DECAY ESTIMATES FOR A PETROVSKI SYSTEM WITH A NONLINEAR DISTRIBUTED FEEDBACK

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Abstract. We consider the uniform stabilizability problem of a Petrovski system, generalizing a simple plate model, by using nonlinear distributed feedbacks. Applying a method of integral inequalities as described in [8], we estimate the energy decay rate in function of the growth properties of the nonlinearity. We apply feedbacks of a different form than e.g. in [6]; this makes it possible to weaken the growth assumptions on the nonlinearity for more regular initial data, thereby obtaining similar results as in [17] for the wave equation.

Key words. Petrovski system, feedback, integral inequality.

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Abbreviated title. Decay estimates for a Petrovski system

1. Introduction and statement of the results

The estimation of the energy decay rate of dissipative systems is the subject of many research, cf. e.g. the proceedings of the annual IEEE Conferences on Decision and Control. One of the most popular methods in this area is the Liapunov method, but special differential or integral inequalities often prove to be very useful, too.

In [8] we proposed for a certain general class of distributed systems a systematic approach based on integral inequalities of the form

$$\int_t^\infty E^{\alpha+1}\ dt \le AE(t), \quad \forall t \in \mathbb{R}_+ := [0, +\infty) \tag{1}$$

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where $E : \mathbf{R}_+ \to \mathbf{R}_+$ is a non-increasing function and α, A are given constants. We proved that if α, A are positive and if E is a nontrivial solution of (1) (i.e. if $E(0) \neq 0$), then

$$E(t) \leq \begin{cases} E(0), & \text{if } t \leq T; \\ E(0)\left(\frac{T+\alpha t}{T+\alpha T}\right)^{-1/\alpha}, & \text{if } t \geq T \end{cases} \tag{2}$$

for all $t \in \mathbf{R}_+$, where we have set

$$T := AE(0)^{-\alpha}. \tag{3}$$

The estimates (2) are optimal in the following sense : given $C > 0$ and $t' > 0$ arbitrarily, there exists a non-increasing function $E : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying (1) and such that $E(0) = C$ and that in (2) equality holds if $t = t'$.

Analogous results were obtained for $-1 < \alpha \leq 0$, too. In particular, for $\alpha = 0$ (2) is replaced by

$$E(t) \leq \begin{cases} E(0), & \text{if } t \leq A; \\ E(0)\exp\left(1 - \frac{t}{A}\right), & \text{if } t \geq A. \end{cases} \tag{4}$$

We showed that integral inequalities of this type arise naturally in a number of distributed or boundary stabilization problems concerning the wave equation or some plate models [8]–[10].

Integral inequalities of type (1) appeared already before ([3], [4], [7], [13]) but their efficiency was not clear comparing for example to the usual Liapunov method as applied e.g. in [1], [2], [5], [6], [12], [13], [14], [18]. In [11] we proved that apart from some trivial cases the Liapunov method never gives the optimal estimates (2), (4) because of the artificially modified energy function and because of artificially introduced parameters.

In the present paper we apply this method for the study of a Petrovski system with distributed feedback. Let Ω be a bounded open set in $\mathbf{R}^n$ having a boundary Γ of class C^4. We shall denote by ν the outward unit normal vector to Γ. Let $g : \mathbf{R} \to \mathbf{R}$ be a non-decreasing, continuous function with $g(0) = 0$, and consider the system

$$u'' + \Delta^2 u - g(\Delta u') = 0 \quad \text{in} \quad \Omega \times \mathbf{R}_+, \tag{5}$$

$$u = \Delta u = 0 \quad \text{on} \quad \Gamma_- \times \mathbf{R}_+ \tag{6}$$

$$u(0) = u_0 \quad \text{and} \quad u'(0) = u_1 \quad \text{on} \quad \Omega. \tag{7}$$

