# Meditations on Eisenstein series

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Devadatta G. Hegde

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## In Gratitude to

Appa and Amma, for their fine upbringing under difficult circumstances.

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## Dedication

To Dharmik

For bringing so much joy to our family.

## Abstract

We determine the locations and the orders of the poles in the half-plane $\mathrm{Re}(s) \geq 0$ of unramified degenerate Eisenstein series attached to a maximal proper parabolic subgroup of a split semi-simple linear algebraic group over a number field.

# Contents

# 1 The historical context and introductory examples

In this chapter, we give an introduction to the spectral decomposition of automorphic forms, some historical context for the present work, and explain the origin of the main idea in this thesis.

## 1.1 Spectral decomposition of automorphic forms: the simplest case

### 1.1.1 The upper half plane

Let

$$\mathfrak{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$$

be the upper half plane. The group $G = SL_2(\mathbb{R})$ acts transitively on $\mathfrak{H}$ by fractional linear transformations:

$$G \curvearrowright \mathfrak{H} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

The stabilizer of $i \in \mathfrak{H}$ is

$$\mathrm{Stab}_G(i) = \mathrm{SO}_2(\mathbb{R}), \quad \mathfrak{H} \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$$

Let

$$d\mu = \frac{dxdy}{y^2} \quad \text{and} \quad \Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The measure $\mu$ and the Laplacian $\Delta$ are invariant under the action of $G$. The simplest case of the spectral decomposition of automorphic forms is the following:

**Problem.** Let $\Gamma = SL_2(\mathbb{Z})$. Decompose the space $L^2(\Gamma\backslash\mathfrak{H})$, or equivalently the space $L^2(\Gamma\backslash G)^K$ of right $K$-invariant functions in $L^2(\Gamma\backslash G)$, with respect to $\Delta$.

### 1.1.2 Discrete decomposition of cuspforms

We need the following subgroups of $G$:

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G \right\}, \ \Gamma_\infty = \Gamma \cap P = \left\{ \begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in G : x \in \mathbb{R} \right\}, \text{ and } \ A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in G : t > 0 \right\}.$$

For $f \in L^2(\Gamma\backslash G)$, the *constant term* of $f$ along $P$ is

$$c_P f(g) = \int_{N\cap\Gamma\backslash N} f(ng)dn$$

where the measure on $N$ is normalized so that the volume of the quotient $N \cap \Gamma\backslash N$ is 1. In the classical setting of $f \in L^2(\Gamma\backslash\mathfrak{H})$, we have

$$c_P f(z) = \int_0^1 f(x+iy)dx$$

The space

$$L^2_0(\Gamma\backslash G) := \left\{ f \in L^2(\Gamma\backslash G) : c_P f = 0 \quad \text{a.e} \right\}$$

of cuspforms is a *closed* subspace of $L^2(\Gamma\backslash G)$.

**Theorem.** *(Gelfand et al. [13, 15]) There exists an orthonormal basis of $\Delta$-eigenfunctions for $L^2_0(\Gamma\backslash G)^K$.*

*Remark.* This result does not say $\dim L^2_0(\Gamma\backslash G) \neq 0$. Existence of cuspforms is proved using other techniques.

### 1.1.3 Continuous spectrum I: an integral representation

Next we characterize $L^2_0(\Gamma\backslash G)^\perp$, the orthogonal complement to cuspforms in $L^2(\Gamma\backslash G)$, in terms of *pseudo-Eisenstein series*.

#### 1.1.3.1 Adjoint relation

For a test function $\varphi \in C_c^\infty(\Gamma_\infty N\backslash G)$, we have the adjoint relation

$$\langle c_P f, \varphi\rangle_{\Gamma_\infty N\backslash G} = \langle f, \Psi_\varphi^P\rangle_{\Gamma\backslash G}$$

where $\langle\cdot,\cdot\rangle_X$ is the usual hermitian inner product on square-integrable functions on a measure space $X$ and

$$\Psi_\varphi^P(g) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\gamma g).$$

The function $\Psi_\varphi^P$ is the *pseudo-Eisenstein series* made from the test function $\varphi$ and the sum is locally finite, that is, for $g$ in a fixed compact set, there are only finitely many non-zero terms in the sum.

#### 1.1.3.2 $L_0^2(\Gamma\backslash G)^\perp$

A function $f \in L^2(\Gamma\backslash G)$ is a cuspform if $c_P f = 0$ as a locally $L^1$ function. By the adjoint relation, the space $V$ orthogonal to cuspforms in $L^2(\Gamma\backslash G)$ is the $L^2$-closure of the space spanned by pseudo-Eisenstein series $\Psi_\varphi^P$ as $\varphi$ varies over $C_c^\infty(\Gamma_\infty N\backslash G)$.

#### 1.1.3.3 Mellin transform on $C_c^\infty(\Gamma_\infty N\backslash G)^K$

By the Iwasawa decomposition $G = NAK$, we write

$$g = n\cdot a(g)\cdot k, \quad n\in N,\ a(g)\in A,\ k\in K$$

where $a(g)$ is uniquely determined by $g$. Let $\delta_P : P \to (0,\infty)$ be the modular character of $P$. We can extend $\delta_P$ to a smooth function in $C^\infty(\Gamma_\infty N\backslash G)^K$. For

$$a_y := \begin{pmatrix} y^{\frac12} & 0 \\ 0 & y^{-\frac12}\end{pmatrix}, \quad (y>0)$$

we have $\delta_P(a_y) = y$.

Give measure 1 for $K$. For $\varphi \in C_c^\infty(\Gamma_\infty N\backslash G)^K$, we have

$$\int_{\Gamma_\infty N\backslash G} \varphi = \int_A \delta_P^{-1}(a)\varphi(a)da$$

Note that $(0,\infty) \overset{\log}{\simeq} \mathbb{R}$ and the Fourier transform on $\mathbb{R}$ becomes Mellin transform on $(0,\infty)$. For a test function $\varphi \in C_c^\infty(\Gamma_\infty N\backslash G)^K$, the *Mellin transform* $\mathcal{M}\varphi$ of $\varphi$ is

$$\mathcal{M}\varphi(s) = \int_0^\infty y^{-\frac{1}{2}-\frac{s}{2}}\varphi(a_y)\frac{dy}{y}, \quad s \in \mathbb{C}$$

The *Mellin inversion* is

$$\varphi(a_y) = \frac{1}{2\pi i}\int_{\Re(s)=0} \mathcal{M}\varphi(s) y^{\frac{1}{2}+\frac{s}{2}} ds$$

Since $\mathcal{M}\varphi$ is rapidly decreasing along the imaginary lines $\{\sigma + it : t \in \mathbb{R}\}$ for any $\sigma \in \mathbb{R}$, using Cauchy's theorem we write

$$\varphi(g) = \frac{1}{2\pi i}\int_{\Re(s)=\sigma} \mathcal{M}\varphi(s)\delta_P^{\frac{1}{2}+\frac{s}{2}}(g) ds, \quad \sigma \in \mathbb{R}$$

#### 1.1.3.4 The Eisenstein series

Let

$$E_s^P(g) = E_s(g) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \delta_P(\gamma\cdot g)^{\frac{1}{2}+\frac{s}{2}}$$

This sum converges for $\Re(s) > 1$ and has a meromorphic continuation to $\mathbb{C}$. Its constant term along $P$ is

$$c_P E_s(g) = \delta_P^{\frac{1}{2}+\frac{s}{2}}(g) + c(s)\cdot\delta_P^{\frac{1}{2}-\frac{s}{2}}(g)$$

where

$$c_s = c(s) = \frac{\xi(s)}{\xi(s+1)}, \ \xi : \text{completed zeta for } \mathbb{Z}$$

The function $E_s(g)$ satisfies the functional equation

$$E_s(g) = c_s E_{-s}(g).$$

#### 1.1.3.5 Integral representation for $\Psi_\varphi^P$

Let $\varphi \in C_c^\infty(\Gamma_\infty N\backslash G)^K$ be right $K$-invariant. Using the Mellin inversion above, we have

$$\Psi_\varphi^P(g) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(a(\gamma g))$$

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{1}{2\pi i} \int_{\Re(s)=0} \mathcal{M}\varphi(s) \delta_P(\gamma \cdot g)^{\frac{1}{2}+\frac{s}{2}} ds$$

For $\sigma := \Re(s) > 1$,

$$\Psi_\varphi^P(g) = \frac{1}{2\pi i} \int_{\Re(s)=\sigma} \mathcal{M}\varphi(s) \left( \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \delta_P(\gamma \cdot g)^{\frac{1}{2}+\frac{s}{2}} \right) ds$$

$$= \frac{1}{2\pi i} \int_{\Re(s)=\sigma} \mathcal{M}\varphi(s) \cdot E_s(g) ds$$

### 1.1.4 Continuous spectrum II: contour deformation

While the integral representation above expresses $\Psi_\varphi^P$ as an integral against the eigenfunctions $E_s(g)$, the coefficient

$$\mathcal{M}\varphi(s) \neq \langle \Psi_\varphi^P, E_s(g) \rangle$$

even up to a constant.

#### 1.1.4.1 Computing the coefficient

We have

$$\langle \Psi_\varphi^P, E_s \rangle = \langle \varphi, c_P E_s \rangle$$

$$= \langle \varphi, \delta_P^{\frac{1}{2}+\frac{s}{2}} + c(s) \cdot \delta_P^{\frac{1}{2}-\frac{s}{2}} \rangle$$

$$= \mathcal{M}\varphi(s) + c(s)\mathcal{M}\varphi(-s)$$

So we deform the contour to $\Re(s) = 0$ to take advantage of the functional equation of the $E_s(g)$.

#### 1.1.4.2 Justifying the contour deformation

Estimating $E_s(g)$ directly is difficult. Instead, for $\varphi, \psi \in C_c^\infty(\Gamma_\infty N \backslash G)^K$, we deform the integral given by the inner product

$$\left\langle \Psi_\varphi^P, \Psi_\psi^P \right\rangle = \frac{1}{2\pi i} \left\langle \int_{\Re(s)=\sigma} \mathcal{M}\varphi(s) \cdot E_s(g) ds, \Psi_\varphi^P \right\rangle$$

$$= \frac{1}{2\pi i} \int_{\Re(s)=\sigma} \mathcal{M}\varphi(s) \langle c_P E_s, \psi \rangle ds \quad \text{(adjoint relation)}$$

Using *Maass-Selberg relations* (see [5]), one can show

$$\left|c_{\sigma+it}\right| \ll_\sigma 1 \quad \text{for } \sigma > 0.$$

For $\sigma = 0$, $|c_{it}| = 1$. These estimates, together with the rapid decay of $\mathcal{M}\varphi$ along the vertical lines $\{\sigma+it : t \in \mathbb{R}\}$ for any fixed $\sigma$ justify the deformation of the contour over the line $\Re(s) = \sigma > 1$ to the line $\Re(s) = 0$. It is known that the poles of $E_s(g)$ in the region $\Re(s) \geq 0$ are all *simple* and are in in the interval $(0, 1]$. We obtain

$$\left\langle \Psi^P_\varphi, \Psi^P_\psi \right\rangle = \sum_{s_o} \mathcal{M}\varphi(s_0)\mathsf{res}_{s=s_0}\langle c_P E_s, \psi\rangle + \frac{1}{2\pi i}\int_{\Re(s)=0} \mathcal{M}\varphi(s)\langle c_P E_s, \psi\rangle ds$$

where the sum over $s_0$ is over the poles of $E_s(g)$ in the interval $(0, 1]$.

### 1.1.4.3 Residues of Eisenstein series

For the case $G = SL_2(\mathbb{R})$ and $\Gamma = SL_2(\mathbb{Z})$, the formula

$$c_P E_s(g) = \delta_P^{\frac{1}{2}+\frac{s}{2}}(g) + c(s)\cdot\delta_P^{\frac{1}{2}-\frac{s}{2}}(g), \quad c(s) = \frac{\xi(s)}{\xi(s+1)}$$

shows that in the region $\Re(s) \geq 0$, the function $c(s)$ has a simple pole at $s = 1$ and is holomorphic at other points. At $s = 1$, we have

$$\mathsf{res}_{s=1} c_P E_s(g) = \frac{1}{\xi(2)} = \text{constant}$$

One can actually show that

$$\mathsf{res}_{s=1} E_s(g) = \frac{1}{\xi(2)}$$

Note that

$$\int_A \delta_P^{-1}(a)\varphi(a)da = \mathcal{M}\varphi(1) = \int_{\Gamma_\infty N\backslash G} \varphi = \langle\varphi, 1\rangle$$

### 1.1.4.4 Conclusion

Using the functional equation of the Eisenstein series and the substitution $s \to -s$, we verify that

$$\int_{\Re(s)=0} \mathcal{M}\varphi(s)\langle c_P E_s, \psi\rangle ds$$

$$= \frac{1}{2}\int_{\Re(s)=0} \mathcal{M}\varphi(s)\langle c_P E_s, \psi\rangle ds + \frac{1}{2}\int_{\Re(s)=0} \mathcal{M}\varphi(-s)c(s)\langle c_P E_s, \psi\rangle ds$$

$$= \frac{1}{2}\int_{\Re(s)=0} \left(\mathcal{M}\varphi(s) + c(s)\mathcal{M}\varphi(-s)\right)\langle c_P E_s, \psi\rangle ds$$

$$= \frac{1}{2}\int_{\Re(s)=0} \langle \Psi_\varphi^P, E_s\rangle \overline{\langle \Psi_\psi^P, E_s\rangle} ds$$

*Remark.* Note that the Eisenstein series has poles from the critical zeros of $\xi(s)$ for $\Re(s) < 0$. Thus, one cannot use the substitution $s \to -s$ for integrals over $\Re(s) \neq 0$.

**Theorem 1.** *For $f \in C_c^\infty(\Gamma\backslash G)^K$, we have*

$$f(g) = \sum_F \langle f, F\rangle F(g) + \frac{\langle f, 1\rangle}{\xi(2)} 1 + \frac{1}{4\pi i}\int_{\Re(s)=0} \langle f, E_s\rangle E_s(g) \cdot ds$$

*where the sum is over an orthonormal basis of $\Delta$-eigenfunction cusp-forms.*

*Remark.* See the comments at the very end of the article [14] by Godement for some historical context.

## 1.2 A higher rank example

We discuss the simplest higher rank example of spectral decomposition of automorphic forms on $\mathrm{SL}_3(\mathbb{R})$ to the extent necessary to explain the origins of this thesis.

### 1.2.1 Spectral decomposition of spherical automorphic forms

Let $G = \mathrm{SL}_n(\mathbb{R})$, $K = \mathrm{SO}_n(\mathbb{R})$, and $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. We restrict our discussion to the space $L^2(\Gamma\backslash G)^K$ of right $K$-invariant (also called *spherical*) functions in $L^2(\Gamma\backslash G)$ where all the phenomena we wish to mention already occur. Let $\mathscr{H}_K := C_c^\infty(K\backslash G/K)$. It is a *commutative* $\mathbb{C}$-algebra (without identity) under convolution and $\varphi \in \mathscr{H}_K$ acts on $L^2(\Gamma\backslash G)^K$ by

$$(\varphi \cdot f)(g) = \int_G \varphi(h) f(gh) dh, \quad f \in L^2(\Gamma\backslash G)^K$$

The *eigenfunction expansion* of a function $f \in L^2(\Gamma\backslash G)^K$ in terms of simultaneous eigenfunctions of $\mathscr{H}_K$ is the problem of the spectral decomposition of spherical automorphic forms. The corresponding problem for the invariant Laplacian $\Omega$ on $G/K$ follows from that of $\mathscr{H}_K$.

### 1.2.2 Constant terms and cuspforms

The standard parabolic subgroups are parametrized by ordered partitions

$$\pi = (n_1, \dots, n_r), \quad n = n_1 + \cdots + n_r$$

of $n$, where $n_i > 0$ and not necessarily in decreasing order. The Borel subgroup $B$ of upper-triangular matrices in $G$ corresponds to $(1, 1, \cdots, 1)$. The parabolic subgroup $P_\pi$ consists of block-upper-triangular matrices in $G$ of size $n_1, \dots, n_r$ in that order. The standard Levi decomposition of $P_\pi$ is

$$P_\pi = N_\pi \rtimes M_\pi$$

where $M_\pi$ is the intersection with $P$ of

$$\mathrm{GL}_{n_1}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{R})$$

embedded diagonally in $\mathrm{GL}_n(\mathbb{R})$ and $N_\pi$ is the unipotent radical of $P_\pi$ given by the group of matrices which differ from the identity by a matrix with entries only above the diagonal blocks of $M_\pi$. If $P$ is understood, we write $N_P, M_P$ etc, instead of the subscript $\pi$.

