SCHAUDER EXPANSION BY SOME QUADRATIC BASE FUNCTION

BY

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Schauder Expansion by Some Quadratic Base Function

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Section 0. Introduction

In the previous paper [2] we found that the so-called Takagi function, which is continuous and has no finite derivative anywhere, satisfies an infinite sequence of finite difference equations. It is the unique solution of a Dirichlet problem in some generalized sense. The present note proposes a generalization of the Shauder expansion which played an important role in the previous paper.

Section 1. Schauder Expansion

Let us consider the quadratic function:

$$F_{\alpha,\beta}(x) = 4(\beta-\alpha)^{-2}(x-\alpha)(x-\beta) \qquad (\alpha \le x \le \beta)$$

$$= 0 \qquad (\text{otherwise}) .$$

We define the function $\xi_{k,i}(x)$:

$$\xi_{k,i}(x) = 2^{2k+2}(x-i/2^k)(i+1/2^k-x) = F_{i/2^k,\,i+1/2^k}(x) .$$

Now all our new Schauder basis is the following:

$$1, x, \{\xi_{k,i}(x)\} \quad \text{for } k=0,1,\dots \text{ and } 0 \le i \le 2^k-1 .$$

Using these basis functions, we get the following expansion theorem:

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Theorem: If $f(x)$ belongs to $C[0,1]$, we can expand it uniquely in the following series:

(1) $$f(x) = c_0+c_1x+c_{0,0}\xi_{0,0}(x) + \sum_{k=1}^{+\infty} \sum_{i=0}^{2^k-1} c_{k,i}\xi_{k,i}(x)$$

(2) $$c_0 = f(0),\ c_1 = f(1)-f(0),\ c_{0,0}=f(1/2)-(1/2)\{f(0)+f(1)\}$$

$$c_{k,i} = f(2i+1/2^{k+1})-(3/8)f(i/2^k)-(3/4)f(i+1/2^k)+(1/8)f(i+2/2^k)$$

$$(i=\text{even},\ 0 \leq i \leq 2^k-1, k=1,2,\ldots)$$

$$= f(2i+1/2^{k+1})-(3/4)f(i/2^k)-(3/8)f(i+1/2^k)+(1/8)f(i-1/2^k)$$

$$(i=\text{odd},\ 0 \leq i \leq 2^k-1, k=1,2,\ldots) .$$

Proof. Put

$$S_n(x)=c_0+c_1x+c_{0,0}\xi_{0,}(x) + \sum_{k=1}^{n} \sum_{i=0}^{2^k-1} c_{k,i}\xi_{k,i}(x) ,$$

For example,

$$S_1(x) = f(0)+\{f(1)-f(0)\}x+\{f(1/2)-(1/2)[f(0)+f(1)]\}\xi_{0,0}(x)$$

$$+ \{f(1/4)-(3/8)f(0)-(3/4)f(1/2)+(1/8)f(1)\}\xi_{1,0}(x)$$

$$+\{f(3/4)-(3/4)f(1/2)-(3/8)f(1)+(1/8)f(0)\}\xi_{1,1}(x) .$$

We can show $S_1(i/2) = f(i/2)$ $(i=0,1,2)$.

In fact, if i=0, $S_1(0) = f(0)$.

$$i=1, S_1(1/2)=f(0)+(f(1)-f(0)\}(1/2)+\{f(1/2)-(1/2)[f(0)+f(1)]\}$$

$$=f(1/2) .$$

$$i=2, S_1(1)=f(1).$$

Next, for k=2, we have to show $S_1(i/2^2)=f(i/2^2)$ for i=0,1,2,3. If i=even, thi is already done. Thus, we only have to show that

$$S_1(2i+1/2^2)=f(2i+1/2^2) \quad \text{for } i=0,1.$$

For example, i=0

$$S_1(1/4)=f(0)+f\{(1)-f(0)\}(1/4)+\{f(1/2)-(1/2)[f(1)+f(0)]\}(3/4)$$

$$+\{f(1/4)-(3/8)f(0)-(3/4)f(1/2)+(1/8)f(1)$$

$$=f(1/4) .$$

Similarly, we get

$$S_1(3/4)=f(3/4) .$$

We now proceed to the Induction.

