \frac{E_1\alpha}{\rho_1}\lambda^{k-2}u_x(0)w_{xx}(0) - \frac{E_1\alpha}{\rho_1}\lambda^{k-2}u(L)w_{xxx}(L)$$
$$+\frac{E_1\alpha}{\rho_1}\lambda^{k-2}u(0)w_{xxx}(0) - \alpha\frac{G_2}{\rho_1 h_1}\lambda^{k-2}\langle[-u+\phi], w_{xx}\rangle - \alpha^2\frac{G_2}{\rho_1 h_1}\lambda^{k-2}w_x^2(L)$$
$$+\alpha^2\frac{G_2}{\rho_1 h_1}\lambda^{k-2}w_x^2(0) = o(1)$$

By GNI it is not hard to see $\|\frac{w_{xx}(0)}{\lambda^{1/4}}\| = O(1)$,$\|\frac{w_{xxx}(L)}{\lambda^{3/4}}\| = O(1)$,$\|\frac{w_{xxx}(0)}{\lambda^{3/4}}\| = O(1)$. With (5.2.3.21) , (5.2.3.10) and (5.2.3.11) the above expression reduces to

$$\alpha\lambda^k Re\langle u_x, w_x\rangle - \frac{E_1\alpha}{\rho_1}\lambda^{k-1}Re\langle u, \frac{w_{xxxx}}{\lambda}\rangle - \alpha\frac{G_2}{\rho_1 h_1}\lambda^{k-2}Re\langle[-u+\phi], w_{xx}\rangle$$
$$-\alpha^2\frac{G_2}{\rho_1 h_1}\lambda^{k-2}w_x^2(L) = o(1) \tag{5.2.3.23}$$

(5.2.3.22) minus (5.2.3.23) implies the following.

$$-\lambda^k\langle\phi, u_x\rangle - \frac{E_1\alpha}{\rho_1}\lambda^{k-1}Re\langle u, \frac{w_{xxxx}}{\lambda}\rangle - \alpha\frac{G_2}{\rho_1 h_1}\lambda^{k-2}Re\langle[-u+\phi], w_{xx}\rangle + o(1)$$
$$= \alpha^2\frac{G_2}{\rho_1 h_1}\lambda^{k-2}w_x^2(L)$$

We can see if we wish to get an estimate on $\lambda^{1/2}w_x(L)$ we need to show the above internal terms tend to zero. At this point we will let $k = 3$, we try to get a higher estimate on our u and ϕ terms to show the internal terms tend to zero, but before that we can already see by (5.2.3.14), (5.2.3.16) and (5.2.3.17) above will reduce to

$$\alpha^2\frac{G_2}{\rho_1 h_1}\lambda w_x^2(L) = -\lambda^3\langle\phi, u_x\rangle - \frac{E_1\alpha}{\rho_1}\lambda^2 Re\langle u, \frac{w_{xxxx}}{\lambda}\rangle + o(1) \tag{5.2.3.24}$$

Now we wish to get a higher estimate on our u and ϕ terms to show the internal terms tend to zero.

We can see from (5.2.3.16) the term that gives us difficulty is the following term $\langle w_x, xu_x\rangle$ so if we can get a higher estimate on this we can get a higher estimate on our u and u_x term. So to do this we take the inner product of (5.2.3.7) with xw_{xx}. We also divide by λ^3

$$-\lambda^2\langle u, xw_{xx}\rangle - \frac{E_1}{\rho_1}\langle u_{xx}, xw_{xx}\rangle - \frac{G_2}{\rho_1 h_1}\langle\tau, xw_{xx}\rangle = \frac{o(1)}{\lambda^3}$$

We can see the last term is of zero. So now we integrate by parts.

$$L\lambda^2 u(L)w_x(L) + \langle\lambda^{3/2}[xu_x+u], \lambda^{1/2}w_x\rangle + \frac{E_1}{\rho_1}\langle u_x, xw_{xxx}+w_{xx}\rangle - \frac{E_1}{\rho_1}Lu_x(L)w_{xx}(L) = o(1)$$

It is not hard to see that the boundary terms tend to zero therefore if we use (5.2.3.16) and (5.2.3.14) we can see it will reduce to

$$\langle\lambda^{3/2}xu_x, \lambda^{1/2}w_x\rangle + \frac{E_1}{\rho_1}\langle\lambda^{1/2}u_x, \frac{xw_{xxx}}{\lambda^{1/2}}\rangle = O(1)$$

From (5.2.3.9) it is not hard to see $\|\frac{w_{xxxx}}{\lambda}\| = O(1)$. Using the interpolation we have

$$\|\frac{w_{xxx}}{\lambda^{1/2}}\| \le K\|w_{xx}\|^{1/2}\|\frac{w_{xxxx}}{\lambda}\|^{1/2} = O(1)$$

Then from (5.2.3.16) we can reduce to

$$\lambda^2\langle xu_x, w_x\rangle = O(1) \tag{5.2.3.25}$$

With (5.2.3.25) taking the L^2 inner product of xu_x it is not hard to see after we take the reals

$$\|\lambda^2 u\|^2 + \frac{E_1}{\rho_1}\|\lambda u_x\|^2 = O(1) \tag{5.2.3.26}$$

Repeating the same exact process for a ϕ we can see

$$\|\lambda^2 \phi\|^2 + \frac{E_3}{\rho_3}\|\lambda \phi_x\|^2 = O(1) \tag{5.2.3.27}$$

Our estimates are not quite high enough so we repeat the process again to get a slightly higher estimate on these terms

We again take the L^2 inner product of (5.2.3.7) with xw_{xx}

$$-\lambda^{5/2}\langle u, xw_{xx}\rangle - \lambda^{1/2}\frac{E_1}{\rho_1}\langle u_{xx}, xw_{xx}\rangle - \frac{G_2}{\rho_1 h_1}\lambda^{1/2}\langle \tau, xw_{xx}\rangle = \frac{o(1)}{\lambda^{5/2}}$$

We can see the last term is of zero. So now we integrate by parts and since $\|w_{xx}\| = O(1)$, and $\|\lambda^{1/2}u_x\| = o(1)$

$$L\lambda^{5/2}u(L)w_x(L) + \langle\lambda^{5/2}[xu_x + u], \lambda^{1/2}w_x\rangle + \frac{E_1}{\rho_1}\lambda^{1/2}\langle u_x, xw_{xxx}\rangle - \frac{E_1}{\rho_1}\lambda^{1/2}Lu_x(L)w_{xx}(L) = o(1)$$

It is not hard to see that the boundary terms tend to zero therefore if we use (5.2.3.26) and (5.2.3.14) we can see it will reduce to

$$\langle\lambda^2 xu_x, \lambda^{1/2}w_x\rangle + \frac{E_1}{\rho_1}\langle\lambda u_x, \frac{xw_{xxx}}{\lambda^{1/2}}\rangle = O(1)$$

Since we already have $\|\frac{w_{xxx}}{\lambda^{1/2}}\| = O(1)$ Then from (5.2.3.26) we can reduce to

$$\lambda^{5/2}\langle xu_x, w_x\rangle = O(1) \tag{5.2.3.28}$$

With (5.2.3.28) taking the L^2 inner product of xu_x it is not hard to see after we take the reals

$$\|\lambda^{9/4} u\|^2 + \frac{E_1}{\rho_1}\|\lambda^{5/4} u_x\|^2 = O(1) \tag{5.2.3.29}$$