Let $P$ be a standard parabolic subgroup. For a function $f$ on $\Gamma\backslash G$, the *constant term of $f$ along $P$* is

$$(c_P f)(g) := \int_{N_P \cap \Gamma \backslash N_P} f(ng) dn$$

It is a function on $N_P(M_P \cap \Gamma)\backslash G$. The space of *spherical cuspforms* on $\Gamma\backslash G$ is

$$L_0^2(\Gamma\backslash G)^K := \left\{ f \in L^2(\Gamma\backslash G)^K : c_P f = 0 \ \forall \text{ parabolic subgroups } P \neq G \right\}$$

Gelfand et al. [13, 12] proved the *discrete decomposition of cuspforms*:

$$L_0^2(\Gamma\backslash G)^K = \hat{\bigoplus_{\sigma : \mathscr{H}_K \to \mathbb{C}}} V_\sigma$$

where $\sigma$ ranges over characters of $\mathscr{H}_K$ (ring homomorphisms), the hat $\hat{}$ on the sum denotes completion, and $\mathscr{H}_K$ acts on the $\sigma$-isotypic component $V_\sigma$ by the character $\sigma$. Further, each $V_\sigma$ is finite dimensional.

Note that the discrete spectrum of Laplacian acting on $L^2(\Gamma\backslash G)^K$ is *larger* than $L_0^2(\Gamma\backslash G)^K$. For example, the constant functions are in the discrete spectrum of $L^2(\Gamma\backslash G)^K$ but not in $L_0^2(\Gamma\backslash G)^K$. There are other functions as well (see Moeglin-Waldspurger [35]). The corresponding continuous spectrum has been completely characterized, in terms of the discrete spectrum of spaces of lower dimension, using Eisenstein series.

### 1.2.3 Eisenstein series and spectral decomposition

Fix a standard parabolic subgroup $P$ of $G$ corresponding to the partition $\pi = (n_1, \dots, n_r)$. Let

$$\mathfrak{a}_P = \left\{ (u_1, \dots, u_r) \in \mathbb{R}^r : \sum_{i=1}^{r} u_i = 0 \right\}$$

Any $g \in G$ can be decomposed as

$$g = nmk, \quad n \in N_P, \ k \in K,$$

and

$$m = m_1 \cdots m_r \in M_P, \quad m_i \in \mathrm{GL}_{n_i}(\mathbb{R}).$$

This decomposition is not unique, but the vector

$$H_P(g) = (\log|\det m_1|, \dots, \log|\det m_r|) \in \mathfrak{a}_P$$

is uniquely determined. Let $A_P$ be the subgroup of elements $m \in M_P$ such that each $m_i$ is a positive multiple of the identity matrix and let $M_P^1 \subset M_P$ be the subgroup

$$M_P^1 := \{ m \in M_P : H_P(m) = 0 \}$$

Then

$$M_P = M_P^1 \times A_P \quad \text{(direct product)}$$

We can write

$$g = nmak, \quad n \in N_P, \ m \in M_P^1, \ a \in A_P, \ k \in K$$

where $a$ is uniquely determined by $g$.

Let $K_P := M_P^1 \cap K$. One can define an *abelian* convolution algebra $\mathscr{H}_{K_P}$ exactly as before. Given an *eigenfunction* $\phi \in L^2(\Gamma \cap M_P \backslash M_P^1)^{K_P}$ of $\mathscr{H}_{K_P}$, one can associate an Eisenstein series $E^P_{\Lambda,\phi}(g)$ attached to $\phi$. Put

$$\phi_P(g) = \phi(m), \quad g = nmak \text{ as above.}$$

Let $\Lambda \in \mathfrak{a}_P^* \otimes \mathbb{C}$ and $\rho_P \in \mathfrak{a}_P^*$. The function $E^P_{\Lambda,\phi}(g)$ is given by the sum

$$E^P_{\Lambda,\phi}(g) = \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} \phi_P(\gamma g) \exp\langle \rho_P + \Lambda, H_P(\gamma g)\rangle$$

The sum converges for $\Lambda$ in some non-empty open set in $\mathfrak{a}_P^* \otimes \mathbb{C}$ and has a meromorphic continuation to $\mathfrak{a}_P^* \otimes \mathbb{C}$; see for example [31],[2]. The parameter $\rho_P$ is chosen to simplify the formulas for the functional equations of $E^P_{\Lambda,\phi}(g)$.

If $\phi$ is a *cuspform* on $\Gamma \cap M_P \backslash M_P^1$, then $E^P_{\Lambda,\phi}$ is an example of a *cuspidal* Eisenstein series [38]. If $\phi$ is a *non-cuspidal* eigenfunction in $L^2(\Gamma\backslash G)^K$, such as the constant function 1 or other *Speh forms* classified by Moeglin and Waldspurger [35], then $E^P_{\Lambda,\phi}$ is an example of a *non-cuspidal* Eisenstein series appearing in the spectral decomposition [31, 36]. The simplest example is when $\phi = 1$, which we call the *unramified degenerate Eisenstein series.*

In this thesis, we determine the poles of unramified degenerate Eisenstein series relevant for spectral decomposition. We restrict our attention to that part of the spectrum where these Eisenstein series appear, namely the unramified Borel part discussed next.

### 1.2.4 The unramified Borel part of the spectrum

Let $B$ be the Borel subgroup of upper triangular matrices in $G$ with the standard Levi decomposition $N \rtimes M$. For $\varphi \in C_c^\infty\left((\Gamma \cap B)N\backslash G\right)^K$, let

$$\Psi_\varphi^B(g) = \sum_{\gamma \in \Gamma \cap B \backslash \Gamma} \varphi(\gamma g)$$

The sum if locally finite. Let

$$V = \text{closure}\left(\text{span}\left\{\Psi_\varphi^B(g) : \varphi \in C_c^\infty((\Gamma \cap B)N\backslash G)^K\right\}\right)$$

It is a closed subspace of $L^2(\Gamma\backslash G)^K$ stable under the $\mathcal{H}_K$-action. We discuss the spectral decomposition of $V$ with respect to the Hecke algebra $\mathcal{H}_K$. This case resembles the $SL_2$ case to some extent.

#### 1.2.4.1 Mellin transform

For the remainder of this section, let

$$a(g)^{\Lambda} := e^{\langle \Lambda, H_B(g) \rangle}$$

for notational simplicity. For $\Lambda \in \mathfrak{a}_P^* \otimes \mathbb{C}$, the *Mellin transform* of $\varphi \in C_c^{\infty}\left((\Gamma \cap B)N \backslash G\right)^K$ is

$$\widehat{\varphi}(\Lambda) = \int_A \delta_B^{-\frac{1}{2}}(a) a(g)^{-\Lambda} \varphi(a) da$$

The function $\widehat{\varphi}(\Lambda)$ is a *Paley-Wiener* function on $\mathfrak{a}_B^* \otimes \mathbb{C}$. The *Mellin inversion formula* is

$$\varphi(g) = \frac{1}{(2\pi i)^{\dim \mathfrak{a}_B}} \int_{\sigma + i\mathfrak{a}_B^*} \widehat{\varphi}(\Lambda) \delta_B(g)^{\frac{1}{2}} a(g)^{\Lambda} d\Lambda, \quad \sigma \in \mathfrak{a}_B^*$$

#### 1.2.4.2 The contour integral

As in the $SL_2(\mathbb{R})$ case, we get the contour integral

$$\left\langle \Psi_{\varphi}^B(g), \Psi_{\psi}^B(g) \right\rangle = \frac{1}{(2\pi i)^{n-1}} \int_{\sigma + i\mathfrak{a}_B^*} \widehat{\varphi}(\Lambda) \left\langle c_B E_{\Lambda}^B(g), \psi \right\rangle d\Lambda$$

The vector $\sigma \in \mathfrak{a}_B^*$ is initially in the convergence region of the Eisenstein series and we need to deform the integral to $\sigma = 0$. The following formula is due to Gelfand et al. [12] (page 82):

$$c_B E_{\Lambda}^B(g) = \sum_{w \in W} c_{w,\Lambda} a(g)^{\rho_B + w \cdot \Lambda}$$

where

$$c_{w,\Lambda} = \prod_{\substack{\alpha \in \Phi^+: \\ w \cdot \alpha < 0}} \frac{\xi(\langle \Lambda, \alpha^{\vee} \rangle)}{\xi(1 + \langle \Lambda, \alpha^{\vee} \rangle)},$$

$\xi$ is the completed Riemann zeta function, $\Phi^+$ is the set of positive roots of $\mathrm{SL}_n$, and $\alpha^{\vee} \in \mathfrak{a}_P$ is the coroot of $\alpha \in \Phi^+$.

### 1.2.5 The $\mathrm{SL}_3$ example

In this case, we can illustrate the contour deformation with the diagram (due to Bill Casselman) 1.2.1. We use the notation in the figure for the following discussion. For a parabolic subgroup $P \neq B, G$, the unramified degenerate Eisenstein series is

$$E_{\Lambda}^P(g) = \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} \exp \langle \rho_P + \Lambda, H_P(\gamma g) \rangle$$

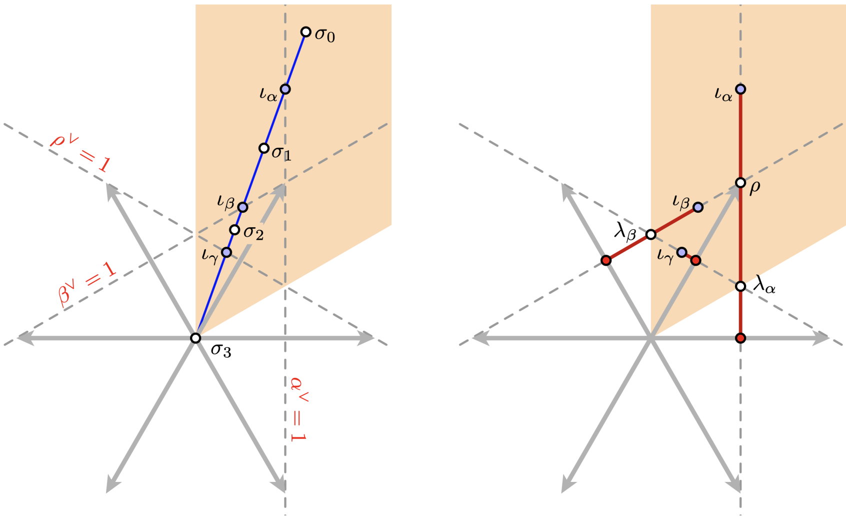



Figure 1.2.1: Contour deformation for $SL_3$ (due to Bill Casselman)

#### 1.2.5.1 Contour deformation: initial remarks

The deformation of the contour integral

$$I_\sigma = \left\langle \Psi_\varphi^B(g), \Psi_\psi^B(g) \right\rangle = \frac{1}{(2\pi i)^2} \int_{\sigma + i\mathfrak{a}_B^*} \widehat{\varphi}(\Lambda) \left\langle c_B E_\Lambda^B(g), \psi \right\rangle d\Lambda$$

happens in three stages.

- First, one deforms the *two-dimensional* integral $I_\sigma$ from $\sigma = \sigma_0$ to the origin $\sigma_3$ (the left-half of figure 1.2.1). We pick up residues along the three (affine) hyperplanes marked $\alpha^\vee = 1, \beta^\vee = 1$, and $\rho^\vee = 1$. The integral at the origin contributes a two dimensional spectrum.

- The three *one-dimensional* residues contribute to the one dimensional spectrum. The residues of $E_\Lambda^B$ along the hyperplanes labelled $\alpha^\vee = 1, \beta^\vee = 1$ in figure 1.2.1 give the degenerate Eisenstein series corresponding to the partitions $(2,1)$ and $(1,2)$ respectively. It is very interesting to see how the residue along $\rho^\vee = 1$ participates, but we will not consider this here. We need to deform the integrals over $\iota_\alpha, \iota_\beta$, and $\iota_\gamma$ as shown in the right-half of figure 1.2.1.

- Finally, one gets the *discrete spectrum* at $\rho, \lambda_\alpha, \lambda_\beta$.

#### 1.2.5.2 Key insight: a miraculous cancellation.

We note the most remarkable cancellation. Even though the Eisenstein series $E_\Lambda^B$ has poles along the three hyperplanes (marked $\alpha^\vee = 1$,$\beta^\vee = 1$, $\rho^\vee = 1$), the residue at $\lambda_\alpha, \lambda_\beta$ in the figure 1.2.1 is zero! In earlier literature, for example Labesse's [27] discussion of Langlands's work, these were explained using local computations.

To clarify the situation, we abstract the main feature and operate heuristically. Let $f(z, w)$ be a meromorphic function $\mathbb{C}^2$ such that its only singularities are along the hyperplanes $z = a$ and $w = b$. Further assume that these singularities are simple, that is, $(z - a)(w - b) \cdot f(z, w)$ extends to an entire function on $\mathbb{C}^2$.

**Problem.** How can it happen that $\mathsf{res}_{w=b} f(z, w)$ has no singularity at the $(a, b)$ on the affine hyperplane $w = b$?

**Solution.** We do our computations in a heuristic fashion to see the phenomena. Let

$$g(z, w) = (z - a)(w - b) \cdot f(z, w)$$

Then $g$ is entire on $\mathbb{C}^2$. We have,

$$\mathsf{res}_{w=b} f(z, w) = \mathsf{res}_{w=b} \frac{g(z, w)}{(z - a)(w - b)} = \frac{g(z, b)}{z - a}$$

If this has no singularity at $z = a$, then $g(a, b)$ must be zero! That is, a zero of $f(z, w)$ passes through $(a, b)$ which makes the singularity disappear.

#### 1.2.5.3 Illustration for $\mathrm{SL}_3$

Let $\alpha, \beta, \gamma$ be the positive roots of $\mathrm{SL}_3$ with coroots $\alpha^\vee, \beta^\vee, \gamma^\vee \in \mathfrak{a}_P$. Explicitly, $\alpha^\vee = (1, -1, 0)$, $\beta^\vee = (0, 1, -1)$, and $\gamma^\vee = (1, 0, -1)$. For a coroot $\delta^\vee$, let

$$s_\delta := \langle \Lambda, \delta^\vee \rangle \in \mathbb{C}$$

In our example, the singularity at the point

$$s_\alpha = 1, \;\; s_\gamma = 1$$

does not contribute to the spectrum. Note that the above conditions imply that

$$s_\beta = s_\gamma - s_\alpha = 0$$

The key to zeros of Eisenstein series is the observation that the $SL_2$ Eisenstein series $E_s(g)$ vanishes at $s = 0$:

$$\lim_{s\to 0} c_B E_s(g) = \lim_{s\to 0}\left(\delta_B^{\frac{1}{2}+\frac{s}{2}}(g) + c(s)\delta_B^{\frac{1}{2}-\frac{s}{2}}(g)\right) = 0$$

since

$$\lim_{s\to 0} c(s) = \lim_{s\to 0}\frac{\xi(s)}{\xi(1+s)} = \frac{\mathsf{res}_{s=0}\xi(s)}{\mathsf{res}_{s=1}\xi(s)} = \frac{-1}{1} = -1.$$

Thus,

$$\lim_{s\to 0} E_s(g) = 0.$$

One can show using this observation, that $E^B_\Lambda$ vanishes along

$$H_\delta = \{\Lambda \in \mathfrak{a}_P^* \otimes \mathbb{C} : \langle \Lambda, \delta^\vee\rangle = s_\delta = 0\}$$

for all positive roots $\delta$.