Using the methods of [5], it is easy to verify that this system is well-posed in the following sense : introducing the Hilbert spaces H, V by

$$H := H_0^1(\Omega) \quad \text{and} \quad V := \{v \in H^3(\Omega) : v = \Delta v = 0 \text{ on } \Gamma\}, \tag{8}$$

for every $(u_0, u_1) \in V \times H$ the system (5)–(7) has a unique solution u satisfying

$$C(\mathbf{R}_+; V) \cap C^1(\mathbf{R}_+; H).$$

Furthermore, the system is dissipative in the sense that its energy $E : \mathbf{R}_+ \to \mathbf{R}_+$ defined by

$$E = \frac{1}{2} \int_\Omega |\nabla u'|^2 + |\nabla \Delta u|^2 \; dx \tag{9}$$

is non-increasing. Indeed, it is easy to verify, at least formally, that

$$E' = - \int_\Omega (\Delta u') g(\Delta u') \; dx \leq 0; \tag{10}$$

we will prove it later (cf. lemma 1).

Our purpose is to estimate the energy decay rate under suitable growth conditions on g. Let

$$1 \leq p < +\infty, \quad 1 \leq q < +\infty \tag{11}$$

and assume that there exist positive constants c_1, c_2, c_3, c_4 such that

$$c_1 |x|^p \leq |g(x)| \leq c_2 |x|^{1/p} \quad if \quad |x| \leq 1 \tag{12}$$

and

$$c_3 |x| \leq |g(x)| \leq c_2 |x|^q \quad if \quad |x| > 1. \tag{13}$$

We shall prove the following result.

THEOREM 1. — *Assume that*

$$(n-2)q \leq n+2. \tag{14}$$

Then for every $(u_0, u_1) \in V \times H$ *the solution of* (5)–(7) *satisfies*

$$E(t) \leq Ct^{\frac{-2}{p-1}} \ \forall t > 0 \tag{15}$$

if $p > 1$, *and*

$$E(t) \leq Ce^{-\omega t} \ \forall t > 0 \tag{16}$$

if $p = 1$.

In (15), (16) *the constant* C *depends only on the initial energy* $E(0)$ *in a continuous way (and the dependence is linear if* $p = 1$*), while the constant* ω *is independent of the particular choice of the initial data.*

If $n = 1$, *then the estimates* (15), (16) *remain valid for* $q = +\infty$, *too, i.e. even if* $|g(x)|$ *blows up arbitrarily fast as* $|x| \to +\infty$.

Let us observe that the decay rate in (15) is determined only by the behavior of g near zero.

If the initial data and g are more regular, then we may weaken our growth assumptions on g :

THEOREM 2. — *Assume that* g *is locally absolutely continuous and*

$$(n-4)q \leq n+4, \tag{17}$$

and set

$$p' := \max\left\{p, \frac{n}{2}\left(1 - \frac{1}{q}\right) - 1\right\}, \tag{18}$$

$$\widetilde{H} := H^2(\Omega) \cap H \quad \textit{and} \quad \widetilde{V} := H^4(\Omega) \cap V. \tag{19}$$

Then for every

$$(u_0, u_1) \in \widetilde{V} \times \widetilde{H} \quad \textit{such that} \quad g(\Delta u_1) \in L^2(\Omega), \tag{20}$$

the solution of (5)–(7) *satisfies*

$$E(t) \leq Ct^{\frac{-2}{p'-1}} \ \forall t > 0 \tag{21}$$

if $p' > 1$, *and*

$$E(t) \leq Ce^{-\omega t} \ \forall t > 0 \tag{22}$$

if $p' = 1$.

The constants C, ω *depend on the norm of* (u_0, u_1) *in* $\widetilde{V} \times \widetilde{H}$, *not only on the initial energy* $E(0)$.

If $n \leq 3$, *then the estimates* (21), (22) *remain valid for* $q = +\infty$, *too.*

Let us observe that now the decay rate depends on the behavior of g both near zero and near infinity. We may explain the simpler situation of theorem 1 by saying that there g did not grow too fast near infinity so as to influence the energy decay rate.