Repeating the same exact process for a ϕ we can see

$$\|\lambda^{9/4} \phi\|^2 + \frac{E_3}{\rho_3}\|\lambda^{5/4} \phi_x\|^2 = O(1) \tag{5.2.3.30}$$

We finally have a high enough estimate on our desired terms so lets turn our attention back to (5.2.3.24) we can see with (5.2.3.29) and (5.2.3.30). Since $\|\frac{w_{xxxx}}{\lambda}\| = O(1)$ we can see

$$|\lambda^{1/2}w_x(L)| = o(1) \tag{5.2.3.31}$$

The hard part is finally done. The next step we take is we take the L^2 inner product of (5.2.3.9) with $\frac{e^{-(\frac{h\rho}{EI})^{1/4}\lambda^{1/2}(L-x)}}{\lambda^{1/2}}$ to get the following

$$(\frac{EI}{h\rho})^{1/4}\lambda w(L) + (\frac{EI}{h\rho})^{1/2}\lambda^{1/2}w_x(L) - (\frac{EI}{h\rho})^{3/4}w_{xx}(0)e^{-\lambda^{1/2}(L)}$$
$$-\frac{1}{\lambda^{1/2}}[w_{xxx}(0)]e^{-\lambda^{1/2}(L)} = o(1)$$

We can then see $\frac{1}{\lambda^{1/4}}w_{xx}(0) = O(1)$ and $\frac{1}{\lambda^{3/4}}w_{xxx}(0) = O(1)$ there since $\lambda^{1/4}e^{-\lambda^{1/2}L} = o(1)$ then this implies

$$|\lambda w(L)| = o(1) \tag{5.2.3.32}$$

Now finally from (5.2.3.18) we can see from (5.2.3.32)

$$\|\lambda w\|^2 + \|w_{xx}\|^2 = o(1) \tag{5.2.3.33}$$

We can see we have reached the promised contradiction. So we can conclude that $\sup_{|\lambda|\to 0}\frac{1}{\lambda^3}\|(i\lambda I-\mathcal{A})^{-1}\| < +\infty$ and from the previous subsection we have finally proved Theorem 5.2.1.2

5.2.4 Proof of Theorem 5.2.1.2 $\sup_{|\lambda|\to 0}\frac{1}{\lambda^3}\|(i\lambda-\mathcal{A})^{-1}\| < +\infty$ with $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ or when $\frac{\rho_3 E_1}{\rho_1 E_3} \notin \mathcal{Q}$

We were actually able to do it for the case when $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ or when $\frac{\rho_3 E_1}{\rho_1 E_3} \notin \mathcal{Q}$ just recently. Section 5.2.3 now appears to be pointless because we are able to solve it for this case which is included in this proof. However, I believe the previous section might help us being able to achieve our final goal of 1-feedback controller. This is why I left it in my thesis.

We again begin the proof by a contradiction we assume that $\sup_{|\lambda|\to 0}\frac{1}{\lambda^3}\|(i\lambda-\mathcal{A})^{-1}\| < +\infty$ is false. this implies that $\exists\lambda \to \infty \ni \lambda^3\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$. This is written out below, we also have a unit norm aka $\|U\|_{\mathcal{H}} = 1$. We now express $\lambda^3\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$.

i.e.

$$\lambda^k[i\lambda u - v] = f_1 = o(1) \in H^1 \tag{5.2.4.1}$$
$$\lambda^k[i\lambda v - \frac{E_1}{\rho_1}u_{xx} - \frac{G_2}{\rho_1 h_1}\tau] = f_2 = o(1) \in L^2 \tag{5.2.4.2}$$
$$\lambda^k[i\lambda\phi - \varphi] = f_3 = o(1) \in H^1 \tag{5.2.4.3}$$
$$\lambda^k[i\lambda\varphi - \frac{E_3}{\rho_3}\phi_{xx} + \frac{G_2}{\rho_3 h_3}\tau] = f_4 = o(1) \in L^2 \tag{5.2.4.4}$$
$$\lambda^k[i\lambda w - \Omega] = f_5 = o(1) \in H^2 \tag{5.2.4.5}$$

$$\lambda^k[i\lambda\Omega + \frac{EI}{\rho h}w_{xxxx} - \frac{G_2\alpha}{\rho h}\tau_x] = f_6 = o(1) \in L^2 \tag{5.2.4.6}$$

We begin by plugging in (5.2.4.1) into (5.2.4.2), (5.2.4.3) into (5.2.4.4), and (5.2.4.5) into (5.2.4.6) to arrive at the three expressions below

$$-\lambda^2 u_1 - \frac{E_1}{\rho_1}u_{1,xx} - \frac{G_2}{\rho_1 h_1}\tau = \frac{g_2 + i\lambda g_1}{\lambda^k} \tag{5.2.4.7}$$

$$-\lambda^2 \phi_1 - \frac{E_3}{\rho_3}\phi_{1,xx} + \frac{G_2}{\rho_3 h_3}\tau = \frac{g_4 + i\lambda g_3}{\lambda^k} \tag{5.2.4.8}$$

$$-\lambda^2 w_1 + \frac{EI}{h\rho}w_{1,xxxx} - \frac{G_2\alpha}{\rho h}\tau_x = \frac{g_6 + i\lambda g_5}{\lambda^k} \tag{5.2.4.9}$$

We can deduce the following from the norm and interpolation

$$\|\lambda u\| = O(1) \; and \; \|\lambda\phi\| = O(1) \tag{5.2.4.10}$$

$$\|w_{xx}\| = O(1) \; \|\lambda^{1/2}w_x\| = O(1) \; \|\lambda w\| = O(1) \tag{5.2.4.11}$$

We can see (5.2.4.10) and (5.2.4.11) implies

$$\|\lambda^{1/2}\tau\| = O(1) \tag{5.2.4.12}$$

Taking the L^2 inner product of xu_x with (5.2.4.7) and then taking the reals, we will arrive at the following with the help of (5.2.4.12)

$$\|\lambda^{3/2}u\|^2 + \frac{E_1}{\rho_1}\|\lambda^{1/2}u_x\|^2 = \langle \frac{G_2}{\rho_1 h_1}\lambda^{1/2}\tau, x\lambda^{1/2}u_x\rangle = O(1) \tag{5.2.4.13}$$

Same thing with (5.2.4.8) and $x\phi_X$ implies

$$\|\lambda^{1/2}\phi_x\| = O(1) \; \|\lambda^{3/2}\phi\| = O(1) \tag{5.2.4.14}$$

To finish the proof we need to show $\|\lambda w\|$ and $\|w_{xx}\|$ tend to zero. We start by taking the inner product of (5.2.4.9) with xw_x to figure out our finish line

$$\|\lambda w\|^2 + \frac{3}{2}\frac{EI}{h\rho}\|w_{xx}\|^2 = L|\lambda w(L)|^2 + \frac{o(1)}{\lambda^k} = O(1) \tag{5.2.4.15}$$

We can now conclude that once we show the boundary term tends to zero the proof is complete.