For a positive root $\delta$, let

$$S_\delta = \{\Lambda \in \mathfrak{a}_P^* \otimes \mathbb{C} : s_\delta = 1\}$$

The relevant zeros and poles of $E^B_\Lambda$ are captured by

$$F(\Lambda) := \frac{s_\alpha}{s_\alpha - 1}\cdot\frac{s_\beta}{s_\beta - 1}\cdot\frac{s_\alpha + s_\beta}{s_\alpha + s_\beta - 1}$$

$$\begin{aligned}\mathsf{Res}_{s_\alpha=1}F(\Lambda) &= (s_\alpha - 1)\frac{s_\alpha}{s_\alpha - 1}\cdot\frac{s_\beta}{s_\beta - 1}\cdot\frac{s_\alpha + s_\beta}{s_\alpha + s_\beta - 1}\Bigg|_{S_\alpha}\\ &= \frac{\cancel{s_\beta}}{s_\beta - 1}\cdot\frac{1 + s_\beta}{\cancel{s_\beta}}\Bigg|_{S_\alpha} = \frac{1 + s_\beta}{s_\beta - 1}\Bigg|_{S_\alpha}\end{aligned}$$

The cancellation above explains why the degenerate Eisenstein series obtained by taking residue along $S_\alpha$ ($\alpha$ a simple root) is *holomorphic* at the point $S_\alpha \cap S_\gamma$, where $\gamma$ is the non-simple positive root.

### 1.2.6 The result for $\mathrm{SL}_n$

Since the general case requires considerable notation from Arthur [1], we first state the result of this thesis for the simplest case of maximal parabolic subgroups of $\mathrm{SL}_n(\mathbb{R})$.

Let $G = \mathrm{SL}_n(\mathbb{R})$ and $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. Let $P$ be the maximal parabolic subgroup of $G$ corresponding to the partition $n = (a, b)$

with $n = a + b$ and $\delta_P$ be the modular function of $P$ extended to $G$ using the Iwasawa decomposition $G = PK$. The corresponding unramified degenerate Eisenstein series is

$$E_s^P(g) = \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} \delta_P^{\frac{1}{2}+\frac{s}{n}}(\gamma g), \quad \Re(s) > \frac{n}{2}$$

The series $E_s^P(g)$ converges absolutely for $\Re(s) > \frac{n}{2}$ and has a meromorphic continuation to $\mathbb{C}$. Let $P = N \rtimes M$ be the standard Levi decomposition of $P$. We denote the Langlands dual groups by left-superscript $L$. The principal $SL_2$ in ${}^L M$ acts on ${}^L\mathfrak{n} = \mathrm{Lie}({}^L N)$ and let

$${}^L\mathfrak{n} = \bigoplus_{i \geq 0} V_i^{\oplus m_i}, \quad V_k = \mathrm{Sym}^k(\mathrm{std})$$

be the decomposition of ${}^L\mathfrak{n}$ into irreducibles where $m_i \in \mathbb{N}$ is the multiplicity of $V_i$.

**Theorem.** *In the region $\Re(s) \geq 0$, the function $E_s^P(g)$ is holomorphic except for a pole of order $m_i$ at $1 + i/2$ where $i \in \mathbb{N}$.*

*Remark.* (a) Explicit calculations show that in the region $\Re(s) \geq 0$, the function $E_s^P$ is holomorphic except for simple poles at

$$s \in \left\{\frac{n}{2}, \frac{n}{2} - 1, \ldots, \frac{n}{2} - (\min\{a, b\} - 1)\right\}$$

This was first shown by Hanzer and Muic [19] by a detailed study of $c_B E_s^P(g)$ and observing cancellation among a sum of intertwining operators.

(b) For $\Re(s) < 0$, the poles are related to the location of the *critical zeros* of zeta functions and are presently beyond reach. Fortunately, these poles do not play any role in the spectral decomposition of automorphic forms.

## 1.3 Some historical context

Let $\mathfrak{H}$ be the usual upper half-plane and $z = x + iy \in \mathfrak{H}$. The simplest example of Eisenstein series is

$$E_s(z) = \frac{1}{2} \sum_{c,d \text{ coprime}} \frac{y^{\frac{1}{2}+\frac{s}{2}}}{|cz + d|^{s+1}}, \quad \mathrm{Re}(s) > 1$$

where the sum is over all integer pairs $(c, d) \in \mathbb{Z}^2$ with $\gcd(c, d) = 1$. The sum converges for $\mathrm{Re}(s) > 1$. The map $s \mapsto E_s$ has a meromorphic continuation to $\mathbb{C}$ as a vector-valued map taking values in the space of smooth functions of uniform moderate growth on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ (see Bernstein-Lapid [2]).

The constant term of $E_s$ is

$$\int_0^1 E_s(x + iy)dx = y^{\frac{1}{2}+\frac{s}{2}} + c(s)y^{\frac{1}{2}-\frac{s}{2}},$$

where

$$c(s) = \frac{\xi(s)}{\xi(1+s)}, \quad \xi(s) = \pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s)$$

The Eisenstein series $E_s$ satisfies the functional equation

$$E_s = c(s)E_{-s}$$

To motivate the objects appearing in the main result 3 of this thesis and to explain the complementary nature of this result to those of Langlands, we recall Langlands's landmark computation of the analogues of function $c(s)$ above appearing in the functional equation of maximal parabolic *cuspidal* Eisenstein series [30]. We follow Casselman's account [6] and some unpublished notes of Erez Lapid. Now we use the standard notation in the subject due to Arthur [1], recalled in section 2.1.

Let $G$ be a split reductive group over $F = \mathbb{Q}$ (for simplicity). Let $T \subset B \subset G$ be a maximal $F$-split torus $T$ contained in a Borel subgroup $B$ defined over $F$. A *standard* parabolic subgroup is an $F$-parabolic subgroup of $G$ containing $B$.

*Remark* 2. As is common in the subject, in what follows, a parabolic subgroup is always understood to be a *proper standard* parabolic subgroup unless explicitly cautioned otherwise. For example, a maximal parabolic subgroup means a standard maximal proper parabolic subgroup. We refer to the standard Levi decomposition of a parabolic subgroup as *the* Levi decomposition (see I.1.4 of Moeglin-Waldspurger [36]).

Let $\mathbb{A}$ be the adele ring of $F$. Let $X(G)$ be the lattice of $F$-characters of $G$ and

$$G(\mathbb{A})^1 = \bigcap_{\chi \in X(G)} \ker|\chi|$$

Let $\mathbf{K}$ be the standard maximal compact subgroup of $G(\mathbb{A})$ (see I.1.4 of Moeglin-Waldspurger [36]).

### 1.3.1 Constant terms and cuspforms

For a function $f$ on $G(F)\backslash G(\mathbb{A})$ and a parabolic subgroup $P$ with the unipotent radical $N$, the *constant term of $f$ along $P$* is

$$c_P f(g) := \int_{N(F)\backslash N(\mathbb{A})} f(ng)dn$$

The space of *cuspforms* is

$$L_0^2(G(F)\backslash G(\mathbb{A})^1)$$

$$= \{f \in L^2(G(F)\backslash G(\mathbb{A})^1) : c_P f = 0 \ \forall \text{ parabolic subgroups } P \neq G\}$$

The space $\mathcal{H}_{\mathbf{K}} := C_c^\infty(G(\mathbb{A})//\mathbf{K})$ of $\mathbf{K}$-bi-invariant test functions on $G(\mathbb{A})$ is a *commutative* algebra under convolution. A basic result is that the space $L_0^2(G(F)\backslash G(\mathbb{A})^1)^{\mathbf{K}}$ of $\mathbf{K}$-invariant cuspforms decomposes discretely with respect to $\mathcal{H}_{\mathbf{K}}$:

$$L_0^2(G(F)\backslash G(\mathbb{A})^1)^{\mathbf{K}} = \widehat{\bigoplus_{\chi:\mathcal{H}_{\mathbf{K}}\to\mathbb{C}}} V_\chi$$

where hat $\hat{}$ on the sum denotes completion and $\mathcal{H}_{\mathbf{K}}$ acts on $V_\chi$ by $\chi$. We note that $\dim V_\chi < \infty$ for each algebra homomorphism $\chi : \mathcal{H}_{\mathbf{K}} \to \mathbb{C}$. We refer to a *non-zero* eigenfunction of $\mathcal{H}_{\mathbf{K}}$ in $L_0^2(G(F)\backslash G(\mathbb{A})^1)^{\mathbf{K}}$ as a *strong sense cuspform* on $G(\mathbb{A})$.

### 1.3.2 Maximal parabolic cuspidal Eisenstein series

For the rest of this introduction, we assume that $G$ is *semi-simple* and $P = N \rtimes M$ is the Levi decomposition of a *maximal* parabolic subgroup $P$ of $G$ (see remark 2). Let $\delta_P : G(\mathbb{A}) \to (0, \infty)$ be the extension to $G(\mathbb{A})$ of the modular character on $P(\mathbb{A})$ using the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})\mathbf{K}$. Let $\varpi$ be the fundamental weight corresponding to $P$. We parameterize the space $\mathfrak{a}_P^* \otimes \mathbb{C}$ by $s\varpi$ for $s \in \mathbb{C}$.

Let $\varphi$ be a strong sense cuspform on $M(\mathbb{A})$. We denote the usual extension of $\varphi$ to $G(\mathbb{A})$ using the Levi-Langlands decomposition

$$\varphi(g) = \varphi(m), \ g = nmak \in N(\mathbb{A})M(\mathbb{A})^1 A_M^+ \mathbf{K}$$

also by the same letter $\varphi$. Let

$$\varphi_s = \delta_P^{\frac{1}{2}} \cdot \varphi \cdot e^{\langle s\varpi, H_P(g)\rangle} \in C^\infty(G(\mathbb{A}))$$

The *Eisenstein series* made from *cuspidal data* $\varphi$ is

$$E^P(s,\varphi,g) := \sum_{\gamma\in P(F)\backslash G(F)} \varphi_s(\gamma g)$$

The sum converges for $\mathrm{Re}(s) \gg 0$. It has a meromorphic continuation to $\mathbb{C}$ [2].

Let $\chi : \mathcal{H}_{\mathbf{K}\cap M(\mathbb{A})} \to \mathbb{C}$ be the character corresponding to $\varphi$ and let $V_\chi$ be the corresponding eigenspace. Further assume that the closure of the space generated by $\varphi$ is an *irreducible* cuspidal automorphic representation $\pi$ of $M(\mathbb{A})$. The Eisenstein series $E^P(s,\varphi,g)$ satisfies a functional equation

$$E^P(s,\varphi,g) = c(s,\pi)E^{\overline{P}}(-s,\varphi',g) \tag{1.3.1}$$

for some $\varphi' \in V_\chi$, where $\overline{P} = \overline{N} \rtimes M$ is the parabolic (non-standard) opposite to $P$. Langlands computed $c(s,\pi)$ as a ratio of products of $L$-functions in [30].

### 1.3.3 Langlands's computation

In the Langlands dual group ${}^LG$, there is a corresponding parabolic subgroup ${}^LP = {}^LN \rtimes {}^LM$ with Levi ${}^LM$ and the unipotent radical ${}^LN$. The maximal torus $A$ in the *center* of ${}^LM$ is one dimensional since ${}^LG$ is semi-simple and ${}^LP$ is a maximal parabolic subgroup of ${}^LG$. Consider the eigenspace decomposition of ${}^L\mathfrak{n}$ under the adjoint action of $A$:

$${}^L\mathfrak{n} := \mathrm{Lie}({}^LN) = r_1 \oplus \cdots \oplus r_m$$

where

$$r_j = \bigoplus_{\alpha^\vee : \langle\varpi,\alpha^\vee\rangle = j} {}^L\mathfrak{g}_{\alpha^\vee} \quad j \geq 1$$

and the sum is over coroots $\alpha^\vee$ of $G$ for which $\langle\varpi,\alpha^\vee\rangle = j$, that is if $\beta^\vee$ be the simple coroot corresponding to ${}^LP$, then

$$\alpha^\vee = \cdots + j\beta^\vee + \cdots$$

when $\alpha^\vee$ is written as a sum of simple coroots. By a theorem of Shahidi [38], each $r_i$ is an irreducible representation of ${}^LM$.

Langlands [30] expressed $c(s,\pi)$ in equation 1.3.1 as a product of ratios of $L$-functions:

$$c(s,\pi)=\prod_{j=1}^{m}\frac{L(js,\pi,r_j)}{L(1+js,\pi,r_j)}$$

The largest possible $m$ occurs for the maximal parabolic subgroup corresponding to the node with three neighbors in the Dynkin diagram for $E_8$ when $m=6$. An illustrative example is when $G=\mathrm{Sp}_{2n}$ and $P=N\rtimes M$ is the Siegel parabolic subgroup with Levi $M\simeq GL_n$. In this case

$$c(s,\pi)=\frac{L(s,\pi)}{L(1+s,\pi)}\cdot\frac{L(2s,\pi,\wedge^2)}{L(1+2s,\pi,\wedge^2)},$$

the Eisenstein series $E^P(g,\varphi,s)$ converges for $\Re(s)>\frac{n+1}{2}$, and has a pole at $s=\frac{1}{2}$ if $L(s,\pi,\wedge^2)$ has a pole at $s=1$ *and* $L(1/2,\pi)\neq 0$ (see [32] for a further discussion of this example).

The case $G=G_2$ provided the extremely striking example of symmetric cube $L$-functions attached to modular forms on the upper half-plane, obtained *without* recourse to Fourier coefficients. This computation was a turning point in the theory of automorphic forms. The $L$-functions appearing in these formulas for $c(s,\pi)$ have been thoroughly investigated by Shahidi [39][38].

### 1.3.4 Main theorem

Let $P=N\rtimes M$ be a maximal parabolic subgroup of $G$. Take $\varphi=1$ and let

$$\varphi_s(g)=\delta_P^{\frac{1}{2}}\cdot\varphi\cdot e^{\langle s\varpi,H_P(g)\rangle}=\delta_P^{\frac{1}{2}}\cdot e^{\langle s\varpi,H_P(g)\rangle}$$

and

$$E^P(s,g):=\sum_{\gamma\in P(F)\backslash G(F)}\varphi_s(\gamma g)$$

This is the simplest example of a *non-cuspidal* Eisenstein series, since $\varphi=1$ is *not* a cuspform on $M(\mathbb{A})$, called the *unramified degenerate Eisenstein series attached to* $P$. The map $s\mapsto E(s,\bullet)$ initially convergent for $\mathrm{Re}(s)\gg 0$ has a meromorphic continuation to $\mathbb{C}$ as a vector-valued function with values in the space of smooth functions of uniform moderate growth on $G(F)\backslash G(\mathbb{A})$ (see Bernstein-Lapid [2]).

In this paper, we obtain a polynomial $p\in\mathbb{C}[s]$ given in terms of the structure of $P$, whose zeros capture the locations and the orders of the poles of $E^P(s,g)$ in the region $\mathrm{Re}(s)\geq 0$. Let $V_r$ be the

$(r+1)$-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ given by the $r$-th symmetric power of the standard representation. The notion of *principal* $\mathfrak{sl}_2\mathbb{C}$ for the Lie algebra of a split *reductive* group (with a fixed pinning) is defined in 2.3.2 following Gross [16].

**Theorem 3.** *Let $G$ be a split semi-simple linear algebraic group over a number field and $P = N \rtimes M$ be the standard Levi decomposition of a standard maximal parabolic subgroup $P$. Let*

$$^L\mathfrak{n} = r_1 \oplus r_2 \oplus \cdots \oplus r_m$$

*where $r_1, \ldots, r_m$ are the irreducible constituents of the adjoint representation of $^LM$ on $^L\mathfrak{n}$ as described in 1.3.3. Let*

$$r_j \simeq \bigoplus_{\ell \geq 0} V_\ell^{m_\ell(j)}, \quad V_k = sym^k(std)$$

*be the decomposition of $r_j$ into irreducible constituents under the action of the* ***principal*** $\mathfrak{sl}_2\mathbb{C} \subset {}^L\mathfrak{m}$. *Let*

$$p(s) = \prod_{j=1}^{m} \prod_{\ell \geq 0} (js - 1 - \ell/2)^{m_\ell(j)} \in \mathbb{C}[s]$$

*In the region $\mathrm{Re}(s) \geq 0$, $p(s) \cdot E^P(s,g)$ is holomorphic. If $s = s_0 > 0$ is a zero of $p(s)$ of order $d$, then*

$$F_{-d}(g) := \lim_{s \to s_o} (s - s_0)^d E^P(s,g)$$

*is a non-zero function on $G(\mathbb{A})$.*

*In other words, the order of the zeros of $p(s)$ is the same as the order of the poles of $E^P(s,g)$ in the region $Re(s) \geq 0$.*

We give a simple example to illustrate the theorem. See Fulton-Harris [10], chapter 11, for a method to decompose a representation of $\mathfrak{sl}_2\mathbb{C}$ (abstractly) into irreducible constituents.