2. Proof of theorem 1

It is sufficient to prove the estimates (15), (16) for smooth solutions corresponding to initial data satisfying (20). Indeed, the general case then follows by an easy density argument, remarking that the convergence of the initial data in $V \times H$ implies the uniform convergence of the corresponding energy functions on $\mathbb{R}$. Since (20) implies by the standard elliptic regularity theory that the solution of (5)–(7) satisfies

$$u \in L^\infty_{loc}(\mathbb{R}_+; \widetilde{V}) \cap W^{1,\infty}_{loc}(\mathbb{R}_+; \widetilde{H}) \cap W^{2,\infty}_{loc}(\mathbb{R}_+; L^2(\Omega)) \tag{23}$$

and

$$g(\Delta u^1) \in L^\infty(\mathbb{R}_+; L^2(\Omega)), \tag{24}$$

we may assume (23) and (24) in our computations below.

We need two identities.

LEMMA 1. — *The function* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *is non-increasing, locally absolutely continuous and*

$$E' = -\int_\Omega (\Delta u') g(\Delta u') \, dx \leq 0 \quad a.e.\ in \quad \mathbb{R}_+. \tag{25}$$

PROOF. — Fixing $0 \leq S < T < \infty$ arbitrarily, we have the following equality by using (5), (6) and the definition of the energy :

$$\begin{aligned} 0 &= \int_S^T \int_\Omega (-\Delta u')(u'' + \Delta^2 u - g(\Delta u')) \, dx \, dt \\ &= \int_S^T \int_\Omega \nabla u' \cdot \nabla u'' + \nabla \Delta u \cdot \nabla \Delta u' + (\Delta u') g(\Delta u') \, dx \, dt \\ &= E(T) - E(S) + \int_S^T \int_\Omega (\Delta u') g(\Delta u') \, dx \, dt \end{aligned}$$

i.e.

$$E(S)-E(T)=\int_S^T\int_\Omega (\Delta u')g(\Delta u')\,dx\,dt,\quad 0\le S<T<+\infty. \tag{26}$$

The right-hand side of (26) is nonnegative by our assumptions on g; hence E is non-increasing. It follows also from (26) that E is locally absolutely continuous and that the equality in (25) is satisfied. □

LEMMA 2. — *We have*

$$\begin{aligned}2\int_S^T E^{\frac{p+1}{2}}\,dt &= \left[E^{\frac{p-1}{2}}\int_\Omega u'\Delta u\,dx\right]_S^T\\ &\quad -\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega u'\Delta u\,dx\,dt\\ &\quad +\int_S^T E^{\frac{p-1}{2}}\int_\Omega 2|\nabla u'|^2-(\Delta u)g(\Delta u')\,dx\,dt\end{aligned} \tag{27}$$

for all $0\le S<T<+\infty$.

PROOF. — We have

$$\begin{aligned}0&=\int_S^T E^{\frac{p-1}{2}}\int_\Omega(-\Delta u)(u''+\Delta^2u-g(\Delta u'))\,dx\,dt\\ &=\left[E^{\frac{p-1}{2}}\int_\Omega -u'\Delta u dx\right]_S^T+\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega u'\Delta u\,dx\,dt\\ &\quad+\int_S^T E^{\frac{p-1}{2}}\int_\Omega u'\Delta u'+|\nabla\Delta u|^2+(\Delta u)g(\Delta u')\,dx\,dt\\ &=\left[E^{\frac{p-1}{2}}\int_\Omega -u'\Delta u dx\right]_S^T+\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega u'\Delta u\,dx\,dt\\ &\quad+\int_S^T E^{\frac{p-1}{2}}\int_\Omega -|\nabla u'|^2+|\nabla\Delta u|^2+(\Delta u)g(\Delta u')\,dx\,dt.\end{aligned}$$

Using the definition of the energy, (27) follows. □

LEMMA 3. — *We have*

$$\begin{aligned}2\int_S^T E^{\frac{p+1}{2}}\,dt &\le cE(S)\\ &\quad+\int_S^T E^{\frac{p-1}{2}}\int_\Omega 2|\nabla u'|^2-(\Delta u)g(\Delta u')\,dx\,dt\end{aligned} \tag{28}$$

for all $0 \leq S < T < +\infty$.