We can see by (5.2.4.9) and our interpolation

$$\|\frac{w_{xxx}}{\lambda^{1/2}}\| = O(1) \; and \; \|\frac{w_{xxxx}}{\lambda}\| = O(1)$$

Lets get an estimate on the other boundary terms for w. Lets take the inner product of $\frac{e^{-(\frac{h\rho}{EI})^{1/4}\sqrt{\lambda}(L-x)}}{\sqrt{\lambda}}$ with (5.2.4.9). Doing so we see

$$[\frac{EI}{h\rho}w_{xxx}(x)-\frac{G_2\alpha}{\rho h}\tau(x)]\frac{e^{-(\frac{h\rho}{EI})^{1/4}\sqrt{\lambda}(L-x)}}{\sqrt{\lambda}}|_{x=0,L}+\sqrt{\lambda}(\frac{EI}{h\rho})^{3/4}w_{xx}(x)\frac{e^{-(\frac{h\rho}{EI})^{1/4}\sqrt{\lambda}(L-x)}}{\sqrt{\lambda}}|_{x=0,L}$$
$$+(\frac{EI}{h\rho})^{1/2}\lambda w_x(x)\frac{e^{-(\frac{h\rho}{EI})^{1/4}\sqrt{\lambda}(L-x)}}{\sqrt{\lambda}}|_{x=0,L}+(\frac{EI}{h\rho})^{1/4}\lambda^{3/2}w(x)\frac{e^{-(\frac{h\rho}{EI})^{1/4}\sqrt{\lambda}(L-x)}}{\sqrt{\lambda}}|_{x=0,L}$$
$$+\frac{G_2\alpha}{h\rho}\langle\lambda^{1/2}\tau,\frac{e^{-(\frac{h\rho}{EI})^{1/4}\sqrt{\lambda}(L-x)}}{\sqrt{\lambda}}\rangle=o(1)$$

We can show with GNI $\frac{w_{xxx}(0)}{\lambda^{3/4}}=O(1)$, and $\frac{w_{xxx}(0)}{\lambda^{1/4}}=O(1)$. Since $\lambda^{1/4}e^{-(\frac{h\rho}{EI})^{1/4}\sqrt{\lambda}(L)}=o(1)$. We can deduce with our boundary conditions.

$$o(1)=\lambda w(L)-\lambda^{1/2}(\frac{EI}{h\rho})^{1/4}w_x(L) \tag{5.2.4.16}$$

Therefore from (5.2.4.15) we can deduce

$$\lambda^{1/2}w_x(L)=O(1) \tag{5.2.4.17}$$

We can also see from our boundary condition and (5.2.4.17) if $k\geq 1$ then.

$$\lambda^{1/2}w_{xxx}(L)=O(1) \tag{5.2.4.18}$$

Taking the L^2 inner product of $2(x-L)w_x$ with (5.2.4.9). then we can see

$$w_{xx}(0)=O(1) \tag{5.2.4.19}$$

Taking the L^2 inner product of (5.2.4.9) with $\frac{e^{-(\frac{EI}{h\rho})^{1/4}\sqrt{\lambda}(x)}}{\sqrt{\lambda}}$ We eventually deduce

$$\lambda^{-1/2}w_{xxx}(0)=O(1) \tag{5.2.4.20}$$

We can see from (5.2.4.13) the term that gives us difficulty is the following term $\langle w_x,xu_x\rangle$ so if we can get a higher estimate on this term we can get a higher estimate on our u. So to do this we take the inner product of (5.2.4.7) with xw_{xx}. We also divide by λ^3

$$-\lambda^2\langle u,xw_{xx}\rangle-\frac{E_1}{\rho_1}\langle u_{xx},xw_{xx}\rangle-\frac{G_2}{\rho_1h_1}\langle\tau,xw_{xx}\rangle=o(1)$$

We can see the last term is of zero. So now we integrate by parts.

$$L\lambda^2u(L)w_x(L)+\langle\lambda^{3/2}[xu_x+u],\lambda^{1/2}w_x\rangle+\frac{E_1}{\rho_1}\langle u_x,xw_{xxx}+w_{xx}\rangle-\frac{E_1}{\rho_1}Lu_x(L)w_{xx}(L)=o(1)$$

It is not hard to see above will reduce to

$$\lambda^2\langle xu_x,w_x\rangle=O(1) \tag{5.2.4.21}$$

Now with the help of (5.2.4.21) we turn our attention back to (5.2.4.7) and if we take the L^2 inner product with xu_x

$$-\lambda^4\langle u, xu_x\rangle - \frac{E_1}{\rho_1}\lambda^2\langle u_{xx}, xu_x\rangle - \frac{G_2}{\rho_1 h_1}\lambda^2\langle[-u+\phi+\alpha w_x], xu_x\rangle = o(1)$$

From (5.2.4.13) and (5.2.4.14) we get the following

$$-\lambda^4\langle u, xu_x\rangle - \lambda^2\frac{E_1}{\rho_1}\langle u_{xx}, xu_x\rangle - \lambda^2\frac{G_2}{\rho_1 h_1}\langle \alpha w_x, xu_x\rangle = O(1)$$

From (5.2.4.21) we have (Taking the reals)

$$-\lambda^4 Re\langle u, xu_x\rangle - \lambda^2 Re\frac{E_1}{\rho_1}\langle u_{xx}, xu_x\rangle = O(1) \tag{5.2.4.22}$$

Now integration by parts we can see this will imply

$$\|\lambda^2 u\|^2 + \|\lambda u_x\|^2 = O(1) \tag{5.2.4.23}$$

We can do the same exact thing with ϕ to get

$$\|\lambda^2 \phi\|^2 + \|\lambda \phi_x\|^2 = O(1) \tag{5.2.4.24}$$

We revisit this whole process again to get a higher approximation on our u and ϕ.

to do this we take the inner product of (5.2.4.7) with xw_{xx}. We also divide by $\lambda^{5/2}$

$$-\lambda^{5/2}\langle u, xw_{xx}\rangle - \frac{E_1}{\rho_1}\lambda^{1/2}\langle u_{xx}, xw_{xx}\rangle - \frac{G_2}{\rho_1 h_1}\langle \tau, xw_{xx}\rangle = o(1)$$

We can see the last term is of zero. So now we integrate by parts.

$$L\lambda^{5/2}u(L)w_x(L) + \langle\lambda^2[xu_x+u], \lambda^{1/2}w_x\rangle + \frac{E_1}{\rho_1}\langle\lambda^{1/2}u_x, xw_{xxx}+w_{xx}\rangle - \frac{E_1}{\rho_1}Lu_x(L)w_{xx}(L) = o(1)$$

It is not hard to see above will reduce to

$$\lambda^{5/2}\langle xu_x, w_x\rangle = O(1) \tag{5.2.4.25}$$

Now with the help of (5.2.4.25) we turn our attention back to (5.2.4.7) and if we take the L^2 inner product with xu_x

$$-\lambda^{9/2}\langle u, xu_x\rangle - \frac{E_1}{\rho_1}\lambda^{5/2}\langle u_{xx}, xu_x\rangle - \frac{G_2}{\rho_1 h_1}\lambda^{5/2}\langle[-u+\phi+\alpha w_x], xu_x\rangle = o(1)$$