**Example 4.** Let $G = PGL_{n+1}$ $(n \geq 2)$ and $P = N \rtimes M$ be the maximal parabolic subgroup corresponding to the ordered partition $(a+1, b+1)$ of $n+1$ with $a+b = n-1$ so that the derived group of the Levi subgroup $M$ is of type $A_a \times A_b$ (Dynkin diagram notation). The dual group $^LG = \mathrm{SL}_{n+1}(\mathbb{C})$. The principal $\mathfrak{sl}_2\mathbb{C} \subset {}^L\mathfrak{m}$ is given by the $\mathfrak{sl}_2$-triple $\{H, X, Y\}$ where the neutral element is

$$H = \mathrm{diag}(a, a-2, \ldots, -a, b, b-2, \ldots, -b)$$

We may identify ${}^L\mathfrak{n} = r_1$ with $(a+1)\times(b+1)$-matrices. We write $H$-eigenvalues of the corresponding coroot vectors in the matrix

$$\begin{bmatrix} a-b & (a-b)-2 & \cdots & (a+b)-2 & (a+b) \\ (a-b)-2 & & & & (a+b)-2 \\ \vdots & & \iddots & & \vdots \\ 2-(a+b) & & & & (b-a)+2 \\ -(a+b) & 2-(a+b) & \cdots & (b-a)-2 & b-a \end{bmatrix}$$

Then

$$r_1 \simeq \bigoplus_{k=a+b-2\min\{a,b\}}^{k=a+b} V_k, \quad k \text{ increments of } 2$$

Note that $a+b=n-1$. The Eisenstein series $E^P(s,g)$ converges for $\mathrm{Re}(s) > \frac{n+1}{2}$ and has simple poles at

$$\frac{n+1}{2}, \frac{n+1}{2}-1, \ldots, \frac{n+1}{2}-\min\{a,b\}$$

in the region $\mathrm{Re}(s) > 0$.

*Remark.* This was first shown by Hanzer and Muic [19] by a detailed study of $c_B E_s^P(g)$ and observing cancellation among a sum of intertwining operators. See also the work of Koecher [23].

### 1.3.5 Applications of the poles of degenerate Eisenstein series

Degenerate Eisenstein series (including the ramified ones) and their poles are of interest for several applications. We mention some of them to indicate some of the previous work on the poles of unramified degenerate Eisenstein series.

#### 1.3.5.1 Siegel-Weil formula

The classical Siegel-Weil formula [42] identifies the integral of a certain theta series as the special value of an Eisenstein series. The automorphic forms appearing in the Laurent expansion at these poles of degenerate Eisenstein series play a central role in the regularized versions of the Siegel-Weil formula (see [25], [11]). Kudla and Rallis determined the poles of Siegel parabolic degenerate Eisenstein series. For an important recent work, see Halawi and Segal [17].

#### 1.3.5.2 Integral representation for $L$-functions

Degenerate Eisenstein series are used to obtain integral representations of some automorphic $L$-functions. The information about the poles can be used to obtain results about the poles of these automorphic $L$-functions (see [7], [26]).

#### 1.3.5.3 Arthur conjecture

Let $G$ be a split simple adjoint group $G$ over a number field. The maximal parabolic unramified degenerate Eisenstein series are functions of one complex variable and the leading term of the Laurent expansion at *some* of these poles is square-integrable. These are used to obtain unitary representations of the adele group $G(\mathbb{A})$ and the local constituents of these representations are unitary. This method was used by Miller [33] to verify Arthur's conjecture that the spherical constituents of principal series representations at certain points of reducibility are unitary.

#### 1.3.5.4 Spectral decomposition

Poles of unramified degenerate Eisenstein series play a central role in understanding Langlands' work [31] on the spectral decomposition of automorphic forms. In this work, Langlands notes that "A number of unexpected and unwanted complications must be taken into account..." One such complication is the *cancellation of residues during the contour deformation*. In particular, the case of $G_2$ was first obtained by Langlands in appendix III of [31]. A first step in understanding this difficult work is to determine the poles of the unramified degenerate Eisenstein series, which we do in this paper. For some recent work on spectral decomposition, see [8][22].

# 2 Poles of unramified degenerate Eisenstein series

In this chapter we prove the main result of this thesis.

## 2.1 Unramified degenerate Eisenstein series

In this section we recall the standard notation due to Arthur [1]. Even though we only deal with maximal parabolic subgroups, it is clarifying to introduce the algebraic preliminaries for the non-maximal cases.

Let $F$ be a number field, $\mathbb{A}$ be the ring of adeles of $F$, and $|\cdot|$ denote the adele norm on $\mathbb{A}$. Let $\mathbf{G}_a$ be the additive group over $F$ and $\mathbf{G}_m = GL_1$ be the multiplicative group over $F$. The group $\mathbf{G}_m(\mathbb{A})$ of ideles for $F$ is denoted $\mathbb{J}$.

### 2.1.1 Homomorphism $H_G$

Let $G$ be a connected linear algebraic group defined over $F$, not necessarily reductive. Let

$$X_F(G) = \mathrm{Hom}_F(G, \mathbf{G}_m)$$

denote the abelian group of $F$-rational characters of $G$ and let

$$\mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(X_F(G), \mathbb{R}) \quad \text{and} \quad \mathfrak{a}_G^* = X_F(G) \otimes_{\mathbb{Z}} \mathbb{R}$$

These are vector spaces over $\mathbb{R}$ and there is a natural pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{a}_G^* \times \mathfrak{a}_G \to \mathbb{R}$$

The homomorphism

$$H_G : G(\mathbb{A}) \to \mathfrak{a}_G$$

is

$$H_G(x) := [\chi \mapsto \log |\chi(x)|]$$

Let

$$G(\mathbb{A})^1 = \ker H_G \subset G(\mathbb{A})$$

The function $H_G$ is trivial on $G(F)$. For $G = N \rtimes L$ a Levi decomposition of $G$, $H_G$ is trivial on $N(\mathbb{A})$ and $L_{\mathrm{der}}(\mathbb{A})$, where $L_{\mathrm{der}}$ is the derived group of $L$.

### 2.1.2 Assumptions on $G$

In discussing Eisenstein series, it is convenient to assume that $G$ is semi-simple so that $\mathfrak{a}_G = 0$. However, the data on the Levi subgroups of the parabolic subgroups must be defined for reductive groups. We therefore begin with a connected reductive group $G$ split over $F$. From subsection 2.1.6 onwards, we put further restriction that $G$ is semi-simple.

Let $G$ be a connected and reductive algebraic group. Let $Z_G$ be the center of $G$. Let $G_{\mathbb{Q}}$ be the restriction of scalars of $F$ to $\mathbb{Q}$ and $A_G^+$ be the connected component of the group of real points of the maximal $\mathbb{Q}$-split torus in the center of $G_{\mathbb{Q}}$. Then

$$A_G^+ \subset Z_G(F \otimes \mathbb{R}) \subset Z_G(\mathbb{A}) \subset G(\mathbb{A})$$

The map

$$H_G : A_G^+ \to \mathfrak{a}_G$$

is an isomorphism. For a parabolic subgroup $P = N \rtimes M$ (Levi decomposition) of $G$, we observe that $X_F(P) = X_F(M)$ and hence

$$\mathfrak{a}_P = \mathfrak{a}_M.$$

### 2.1.3 Roots and Coroots

Let $G$ be a connected reductive group split over $F$. For the rest of the paper, fix a maximal split torus $T \subset G$ with Lie algebra $\mathfrak{t}$. Let

$$X_F(T) = \mathrm{Hom}(T, \mathbf{G}_m) \quad \text{and} \quad X_F^\vee(T) = \mathrm{Hom}(\mathbf{G}_m, T)$$

There is a natural pairing

$$X_F(T) \times X_F^\vee(T) \to \mathbb{Z}; \quad (\chi, \eta) \mapsto \langle \chi, \eta \rangle$$

defined by

$$\chi \circ \eta(x) = x^{\langle \chi, \eta \rangle}, \quad \forall x \in \mathbf{G}_m$$

Let $C_G(T)$ and $N_G(T)$ denote the centralizer and the normalizer of $T$ in $G$. The Weyl group of the pair $(G, T)$ is

$$W = W(G, T) = N_G(T)/C_G(T)$$

The adjoint action Ad of $T$ on $\mathfrak{g} = \mathrm{Lie}(G)$ is diagonalizable and

$$\mathfrak{g} = \mathfrak{t} \oplus \left( \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right)$$

where $\Phi \subset X_F(T)$ is the finite set of *roots* and

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \mathrm{Ad}(t) \cdot x = \alpha(t)x \text{ for all } t \in T\}$$

are $T$-eigenspaces. The only rational multiples of $\alpha$ in $\Phi$ are $\pm\alpha$.

For each root $\alpha \in \Phi$, the subtorus $T_\alpha := (\ker \alpha)^\circ$ of $T$ has codimension 1, where $\circ$ means the connected component of the identity. Then $G_\alpha := C_G(T_\alpha)$ is connected and

$$\mathrm{Lie}(G_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

The Weyl group of the pair $(G_\alpha, T)$ has order 2, and embeds in $W(G, T)$. Let $w_\alpha \in W(G, T)$ be the non-identity element of $W(G_\alpha, T)$; then $w_\alpha$ acts on $X_F(T)$ as

$$w_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha$$

for a unique *coroot* $\alpha^\vee \in X_F^\vee(T)$. Since $w_\alpha^2 = 1$, we must have $\langle \alpha, \alpha^\vee \rangle = 2$. See [4][40], for example.

### 2.1.4 Root Groups and a pinning

For $\alpha \in \Phi$, there is a unique algebraic subgroup $U_\alpha \simeq \mathbf{G}_a$ of $G$, called the *root group* corresponding to $\alpha$, which is normalized by $T$ and on which the adjoint action of $T$ is through the character $\alpha$. The Lie algebra of $U_\alpha$ is $\mathfrak{g}_\alpha$.

We fix a Borel subgroup $B$ defined over $F$ for the rest of this paper. The *root datum* attached to $T \subset B \subset G$ is the quadruple

$$(X_F(T), \Delta_B, X_F^\vee(T), \Delta_B^\vee)$$

A *pinning* (or splitting [40], pg. 9) of $G$ for $T \subset B \subset G$ is a collection of isomorphisms

$$\{e_\alpha : \mathbf{G}_a \to U_\alpha \mid \alpha \in \Delta_B\}$$

### 2.1.5 Parabolic subgroups

Any $F$-subgroup $P$ of $G$ containing $B$ is a *standard* parabolic subgroup (relative to $B$). A standard parabolic subgroup $P$ has a unique *standard* Levi decomposition $P = N_P \rtimes M_P$ where $M_P$ contains $T$.

*Remark.* Since we only consider standard parabolic subgroups relative to $B$ and standard Levi decompositions, we drop the adjective *standard* and refer to *the* Levi decomposition.

Observe that $X_F(P) = X_F(M_P)$ and $\mathfrak{a}_P = \mathfrak{a}_{M_P}$. For notational convenience, we write $A_P := A_{M_P}$ and note that $A_P$ is *not* central in $P$ to avoid any potential confusion. The action of $A_P$ on $\mathfrak{n}_P := \text{Lie} N_P$ is diagonalizable and

$$\mathfrak{n}_P = \bigoplus_{\beta \in \Phi_P} \mathfrak{n}_\beta$$

where $\Phi_P$ is a finite subset of $\mathfrak{a}_P^*$ and

$$\mathfrak{n}_\beta = \{X \in \mathfrak{n}_P : \text{Ad}(a)X = \beta(a)X,\ \forall a \in A_P\}$$

Note that $\Phi_B$ is a set of positive roots in $\Phi$, and let $\Delta_B$ be the corresponding set of simple roots.

For each parabolic subgroup, let $\Delta_B^P \subset \Delta_B$ denote the subset of $\alpha \in \Delta_B$ appearing in the action of $T$ in the unipotent radical of $B \cap M_P$. The correspondence $P \to \Delta_B^P$ is a bijection between the set of standard parabolic subgroups of $G$ and the set of subsets of $\Delta_B$.

Let $\Delta_P$ be the set of linear forms on $\mathfrak{a}_P$ obtained by the restriction of the elements in $\Delta_B - \Delta_B^P$. Then $\Delta_P$ is in bijection with $\Delta_B - \Delta_B^P$, and any root in $\Phi_P$ can be written uniquely as a nonnegative integral linear combination of elements in $\Delta_P$.

### 2.1.6 Decompositions of $\mathfrak{a}_B$ and $\mathfrak{a}_B^*$

For the rest of this section, assume that $G$ is semi-simple so that $\mathfrak{a}_G = 0$. Let $P \supset B$ be a parabolic subgroup. The inclusions

$$A_P \subset A_B \subset M_B \subset M_P$$

give canonical decompositions

$$\mathfrak{a}_B = \mathfrak{a}_P \oplus \mathfrak{a}_B^P, \quad \mathfrak{a}_B^* = \mathfrak{a}_P^* \oplus \left(\mathfrak{a}_B^P\right)^*$$

For any $\Lambda \in \mathfrak{a}_B^*$ and $H \in \mathfrak{a}_B$, we write

$$\Lambda = \Lambda_P + \Lambda_B^P \quad \text{where } \Lambda_P \in \mathfrak{a}_P^*,\ \Lambda_B^P \in \left(\mathfrak{a}_B^P\right)^* \tag{2.1.1}$$

and

$$H = H_P + H_B^P \quad \text{where } H_P \in \mathfrak{a}_P,\ H_B^P \in \mathfrak{a}_B^P \tag{2.1.2}$$

### 2.1.7 Relation between $H_B$ and $H_P$

Let $K$ be the standard special maximal compact subgroup which provides the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K(\mathbb{A})$ for any standard parabolic subgroup $P$. We extend $H_P$ from $P_\mathbb{A}$ to $G_\mathbb{A}$ by

$$H_P : G_\mathbb{A} \to \mathfrak{a}_P, \quad H_P(pk) = H_P(p) \quad p \in P_\mathbb{A}, k \in K_\mathbb{A}$$

The decomposition $\mathfrak{a}_B = \mathfrak{a}_P \oplus \mathfrak{a}_B^P$ gives

$$H_B(x) = H_P(x) + H_B^P(x) \quad \forall x \in G_\mathbb{A}$$

and the two definitions—one by extension of $H_P$ to $G_\mathbb{A}$ and the other as the projection of $H_B$ to $\mathfrak{a}_P$—are the same object.

### 2.1.8 Basis for $\mathfrak{a}_P^*$ and $\mathfrak{a}_P$

Let $\widehat{\Delta}_B = \{\varpi_\alpha : \alpha \in \Delta_B\}$ be the set of *fundamental weights*, defined by

$$\langle \varpi_\alpha, \beta^\vee \rangle = \delta_{\alpha\beta} \quad \text{(Kronecker delta) } \forall \alpha, \beta \in \Delta_B$$

Then

$$\widehat{\Delta}_P = \left\{\varpi_\alpha : \alpha \in \Delta_B - \Delta_B^P\right\}$$

is a basis for $\mathfrak{a}_P^*$. Let

$$\Delta_P^\vee = \{\alpha^\vee : \alpha \in \Delta_P\}$$

be the dual basis of $\widehat{\Delta}_P$. For $\alpha \in \Delta_P$, let $\beta \in \Delta_B - \Delta_B^P$ be the simple root whose restriction to $\mathfrak{a}_P$ is $\alpha$. Then $\alpha^\vee$ is the canonical projection of $\beta^\vee \in \mathfrak{a}_B$ onto $\mathfrak{a}_P$.

### 2.1.9 Unramified degenerate Eisenstein series

Let $P$ be a parabolic subgroup and let

$$\rho_P = \frac{1}{2} \sum_{\alpha \in \Phi_P} (\dim \mathfrak{n}_\alpha) \alpha$$

This defines $\rho_P$ and $\rho_B$ and the notation is consistent with equation 2.1.1 giving $\rho_B = \rho_P + \rho_B^P$. Let

$$E_\Lambda^P(g) = \sum_{\gamma \in P_k \backslash G_k} e^{\langle \rho_P + \Lambda, H_P(\gamma g) \rangle}, \quad \Lambda \in \mathfrak{a}_P^* \otimes \mathbb{C}.$$

It converges absolutely and uniformly on compact subsets of $G_{\mathbb{A}}$ when $\Lambda$ is in the tube $\rho_P + T_P$, where

$$T_P := \{\Lambda \in \mathfrak{a}_P^* \otimes \mathbb{C} : \Re\langle \Lambda, \alpha^\vee \rangle > 0, \ \forall \alpha \in \Delta_P\}$$

is the *tube* over the *positive cone* $C_P := T_P \cap \mathfrak{a}_P^*$. We call this the *positive tube* $T_P$ to simplify terminology. The function $E_\Lambda^P(g)$ has a meromorphic continuation to $\mathfrak{a}_P^* \otimes \mathbb{C}$ (see [2]). For $P \neq B, G$, we call $E_\Lambda^P(g)$ *the unramified degenerate Eisenstein series attached to* $P$.