Here and in the sequel c will denote diverse positive constants depending only on $E(0)$.

Proof. — Using the obvious estimates

$$\|u'\|_{L^2(\Omega)} \leq c\|\nabla u'\|_{L^2(\Omega)} \tag{29}$$

and

$$\|\Delta u\|_{L^2(\Omega)} \leq c\|\nabla \Delta u\|_{L^2(\Omega)} \tag{30}$$

(we recall that $u' = u = \Delta u = 0$ on Γ), we have

$$\begin{aligned}\left|E^{\frac{p-1}{2}} \int_\Omega u'\Delta u \ dx\right| &\leq cE^{\frac{p-1}{2}} \|\nabla u'\|_{L^2(\Omega)}\|\nabla \Delta u\|_{L^2(\Omega)} \\ &\leq cE^{\frac{p-1}{2}} \int_\Omega |\nabla u'|^2 + |\nabla \Delta u|^2 \ dx = cE^{\frac{p+1}{2}} \\ &\leq cE(0)^{\frac{p-1}{2}} E = cE.\end{aligned}$$

(We used the non-increasingness of E.) Hence

$$\left[E^{\frac{p-1}{2}} \int_\Omega u'\Delta u \ dx\right]_S^T \leq cE(S) + cE(T) \leq cE(S).$$

Furthermore, using (29) and (30) again,

$$\begin{aligned}\left|\frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \int_\Omega u'\Delta u \ dx \ dt\right| &\leq c \int_S^T E^{\frac{p-3}{2}}(-E')E \ dt \\ &= cE^{\frac{p+1}{2}}(S) - cE^{\frac{p+1}{2}}(T) \leq cE^{\frac{p+1}{2}}(S) \leq cE(S).\end{aligned}$$

Using these two estimates, (28) follows from (27). □

Lemma 4. — *We have the estimate*

$$\int_\Omega |\nabla u'|^2 \ dx \leq -cE' + c(-E')^{\frac{2}{p+1}}. \tag{31}$$

Proof. — For $t \in \mathbf{R}_+$ fixed arbitrarily, set

$$\Omega_1 = \{x \in \Omega : |\Delta u'(x)| \leq 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega : |\Delta u'(x)| > 1\}. \tag{32}$$

Since

$$\|\nabla v\|_{L^2(\Omega)} \le c\|\Delta v\|_{H^{-1}(\Omega)} \le \|\Delta v\|_{L^2(\Omega)}$$

for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\int_\Omega |\nabla u'|^2 \, dx \le c \int_\Omega |\Delta u'|^2 \, dx = c \int_{\Omega_1} |\Delta u'|^2 \, dx + c \int_{\Omega_2} |\Delta u'|^2 \, dx. \tag{33}$$

Using the first inequality in (12) we obtain that

$$\begin{aligned} \int_{\Omega_1} |\Delta u'|^2 \, dx &\le c \int_{\Omega_1} \big((\Delta u')g(\Delta u')\big)^{\frac{2}{p+1}} \, dx \\ &\le c\Big(\int_{\Omega_1} (\Delta u')g(\Delta u') \, dx\Big)^{\frac{2}{p+1}} \\ &\le c\Big(\int_{\Omega} (\Delta u')g(\Delta u') \, dx\Big)^{\frac{2}{p+1}} = c(-E)^{\frac{2}{p+1}}. \end{aligned} \tag{34}$$