From (5.2.4.23) and (5.2.4.24) we get the following

$$-\lambda^{9/2}\langle u, xu_x\rangle - \lambda^{5/2}\frac{E_1}{\rho_1}\langle u_{xx}, xu_x\rangle - \lambda^{5/2}\frac{G_2}{\rho_1 h_1}\langle \alpha w_x, xu_x\rangle = O(1)$$

From (5.2.4.25) we have (Taking the reals)

$$-\lambda^{9/2}Re\langle u, xu_x\rangle - \lambda^{5/2}Re\frac{E_1}{\rho_1}\langle u_{xx}, xu_x\rangle = O(1) \qquad (5.2.4.26)$$

Now integration by parts we can see this will imply

$$\|\lambda^{9/4}u\|^2 + \|\lambda^{5/4}u_x\|^2 = O(1) \qquad (5.2.4.27)$$

We can do the same exact thing with ϕ to get

$$\|\lambda^{9/4}\phi\|^2 + \|\lambda^{5/4}\phi_x\|^2 = O(1) \qquad (5.2.4.28)$$

Now that we have those approximations we can start the rest of the proof. To begin we take the inner product of (5.2.4.7) with u_x to get the following

$$+\lambda^3\frac{E_1}{\rho_1}u_x^2(0) - \lambda^3\frac{\alpha G_2}{\rho_1 h_1}\langle w_x, u_x\rangle = o(1) \qquad (5.2.4.29)$$

Now we take (5.2.4.7) with $\frac{\alpha G_2}{\rho_1 h_1}\frac{w_{xx}}{\lambda^2}$ To get the following.

$$-\frac{\alpha G_2}{\rho_1 h_1}\lambda^3\langle u, w_{xx}\rangle - \lambda\frac{E_1}{\rho_1}\frac{\alpha G_2}{\rho_1 h_1}\langle u_{xx}, w_{xx}\rangle - \lambda[\frac{G_2\alpha}{\rho_1 h_1}]^2\langle w_x, w_{xx}\rangle = o(1)$$

With integration by parts we get

$$\frac{\alpha G_2}{\rho_1 h_1}\lambda^3\langle u_x, w_x\rangle + \lambda\frac{E_1}{\rho_1}\frac{\alpha G_2}{\rho_1 h_1}u_x(0)w_{xx}(0) + \lambda\frac{E_1}{\rho_1}\frac{\alpha G_2}{\rho_1 h_1}\langle u_x, w_{xxx}\rangle$$
$$-\lambda[\frac{G_2\alpha}{\rho_1 h_1}]^2 w_x^2(L) = o(1)$$

we can integrate by parts again to show

$$\frac{\alpha G_2}{\rho_1 h_1}\lambda^3\langle u_x, w_x\rangle + \lambda\frac{E_1}{\rho_1}\frac{\alpha G_2}{\rho_1 h_1}u_x(0)w_{xx}(0) - \lambda^2\frac{E_1}{\rho_1}\frac{\alpha G_2}{\rho_1 h_1}\langle u, \frac{w_{xxxx}}{\lambda}\rangle$$
$$-\lambda[\frac{G_2\alpha}{\rho_1 h_1}]^2 w_x^2(L) = o(1)$$

We can see we will be left with the following.

$$\frac{\alpha G_2}{\rho_1 h_1}\lambda^3\langle u_x, w_x\rangle + \lambda\frac{E_1}{\rho_1}\frac{\alpha G_2}{\rho_1 h_1}u_x(0)w_{xx}(0) - \lambda[\frac{G_2\alpha}{\rho_1 h_1}]^2 w_x^2(L) = o(1) \qquad (5.2.4.30)$$

Adding (5.2.4.29) with (5.2.4.30) we get

$$\lambda^3\frac{E_1}{\rho_1}u_x^2(0)+\lambda\frac{E_1}{\rho_1}\frac{\alpha G_2}{\rho_1 h_1}u_x(0)w_{xx}(0)-\lambda[\frac{G_2\alpha}{\rho_1 h_1}]^2w_x^2(L)=o(1)$$

Looking at above since $w_{xx}(0)=O(1)$ and $\lambda^{1/2}w_x(L)=O(1)$ we should be able to deduce that. $\lambda^{3/2}u_x(0)=O(1)$ Therefore we can see the middle boundary term tends to zero. So we will arrive at the following.

$$\lambda^3\frac{E_1}{\rho_1}u_x^2(0)=\lambda[\frac{G_2\alpha}{\rho_1 h_1}]^2w_x^2(L) \tag{5.2.4.31}$$

(5.2.4.31) is important we will revisit this expression later. We can repeat the same process with ϕ to get the following. To begin we take the inner product of (5.2.4.8) with ϕ_x to get the following

$$+\lambda^5\phi^2(0)+\lambda^3\frac{\alpha G_2}{\rho_3 h_3}\langle w_x,\phi_x\rangle=o(1) \tag{5.2.4.32}$$

Now we take (5.2.4.8) with $-\frac{\alpha G_2}{\rho_3 h_3}\frac{w_{xx}}{\lambda^2}$ To get the following.

$$+\frac{\alpha G_2}{\rho_3 h_3}\lambda^3\langle\phi,w_{xx}\rangle+\lambda\frac{E_3}{\rho_3}\frac{\alpha G_2}{\rho_3 h_3}\langle\phi_{xx},w_{xx}\rangle+\lambda[\frac{G_2\alpha}{\rho_3 h_3}]^2\langle w_x,w_{xx}\rangle=o(1)$$

With integration by parts we get

$$-\lambda^3\frac{\alpha G_2}{\rho_3 h_3}\phi(0)w_x(0)-\frac{\alpha G_2}{\rho_3 h_3}\lambda^3\langle\phi_x,w_x\rangle-\lambda\frac{E_3}{\rho_3}\frac{\alpha G_2}{\rho_3 h_3}\langle\phi_x,w_{xxx}\rangle$$
$$-\lambda[\frac{G_2\alpha}{\rho_3 h_3}]^2w_x^2(L)=o(1)$$

we can integrate by parts again to show

$$-\lambda^3\frac{\alpha G_2}{\rho_3 h_3}\phi(0)w_x(0)-\frac{\alpha G_2}{\rho_3 h_3}\lambda^3\langle\phi_x,w_x\rangle-\lambda[\frac{G_2\alpha}{\rho_3 h_3}]^2w_x^2(L)=o(1) \tag{5.2.4.33}$$

Adding (5.2.4.32) with (5.2.4.33) we get

$$\lambda^5\phi^2(0)=\lambda[\frac{G_2\alpha}{\rho_3 h_3}]^2w_x^2(L)+o(1) \tag{5.2.4.34}$$

We can see (5.2.4.31) implies

$$\lambda^{3/2}u_x(0)=\pm\sqrt{\frac{\rho_1}{E_1}}\frac{G_2\alpha}{\rho_1 h_1}\lambda^{1/2}w_x(L) \tag{5.2.4.35}$$

We can see (5.2.4.34) implies

$$\lambda^{5/2}\phi(0)=\pm\frac{G_2\alpha}{\rho_3 h_3}\lambda^{1/2}w_x(L) \tag{5.2.4.36}$$