### 2.1.10 A remark on terminology

We do *not* consider the ramified cases in this paper. We drop the word *unramified* to lighten the presentation, and all further references to degenerate Eisenstein series shall be taken to mean unramified degenerate Eisenstein series.

## 2.2 Minimal parabolic Eisenstein series

Throughout this section, let $G$ be an *$F$-split semi-simple* linear algebraic group over $F$ with $T \subset B \subset G$ for a maximal $F$-split torus contained in a Borel subgroup $B$ defined over $F$.

In general, *cuspidal* Eisenstein series are more tractable than general Eisenstein series in their analytic behavior since their constant terms—from which many analytic properties of Eisenstein series follow—are easier to compute. The main result of the paper provides evidence that the poles of *non-cuspidal* Eisenstein series occurring in the spectral decomposition can be understood from the *zeros* and *poles* of cuspidal Eisenstein series.

### 2.2.1 Unramified Borel Eisenstein series

Let $B = N \rtimes M$ be the Levi decomposition of $B$. We write

$$a(g)^\Lambda := e^{\langle \Lambda, H_B(g) \rangle}$$

to simplify the notation. For $P = B$, we call the Eisenstein series $E^B_\Lambda(g)$ *the* minimal parabolic (or *the* Borel) Eisenstein series:

$$E^B_\Lambda(g) = \sum_{\gamma \in B(F)\backslash G(F)} a(\gamma g)^{\rho_B + \Lambda} \quad \Lambda \in T_B,\ g \in G(\mathbb{A})$$

The function $E^B_\Lambda(g)$ converges absolutely in the positive tube $\rho_B + T_B$ and has a meromorphic continuation to $\mathfrak{a}^*_B \otimes \mathbb{C}$. For $w \in W$, it satisfies the functional equation

$$E^B_\Lambda(g) = c_{w,\Lambda} E^B_{w\cdot\Lambda}(g)$$

where

$$c_{w,\Lambda} = \prod_{\substack{\alpha \in \Phi_B: \\ w\cdot\alpha < 0}} \frac{\xi(\langle \Lambda, \alpha^\vee \rangle)}{\xi(1 + \langle \Lambda, \alpha^\vee \rangle)}$$

and

$$\xi(s) := \xi_F(s) \text{ is the completed zeta function of } F$$

Unlike $P \neq B$, when $P = B$, we get a *cuspidal* Eisenstein series in a vacuous, but meaningful, sense since the trivial character on $k^\times \backslash \mathbb{J}^1_k$ is a cuspform for $GL_1$.

### 2.2.2 Poles of $E^B_\Lambda(g)$

The constant term $c_B E^B_\Lambda$ of $E^B_\Lambda$ along $B$ is

$$c_B E^B_\Lambda(g) := \int_{N(F)\backslash N(\mathbb{A})} E^B_\Lambda(ng)dn$$

It is a function on $N(\mathbb{A})M(F)\backslash G(\mathbb{A})$. It was first computed by Gelfand et al. [12] (page 82):

$$c_B E^B_\Lambda(g) = \sum_{w \in W} c_{w,\Lambda} a(g)^{\rho_B + w\cdot\Lambda}$$

From the above formula, we see that the singularities of $c_B E^B_\Lambda(g)$ are hyperplanes of the form

$$S(\alpha, c) := \{\Lambda \in \mathfrak{a}^*_B \otimes \mathbb{C} : \langle \Lambda, \alpha^\vee \rangle = c\}$$

where $\alpha \in \Phi_B$ and $c \in \mathbb{C}$. Following Langlands, we say that the singularity along $S(\alpha, c)$ is *real* if $c \in \mathbb{R}$. In the positive tube $T_B$, the singularities of $E^B_\Lambda(g)$ and $c_B E^B_\Lambda$ are the same, are real, and are given by

$$S_\gamma := S(\gamma, 1) \quad (\gamma \in \Phi_B) \tag{2.2.1}$$

These singularities are *simple* in the sense that

$$\Lambda \mapsto \prod_{\gamma \in \Phi_B} (\langle \Lambda, \gamma^\vee \rangle - 1)\, E^B_\Lambda(g)$$

extends to a holomorphic function on the positive tube $T_B$.

### 2.2.3 Zeros of $E^B_\Lambda(g)$

#### 2.2.3.1 The $SL_2$ Eisenstein series

As a prelude to the more general case, consider the $\mathrm{SL}_2$ Eisenstein series $E_s(z)$ in the introduction. Its constant term is

$$\int_0^1 E_s(x+iy)dx = y^{\frac{1}{2}+\frac{s}{2}} + c(s) y^{\frac{1}{2}-\frac{s}{2}},$$

The constant term vanishes at $s_0$ as a function of $y$ only if

$$c(s_0) = -y^{s_0} \quad \text{for all } y > 0$$

This can happen only for $s_0 = 0$ and if

$$\lim_{s \to 0} c(s) = -1.$$

This is indeed true, since for any number field $F$, the corresponding completed zeta function $\xi := \xi_F$ satisfies

$$\lim_{s \to 0} \frac{\xi(s)}{\xi(s+1)} = \frac{\mathsf{res}_{s=0}\xi(s)}{\mathsf{res}_{s=1}\xi(s)} = -1.$$

Further, the zero of $E_s(z)$ at 0 is simple.

#### 2.2.3.2 The general case

We say that $\Lambda \in \mathfrak{a}^*_B \otimes \mathbb{C}$ is *regular* if $\Lambda$ is not fixed by any $w \in W$. For regular $\Lambda \in \mathfrak{a}^*_B \otimes \mathbb{C}$, the set

$$\left\{ a(g)^{w \cdot \Lambda} : w \in W \right\}$$

is a linearly independent set of functions on $G(\mathbb{A})$ and

$$c_B E^B_\Lambda(g) = 0 \iff \sum_{w\in W} c_{w,\Lambda} a(g)^{w\cdot\Lambda}$$

$$\iff c_{w,\Lambda} = 0 \text{ for all } w \in W$$

Since $c_{1,\Lambda} = 1$, we conclude that the Eisenstein series $E^B_\Lambda \neq 0$ for regular $\Lambda \in \mathfrak{a}^*_B \otimes \mathbb{C}$. Let

$$H_\alpha := \{\Lambda \in \mathfrak{a}^*_B \otimes \mathbb{C} : \langle \Lambda, \alpha^\vee \rangle = 0\}, \quad \alpha \in \Phi_B$$

Note that every $\Lambda$ in the complement of $\cup_{\alpha\in\Phi_B} H_\alpha$ in $\mathfrak{a}^*_B \otimes \mathbb{C}$ is regular. The set of regular elements is an open dense subset of $\mathfrak{a}^*_B \otimes \mathbb{C}$.

The following result of Jacquet [21] computes all possible zeros of the Borel Eisenstein series $E^B_\Lambda$.

**Proposition 5.** *The Eisenstein series $E^B_\Lambda(g)$ has a simple zero along the hyperplanes $H_\alpha$ for $\alpha \in \Phi_B$.*

*Proof.* (Jacquet) From the general theory of Eisenstein series, it is enough to show that for generic $\Lambda \in H_\alpha$

$$c_B E^B_\Lambda(g) = 0$$

for each $\alpha \in \Phi_B$ and that the zero along $H_\alpha$ is simple. We first prove this for a simple root $\alpha \in \Delta_B$ and deduce the case when $\alpha$ is not simple using the functional equations of $E^B_\Lambda(g)$.

**Step 1:** $\alpha$ simple. Let $w_\alpha \in W$ be the reflection corresponding to a simple root $\alpha \in \Delta_B$. The group $W_\alpha = \{1, w_\alpha\}$ acts on $W$ on the right with orbits of the form $\{w, w \cdot w_\alpha\}$ for $w \in W$. Using

$$c_{ww_\alpha,\Lambda} = c_{w,w_\alpha\Lambda} \cdot c_{w_\alpha,\Lambda} \quad \text{(cocycle relation)},$$

$$w_\alpha \cdot \Lambda = \Lambda \quad \text{on } H_\alpha,$$

and

$$\lim_{\langle\Lambda,\alpha^\vee\rangle\to 0} c_{w_\alpha,\Lambda} = \lim_{\langle\Lambda,\alpha^\vee\rangle\to 0} \frac{\xi(\langle\Lambda,\alpha^\vee\rangle)}{\xi(1+\langle\Lambda,\alpha^\vee\rangle)} = -1,$$

we get

$$\lim_{\langle\Lambda,\alpha^\vee\rangle\to 0} \left(c_{w,\Lambda} a(g)^{\rho_B + w\cdot\Lambda} + c_{ww_\alpha,\Lambda} a(g)^{\rho_B + ww_\alpha\cdot\Lambda}\right)$$

$$= \lim_{\langle\Lambda,\alpha^\vee\rangle\to 0} \left(c_{w,\Lambda} a(g)^{\rho_B + w\cdot\Lambda} + c_{w,w_\alpha\cdot\Lambda} c_{w_\alpha,\Lambda} a(g)^{\rho_B + ww_\alpha\cdot\Lambda}\right) = 0$$

away from the singularities of $c_{w,\Lambda}$. Breaking up the following sum over $W$ by the orbits of the $W_\alpha$ action,

$$\lim_{\langle\Lambda,\alpha^\vee\rangle\to 0} c_B E^B_\Lambda(g) = \lim_{\langle\Lambda,\alpha^\vee\rangle\to 0} \sum_{w\in W} c_{w,\Lambda} a(g)^{\rho_B + w\cdot\Lambda}$$

$$= \sum_{\dot w\in W/W_\alpha} \lim_{\langle\Lambda,\alpha^\vee\rangle\to 0} \left(c_{\dot w,\Lambda} a(g)^{\rho_B+\dot w\cdot\Lambda} + c_{\dot w w_\alpha,\Lambda} a(g)^{\rho_B+\dot w w_\alpha\cdot\Lambda}\right) = 0$$

For generic $\Lambda \in H_\alpha$, the set $\{a^{\dot w\Lambda} : \dot w \in W/W_\alpha\}$ is a linearly independent set of functions on $G(\mathbb{A})$. The simplicity of the zero along $H_\alpha$ follows from the observation that

$$\lim_{\langle\Lambda,\alpha^\vee\rangle\to 0} \frac{1}{\langle\Lambda,\alpha^\vee\rangle} \left(a(g)^{\rho_B+\dot w\cdot\Lambda} + c_{w_\alpha,\Lambda} a(g)^{\rho_B+w_\alpha\cdot\Lambda}\right) \neq 0$$

for generic $\Lambda \in H_\alpha$ and that the term corresponding to $\dot w = 1$ is non-zero. The non-vanishing above is similar to the fact that the $\mathrm{SL}_2$ Eisenstein series $E_s(z)$ has a simple zero at $s = 0$.

**Step 2.** $\alpha$ non-simple. Now we prove the vanishing for a general positive root. Given a positive root $\beta \in \Phi_B$, there exists $w \in W$ and a simple root $\alpha \in \Delta_B$ such that $w\cdot\beta = \alpha$. We have the functional equation

$$E^B_\Lambda(g) = c_{w,\Lambda} E^B_{w\cdot\Lambda}(g),$$

Since $w\cdot\beta = \alpha > 0$, the formula

$$c_{w,\Lambda} = \prod_{\substack{\gamma\in\Phi_B:\\ w\cdot\gamma<0}} \frac{\xi(\langle\Lambda,\gamma^\vee\rangle)}{\xi(1+\langle\Lambda,\gamma^\vee\rangle)}$$

shows that $H_\beta$ is not a singular hyperplane of $c_{w,\Lambda}$. Since

$$\lim_{s\to 0} \frac{\xi(s)}{\xi(s+1)} \neq 0,$$

it follows that $c_{w,\Lambda}$ does not vanish along $H_\beta$. Using $w\cdot H_\beta = H_\alpha$, we conclude that $c_B E^B_\Lambda(g)$ and $E^B_\Lambda(g)$ have simple zeros along $H_\beta$. $\square$

### 2.2.4 $E^P_{\Lambda_P}$ as a residue of $E^B_\Lambda$

The result of this subsection is the well-known theorem of Langlands that the non-cuspidal Eisenstein series occurring in the spectral decomposition of automorphic forms are "residues" of cuspidal

Eisenstein series (see Moeglin [37]). The general notion of residue required to prove this result is discussed in chapter 7 of Langlands [31] and section V.1 of Moeglin-Waldspurger [36].

The case we need is the simplest and occurs without any of the complications of the general case (see Langlands [28, 29]). For a parabolic subgroup $P$ of $G$, the set

$$S_P := \bigcap_{\alpha \in \Delta_B^P} S_\alpha \quad \text{(see 2.2.1 for the definition of } S_\alpha\text{)}$$

is an affine subspace of $\mathfrak{a}_B^* \otimes \mathbb{C}$. The function

$$\prod_{\alpha \in \Delta_B^P} (\langle \Lambda, \alpha^\vee \rangle - 1) \cdot E_\Lambda^B$$

extends to a meromorphic function on $S_P$. The *residue* of $E_\bullet^B$ along $S_P$ is

$$\left(\mathsf{Res}_{S_P} E_\bullet^B\right) := \left( \prod_{\alpha \in \Delta_B^P} (\langle \Lambda, \alpha^\vee \rangle - 1) \cdot E_\Lambda^B \right) \Bigg|_{S_P}$$

It is a meromorphic function on $S_P = \rho_B^P + \mathfrak{a}_P^* \otimes \mathbb{C}$.

The following well-known result shows how the degenerate Eisenstein series occur as residues of the minimal parabolic Eisenstein series.

**Proposition 6.** *(Langlands) For the decomposition 2.1.6*

$$\Lambda = \Lambda_P + \Lambda_B^P,$$

*we have*

$$\left(\mathsf{Res}_{S_P} E_\bullet^B\right)(\rho_B^P + \Lambda_P) = c \cdot E_{\Lambda_P}^P$$

*for some $c \neq 0$.*

## 2.3 Poles of maximal parabolic degenerate Eisenstein series

Let $T \subset B \subset G$ be as before for a connected semi-simple algebraic group $G$. Let $P$ be a maximal $F$-parabolic subgroup $P \supset B$. We have $\Delta - \Delta_B^P = \{\beta\}$ for some simple root $\beta \in \Delta_B$. Let $\varpi \in \mathfrak{a}_B^*$ be the fundamental weight dual vector to the coroot $\beta^\vee$.

The vector $\varpi \in \mathfrak{a}_P^*$ and we parametrize $\mathfrak{a}_P^* \otimes \mathbb{C}$ by $s\varpi$. Let

$$E^P_{s\varpi}(g) = \sum_{\gamma \in P(F)\backslash G(F)} e^{\langle \rho_P + s\varpi, H_P(g)\rangle}$$

It converges for $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to $\mathbb{C}$.

In this section, we show that the poles of $E^P_{s\varpi}(g)$ in the region $\mathrm{Re}(s) \geq 0$ are determined by the zeros of a polynomial $p \in \mathbb{C}[s]$ obtained using the structure of $P$. Before we treat the general case, we discuss the simplest example of $SL_3$ which highlights the issues we must address.

### 2.3.1 The $SL_3$ example

The poles of $E^B_\Lambda$ *outside the positive tube* can come from the *critical zeros* of $\xi := \xi_F$ the completed zeta function of the number field $F$, as can be seen from the formula

$$c_B E^B_\Lambda(g) = \sum_{w \in W} c_{w,\Lambda} a(g)^{\rho_B + w\cdot\Lambda}, \quad c_{w,\Lambda} = \prod_{\substack{\alpha>0:\\ w\cdot\alpha<0}} \frac{\xi(\langle \Lambda, \alpha^\vee\rangle)}{\xi(1 + \langle \Lambda, \alpha^\vee\rangle)}$$

Casselman [5] brought attention to a curious phenomena where the poles from the critical zeros do not contribute to the poles meeting the positive tube of the degenerate Eisenstein series $E^P_{\Lambda_P}$, even though the point $\rho^P_B \in S_P \subset \mathfrak{a}_B^* \otimes \mathbb{C}$ is outside the positive cone. Following Casselman, we illustrate this phenomenon for $SL_3$.