Furthermore, using the first inequality in (13), we have

$$\begin{aligned} \int_{\Omega_2} |\Delta u'|^2 \, dx &\le c \int_{\Omega_2} (\Delta u')g(\Delta u') \, dx \\ &\le \int_{\Omega} (\Delta u')g(\Delta u') \, dx = -cE'. \end{aligned} \tag{35}$$

The inequality (31) follows from (33), (34) and (35). $\square$

Lemma 5. — *We have*

$$\Big|\int_\Omega (\Delta u)g(\Delta u') \, dx\Big| \le cE^{1/2}(-E')^{\frac{1}{p+1}} + cE^{1/2}(-E')^{\frac{1}{1+q^{-1}}}. \tag{36}$$

Proof. — We define Ω_1 and Ω_2 by (32) again. Using the second inequality in (12) we obtain

$$\int_{\Omega_1} |g(\Delta u')|^2 \, dx \le c \int_{\Omega_1} \big((\Delta u')g(\Delta u')\big)^{\frac{2}{p+1}} \, dx \le c(-E')^{\frac{2}{p+1}}$$

as in (34), whence

$$\begin{aligned} &\Big|\int_{\Omega_1} (\Delta u)g(\Delta u') \, dx\Big| \le c\|\Delta u\|_{L^2(\Omega)}(-E')^{\frac{1}{p+1}} \\ &\le c\|\nabla \Delta u\|_{L^2(\Omega)}(-E')^{\frac{1}{p+1}} \le cE^{1/2}(-E')^{\frac{1}{p+1}}. \end{aligned} \tag{37}$$

Furthermore, using the second inequality in (13) we have, if $q < +\infty$,

$$\int_{\Omega_2} |g(\Delta u')|^{1+q^{-1}}\ dx \le c \int_{\Omega_2} (\Delta u')g(\Delta u')\ dx$$
$$\le c \int_{\Omega} (\Delta u')g(\Delta u')\ dx = -cE'$$

and therefore, using also the imbedding (cf. (14))

$$H^3(\Omega) \subset W^{2,1+q}(\Omega),$$

$$\left| \int_{\Omega_2} (\Delta u)g(\Delta u')\ dx \right| \le c\|\Delta u\|_{L^{1+q}(\Omega)}(-E')^{\frac{1}{1+q^{-1}}}$$
$$\le c\|\nabla \Delta u\|_{L^2(\Omega)}(-E')^{\frac{1}{1+q^{-1}}} \le cE^{1/2}(-E')^{\frac{1}{1+q^{-1}}}. \tag{38}$$

This inequality also holds if $q = +\infty$: using the imbedding

$$H^3(\Omega) \subset W^{2,\infty}(\Omega),$$

in this case we have

$$\left| \int_{\Omega_2} (\Delta u)g(\Delta u')\ dx \right| \le c\|\Delta u\|_{L^\infty(\Omega)} \int_{\Omega_2} |g(\Delta u')|\ dx$$
$$\le c\|\nabla \Delta u\|_{L^2(\Omega)} \int_{\Omega_2} (\Delta u')g(\Delta u')\ dx \le cE^{1/2}(-E').$$

Finally, (37) and (38) imply (36). $\square$

LEMMA 6. — *We have*

$$\int_S^T E^{\frac{p+1}{2}}\ dt \le cE(S), \quad 0 \le S < T < +\infty. \tag{39}$$

PROOF. — We estimate the last term of (28) by using the preceeding two lemmas. We fix $\epsilon > 0$ arbitrarily and we apply the Young inequality as follows :