So we will keep these two equations in mind. But first lets take the inner product of (5.2.4.7) with $cos(\sqrt{\frac{\rho_1}{E_1}}\lambda x)$. Doing this we will arrive at the following

$$\frac{E_1}{\rho_1}\lambda^{3/2}u_x(0) = \sqrt{\frac{E_1}{\rho_1}}\frac{G_2\alpha}{\rho_1 h_1}\lambda^{1/2}w_x(L)sin(\sqrt{\frac{\rho_1}{E_1}}\lambda L)$$

$$\lambda^{-1/2}w_{xx}(0) + \lambda^{-3/2}w_{xxx}(x)sin(\sqrt{\frac{\rho_1}{E_1}}\lambda x)|_{x=0,L} + \langle\frac{w_{xxxx}}{\lambda^{3/2}}, cos(\sqrt{\frac{\rho_1}{E_1}}\lambda x)\rangle$$

We can see everything will tend to zero besides the two following terms

$$\frac{E_1}{\rho_1}\lambda^{3/2}u_x(0) = \sqrt{\frac{E_1}{\rho_1}}\frac{G_2\alpha}{\rho_1 h_1}\lambda^{1/2}w_x(L)sin(\sqrt{\frac{\rho_1}{E_1}}\lambda L) \tag{5.2.4.37}$$

Hence we will arrive at

$$\lambda^{3/2}u_x(0) = \sqrt{\frac{\rho_1}{E_1}}\frac{G_2\alpha}{\rho_1 h_1}\lambda^{1/2}w_x(L)sin(\sqrt{\frac{\rho_1}{E_1}}\lambda L) \tag{5.2.4.38}$$

In the view of (5.2.4.35) we arrive at

$$\pm\sqrt{\frac{\rho_1}{E_1}}\frac{G_2\alpha}{\rho_1 h_1}\lambda^{1/2}w_x(L) = \sqrt{\frac{\rho_1}{E_1}}\frac{G_2\alpha}{\rho_1 h_1}\lambda^{1/2}w_x(L)sin(\sqrt{\frac{\rho_1}{E_1}}\lambda L) \tag{5.2.4.39}$$

So we can see

$$\pm 1 = sin(\sqrt{\frac{\rho_1}{E_1}}\lambda L) \tag{5.2.4.40}$$

Case i) $\frac{\rho_3}{\rho_1} \neq \frac{E_3}{E_1}$ **and** $\frac{\rho_3 E_1}{\rho_1 E_3} \notin \mathcal{Q}$

This implies that λ has to be the following sequence otherwise it implies $\lambda^{1/2}w_x(L) = o(1)$. Therefore assume λ is the following sequence and show by contradiction that $\lambda^{1/2}w_x(L) = o(1)$ if $\frac{\rho_3}{\rho_1} \neq \frac{E_3}{E_1}$ and $\frac{\rho_3 E_1}{\rho_1 E_3} \notin \mathcal{Q}$. Let $\beta_n \in 2\mathcal{N}+1$, This sequence can be any sequence as long as every element of $\beta_n \in 2\mathcal{N}+1$

$$\sqrt{\frac{E_1}{\rho_1}}\frac{\beta_n \pi}{2L} = \lambda_n \tag{5.2.4.41}$$

We will keep (5.2.4.41) In mind to show we can achieve a contradiction. Now we take the inner product of (5.2.4.8) with $sin(\sqrt{\frac{\rho_3}{E_3}}\lambda x)$.

$$\sqrt{\frac{E_3}{\rho_3}}\lambda^{5/2}\phi(0) = \sqrt{\frac{E_3}{\rho_3}}\frac{G_2\alpha}{\rho_3 h_3}\lambda^{1/2}w_x(L)cos(\sqrt{\frac{\rho_3}{E_3}}\lambda L) \tag{5.2.4.42}$$

$$\lambda^{5/2}\phi(0) = \frac{G_2\alpha}{\rho_3 h_3}\lambda^{1/2}w_x(L)cos(\sqrt{\frac{\rho_3}{E_3}}\lambda L) \tag{5.2.4.43}$$

Now in the view of (5.2.4.36)

$$\pm\frac{G_2\alpha}{\rho_3 h_3}\lambda^{1/2}w_x(L) = \frac{G_2\alpha}{\rho_3 h_3}\lambda^{1/2}w_x(L)cos(\sqrt{\frac{\rho_3}{E_3}}\lambda L) \tag{5.2.4.44}$$

Therefore

$$\pm 1 = cos(\sqrt{\frac{\rho_3}{E_3}}\lambda L) \tag{5.2.4.45}$$

However this implies $\exists \alpha_n$: { all elements of the sequence are elments of $2\mathcal{N}$} where

$$\lambda_n = \sqrt{\frac{E_3}{\rho_3}}\frac{\alpha_n \pi}{2L} \tag{5.2.4.46}$$

so we can see (5.2.4.46) and (5.2.4.41) implies

$$\sqrt{\frac{E_3}{\rho_3}}\alpha_n = \sqrt{\frac{E_1}{\rho_1}}\beta_n \tag{5.2.4.47}$$

Therefore we can see,

$$\frac{\alpha_n^2}{\beta_n^2} = \frac{E_1\rho_3}{E_3\rho_1} \tag{5.2.4.48}$$

Since $\frac{\alpha_n^2}{\beta_n^2} \in \mathcal{Q}$ and $\frac{E_1\rho_3}{E_3\rho_1} \notin \mathcal{Q}$. We have reached a contradiction therefore.

$$\lambda^{1/2}w_x(L) = o(1)$$

Case ii) $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$

All we have to do different is we can just square (5.2.4.45) and (5.2.4.40) and then add them together, we get

$$2 = sin^2(\sqrt{\frac{\rho_1}{E_1}}\lambda L) + cos^2(\sqrt{\frac{\rho_3}{E_3}}\lambda L)$$

We can see when $\frac{\rho_1}{E_1} = \frac{\rho_3}{E_3}$ we have $2 = 1$ which is a contradiction therefore

$$\lambda^{1/2}w_x(L) = o(1)$$

Now completing the rest of the proof to reiterate when $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ or when $\frac{\rho_3 E_1}{\rho_1 E_3} \notin \mathcal{Q}$, we have

$$\lambda^{1/2}w_x(L) = o(1) \tag{5.2.4.49}$$

By (5.2.4.49) and (5.2.4.16)

$$\lambda w(L) = o(1) \tag{5.2.4.50}$$

and by (5.2.4.15)

$$\|\lambda w\| = o(1)\ and\ \|w_{xx}\| = o(1) \tag{5.2.4.51}$$

So we have reached the promised contradiction when $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ or when $\frac{\rho_3 E_1}{\rho_1 E_3} \notin \mathcal{Q}$. Hence the proof is complete

5.3 $k_1 > 0$, $k_2 = 0$, $k_3 > 0$ *and* $k_4 = 0$

5.3.1 Comments

With the certain placements of our feedback controllers we can see our boundary conditions are the following

$$\begin{gathered} u_x(L) = -k_1 u_t(L) \ , \ w_{xt}(L) = -k_3 w_{xx}(L) \\ \phi_x(L) = w_x(0) = w(0) = w_{xxx}(L) - \tau(L) = u(0) = \phi(0) = 0 \end{gathered} \tag{5.3.1.1}$$

Everything is very similar to section 5.2 the only thing that drastically changes is the last part of showing that it is polynomial stable of a certain order is different. In order to save time we only turn our attention to the non trival part of the following Theorem

Theorem 5.3.1.1 *Let $e^{\mathcal{A}t}$ be the semigroup associated with our system then it is polynomially stable of order 1/2 when $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$*

This Theorem will be shown in the following section. We use Theorem 2.4.0.1 to do so.