Let $G = SL_3$. Let $\alpha, \beta \in \Delta_B$ be the simple roots. Let

$$s_\alpha = \langle \Lambda, \alpha^\vee \rangle \quad \text{and} \quad s_\beta = \langle \Lambda, \beta^\vee \rangle$$

**Example 7.** For $w \in W$ the *longest* Weyl element,

$$c_{w,\Lambda} = \frac{\xi(s_\alpha)\xi(s_\beta)\xi(s_\alpha + s_\beta)}{\xi(1 + s_\alpha)\xi(1 + s_\beta)\xi(1 + s_\alpha + s_\beta)}.$$

For $\Delta^P_B = \{\alpha\}$, we have $\rho^P_B = \frac{1}{2}\alpha$, $\langle \rho^P_B, \beta^\vee\rangle = -\frac{1}{2}$, $\langle \varpi, \alpha^\vee \rangle = 0$, and $\langle \varpi, \beta^\vee \rangle = 1$.

For $\Lambda = \rho^P_B + s\varpi$, we have

$$s_\beta = \langle \Lambda, \beta^\vee \rangle = \langle \rho^P_B + s\varpi, \beta^\vee \rangle = -\frac{1}{2} + s$$

and the term $\xi(1+s_\beta)$ in the denominator of $c_{w,\Lambda}$ could contribute poles from the *critical zeros* of $\xi(s)$ to $E^P_{s\varpi}(g)$ in the region $\mathrm{Re}(s) \geq 0$.

However,

$$\mathsf{Res}_{S_P} c_{w,\Lambda} = \frac{\mathsf{Res}_{z=1}\xi(z)}{\xi(2)} \cdot \frac{\xi(s_\beta)\cancel{\xi(1+s_\beta)}}{\cancel{\xi(1+s_\beta)}\xi(2+s_\beta)}$$

and the troublesome term is cancelled.

The reader should consult Casselman's [5] account for further examples of cancellations of this sort. In fact, this paper is the inspiration to all the ideas in this paper. This cancellation is *not* sufficient to determine the poles of degenerate Eisenstein series, not even for $SL_3$. The following example illustrates what is going on.

**Example 8.** The poles and zeros of $E^B_\Lambda(g)$ relevant for determining the poles of degenerate Eisenstein series in our region of interest are captured by the meromorphic function

$$F(\Lambda) := \frac{s_\alpha}{s_\alpha - 1} \cdot \frac{s_\beta}{s_\beta - 1} \cdot \frac{s_\alpha + s_\beta}{s_\alpha + s_\beta - 1}$$

Note that when $s_\alpha = 1$, we can have a pole at

$$(s_\alpha, s_\beta) = (1,1) \text{ and } (1,0).$$

However,

$$\mathsf{Res}_{S_\alpha} F(\Lambda) = (s_\alpha - 1)\frac{s_\alpha}{s_\alpha - 1} \cdot \frac{s_\beta}{s_\beta - 1} \cdot \frac{s_\alpha + s_\beta}{s_\alpha + s_\beta - 1}\bigg|_{S_\alpha}$$

$$= \frac{\cancel{s_\beta}}{s_\beta - 1} \cdot \frac{1+s_\beta}{\cancel{s_\beta}}\bigg|_{S_\alpha} = \frac{1+s_\beta}{s_\beta - 1}\bigg|_{S_\alpha}$$

The cancellation above explains why the degenerate Eisenstein series obtained by taking residue along $S_\alpha$ ($\alpha$ a simple root) is *holomorphic* at the point $S_\alpha \cap S_\gamma$, where $\gamma$ is the non-simple positive root.

This simple observation is sufficient to obtain both the locations and the order of the poles. To explicate the above cancellations, we need the principal $\mathfrak{sl}_2\mathbb{C}$ subalgebras of ${}^L\mathfrak{g}$ discussed below. In subsections 2.3.2 and 2.3.3, we drop the hypothesis that $G$ be semi-simple.

### 2.3.2 Principal homomorphism $\mathbf{SL}_2\mathbb{C} \to {}^LG$

For a connected reductive group $G$ split over $k$ with Lie algebra $\mathfrak{g}$, its root datum consists of a maximal torus $T \subset B$ and the quadruple

$$(X_k(T), \Delta_B, X_k^\vee(T), \Delta_B^\vee)$$

A *pinning/splitting* of $G$ for $T \subset B \subset G$ is a collection of isomorphisms

$$\{e_\alpha : \mathbf{G}_a \to U_\alpha \mid \alpha \in \Delta_B\}$$

The construction of the *dual group* ${}^LG$ of $G$ gives a $k$-split torus ${}^LT$, a Borel ${}^LB \supset {}^LT$, and the root datum

$$\big(X_k({}^LT) = X_k^\vee(T),\ \Delta_{{}^LB} \simeq \Delta_B^\vee,\ X_k^\vee({}^LT) = X_k(T),\ \Delta_{{}^LB}^\vee \simeq \Delta_B\big)$$

and root vectors

$$\{e_{\alpha^\vee} : \mathbf{G}_a \to U_{\alpha^\vee} \mid \alpha \in \Delta_B\}$$

We have an identification between the positive roots of ${}^LB$ in $\mathrm{Hom}({}^LT, \mathbf{G}_m)$ and the positive coroots for $B$ in $\mathrm{Hom}(\mathbf{G}_m, T)$ (see Springer [40]).

Let $\Delta := \Delta_B$, $\Phi^+ := \Phi_B$, and ${}^L\mathfrak{g} := \mathrm{Lie}({}^LG)$. Let

$$x_{\alpha^\vee} := \mathrm{Lie}(e_{\alpha^\vee})(1) \quad \text{in } {}^L\mathfrak{g}_{\alpha^\vee} = \mathrm{Lie}(U_{\alpha^\vee})$$

and

$$X := \sum_{\alpha\in\Delta} x_{\alpha^\vee}$$

Then $X$ is a principal nilpotent element in ${}^L\mathfrak{g} = \mathrm{Lie}({}^LG)$ (sometimes also referred to as regular nilpotent element). For each $\alpha^\vee \in \Phi^\vee$, let $h_{\alpha^\vee} \in {}^L\mathfrak{g}$ be the vector determined by the coroot $\alpha^\vee : \mathbb{G}_m \to T$, and let

$$H = \sum_{\gamma\in\Phi^+} h_{\gamma^\vee} = \sum_{\alpha\in\Delta} c_\alpha h_{\alpha^\vee}$$

The coefficients $c_\alpha$ are *positive* integers. Finally, for each $\alpha \in \Delta$, let $y_{\alpha^\vee}$ be the unique basis of $\mathrm{Lie}(U_{-\alpha^\vee})$ such that

$$[x_{\alpha^\vee}, y_{\alpha^\vee}] = h_{\alpha^\vee}$$

Let

$$Y = \sum_{\alpha\in\Delta} c_\alpha y_{\alpha^\vee}$$

A simple calculation shows that $\{H, X, Y\}$ is a standard $\mathfrak{sl}_2$-triple:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H$$

There is a homomorphism

$$\phi : \mathfrak{sl}_2(\mathbb{C}) \to {}^L\mathfrak{g}$$

given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto Y$$

and since $SL_2(\mathbb{C})$ is simply connected, we have a homomorphism of reductive groups $\varphi : SL_2(\mathbb{C}) \to {}^LG$. We refer to it as *the principal homomorphism* $SL_2 \to {}^LG$. The co-character $\mathbb{G}_m \to {}^LT$ given by the restriction of $\varphi$ to the maximal torus

$$\mathbf{G}_m \simeq \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \subset SL_2 : t \in \mathbf{G}_m \right\}$$

is $2\rho_B$ in $\mathrm{Hom}(\mathbb{G}_m, {}^LT) = \mathrm{Hom}(T, \mathbb{G}_m)$ (see Gross [16]).

### 2.3.3 The action of the principal $\mathfrak{sl}_2\mathbb{C}$ in $\mathfrak{m}$ on $\mathfrak{n}$

Let $G$ be a split connected reductive group and $P = N \rtimes M$ be the standard Levi decomposition of a parabolic subgroup $P$. To explicate in general the cancellations illustrated in 2.3.1 for $SL_3$, we need to study the action of the principal $SL_2$ in $M$ on $\mathfrak{n}$. We first discuss an example.

**Example 9.** Let $G = GL_4$ with standard Levi $M = GL_2 \times GL_2$ of the standard parabolic $P$. The space $\mathfrak{n}$ is the space of $2 \times 2$ matrices (abelian Lie algebra). Let $\alpha_1, \alpha_2, \alpha_3$ be the standard numbering for the simple roots of the standard maximal torus of $G$.

For $i, j \in \{1, 2\}$, denote $E_{ij}$ the matrix whose $(i', j')$-coefficient is 1 if $(i', j') = (i, j)$ and 0 if $(i', j') \neq (i, j)$. The lines $\mathbb{C}E_{ij}$ are eigenspaces for the action of the standard torus of $G$, associated to the root $\alpha_2$ for $(i, j) = (2, 1)$, $\alpha_1 + \alpha_2$ for $(i, j) = (1, 1)$, $\alpha_2 + \alpha_3$ for $(i, j) = (2, 2)$, and $\alpha_1 + \alpha_2 + \alpha_3$ for $(i, j) = (1, 2)$.

The principal $SL_2$ is the diagonal embedding of $SL_2 \to GL_2 \times GL_2$ and $SL_2$ acts by conjugation on $\mathfrak{n}$. This representation decomposes as

$$\mathfrak{n} \simeq V_0 \oplus V_2, \quad V_k \simeq \mathrm{Sym}^k(\mathrm{std})$$

where

$$V_0 = \mathbb{C}(E_{11} + E_{22}) \quad \text{and} \quad V_2 = \mathbb{C}E_{21} \oplus \mathbb{C}(E_{11} - E_{22}) \oplus \mathbb{C}E_{12}$$

Note that $V_0$ is *not* a root space.

The following simple observation is critical in the proof of proposition 12.

**Lemma 10.** *Let $P = N \rtimes M$ be the standard Levi decomposition of a standard proper parabolic subgroup $P$ of $G$. Let $X$ be the principal nilpotent element in $\mathfrak{m}$ and $\alpha \in \Phi_B$ be a positive root with $\mathfrak{g}_\alpha \subset \mathfrak{n}$. Then $X \cdot \mathfrak{g}_\alpha \subset \bigoplus_{\theta \in \Delta_B^P} \mathfrak{g}_{\alpha+\theta}$. In particular, if $X \cdot \mathfrak{g}_\alpha \neq 0$ then there exists $\theta \in \Delta_B^P$ such that $\alpha + \theta \in \Phi_B$.*

*Proof.* Let $\Theta = \Delta_B^P$ for notational convenience. We have $X = \sum_{\theta \in \Theta} x_\theta$ and

$$X \cdot x_\alpha = \sum_{\theta \in \Theta} [x_\theta, x_\alpha] = \sum_{\theta \in \Theta} a_\theta x_{\alpha+\theta} \subset \bigoplus_{\theta \in \Theta} \mathfrak{g}_{\alpha+\theta}$$

for some constants $a_\theta$ for $\theta \in \Theta$. □

The action of the principal $SL_2$ in $M$ commutes with the central torus $A_P \subset M$. For

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi_P} \mathfrak{n}_\alpha$$

we get an action of the principal $SL_2$ in $M$ on each $\mathfrak{n}_\alpha$ ($\alpha \in \Phi_P$). This action plays a central role in the arguments below.

### 2.3.4 No contribution from the critical zeros of $\xi$

We first make a comment about the poles of residues. Let $f$ be a meromorphic function on a complex vector space whose singularities are a locally finite collection $\{L_\mu : \mu \in I\}$ ($I$ is an indexing set) of affine hyperplanes. Assume that the singularity along a hyperplane $L$ is *simple* so that the notion of residue is straightforward. The residue of $f$ along $L$ is a meromorphic function on $L$ and can only have singularities along $L \cap L_\mu$ for $\mu \in I$ such that $L_\mu \neq L$.

We need the following result of Kostant [24].

**Proposition 11.** *(Kostant) Let $G$ be a connected split reductive group over $k$. The list of numbers, with multiplicities, in*

$$\{\langle \rho_B, \alpha^\vee \rangle + 1 : \alpha \in \Phi_B\}$$

*is the same as the list of numbers, with multiplicities, in*

$$\{\langle \rho_B, \alpha^\vee \rangle : \alpha \in \Phi_B - \Delta_B\}$$

*together with positive integers $a_1, \dots, a_n \geq 2$ where $n$ is the cardinality of $\Delta_B$.*

*Remark.* The numbers $a_1, \dots, a_n$ can be explicitly determined in terms of the Poincaré polynomial of the Weyl group of $G$ (see Humphreys [20], chapters 1 and 3). However, we do not need this.

For the remainder of this section, we assume that $G$ is semi-simple and we fix a *maximal* parabolic subgroup $P \supset B$. Let $\varpi \in \mathfrak{a}_P^*$ be the fundamental weight corresponding to $P$. The standard Levi decomposition $P = N \rtimes M$ gives

$$\Phi_B = \Phi_M^+ \sqcup \Phi_N$$

where $\Phi_M^+$ and $\Phi_N$ are the roots with root groups in $M \cap B$ and $N$ respectively. In particular, $\Delta_B^P \subset \Phi_M^+$. Let $\Lambda = \Lambda_P + \Lambda_B^P \in \mathfrak{a}_B^* \otimes \mathbb{C}$ as in subsection 2.1.6.

**Proposition 12.** *Let $\Lambda_P = s\varpi$ and $\rho_B^P + s\varpi$ for $s \in \mathbb{C}$ be a parametrization of $S_P = \rho_B^P + \mathfrak{a}_P^* \otimes \mathbb{C}$. The poles of the Eisenstein series $E_{\varpi s}^P(g)$ in the region $\mathsf{Re}(s) \geq 0$ are real and contained in the set*

$$\{S_P \cap S_\alpha : \alpha \in \Phi_N\}$$

*Proof.* We prove the result in several steps. We begin with some preliminary remarks.

By the general theory of Eisenstein series, it is enough to prove this for the constant term $c_B E_{\Lambda_P}^P$, or the equivalently the corresponding statement for

$$\mathsf{Res}_{S_P} c_B E_\bullet^B(g) = \mathsf{Res}_{S_P} \left( \sum_{w \in W} c_{w,\Lambda} a(g)^{\rho_B + w \cdot \Lambda} \right)$$

The term $\mathsf{Res}_{S_P} c_{w,\Lambda} \neq 0$ only if $w \cdot \Delta_B^P \subset -\Phi_B$. Fix a $w \in W$ satisfying this property.

The product formula for $c_{w,\Lambda}$ contains terms $\xi(1+\langle\Lambda,\gamma^\vee\rangle)$ that can contribute poles from the critical zeros only if $\mathrm{Re}\langle\Lambda,\gamma^\vee\rangle\in(-1,0)$. This happens at $\Lambda=\rho_B^P+s\varpi\in S_P$ if

$$-1<\langle\rho_B^P,\gamma^\vee\rangle+\langle\varpi,\gamma^\vee\rangle\mathrm{Re}(s)<0$$

Note that $\langle\varpi,\gamma^\vee\rangle\geq 0$ for all positive roots $\gamma$. In the region $\mathrm{Re}(s)\geq 0$, the term $\xi(1+\langle\rho_B^P,\gamma^\vee\rangle+s\langle\varpi,\gamma^\vee\rangle)$ can contribute poles from the critical zeros only if $\langle\rho_B^P,\gamma^\vee\rangle<0$.

We prove the theorem by showing that if $w\cdot\gamma<0$ so that it occurs in the product formula of $c_{w,\Lambda}$ and $\langle\rho_B^P,\gamma^\vee\rangle<0$, it is cancelled as in example 2.3.1 above for $SL_3$.

