High-order filtered schemes for the Hamilton-Jacobi continuum limit of nondominated sorting[1]

Warut Thawinrak thawi001@umn.edu **Jeff Calder** jcalder@umn.edu

Department of Mathematics, University of Minnesota Twin Cities

Summary

We show how to construct filtered schemes for the Hamilton-Jacobi equation continuum limit of nondominated sorting by combining high order possibly unstable schemes with first order monotone and stable schemes. We prove that the filtered schemes are stable and convergent for all orders. We then investigat both high-order unfiltered and filtered schemes for the Hamilton-Jacobi equation by implementing both schemes for order $k = 1, 2, 3, 5, 8,$ and 13 numerically solving the equations in various mesh sizes. The errors from their numerical solutions compared to the known solutions were measured in the L^1 norm and the L^∞ norm. Our results suggest that the unfiltered schemes of order higher than 2 are unstable while the 1^{st} order and 2^{nd} order unfiltered schemes remain stable. Moreover, we see that the 2^{nd} order unfiltered scheme shows 2^{nd} order accuracy. Similarly to the unfiltered schemes, we see that the 2^{nd} order filtered scheme seems to show 2^{nd} order accuracy. However, it turns out that the filtered schemes of order higher than 2 only exhibit a 1^{st} order convergence rate. Upon further investigation, this appears to be due to fact that the filtering relies too often on the 1^{st} order scheme.

Introduction

We investigate high-order finite difference schemes for the two-dimensional Hamilton-Jacobi equation

$$\left.\begin{aligned} u_{x_1}u_{x_2} &= f \ \text{ in } (0,1]^2 \\ u &= 0 \ \text{ on } \partial[0,1]^2 \setminus (0,1]^2, \end{aligned}\right\} \tag{1}$$

where $f \geq 0$. This Hamilton-Jacobi equation (1) has a unique non-decreasing viscosity solution. In order to select the viscosity solution of (1), the finite difference scheme is required to be monotone [1]. Unfortunately, all monotone schemes are necessarily first order at best [2].

It has been observed [1, 4] that the monotonicity property can be relaxed to hold only approximately, with a residual error that vanishes as the grid is refined, while still ensuring the scheme converges to the viscosity solution. This allows one to design so-called filtered schemes, which blend together high-order nonmonotone schemes with monotone first-order schemes in such a way that the resulting filtered scheme is approximately monotone.

In this paper, we show how to construct arbitrary order filtered upwind finite schemes for the Hamilton-Jacobi equation (1) and prove that they are stable and convergent for any order. We then present numerical simulations on both filtered schemes and nonfiltered schemes investigating rates of convergence.

Filtered Schemes

Let u be the viscosity solution of (1) and define

$$w(x) = \frac{u(x)}{2\,(x_1x_2)^{1/2}}. \tag{2}$$

We can also show that w is the unique bounded viscosity solution of the Hamilton-Jacobi equation

$$(w + 2x_2w_{x_1})(w + 2x_2w_{x_2}) = f \quad \text{on } (0,1]^2. \tag{3}$$

We define for $u : [0,1]^2 \to \mathbb{R}$ the k^{th}-order backward difference quotient to be

$$\nabla_i^{k,-}u(x) = \frac{1}{h}\sum_{j=0}^{k} d_j u(x - jhe_i). \tag{4}$$

Here, $h > 0$ is the grid resolution, d_i is the backward difference quotient, and as a convention we take $u(x) = 0$ whenever $x \notin [0,1]^2$. The boundary value is irrelevant and does not enter into the scheme. We define the first order upwind scheme approximating the left-hand side of (3) to be

$$F_1(x,w) = \begin{cases} \displaystyle\prod_{i=1}^{2}\left(w(x) + 2x_i\nabla_i^{1,-}w(x)\right), & \text{if } \forall i,\ w(x) + 2x_i\nabla_i^{1,-}w(x) \geq 0 \\ -\infty, & \text{otherwise.} \end{cases} \tag{5}$$

The first order scheme from [3] corresponds to solving

$$F_1(x,w) = f(x) \ \text{ in } [0,1]_h^2,$$

where $\Omega_h := \Omega \cap h\mathbb{Z}^2$ for any $\Omega \subseteq \mathbb{R}^2$. This scheme has a unique solution, and is monotone, stable, and convergent to the viscosity solution.