5.3.2 Proof of Theorem 5.2.1.2 $\sup_{|\lambda|\to 0} \frac{1}{\lambda^2}\|(i\lambda - \mathcal{A})^{-1}\| < +\infty$ with $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$

We do the proof by a contradiction we assume that $\sup_{|\lambda|\to 0} \frac{1}{\lambda^2}\|(i\lambda - \mathcal{A})^{-1}\| < +\infty$ is false. this implies that $\exists \lambda \to \infty \ni \lambda^2\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$. This is written out below, we also have a unit norm aka $\|U\|_{\mathcal{H}} = 1$. We now express $\lambda^2\|(i\lambda I - \mathcal{A})U\|_{\mathcal{H}} = o(1)$.

i.e.

$$\lambda^k[i\lambda u - v] = f_1 = o(1) \in H^1 \tag{5.3.2.1}$$

$$\lambda^k[i\lambda v - \frac{E_1}{\rho_1}u_{xx} - \frac{G_2}{\rho_1 h_1}\tau] = f_2 = o(1) \in L^2 \tag{5.3.2.2}$$

$$\lambda^k[i\lambda \phi - \varphi] = f_3 = o(1) \in H^1 \tag{5.3.2.3}$$

$$\lambda^k[i\lambda \varphi - \frac{E_3}{\rho_3}\phi_{xx} + \frac{G_2}{\rho_3 h_3}\tau] = f_4 = o(1) \in L^2 \tag{5.3.2.4}$$

$$\lambda^k[i\lambda w - \Omega] = f_5 = o(1) \in H^2 \tag{5.3.2.5}$$

$$\lambda^k[i\lambda \Omega + \frac{EI}{\rho h}w_{xxxx} - \frac{G_2\alpha}{\rho h}\tau_x] = f_6 = o(1) \in L^2 \tag{5.3.2.6}$$

We begin by plugging in (5.3.3.1) into (5.3.3.2), (5.3.3.3) into (5.3.3.4), and (5.3.3.5) into (5.3.3.6) to arrive at the three expressions below

$$-\lambda^2 u_1 - \frac{E_1}{\rho_1}u_{1,xx} - \frac{G_2}{\rho_1 h_1}\tau = \frac{g_2 + i\lambda g_1}{\lambda^k} \tag{5.3.2.7}$$

$$-\lambda^2\phi_1 - \frac{E_3}{\rho_3}\phi_{1,xx} + \frac{G_2}{\rho_3 h_3}\tau = \frac{g_4 + i\lambda g_3}{\lambda^k} \tag{5.3.2.8}$$

$$-\lambda^2 w_1 + \frac{EI}{h\rho}w_{1,xxxx} - \frac{G_2\alpha}{\rho h}\tau_x = \frac{g_6 + i\lambda g_5}{\lambda^k} \tag{5.3.2.9}$$

It is easy to see from our dissipation with (5.3.2.1) , and (5.3.2.5)

$$\lambda^{k/2+1}u(L) = \lambda^{k/2}u_x(L) = \lambda^{k/2+1}w_x(L) = \lambda^{k/2}w_{xx}(L) = o(1) \tag{5.3.2.10}$$

From our norm it is not hard to deduce

$$\begin{aligned} \|\lambda u\| = O(1)\ ,\ \|u_x\| = O(1)\ ,\ \|\lambda\phi\| = O(1)\ ,\ \|\phi_x\| = O(1) \\ \|\lambda w\| = O(1)\ ,\ \|w_{xx}\| = O(1) \end{aligned} \tag{5.3.2.11}$$

Using the interpolation we have

$$\|\lambda^{1/2}w_x\| \le K\|\lambda w\|^{1/2}\|w_{xx}\|^{1/2} = O(1) \tag{5.3.2.12}$$

Therefore we can see,

$$\|\lambda^{1/2}\tau\| = O(1) \tag{5.3.2.13}$$

From (5.3.2.9) we can see

$$\|\frac{w_{xxxx}}{\lambda}\| = O(1) \tag{5.3.2.14}$$

Use interpolation we also have

$$\|\frac{w_{xxx}}{\lambda^{1/2}}\| \le K\|w_{xx}\|^{1/2}\|\frac{w_{xxxx}}{\lambda}\|^{1/2} = O(1) \tag{5.3.2.15}$$

We can also deduce from (5.3.2.4) and (5.3.2.5)

$$\|\frac{u_{xx}}{\lambda}\| = O(1)\ and\ \|\frac{\phi_{xx}}{\lambda}\| = O(1) \tag{5.3.2.16}$$

Lets first figure out the direction we need to head to arrived at our contradiction. To figure out the direction we wish to head we first take the L^2 inner product of (5.3.2.7) with xu_x , (5.3.2.8) with $x\phi_x$ and (5.3.2.9) with xw_x with (5.3.2.10) and our boundary conditions after taking the reals, integrating by parts, we can arrive at the following 3 expression.

$$\begin{aligned} \|\lambda^{3/2}u\|^2 + \frac{E_1}{\rho_1}\|\lambda^{1/2}u_x\|^2 = 2Re\langle\lambda^{1/2}\frac{G_2}{\rho_1 h_1}\tau, \lambda^{1/2}xu_x\rangle \\ +\langle\frac{g_2 + i\lambda g_1}{\lambda}, xu_x\rangle = O(1) \end{aligned}$$

Which reduces to

$$\|\lambda^{3/2}u\|^2 + \frac{E_1}{\rho_1}\|\lambda^{1/2}u_x\|^2 = O(1) \tag{5.3.2.17}$$

Now for the ϕ

$$\|\lambda\phi\|^2 + \frac{E_1}{\rho_1}\|\phi_x\|^2 = L|\phi(L)|^2 + 2Re\langle\frac{G_2}{\rho_3 h_3}\tau, x\phi_x\rangle + \langle\frac{g_4 + i\lambda g_3}{\lambda^2}, x\phi_x\rangle$$

Which reduces to

$$\|\lambda\phi\|^2 + \frac{E_1}{\rho_1}\|\phi_x\|^2 = L|\phi(L)|^2 \tag{5.3.2.18}$$

Now for the w

$$\|\lambda^{3/2}w\|^2 + 3\frac{EI}{\rho h}\|\lambda^{1/2}w_{xx}\|^2 + \frac{G_2\alpha}{\rho h}\|\lambda^{1/2}w_x\|^2 = \langle g_6 + i\lambda g_5, xw_x\rangle + L|\lambda^{3/2}w(L)|^2 - 2Re\langle\frac{G_2\alpha}{\rho h}\lambda^{1/2}\tau, \lambda^{1/2}xw_{xx}\rangle \tag{5.3.2.19}$$

We look to get and approximation on $|\lambda^{3/2}w(L)|$ to get an approximation on our internal w terms. We now go through the process of getting a approximation on $|\lambda^{3/2}w(L)|$