**Step 1:** We first deal with $\gamma^\vee$ for $\gamma\in\Phi_M^+$. From $w\cdot\Delta_B^P\subset-\Phi_B$, we know that $w$ maps all the roots in $\Phi_M^+$ to negative roots. For $\gamma\in\Phi_M^+$,

$$\langle\Lambda_P,\gamma^\vee\rangle=0,\qquad\text{for }\Lambda_P\in\mathfrak{a}_P^*\otimes\mathbb{C}$$

On $S_P=\rho_B^P+\mathfrak{a}_P^*\otimes\mathbb{C}$, we have

$$\langle\rho_B^P+\Lambda_P,\gamma^\vee\rangle=\langle\rho_B^P,\gamma^\vee\rangle,\qquad\text{for }\gamma\in\Phi_M^+,\Lambda_P\in S_P$$

By applying the above result of Kostant to $M$, we get

$$\left.\frac{\prod_{\alpha\in\Phi_M^+-\Delta_B^P}\xi(\langle\rho_B^P,\alpha^\vee\rangle)}{\prod_{\alpha\in\Phi_M^+}\xi(1+\langle\rho_B^P,\alpha^\vee\rangle)}\right|_{S_P}=\frac{1}{\prod_{i=1}^p\xi(a_i)}\neq 0$$

for some integers $a_1,\ldots,a_p\geq 2$.

Thus, in the product for $\mathsf{Res}_{S_P}c_{w,\Lambda}\neq 0$, we need only concern with poles from the critical zeros of terms $\xi(1+\langle\Lambda,\gamma^\vee\rangle)$ for $\gamma\in\Phi_N$. We do this in the next few steps.

**Step 2:** We now characterize the roots that may contribute poles from the critical zeros of $\xi$ in terms of the structure of $P$. Let $H,X,Y$ be the standard notation for the principal $\mathfrak{sl}_2$ triple in ${}^L\mathfrak{m}$. For $\gamma\in\Phi_N$, the one-dimensional space ${}^L\mathfrak{g}_{\gamma^\vee}$ is a $H$-eigenspace with eigenvalue $\langle 2\rho_B^P,\gamma^\vee\rangle$ under the adjoint action. We need to prove a cancellation for $\gamma^\vee$ with negative $H$-eigenvalue.

**Step 3:** Now we obtain the coroots that give cancellation to the troublesome roots of the previous step. To do this, let

$${}^L\mathfrak{n}=r_1\oplus r_2\oplus\cdots\oplus r_m$$

where $r_1,\ldots,r_m$ are the irreducible constituents of the adjoint representation of ${}^LM$ on ${}^L\mathfrak{n}$ as described in 1.3.3. We have the $H$-eigenspace decomposition

$$r_j=\bigoplus_{\ell\in\mathbb{Z}}W_\ell(j),\quad W_\ell(j):=\{v\in r_j:H\cdot v=\ell v\}$$

Fix $j \in \{1, \ldots, m\}$ and $k < 0$. Let

$$\Gamma_k(j) := \left\{\gamma \in \Phi_B : {}^L\mathfrak{g}_{\gamma^\vee} \subset W_k(j) \text{ and } w \cdot \gamma < 0\right\}$$

For any $\delta^\vee = \gamma^\vee + \theta^\vee$ for some $\gamma \in \Gamma_k(j)$ and $\theta \in \Delta_B^P$, we have $w \cdot \delta < 0$ since

$$\delta = \frac{|\gamma^\vee|^2}{|\delta^\vee|^2}\gamma + \frac{|\theta^\vee|^2}{|\delta^\vee|^2}\theta \quad \text{and} \quad w \cdot \gamma,\ w \cdot \theta < 0.$$

By lemma 10,

$$X \cdot \left(\bigoplus_{\gamma \in \Gamma_k(j)} {}^L\mathfrak{g}_{\gamma^\vee}\right) \subset \bigoplus_{\delta \in \Gamma_{k+2}(j)} {}^L\mathfrak{g}_{\delta^\vee}$$

Since $W_k(j)$ is an $H$-eigenspace of *negative* eigenvalue, the action of $X$ is injective and the cardinalities satisfy

$$\#\Gamma_k(j) \leq \#\Gamma_{k+2}(j)$$

**Step 4:** We now prove the cancellation for $\gamma \in \Gamma_k(j)$ when $k < 0$. Let $\delta^\vee = \gamma^\vee + \theta^\vee$ for some $\gamma \in \Gamma_k(j)$ and $\theta \in \Delta_B^P$. We have

$$\begin{aligned} \langle \rho_B^P + \varpi s, \delta^\vee \rangle &= \langle \rho_B^P, \delta^\vee \rangle + js \\ &= \langle \rho_B^P, \gamma^\vee \rangle + \langle \rho_B^P, \theta^\vee \rangle + js \\ &= \frac{k}{2} + 1 + js \end{aligned}$$

Thus,

$$\frac{\xi\left(\langle \rho_B^P + s\varpi, \delta^\vee \rangle\right)}{\xi\left(\langle \rho_B^P + s\varpi, \gamma^\vee \rangle + 1\right)} = \frac{\xi(\frac{k}{2} + 1 + js)}{\xi(\frac{k}{2} + 1 + js)} = 1$$

By $\#\Gamma_k(j) \leq \#\Gamma_{k+2}(j)$ (for $k < 0$), we have the required cancellation. We have shown that for $\mathrm{Re}(s) \geq 0$, the critical zeros of $\xi$ do not yield poles of $E_{s\varpi}^P$.

The remaining poles are given by

$$S_P \cap S_\gamma \quad (\gamma \in \Phi_N)$$

by the remark about residues at the beginning of this subsection. □

*Remark.* Similar arguments appear in justifying the contour deformation of Langlands (see [8][34, 35]) and is, in principle, known in great generality [9]. However, this cancellation is *not* sufficient to determine the poles of degenerate Eisenstein series, even for $SL_3$ as shown in 2.3.1.

### 2.3.5 A simple function determining the poles of $E^P_\bullet(g)$

Next we show that the poles of $E^P_{s\varpi}$ in the region $\mathrm{Re}(s) \geq 0$ are determined by a simple fraction with the numerator and the denominators from the zeros and the poles of $E^B_\Lambda$.

$$E^B_\Lambda(g) = \prod_{\alpha\in\Phi_B} \frac{\langle \Lambda, \alpha^\vee\rangle}{\langle \Lambda, \alpha^\vee\rangle - 1} \cdot E^*_\Lambda(g)$$

**Proposition 13.** *Let*

$$F(s) = \frac{\prod_{\alpha\in\Phi_B}\langle \rho^P_B + s\varpi, \alpha^\vee\rangle}{\prod_{\alpha\in\Phi_B-\Delta^P_B}(\langle \rho^P_B + s\varpi, \alpha^\vee\rangle - 1)},$$

*For $Re(s) \geq 0$, the map $s \mapsto F(s)^{-1} \cdot E^P_{s\varpi}(g)$ is a holomorphic function taking values in the space of smooth functions of uniform moderate growth on $G(F)\backslash G(\mathbb{A})$.*

*Proof.* By the general theory of Eisenstein series, it is enough to show this for $c_B\left(F(s)^{-1}E^P_{s\varpi}\right)$. From proposition 12, we know that the poles can occur only along the intersections $S_\alpha \cap S_P$ $(\alpha \in \Phi_N)$.

The function $c_B E^B_\Lambda(g)$ has a simple zero along the hyperplanes

$$H_\alpha = \{\Lambda \in \mathfrak{a}^*_B \otimes \mathbb{C} : \langle \Lambda, \alpha^\vee\rangle = 0\}, \quad (\alpha \in \Phi_B),$$

and has simple poles along $S_\alpha$ for $\alpha \in \Phi_B$. We have

$$c_B E^B_\Lambda(g) = \prod_{\alpha\in\Phi_B} \frac{\langle \Lambda, \alpha^\vee\rangle}{\langle \Lambda, \alpha^\vee\rangle - 1} \cdot c_B E^*_\Lambda(g)$$

Then $c_B E^*_\Lambda(g)$ extends to a meromorphic function of $S_P$ and is *holomorphic* at $\rho^P_B + s\varpi$ for $\mathrm{Re}(s) \geq 0$ by proposition 12. We have (the constant $c \neq 0$ below is from Langlands' residue formula 6),

$$c \cdot c_B E^P_{s\varpi}(g) = \mathsf{Res}_{S_P} c_B E^B_\Lambda(g)$$

$$= \mathsf{Res}_{S_P}\left(\prod_{\alpha\in\Phi_B} \frac{\langle \Lambda, \alpha^\vee\rangle}{\langle \Lambda, \alpha^\vee\rangle - 1} \cdot c_B E^*_\Lambda(g)\right)$$

$$= \mathsf{Res}_{S_P}\left(\prod_{\alpha\in\Phi_B} \frac{\langle \Lambda, \alpha^\vee\rangle}{\langle \Lambda, \alpha^\vee\rangle - 1}\right) \cdot c_B E^*_\Lambda(g)\Big|_{S_P}$$

Note that

$$\mathsf{Res}_{S_P} \prod_{\alpha\in\Phi_B} \frac{\langle\Lambda,\alpha^\vee\rangle}{\langle\Lambda,\alpha^\vee\rangle-1} = \frac{\prod_{\alpha\in\Phi_B}\langle\Lambda,\alpha^\vee\rangle}{\prod_{\alpha\in\Phi_B-\Delta_B^P}\langle\Lambda,\alpha^\vee\rangle-1}\Big|_{S_P}$$

$$= \frac{\prod_{\alpha\in\Phi_B}\langle\rho_B^P+\Lambda_P,\alpha^\vee\rangle}{\prod_{\alpha\in\Phi_B-\Delta_B^P}\langle\rho_B^P+\Lambda_P,\alpha^\vee\rangle-1} = \frac{\prod_{\alpha\in\Phi_B}\langle\rho_B^P+s\varpi,\alpha^\vee\rangle}{\prod_{\alpha\in\Phi_B-\Delta_B^P}\langle\rho_B^P+s\varpi,\alpha^\vee\rangle-1} = F(s)$$

Thus,

$$c\cdot F(s)^{-1}\cdot E^P_{s\varpi} = E^*_{\rho_B^P+s\varpi}$$

The function $E^*_{\rho_B^P+s\varpi}$ is holomorphic for $\mathrm{Re}(s)\geq 0$. □

*Remark* 14. Already the $\mathrm{SL}_2$ Eisenstein series $E_s(z)$ in the introduction has poles from the critical zeros of $\xi(s)$ when $\mathrm{Re}(s)<0$, since poles of $c(s)=\xi(s)/\xi(1+s)$ are poles of $E_s(z)$. However, these poles do not play a role in contour deformation of Langlands (see Godement [14] and Moeglin [37]).

### 2.3.6 Main theorem

Now we prove the main theorem of this paper:

**Theorem 15.** *Let $G$ be a split semi-simple linear algebraic group over a number field and $P=N\rtimes M$ be the standard Levi decomposition of a standard maximal parabolic subgroup $P$. Let*

$$^L\mathfrak{n} = r_1\oplus r_2\oplus\cdots\oplus r_m$$

*where $r_1,\ldots,r_m$ are the irreducible constituents of the adjoint representation of $^LM$ on $^L\mathfrak{n}$ as described in 1.3.3. Let*

$$r_j \simeq \bigoplus_{\ell\geq 0} V_\ell^{m_\ell(j)},\quad V_k = sym^k(std)$$

*be the decomposition into irreducible constituents of $r_j$ under the action of the principal $\mathfrak{sl}_2\mathbb{C}\subset {}^L\mathfrak{m}$. Let*

$$p(s) = \prod_{j=1}^m\prod_{\ell\geq 0}(js-1-\ell/2)^{m_\ell(j)} \in \mathbb{C}[s]$$

*In the region $\mathsf{Re}(s)\geq 0$, $p(s)\cdot E^P_{s\varpi}(g)$ is holomorphic and is not identically zero as a function on $G(\mathbb{A})$ when $Re(s)>0$.*

*Proof.* We use proposition 13. Let

$$F(s) = \frac{\prod_{\alpha \in \Phi_B} \langle \rho_B^P + s\varpi, \alpha^\vee \rangle}{\prod_{\alpha \in \Phi_B - \Delta_B^P} \langle \rho_B^P + s\varpi, \alpha^\vee \rangle - 1}$$

We show that there are lots of cancellations as in the $\mathrm{SL}_3$ case 2.3.1. The decomposition $P = N \rtimes M$ gives

$$\Phi_B = \Phi_M^+ \sqcup \Phi_N$$

where $\Phi_M^+$ and $\Phi_N$ are the roots with root groups in $M \cap B$ and $N$ respectively.

**Step 1:** Terms from $\Phi_M^+$ do not contribute. For any $\eta \in \Phi_M^+$, the term $\langle \rho_B^P + \Lambda_P, \eta^\vee \rangle = \langle \rho_B^P, \eta^\vee \rangle$ is a non-zero constant. If $\eta \notin \Delta_B^P$, then $\langle \rho_B^P, \eta^\vee \rangle > 1$ and

$$\frac{\prod_{\eta \in \Phi_M^+} \langle \rho_B^P + \Lambda_P, \eta^\vee \rangle}{\prod_{\eta \in \Phi_M^+ - \Delta_B^P} \langle \rho_B^P + \Lambda_P, \eta^\vee \rangle - 1} = \text{constant } c > 0$$

Thus, only the terms from $\Phi_N$ need to be considered.

**Step 2:** Grouping the terms in $\Phi_N$ for cancellation in the next step. Let

$${}^L\mathfrak{n} = r_1 \oplus r_2 \oplus \cdots \oplus r_m$$

and

$$\Gamma_j := \left\{ \gamma^\vee \in \Phi_B^\vee : {}^L\mathfrak{g}_{\gamma^\vee} \subset r_j \right\}$$

That is, $\langle \varpi, \gamma^\vee \rangle = j$ for all $\gamma^\vee \in \Gamma_j$. Let $\{H, X, Y\}$ be the principal $\mathfrak{sl}_2\mathbb{C}$ triple in ${}^L\mathfrak{m}$. Note that

$$r_j = \bigoplus_{\gamma \in \Gamma_j} {}^L\mathfrak{g}_{\gamma^\vee}$$

is a decomposition of $r_j$ into $H$-eigenspaces (although not under the full $\mathfrak{sl}_2\mathbb{C}$, see example 9).

**Step 3:** Cancellations. For $\gamma \in \Gamma_j$ and $\Lambda = \rho_B^P + s\varpi \in S_P$, we have

$$\langle \Lambda, \gamma^\vee \rangle = \langle \rho_B^P, \gamma^\vee \rangle + \langle s\varpi, \gamma^\vee \rangle = \langle \rho_B^P, \gamma^\vee \rangle + js$$

Thus, only $H$ eigenvalues $\langle \rho_B^P, \bullet \rangle$ play a role in the cancellation. For roots $\gamma_0, \gamma_1, \ldots, \gamma_k \in \Gamma$ with

$$\left\{ \langle 2\rho_B^P, \gamma_i^\vee \rangle : i = 0, \ldots, k \right\} = \{-k, -k+2, \ldots, k\},$$

we have

$$\prod_{i=0}^{k} \frac{\langle \rho_B^P + s\varpi, \gamma_i^\vee \rangle}{\langle \rho_B^P + s\varpi, \gamma_i^\vee \rangle - 1} = \frac{js + k/2}{js - k/2 - 1}$$

**Step 4:** Conclusion. We have

$$F(s) = \frac{\prod_{j=1}^{m} \prod_{\ell \geq 0} (js + \ell/2)^{m_\ell(j)}}{\prod_{j=1}^{m} \prod_{\ell \geq 0} (js - 1 - \ell/2)^{m_\ell(j)}} \in \mathbb{C}(s)$$

Since the terms in the numerator vanish only for $s \leq 0$ and the terms in the denominator vanish only for $s > 0$, there can be no further cancellation. If $p(s)$ is the denominator in the above expression of $F(s)$, then by proposition 13, we have $p(s) \cdot E_{s\varpi}^P(g)$ is holomorphic for $\mathrm{Re}(s) \geq 0$. Since the numerator of $F(s) \neq 0$ when $\mathrm{Re}(s) > 0$, the non-vanishing assertion follows from the non-vanishing of $E^*_{\rho_B^P + s\varpi}$ which itself follows from the simplicity of the zeros of $E_\Lambda^B$ . □

### 2.3.7 Explicit computations illustrated for $A_n$ and $G_2$

Let $P$ be a maximal parabolic subgroup of $G$ with $\Theta := \Delta_B^P = \Delta_B - \{\beta\}$ for some $\beta \in \Delta_B$. To compute the poles of $E_\Lambda^P(g)$ in the positive tube, we need to decompose ${}^L\mathfrak{n}$ under the action of the principal $\mathfrak{sl}_2\mathbb{C}$ in ${}^L\mathfrak{m}$.