We define the k^{th}-order upwind finite difference scheme approximating the left-hand side of (3) by

$$F_k(x,w) = \prod_{i=1}^{2}\left(w(x) + 2x_i\nabla_i^{k,-}w(x)\right). \tag{6}$$

The k^{th}-order upwind finite difference scheme is then given by

$$F_k(x,w) = f(x) \ \text{ in } [0,1]_h^2. \tag{7}$$

The k^{th}-order *filtered* upwind finite difference approximation of the left-hand side of (3) is given by

$$G_k(x,w) = \begin{cases} F_k(x,w), & \text{if } |F_k(x,w) - F_1(x,w)| \leq \sqrt{h} \text{ and } x \in [kh,1]^2, \\ F_1(x,w), & \text{otherwise.} \end{cases} \tag{8}$$

The k^{th}-order *filtered* upwind finite difference scheme is then given by

$$G_k(x,w) = f(x) \ \text{ in } [0,1]_h^2. \tag{9}$$

The key property of filtered schemes is that any solution w_h of (9) also satisfies

$$f(x) - \sqrt{h} \leq F_1(x,w_h) \leq f(x) + \sqrt{h} \ \text{ in } [0,1]_h^2. \tag{10}$$

Note that we prove in the paper that filtered schemes are stable and convergent for any order.

Numerical Simulations

We run simulations on both backward difference schemes and filtered schemes of orders 1, 2, 3, 5, 8, and 13 with two probability density functions f_1 and f_2 that were introduced before in [3]. The function f_1 is defined as follows:

$$f_1(x) = \frac{1}{4(k+1)^2}\prod_{i=1}^{2}\left(\sum_{j=1}^{2}\sin(kx_j)^2 + 2k + 2kx_i\sin(2kx_i)\right),$$

where $k > 0$. In the simulations, we set $k = 20$. The solution of (1) in this case is known to be

$$u_1(x) = \frac{1}{k+1}\sqrt{x_1x_2}\left(\sin(kx_1)^2 + \sin(kx_2)^2 + 2k\right).$$

We note that the solution u_1 is smooth on $(0,1]^2$.

The function f_2 is defined as follows:

$$f_2(x) = \frac{1}{(C+2)^2}\left(w_2(x) + 2(1+C)x(2)\right)\left(w_2(x) + 2x(1)\right).$$

where $x(i) = x_{\pi_x(i)}$ for a permutation π_x such that $x(1) \leq x(2)$, and $w_2(x) = C\max\{x_1,x_2\} + x_1 + x_2$. We set $C = 10$ in the simulations. The solution in this case is known to be

$$u_2(x) = 2\sqrt{x_1x_2}\,w_2(x).$$

Given these f_1 and f_2, we gather the errors from their numerical solutions compared to the known solutions for each order of each scheme in different mesh sizes h. These errors are measured in both the L^1 norm and the L^∞ norm as numerical evidence of the rate of convergence. We provide here some of the results from our simulations in Figure 1 - 2. Note that we use "S" to denote "backward difference scheme" and "FS" to denote "filtered scheme".

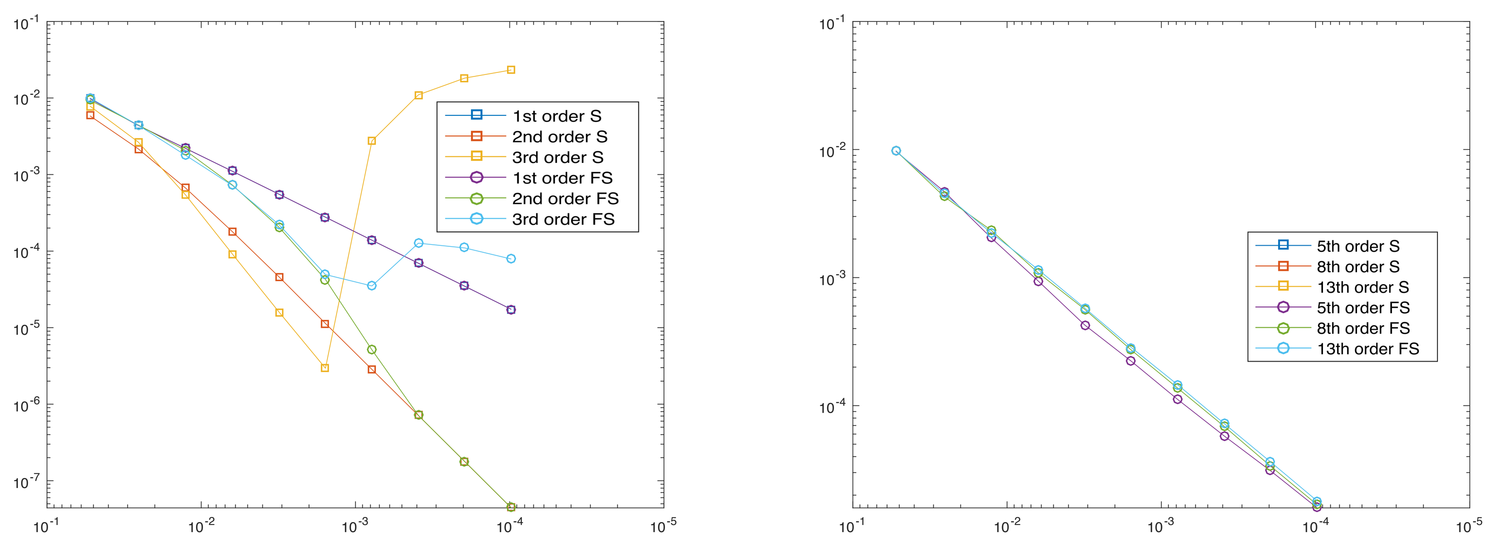

Figure 1: Errors in the L^1 norm from order 1, 2, 3 (left) and 5, 8, 13 (right) schemes when $f = f_1$