Using GNI we can show the following terms are bounded

$$|\frac{w_{xxx}(0)}{\lambda^{3/4}}| \le \|\frac{w_{xxx}}{\lambda^{3/4}}\|_{L^\infty} \le K\|\frac{w_{xxx}}{\lambda^{1/2}}\|^{1/2}\|\frac{w_{xxxx}}{\lambda}\|^{1/2} = O(1) \tag{5.3.2.20}$$

$$|\frac{w_{xx}(0)}{\lambda^{1/4}}| \le \|\frac{w_{xx}}{\lambda^{1/4}}\|_{L^\infty} \le K\|\frac{w_{xxx}}{\lambda^{1/2}}\|^{1/2}\|w_{xx}\|^{1/2} = O(1) \tag{5.3.2.21}$$

Now we can get an estimate on $|\lambda^{3/2}w(L)|$. To do this we take the L^2 inner product of (5.3.2.9) with $\lambda^{-1/2}exp(-(\lambda^2\frac{h\rho}{3EI})^{1/4}(L-x))$, integrating by parts we get

$$3\frac{EI}{h\rho}w_{xxx}(0)exp(-(\lambda^2\frac{h\rho}{3EI})^{1/4}L) - (3\frac{EI}{h\rho})^{3/4}\lambda^{1/2}w_{xx}(0)exp(-(\lambda^2\frac{h\rho}{3EI})^{1/4}L) - (3\frac{EI}{h\rho})^{1/2}\lambda w_x(L) + (3\frac{EI}{h\rho})^{1/4}\lambda^{3/2}w(L) + \langle(\frac{3EI}{h\rho})^{1/4}\frac{G_2\alpha}{\rho h}\lambda^{1/2}\tau, exp(-(\lambda^2\frac{h\rho}{3EI})^{1/4}(L-x))\rangle = o(1)$$

Its easy to see the last term is bounded since $exp(-(\lambda^2\frac{h\rho}{3EI})^{1/4}(L-x)) = O(1)$ and $\tau = o(1)$, then we can see by (5.3.2.20) , (5.3.2.21) and since $\lambda^{3/4}exp(-(\lambda^2\frac{h\rho}{3EI})^{1/4}(L)) = o(1)$ we can see the above expression reduces to

$$-(3\frac{EI}{h\rho})^{1/2}\lambda w_x(L) + (3\frac{EI}{h\rho})^{1/4}\lambda^{3/2}w(L) = \langle(\frac{3EI}{h\rho})^{1/4}\frac{G_2\alpha}{\rho h}\lambda^{1/2}\tau, exp(-(\lambda^2\frac{h\rho}{3EI})^{1/4}(L-x))\rangle = O(1) \tag{5.3.2.22}$$

Finally from (5.3.2.10) we can see

$$\lambda^{3/2}w(L) = O(1) \tag{5.3.2.23}$$

So from (5.3.2.19) we can conclude

$$\|\lambda^{3/2}w\| = O(1)\ ,\ \|\lambda^{1/2}w_{xx}\| = O(1) \tag{5.3.2.24}$$

Then from interpolation

$$\|\lambda w_x\| = O(1) \tag{5.3.2.25}$$

Now it is easy to see

$$\|\lambda^{1/2}\tau\| = o(1) \tag{5.3.2.26}$$

However with this if we revisit (5.3.2.22) Now we can show

$$\|\lambda^{3/2}w\| = o(1)\ ,\ \|\lambda^{1/2}w_{xx}\| = o(1)\ ,\ \|\lambda w_x\| = o(1) \tag{5.3.2.27}$$

Looking at (5.3.2.7) we can see now

$$\|\lambda^{-1/2}u_{xx}\| = O(1) \tag{5.3.2.28}$$

We have finally acquired all the estimates we need to proceed to our last steps of our proof. We begin by taking the L^2 inner product of (5.3.2.7) with $\frac{\phi_{xx}}{\lambda}$ added with the L^2 inner product of (5.3.2.8) with $\frac{-u_{xx}}{\lambda}$. Doing so we arrive at the following

$$-\lambda^2\langle u, \phi_{xx}\rangle - \frac{E_1}{\rho_1}\langle u_{xx}, \phi_{xx}\rangle - \frac{G_2}{\rho_1 h_1}\langle \lambda[-u+\phi+\alpha w_x], \frac{\phi_{xx}}{\lambda}\rangle$$
$$\lambda^2\langle \phi, u_{xx}\rangle + \frac{E_3}{\rho_3}\langle \phi_{xx}, u_{xx}\rangle - \frac{G_2}{\rho_3 h_3}\langle \lambda^{1/2}\tau, \frac{u_{xx}}{\lambda^{1/2}}\rangle = o(1)$$

Since $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$, taking the reals we can reduce above to the following after integrating by parts.

$$\lambda^2 u_x(L)\phi(L) + \frac{G_2}{\rho_1 h_1}\|\phi_x\|^2 = o(1)$$

We can see when $k \geq 2$ from (5.3.2.7) since $|\lambda\phi(L)| = O(1)$, the boundary term tends to zero. Therefore,

$$\|\phi_x\| = o(1) \tag{5.3.2.29}$$

Finally, if we take the L^2 inner product of (5.3.2.8) with ϕ we arrive at

$$\|\lambda\phi\| = o(1) \tag{5.3.2.30}$$

We can see by (5.3.2.17) , (5.3.2.27) , (5.3.2.29), and (5.3.2.30) we have reached a contradiction of a unit norm whenever $k \geq 2$. Hence, the proof is complete.

5.4 Conclusion

In summary we have proved polynoimally stable of order 1/3 for the case $k_1 > 0$ $k_2 = 0$ $k_3 = 0$ $k_4 > 0$ for the two following restrictions $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ or when $\frac{\rho_3 E_1}{\rho_1 E_3} \notin \mathcal{Q}$. We also have shown polynoimally stable of order 1/2 for the case $k_1 > 0$ $k_2 = 0$ $k_3 > 0$ $k_4 = 0$ when $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$.

Bibliography

[1] N. Bazarra, M. Campo, J. R. Fernández, A thermoelastic problem with diffusion, microtemperatures, and microconcentrations, Acta Mech. 230 (2019) 31–48.

[2] N. Bazarra, M.I.M Copetti, J.R. Fernández, R. Quintanilla Numerical analysis of a dual-phase-lag model with microtemperatures, Applied Numerical Mathematics 166 (2021) 1–25.

[3] K. Borgmeyer, R. Quintanilla and R. Racke, Phase-lag heat conduction: decay rates for limit problems and well-posedness, J. Evolution Equations, 14 (2014) 863-884.

[4] A. Borichev, Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, Mathematische Annalen, 347, (2010) 455–478.

[5] P. Casas, R. Quintanilla, Exponential stability in thermoelasticity with microtemperatures, Internat. J. Engrg. Sci. 43 (2005) 33–47.

[6] Cosserat, E., Cosserat, F. Théorie des Corps Déformables. Hermannn. Paris, 1909.

[7] S. C. Cowin, The viscoelastic behavior of linear elastic materials with voids, J. Elasticity 15 (1985) 185–191.

[8] S. C. Cowin, J. W. Nunziato, Linear elastic materials with voids, J. Elasticity 13 (1983) 125–147.

[9] C. Eringen, Microcontinuum Field Theories I: Foundations and Solids. Springer, New York 1999.

[10] B. Feng, T. A. Apalara, Optimal decay for a porous elasticity system with memory, J. Math. Anal. Appl. 470 (2019) 1108–1128.