$$\left| E^{\frac{p-1}{2}} \int_\Omega 2|\nabla u'|^2 - (\Delta u)g(\Delta u')\ dx \right|$$
$$\le -cE^{\frac{p-1}{2}}E' + cE^{\frac{p-1}{2}}(-E')^{\frac{2}{p+1}} + cE^{\frac{p}{2}}(-E')^{\frac{1}{p+1}} + cE^{\frac{p}{2}}(-E')^{\frac{1}{1+q^{-1}}}$$
$$\le (-cE') + \left(\epsilon E^{\frac{p+1}{2}} - c(\epsilon)E'\right) + \left(\epsilon E^{\frac{p+1}{2}} - c(\epsilon)E'\right) + \left(\epsilon E^{\frac{p(q+1)}{2}} - c(\epsilon)E'\right)$$
$$\le c_1\epsilon E^{\frac{p+1}{2}} - c(\epsilon)E';$$

here c_1 is some positive constant, independent of ϵ. (We used here the trivial inequality $p(q+1) \geq p+1$.)

Choosing $\epsilon > 0$ such that $c_1 \epsilon \leq 1$, hence we conclude that

$$\left| E^{\frac{p-1}{2}} \int_\Omega 2|\nabla u'|^2 - (\Delta u) g(\Delta u') \, dx \right| \leq E^{\frac{p+1}{2}} - cE'$$

and therefore

$$\begin{aligned} &\left| \int_S^T E^{\frac{p-1}{2}} \int_\Omega 2|\nabla u'|^2 - (\Delta u) g(\Delta u') \, dx \, dt \right| \\ &\leq \int_S^T E^{\frac{p+1}{2}} \, dt + c\big(E(S) - E(T)\big) \leq \int_S^T E^{\frac{p+1}{2}} \, dt + cE(S). \end{aligned}$$

Substituting this into (28) we find (39). □

It follows from lemmas 1 and 6 that $E : \mathbf{R}_+ \to \mathbf{R}_+$ is a non-increasing function satisfying (1) with $\alpha = (p-1)/2$ and $A = c$. Applying the estimates (2), (4) mentioned in the introduction, hence the theorem follows. □

3. Proof of theorem 2

Let us begin by observing that the inequalities (12) remain valid if we replace p by p'. Therefore the estimate (31) of lemma 4 and the estimate (37) in the proof of lemma 5 remains valid with p replaced by p'. We shall replace the estimate (38) above by the estimate

$$\left| \int_{\Omega_2} (\Delta u) g(\Delta u') \, dx \right| \leq c \|\Delta u\|_{L^{q+1}(\Omega)} (-E')^{\frac{1}{1+q^{-1}}}, \tag{40}$$

thereby proving the following version of lemma 5 :

LEMMA 7. — *We have*

$$\left| \int_\Omega (\Delta u) g(\Delta u') \, dx \right| \leq cE^{1/2} (-E')^{\frac{1}{p'+1}} + c \|\Delta u\|_{L^{q+1}(\Omega)} (-E')^{\frac{1}{1+q^{-1}}}. \tag{41}$$

First we establish a new identity :

LEMMA 8. — *We have*

$$\left[\frac{1}{2} \int_\Omega (\Delta u')^2 + (\Delta^2 u)^2 \, dx \right]_T^S = \int_S^T \int_\Omega g'(\Delta u') |\nabla \Delta u'|^2 \, dx \, dt \tag{42}$$

for all $0 \leq S < T < +\infty$.

PROOF. — We have, integrating by parts and using (5), (6) :

$$0 = \int_S^T \int_\Omega (\Delta^2 u')(u'' + \Delta^2 u - g(\Delta u'))\ dx\ dt$$

$$= \int_S^T \int_\Omega (\Delta u')(\Delta u'') + (\Delta^2 u')(\Delta^2 u) + g'(\Delta u')|\nabla \Delta u'|^2\ dx\ dt$$

$$= \Big[\frac{1}{2}\int_\Omega (\Delta u')^2 + (\Delta^2 u)^2\ dx\Big]_S^T + \int_S^T \int_\Omega g'(\Delta u')|\nabla \Delta u'|^2\ dx\ dt. \quad \square$$