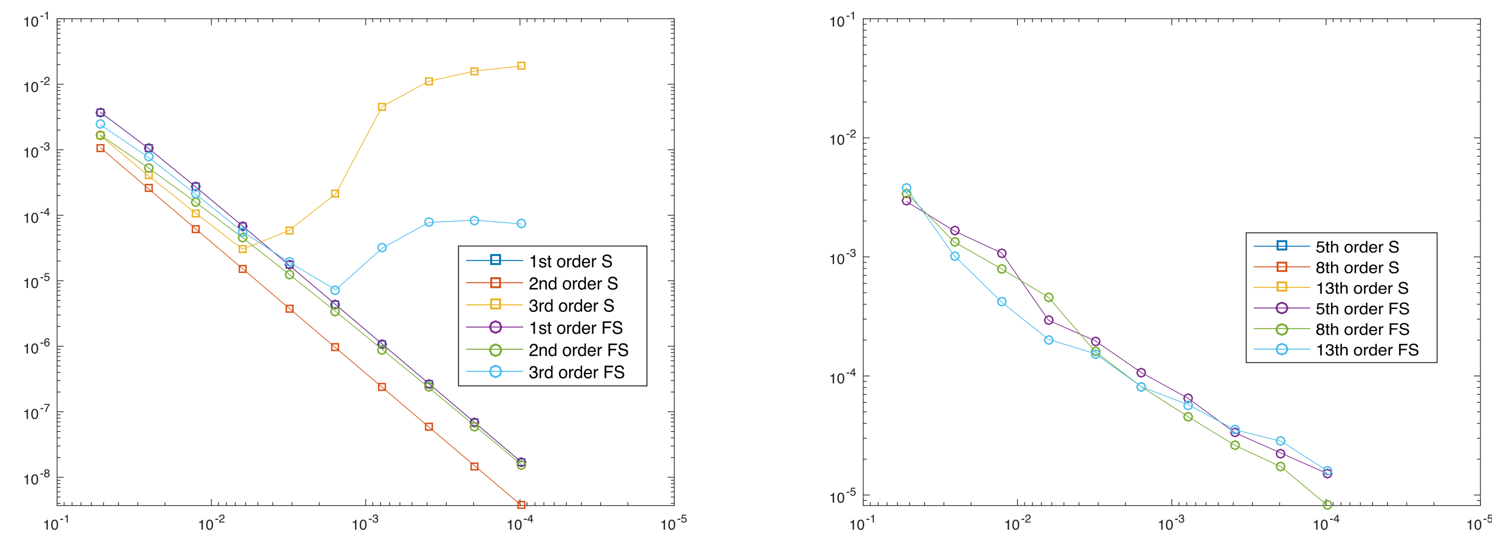

Figure 2: Errors in the L^1 norm from order 1, 2, 3 (left) and 5, 8, 13 (right) schemes when $f = f_2$

Rates of Convergence

The results from the simulations suggest that the unfiltered 2^{nd} order backward difference scheme is convergent with a second order rate. On the other hand, the backward difference schemes of order higher than two appear to be unstable. In fact, the errors for the unfiltered schemes for order $k = 5, 8, 13$ are so large they are not shown in the figures.

Observing the errors from filtered schemes in both the L^1 norm and the L^∞ norm, we see that the 2^{nd} order filtered scheme also tends to give better accuracy than the 1^{st} order one, but other higher order filtered schemes only give comparable accuracies to the 1^{st} order scheme. A further investigation shows that high order filtered schemes rely most of the time on the first order scheme in solving (1). This explains why higher order filtered schemes do not produce better accuracy than lower order ones. To give a better idea, we show the fraction of grid points for which the k^{th} order scheme is being used for various mesh sizes and orders when setting $f = f_1$ in Table 1.

Mesh size h	1^{st} order	2^{nd} order	3^{rd} order	5^{th} order	8^{th} order	13^{th} order
3.33×10^{-2}	95.06%	54.31%	45.00%	44.81%	22.69%	4.88%
6.67×10^{-3}	98.75%	81.09%	81.61%	75.41%	33.95%	7.59%
1.59×10^{-3}	99.69%	97.69%	97.42%	85.61%	37.92%	8.78%
3.92×10^{-5}	99.92%	99.84%	98.57%	87.29%	39.15%	8.97%
9.78×10^{-6}	99.98%	99.96%	97.82%	87.03%	39.43%	9.00%

Table 1: Fraction of grid points for which the k^{th} order scheme being used in the filtered schemes.

This explains why filtering is not successful for higher order schemes. It would be interesting to determine why this is happening and whether it can be improved by using different schemes or a different type of filtering.

References

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