[11] B. Feng, L. Yan, D. S. Almeida Junior, Stabilizatioin for an inhomogeneous porous-elastic system with temperature and microtemperature, Journal Applied Mathematics mechanics (ZAMM) (2020) https://doi.org/10.1002/zamm.202000058.

[12] B. Feng, M. Yin, Decay of solutions for a one-dimensional porous elasticity system with memory: the case of non-equal wave speeds, Math. Mech. Solids 24 (2019) 2361–2373.

[13] Goodman, M. A., Cowin, S. C. A continuum theory for granular materials, Arch. Rat. Mech. Anal., 44, pp. 249–266, 1972.

[14] R. Grot, Thermodynamics of a continuum with microstructure, Internat. J. Engrg. Sci. 7 (1969) 801–814.

[15] D. Ieşan, Thermoelasticity of bodies with microstructure and microtemperatures, Internat J. Solids Structures 44 (2007) 8648–8653.

[16] Ieşan, D. A theory of thermoelastic materials with voids. Acta Mechanica, 60, 67–89, 1986.

[17] Ieşan, D. Thermoelastic Models of Continua, Springer, 2004.

[18] Ieşan, D., Nappa, L. Thermal stresses in plane strain of porous elastic bodies. Meccanica, 39, 125–138, 2004.

[19] D. Ieşan, R. Quintanilla, On a theory of thermoelasticity with microtemperatures, J. Thermal Stresses 23 (2000) 195–215

[20] M. C. Leseduarte, A. Magaña, R. Quintanilla, On the time decay of solutions in porous-thermo-elasticity of type II, Discrete Cont. Dyn. Systems - B 13 (2010) 375–391.

[21] Z. Liu and R. Quintanilla, Time decay in dual-phase-lag thermoelasticity:critical case, Comm. Pure Appl. Anal.. 17 (2018) 177-190.

[22] Z. Liu, R. Quintanilla and Y. Wang, On the phase-lag equation with spatial dependent, Jour. Math. anal. Appl. 455 (2017) 422-438.

[23] Z. Liu, R. Quintanilla, Y. Wang, Dual-phase-lag heat conduction with microtemperature, Submitted (2020).

[24] Z. Liu, R. Quintanilla, Dual-phase-lag one-dimensional thermo-porous-elasticity with microtemperatures. Comput. Appl. Math. 40 (2021), no. 6, Paper No. 231, 12 pp.

[25] Z. Liu, S. Zheng, Semigroups Associated with Dissipative Systems, Chapman & Hall/CRC Research Notes in Mathematics, vol. 398, Chapman & Hall/CRC, Boca Raton, FL, 1999.

[26] A. Magaña, R. Quintanilla, On the exponential decay of solutions in one-dimensional generalized porous-thermo-elasticity, Asymptotic Anal. 49 (2006) 173–187.

[27] A. Magaña, R. Quintanilla, On the time decay of solutions in porous-elasticity with quasi-static microvoids, J. Math. Anal. Appl. 331 (2007) 617–630.

[28] A. Magaña, R. Quintanilla, Exponential stability in type III thermoelasticity with microtemperatures, Z. Angew. Math. Phys. 69(5)(2018) 129(1)–129(8).

[29] A. Magaña, R. Quintanilla, On the existence and uniqueness in phase-lag thermoelasticity, Meccanica 53, (2018) 125-134.

[30] A. Miranville, R. Quintanilla, Exponential decay in one-dimensional type III thermoelasticity with voids, Appl. Math. Lett. 94 (2019) 30–37.

[31] A. Miranville, R. Quintanilla, Exponential decay in one-dimensional type II thermoviscoelasticity with voids, J. Comput. Appl. Math. 368 (2020) 112573.

[32] J. W. Nunziato, S. C. Cowin, A nonlinear theory of elastic materials with voids, Arch. Rational Mech. Anal. 72 (1979) 175–201.

[33] P. X. Pamplona, J. E. Muñoz-Rivera, R. Quintanilla, On the decay of solutions for porous-elastic systems with history, J. Math. Anal. Appl. 379 (2011) 682–705.

[34] F. Passarella, V. Tibullo, G. Viccione, Rayleigh waves in isotropic strongly elliptic thermoelastic materials with microtemperatures, Meccanica 52 (2017) 3033–3041.

[35] R. Quintanilla, A condition on the delay parameters in the one-dimensional dual-phase-lag thermoelastic theory, Journal of Thermal Stresses, 26 (2003), no.7, 713-721.

[36] R. Quintanilla, R. Racke Qualitative aspects in dual-phase-lag thermoelasticity. SIAM journal on applied mathematics, 66 (2006), 977-1001.

[37] A.J.A. Ramos, D.S. Almeida Júnior, M.M. Freitas, M.J. Dos Santos A new exponential decay result for one-dimensional porous dissipation elasticity from second spectrum viewpoint. Applied Mathematics Letters 101 (2020) 106061.

[38] P. Riha, On the theory of heat-conducting micropolar fluids with microtemperatures, Acta Mech. 23 (1975) 1–8.

[39] P. Riha, On the microcontinuum model of heat conduction in materials with inner structure, Internat. J. Engrg. Sci. 14 (1976) 529–535.

[40] M. L. Santos, A. D. S. Campelo, D. S. Almeida Junior, On the decay rates of porous elastic systems, Jour. Elasticity 127 (2017) 79-101.

[41] D. Y. Tzou, A unified approach for heat conduction from macro to microscales, ASME J. Heat Transfer, 117 (1995), 8-16.

[42] Wyler, Andreas. "Stability of wave equations with dissipative boundary conditions in a bounded domain." Differential and Integral Equations 7.2 (1994): 345-366.

[43] Lagnese, John E. Boundary stabilization of thin plates. Society for industrial and Applied Mathematics, 1989.

[44] Liu, Kangsheng, and Zhuangyi Liu. "Boundary stabilization of a nonhomogeneous beam with rotatory inertia at the tip." Journal of Computational and Applied mathematics 114.1 (2000): 1-10.

[45] Akil, Mohammad, et al. "Stabilization of coupled wave equations with viscous damping on cylindrical and non-regular domains: Cases without the geometric control condition." arXiv preprint arXiv:2111.14554 (2021).

[46] Wang, Yang. "Boundary feedback stabilization of a Rao-Nakra sandwich beam." Journal of Physics: Conference Series. Vol. 1324. No. 1. IOP Publishing, 2019.

[47] Liu, Z., S. A. Trogdon, and Yong Jiongmin. "Modeling and analysis of a laminated beam." Mathematical and computer modelling 30.1-2 (1999): 149-167.

[48] Liu, Zhuangyi, Bopeng Rao, and Qiong Zhang. "Polynomial stability of the Rao-Nakra beam with a single internal viscous damping." Journal of Differential Equations 269.7 (2020): 6125-6162.

[49] Akil, Mohammad, and Zhuangyi Liu. "Stabilisation of the generalised Rao-Nakra beam by partial viscous damping." arXiv preprint arXiv:2106.01189 (2021).

[50] Y. Sadasiva Rao and B. Nakra. Vibrations of unsymmetrical sandwich beams and plates with viscoelastic cores. Journal of Sound and Vibration, 34(3):309–326, 197