Since the right-hand side of (42) is nonnegative, lemma 7 shows that

$$\int_\Omega (\Delta u')^2 + (\Delta^2 u)^2\ dx$$

is a non-increasing function of $t \in \mathbf{R}_+$. Since $\Delta^2 u^0$ and Δu^1 belong to $L^2(\Omega)$ by hypothesis, it follows that $\Delta^2 u$ and $\Delta u'$ belong to $L^\infty(\mathbf{R}_+; L^2(\Omega))$ and therefore

$$u \in L^\infty(\mathbf{R}_+; H^4(\Omega)) \cap W^{1,\infty}(\mathbf{R}_+; H^2(\Omega)) \tag{43}$$

by the elliptic regularity theory.

Now we are ready to prove lemma 7 :

PROOF OF LEMMA 7. — . As we showed at the beginning of this section, it is sufficient to prove the estimate (40).

If $q = +\infty$, then $n \leq 3$ by hypothesis; therefore (43) follows easily from (41) and from the definition of Ω_2 :

$$\Big|\int_{\Omega_2} (\Delta u) g(\Delta u')\ dx\Big| \leq c\|\Delta u\|_{L^\infty(\Omega)} \int_{\Omega_2} |g(\Delta u')|\ dx$$

$$\leq c\|\Delta u\|_{L^\infty(\Omega)} \int_{\Omega_2} (\Delta u') g(\Delta u')\ dx = c\|\Delta u\|_{L^\infty(\Omega)}(-E').$$

If $q < +\infty$, then we have, using also the second inequality in (13), and writing for brevity $\|\cdot\|_r$ instead of $\|\cdot\|_{L^r(\Omega)}$,

$$\Big|\int_{\Omega_2} (\Delta u) g(\Delta u')\ dx\Big| \leq c\|\Delta u\|_{q+1} \|g(\Delta u')\|_{L^{1+q^{-1}}(\Omega_2)}$$

$$\leq c\|\Delta u\|_{q+1} \Big\| \Big((\Delta u') g(\Delta u')\Big)^{\frac{1}{1+q^{-1}}} \Big\|_{L^{1+q^{-1}}(\Omega_2)}$$

$$\leq c\|\Delta u\|_{q+1} \|(\Delta u') g(\Delta u')\|_{L^1(\Omega_2)}^{\frac{1}{1+q^{-1}}} = c\|\Delta u\|_{q+1}(-E')^{\frac{1}{1+q^{-1}}}. \quad \square$$

Now we prove, using the assumptions (17), (18), the following version of lemma 6 :

LEMMA 9. — *We have*

$$\int_S^T E^{\frac{p'+1}{2}}\, dt \leq cE(S), \quad 0 \leq S < T < +\infty. \tag{44}$$

PROOF. — Repeating the proof of lemma 6 (replacing p by p' everywhere), it only remains to prove an estimate of the form

$$E^{\frac{p'-1}{2}} \|\Delta u\|_{q+1} (-E')^{\frac{1}{1+q^{-1}}} \leq c_2 \epsilon E^{\frac{p'+1}{2}} - c(\epsilon) E' \tag{45}$$

for all $\epsilon > 0$, with some positive constant c_2, independent of ϵ.

Consider first the case $n < 4$. Then we have

$$H^4(\Omega) \subset W^{2,\infty}(\Omega) \subset W^{2,q+1}(\Omega)$$

and therefore we deduce from (43) that the left-hand side of (45) is majorized by

$$cE^{\frac{p'-1}{2}} (-E')^{\frac{1}{1+q^{-1}}}.$$

The case $q = +\infty$ hence follows at once :

$$cE^{\frac{p'-1}{2}} (-E')^{\frac{1}{1+q^{-1}}} = -cE^{\frac{p'-1}{2}} E' \leq -cE'.$$

If $q < +\infty$, then we apply the Young inequality as follows :

$$cE^{\frac{p'-1}{2}} (-E')^{\frac{1}{1+q^{-1}}} \leq cE^{\frac{(p'-1)(q+1)}{2}} (-E') \leq -cE'.$$