Induction Assumption:

$$S_n(i/2^{k+1}) = f(i/2^{k+1}) \quad \text{for } 0 \leq 2^{k+1}-1, 0 \leq k \leq n .$$

Under this assumption, we are going to show that

$$S_{n+1}(2i+1/2^{k+1})=f(2i+1/2^{k+1}) \quad \text{for } 0 \leq i \leq 2^k-1, 0 \leq k \leq n+1 .$$

First, we have the following equality:

$$S_{n+1}(x)=S_n(x) + \sum_{i=0}^{2^{n+1}-1} c_{n+1,i}\xi_{n+1,i}(x)$$

$\xi_{n+1,i}(i/2^{n+1})=0$ for $i=0,1,..2^{n+1}$). $S_n(i/2^{n+1})=f(i/2^{n+1})$.

Therefore

$$S_{n+1}(2i+1/2^{n+2})=S_n(2i+1/2^{n+2})$$

$$+ \sum_{i=0}^{2^{n+1}-1} c_{n+1,i}\xi_{n+1,i}(2i+1/2^{n+2})$$

$$= S_n(2i+1/2^{n+2})$$

$$+ f(2i+1/2^{n+2})-(3/8)f(i/2^{n+1})-(3/4)f(i+1/2^{n+1})$$

$$+ (1/8)f(i+2/2^{n+1})$$

$$= f(2i+1/2^{n+2})$$

$$+ S_n(2i+1/2^{n+2})-(3/8)S_n(i/2^{n+2})-(3/4)S_n(i+2^{n+1})$$

$$+ (1/8)S_n(i+2/2^{n+1}) .$$

If we take the function ϕ to be a constant, x x^2 it is easy to show the following identity:

$$\phi(2i+1/2^{n+2})=(3/8)\phi(i/2^{n+1})+(3/4)\phi(i+1/2^{n+1})-(1/8)\phi(i+2/2^{n+1}) .$$

Also,

$$\phi(2i+1/2^{n+2})=(3/4)\phi(i+2^{n+1})+(3/8)\phi(i+1/2^{n+1})-(1/8)\phi(i-1/2^{n+1}) .$$

Therefore, because $S_n(x)$ is a linear combination of constants, x and the function x^2, we get

$$S_n(2i+1/2^{n+2})=(3/8)S_n(i/2^{n+1})+(3/4)S_n(i+1/2^{n+1})$$
$$-(1/8)S_n(i+2/^{n+1}) \ .$$

Section 2. An Application

Let us consider the following function S(x) which is continuous and it has no finite derivative everywhere:

$$S(x)=\sum(1/2^{n+1})F_{0,1}(\phi^n(x))$$

$$(\text{here, } \phi(x)=1-|2x-1| \quad 0 \le x \le 1$$
$$=0 \ (\text{otherwise})) \ .$$

First we remark that

$$\xi_{k,i}(x)=F_{i/1,i+1/2}(x) = \chi_{[\frac{i}{2^k},\frac{i+1}{2^k}]}(x)F_{0,1}(\phi^k(x)) \ .$$

We can expand S(x) in the series mentioned in Section 1.

$$S(x)=S(0)+S(1)-S(0))x+\{S(1/2)-(1/2)[S(0)+S(1)]\}F_{0,1}(x)$$
$$+\sum_{k=1}^{+\infty}\sum_{j=0}^{2^k-1} c_{k,j}(x) \ .$$
$$S(0)))=0, S(1)=0, S(1/2)=1.$$

On the other hand, we have

$$S(x)=F_{0,1}(x) + \sum_{k=1}^{+\infty}\sum_{j=0}^{2^k-1} ((1/2^{k+1})\xi_{k,j}(x) \ .$$

Therefore, we get the following infinite set of finite difference equations for S(x),

$S(2i+1/2^{k+1})-(3/8)S(i/2^k)-(3/4)S(i+1//2^k)+(1/8)S(i+2/2^k)=1/2^{k+1}$

(i=even)

$S(2i+1/2^{k+1})-(3/4)S(i/2^k)-(3/8)S(i+1/2^k)+(1/8)S(i-1/2^k)=1/2^{k+1}$

(i=odd) .

Remark. This function $S(x)$ is simply equal to $2T(x)-2(1-x)x$, because we already know that

$$\sum_{k=0}^{+\infty} (1/2^k)[\phi^k(x)]^2 = \frac{1}{2}x(1-x) \ . \ [1] \ .$$

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