Now consider the case $n > 4$ (then $q < +\infty$). Let us recall that the interpolational inequality

$$\|v\|_{q+1} \leq c \|v\|_{2n/(n-2)}^t \|v\|_{2n/(n-4)}^{1-t}$$

is valid whenever

$$0 \leq t \leq 1 \quad \text{and} \quad \frac{1}{q+1} \geq t\frac{n-2}{2n} + (1-t)\frac{n-4}{2n}. \tag{46}$$

Using the Sobolev imbeddings

$$H^1(\Omega) \subset L^{2n/(n-2)}(\Omega) \quad \text{and} \quad H^2(\Omega) \subset L^{2n/(n-4)}(\Omega)$$

and (43) hence we deduce that

$$\|\Delta u\|_{q+1} \le c\|\Delta u\|^t_{H^1(\Omega)}\|\Delta u\|^{1-t}_{H^2(\Omega)} \le c\|\Delta u\|^t_{H^1(\Omega)} \le cE^{t/2}$$

and therefore the left-hand side of (45) may be estimated as follows :

$$\begin{aligned} cE^{\frac{p'-1}{2}}\|\Delta u\|_{q+1}(-E')^{\frac{1}{1+q^{-1}}} &\le cE^{\frac{p'-1+t}{2}}(-E')^{\frac{1}{1+q^{-1}}} \\ &\le \epsilon E^{\frac{(p'-1+t)(q+1)}{2}} - c(\epsilon)E'. \end{aligned} \tag{47}$$

Now let us choose

$$t := \max\Big\{\frac{p'+1}{q+1} + 1 - p', 0\Big\} = \max\Big\{\frac{2+q(1-p')}{q+1}, 0\Big\}. \tag{48}$$

Then

$$(p'-1+t)(q+1) \ge p'+1$$

and therefore (47) will imply (45) as soon as we show that t satisfies the condition (46).

The property $0 \le t \le 1$ is obvious. Indeed, we have $t \ge 0$ by definition and the inequality $t \le 1$ follows from the inequlity

$$2 + q(1-p') = q + 1 + (1 - qp') \le q + 1.$$

The second condition in (46) may also be written in the form

$$\frac{1}{q+1} \ge \frac{n-4}{2n} + \frac{t}{n}. \tag{49}$$

If $2+q(1-p') \le 0$, then $t = 0$ and (49) follows from (17). If $2+q(1-p') > 0$, then $t = \big(2+q(1-p')\big)/(q+1)$ and (49) is easily seen to be equivalent to

$$p' \ge \frac{n}{2}\Big(1 - \frac{1}{q}\Big) - 1$$

which is satisfied by (18).

Consider finally the case $n = 4$ ($q < +\infty$ in this case, too). Let us choose $t \in [0,1]$ again by (48) and choose $s > 2(q+1)$ arbitrarily. Then

$$\frac{1}{q+1} \geq \frac{1}{2(q+1)} + \frac{1}{2(q+1)} \geq \frac{t}{4} + \frac{1-t}{s}$$

whence

$$\|v\|_{q+1} \leq c\|v\|_4^t \|v\|_s^{1-t}.$$

Using the Sobolev imbeddings

$$H^1(\Omega) \subset L^4(\Omega) \quad \text{and} \quad H^2(\Omega) \subset L^s(\Omega)$$

hence we deduce that

$$\|\Delta u\|_{q+1} \leq c\|\Delta u\|^t_{H^1(\Omega)} \|\Delta u\|^{1-t}_{H^2(\Omega)} \leq c\|\Delta u\|^t_{H^1(\Omega)} \leq cE^{t/2}$$

and the proof of (45) may be finished as in the case $n > 4$ (cf. (47)). $\square$

Theorem 2 now follows from lemmas 1 and 9, by using the estimates (2), (4) with $\alpha = (p'-1)/2$ and $A = c$. $\square$

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