Let $H, X, Y$ be the standard $\mathfrak{sl}_2$ triple for the principal $\mathfrak{sl}_2\mathbb{C}$ in ${}^L\mathfrak{m}$. Let $\gamma^\vee$ be such that

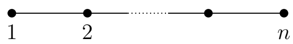
$$\gamma^\vee = \cdots + j\beta^\vee + \cdots$$

Then ${}^L\mathfrak{g}_{\gamma^\vee} \subset r_j$. Since $\langle 2\rho_B^P, \theta^\vee \rangle = 2$ for $\theta \in \Theta$, if $\gamma^\vee + \theta^\vee$ is a coroot, then the $H$-eigenvalue corresponding to $\gamma^\vee + \theta^\vee$ is $2 + \langle 2\rho_B^P, \gamma^\vee \rangle$. Thus, one can decompose ${}^L\mathfrak{n}$ by counting the number of roots $\gamma^\vee$ with ${}^L\mathfrak{g}_{\gamma^\vee} \subset r_j$ of a given height. This gives us a list of $H$-eigenvalues on $r_j$, which is enough to decompose $r_j$ abstractly in terms of symmetric powers under the principal $\mathfrak{sl}_2\mathbb{C}$ action.

*Remark* 16. Note that ordering of roots by height and the $j$s occurring in the decomposition of ${}^L\mathfrak{n}$ does not depend on the lattice of characters. Thus, the result only depends on the *root system* of the dual group. That is, the results are identical for different isogeny classes of $G$.

**Example 17.** We illustrate the procedure for $A_n$ with the Dynkin diagram labelling:

$A_n$:




Let $\alpha_1, \dots, \alpha_n$ be the corresponding simple roots. The positive roots are of the form

$$r_{ij} := \alpha_i + \cdots + \alpha_j, \quad 1 \le i \le j \le n$$

Height of $r_{ij} = j - i + 1$. There are $p_i := n - i + 1$ roots of height $i$.

Let $P = N \rtimes M$ be the standard Levi decomposition of a standard *maximal* parabolic subgroup $P$. Thus $\Delta_B - \Delta_B^P$ has cardinality one. Let ${}^L M$ be of type $A_a \times A_b$ where $a + b = n - 1$. The number $m_i$ of positive roots of height $i$ with root spaces in ${}^L\mathfrak{m}$ is

$$m_i = \begin{cases} (a - i + 1) + (b - i + 1) = n - 2i + 1 & i \le \min\{a, b\} \\ \max\{a, b\} - i + 1 & \min\{a, b\} < i \le \max\{a, b\} \\ 0 & i > \max\{a, b\} \end{cases}$$

The number $n_i$ of roots of height $i$ with root spaces contained in ${}^L\mathfrak{n}$ is

$$n_i = p_i - m_i = \begin{cases} i & i \le \min\{a, b\} \\ n - \max\{a, b\} = \min\{a, b\} + 1 & \min\{a, b\} < i \le \max\{a, b\} \\ n - i + 1 & i > \max\{a, b\} \end{cases}$$

Thus,

$${}^L\mathfrak{n} \simeq \bigoplus_{k = n-1-2\min\{a,b\}}^{(n-1)} V_k, \quad k \text{ increments of } 2$$

When $\mathrm{Re}(s) \ge 0$, the Eisenstein series $E^P_{s\varpi}(g)$ has simple poles for

$$s = 1 + \frac{n-1}{2}, \cdots, 1 + \frac{n - 1 - 2\min\{a, b\}}{2}$$

or

$$s = \frac{n+1}{2}, \frac{n-1}{2}, \cdots, \frac{n+1}{2} - \min\{a, b\}$$

**Example 18.** This example illustrates some features not present in type $A_n$. Let $G = G_2$, the exceptional $k$-group split over $k$. Let $\alpha$

be the short simple root and $\beta$ be the long simple root of $G_2$. The positive roots are

$$\Phi_B = \{\alpha,\ \beta,\ \alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta,\ 3\alpha+2\beta\}$$

There are two maximal parabolic subgroups $P = N_P \rtimes M_P$ and $Q = N_Q \rtimes M_Q$. Let $\Delta_B^P = \{\alpha\}$ and $\Delta_B^Q = \{\beta\}$.

Note that in ${}^LG \simeq G_2$, the root $\alpha^\vee$ is the long simple root and the root $\beta^\vee$ is the short simple root.

For $P$: We have

$${}^L\mathfrak{n}_P = r_1 \oplus r_2 \oplus r_3, \quad r_1, r_3 \simeq V_1 \text{ and } r_2 \simeq V_0$$

and

$$s = \frac{3}{2}, \frac{1}{2}\left(1+\frac{0}{2}\right), \frac{1}{3}\left(1+\frac{1}{2}\right)$$

That is, $E_s^P(g)$ has a simple pole at $s = \frac{3}{2}$ and a double pole at $s = \frac{1}{2}$.

For $Q$: We have

$${}^L\mathfrak{n}_Q = r_1 \oplus r_2, \quad r_1 \simeq V_3,\ r_2 \simeq V_0$$

and

$$s = 1+\frac{3}{2}, \frac{1}{2}\left(1+\frac{0}{2}\right)$$

That is, $E_s^Q(g)$ has a simple poles at $s = \frac{5}{2}, \frac{1}{2}$.

*Remark.* These results have been known at least since 1976 (see appendix III of Langlands [31]).

## 2.4 Computations for classical groups

Throughout this section, we denote by $V_k$ the $(k+1)$-dimensional irreducible representation of $\mathfrak{sl}_2\mathbb{C}$. We begin by recalling the following fact.

**Fact 19.** *For a semi-simple Lie algebra $\mathfrak{g}$ with a chosen basis of simple roots $\Delta$, the principal $\mathfrak{sl}_2 = span\{H, X, Y\}$ satisfies*

$$\alpha(H) = 2 \quad \textit{for all } \alpha \in \Delta.$$

This characterization allows us to explicitly write $H$ in all our computations using the data given in Bourbaki [3] as appendices (called "plates").

Throughout this section, let $P = N \rtimes M$ be the standard Levi decomposition of a standard *maximal* parabolic subgroup $P$. To compute the poles of $E_\Lambda^P(g)$ in the right-half space, we need to compute the decomposition of ${}^L\mathfrak{n}$ under the action of the principal $\mathfrak{sl}_2$ with standard triples $\{H, X, Y\}$ in ${}^L\mathfrak{m}$. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$.

We write $(\mu; m)$ to indicate that the Eisenstein series has a pole of order $m$ at $s = \mu$. If the pole is simple, we only write $\mu$.

### 2.4.1 Description of the groups

Let $k$ be a number field. We view $V = k^n$ as column vectors. For an $n \times n$ matrix $\Omega$, let

$$G_\Omega(V) = \left\{g \in GL(V) : g^T \Omega g = \Omega\right\}$$

Let

$$\mathrm{SL}_n(V) = \{g \in GL(V) : \det g = 1\}$$

Now we define the isometry groups. Let

$$\omega_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad n \times n \text{ matrix}$$

Let

$$\mathrm{Sp}_{2n}(V) = G_\Omega(V), \quad V = k^{2n},\ \Omega = \begin{pmatrix} 0 & \omega_n \\ -\omega_n & 0 \end{pmatrix}$$

and (for $n \geq 2$)

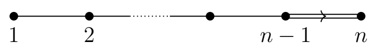
$$\mathrm{SO}_n(V) = G_\Omega(V) \bigcap \mathrm{SL}(V), \quad V = k^n,\ \Omega = \omega_n$$

We take the standard choice of maximal isotropic flags to define our Borel subgroup as upper-triangular matrices in these groups.

### 2.4.2 $B_n$: odd special orthogonal groups

The Dynkin diagram is

$B_n$:




The dual group is $\mathrm{Sp}_{2n}(\mathbb{C})$.

#### 2.4.2.1 Siegel parabolic case

For $\Delta - \Delta_B^P = \{\alpha_n\}$, the derived group of ${}^L M$ is of type $A_{n-1}$. The corresponding $H$ is

$$H = \mathrm{diag}(n-1, n-3, \ldots, 1-n, n-1, n-3, \ldots, 1-n)$$

Note that $\alpha(H) = 2$ for all $\alpha \in \Delta_B^P$. We identify ${}^L\mathfrak{n}$ with $n \times n$ matrices symmetric about the non-principal diagonal. Then

$${}^L\mathfrak{n} = r_1 \simeq V_{2(n-1)} \oplus V_{2(n-3)} \oplus \cdots \oplus \begin{cases} V_0 & n \text{ odd} \\ V_2 & n \text{ even} \end{cases}$$

Poles at

$$s = n, n-2, n-4, \ldots, \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}.$$

*Remark* 20. We note that if $m \in M(\mathbb{A}) \simeq GL_n(\mathbb{A})$, then

$$e^{\langle s\varpi, H_P(m)\rangle} = |\det m|^{s/2}$$

A common parametrization is $|\det m|^s$ in which case we have simple poles at

$$s = \frac{n}{2}, \frac{n}{2} - 1, \frac{n}{2} - 2, \ldots, \begin{cases} 1/2 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

#### 2.4.2.2 Non-Siegel parabolic subgroups

For $\Delta - \Delta_B^P = \{\alpha_a\}$ for $1 \leq a < n$, the derived group of ${}^L M$ is of type $A_{a-1} \times C_b$ with $a + b = n$ and $b \geq 1$. The corresponding $H$ is

$$H = (a-1, \ldots, 1-a, 2b-1, \ldots, 3, 1, -1, -3, \ldots, 1-2b, a-1 \ldots, 1-a)$$

using fact 19. We have

$$^{L}\mathfrak{n} = r_1 \oplus r_2$$

We parametrize $r_2$ with $a \times a$ matrices symmetric about the non-principal diagonal. We have

$$r_2 \simeq V_{2(a-1)} \oplus V_{2(a-3)} \oplus \cdots \oplus \begin{cases} V_0 & a \text{ odd} \\ V_2 & a \text{ even} \end{cases}$$

and this contributes at a simple pole at the points

$$s = \frac{a}{2}, \frac{a}{2} - 1, \frac{a}{2} - 1 \ldots, \begin{cases} 1/2 & a \text{ odd} \\ 1 & a \text{ even} \end{cases}$$

We parametrize $r_1$ with $a \times 2b$ matrices. The corresponding $H$-eigenvalues are given by

$$\begin{bmatrix} a-2b & (a-2b)+2 & \cdots & (a+2b)-2 \\ (a-2b)+2 & & & (a+2b)-4 \\ \vdots & & \iddots & \vdots \\ 4-(a+2b) & 6-(a+2b) & & \\ 2-(a+2b) & 4-(a+2b) & \cdots & 2b-a \end{bmatrix}$$

The structure is similar to the $A_n$ computation and we have

$$r_1 \simeq \bigoplus_{k=a+2b-2-2(\min\{a,2b\}-1)}^{k=a+2b-2} V_k, \quad k \text{ increments of } 2$$

This contributes a simple pole at

$$s = \frac{a+2b}{2}, \frac{a+2b}{2} - 1, \ldots, \frac{a+2b}{2} - (\min\{a, 2b\} - 1)$$

$E^{P}_{s\varpi}$ has double poles when

$$\frac{a}{2} \geq \frac{a+2b}{2} - (\min\{a, 2b\} - 1) \iff \min\{a, 2b\} \geq b + 1$$

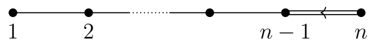
at

$$s = \frac{a}{2}, \ldots, \frac{a+2b+2}{2} - \min\{a, 2b\}$$

### 2.4.3 $C_n$: symplectic groups

The Dynkin diagram is

$C_n$:




#### 2.4.3.1 Siegel parabolic case

For $\Delta - \Delta_B^P = \{\alpha_n\}$, the derived group of ${}^LM$ is of type $A_{n-1}$. The corresponding $H$ is

$$H = \mathrm{diag}(n-1, n-3, \ldots, 1-n, 0, n-1, n-3, \ldots, 1-n)$$

We have

$${}^L\mathfrak{n} = r_1 \oplus r_2$$

with $r_1 \simeq V_{n-1}$ which gives a simple pole at $s = \frac{n+1}{2}$. We can parametrized $r_2$ by $n \times n$ matrices skew-symmetric about the non-principal diagonal and

$$r_2 \simeq V_{2(n-2)} \oplus V_{2(n-4)} \oplus \cdots \oplus \begin{cases} V_0 & n \text{ even} \\ V_2 & n \text{ odd} \end{cases}$$

We have simple poles at

$$s = \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \ldots, \begin{cases} \frac{1}{2} & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

*Remark.* This result was proved by Kudla-Rallis [26].

#### 2.4.3.2 Non-Siegel parabolic subgroups

For $\Delta - \Delta_B^P = \{\alpha_a\}$ for $1 \leq a < n$, the derived group of ${}^LM$ is of type $A_{a-1} \times B_b$ with $a + b = n$ and $b \geq 1$. The corresponding $H$ is

$$H = (a-1, \ldots, 1-a, 2b, \ldots, 2, 0, -2, \ldots, -2b, a-1 \ldots, 1-a)$$

using fact 19. We have ${}^L\mathfrak{n} = r_1 \oplus r_2$ where

$$r_2 \simeq V_{2(a-2)} \oplus V_{2(a-4)} \oplus \cdots \oplus \begin{cases} V_0 & a \text{ even} \\ V_2 & a \text{ odd} \end{cases}$$

This contributes simple poles at

$$s = \frac{a-1}{2}, \frac{a-3}{2}, \ldots, \begin{cases} \frac{1}{2} & a \text{ even} \\ 1 & a \text{ odd} \end{cases}$$

We parametrize $r_1$ with $a \times (2b+1)$ matrices. The corresponding $H$-eigenvalues are given by

$$\begin{bmatrix} (a-2b)-1 & (a-2b)+1 & \cdots & (a+2b)-1 \\ (a-2b)-3 & & & (a+2b)-3 \\ \vdots & & \iddots & \vdots \\ 3-(a+2b) & 5-(a+2b) & & \\ 1-(a+2b) & 3-(a+2b) & \cdots & (2b-a)+1 \end{bmatrix}$$

Thus,

$$r_1 \simeq \bigoplus_{k=a+2b-1-2(\min\{a,2b+1\}-1)}^{k=a+2b-1} V_k, \quad k \text{ increments of } 2$$

This contributes simple poles at

$$s = \frac{a+2b+1}{2}, \frac{a+2b+1}{2} - 1, \ldots, \frac{a+2b+1}{2} - (\min\{a, 2b+1\} - 1)$$

We have double poles when

$$\frac{a-1}{2} \geq \frac{a+2b+1}{2} - (\min\{a, 2b+1\} - 1) \iff \min\{a, 2b+1\} \geq b+2$$

at

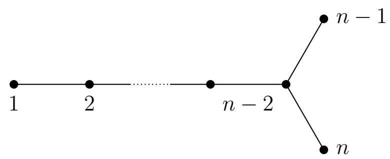
$$\frac{a-1}{2}, \frac{a-3}{2}, \ldots, \frac{a+2b+3}{2} - \min\{a, 2b+1\}$$

*Remark.* See Hanzer [18] for a more extensive discussion of this case.

### 2.4.4 $D_n$: even special orthogonal groups

We take $n \geq 3$. The Dynkin diagram is

$D_n$:



The simple roots are denoted $\alpha_1, \ldots, \alpha_n$ with the subscript corresponding to the labelled node in the Dynkin diagram. The positive roots are $\alpha_{ij}, \alpha'_{ij}$ and $\alpha'_i$, where

$$\alpha_{ij} = \alpha_i + \cdots + \alpha_j \quad (1 \le i \le j \le n-1)$$

$$\alpha'_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \quad (1 \le i < j \le n-1)$$

$$\alpha'_i = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n \quad (1 \le i \le n-1)$$

The notation gives the convenience $\alpha'_{n-1} = \alpha_n$.

The structure of parabolic subgroups of this case is a bit different. There are two "Siegel-type" parabolic subgroups corresponding to $\Delta - \Delta_B^P$ equals $\{\alpha_{n-1}\}$ or $\{\alpha_n\}$. The derived group of the Levi in both cases in of type $A_{n-1}$. We have ${}^L\mathrm{SO}_{2n} = \mathrm{SO}_{2n}(\mathbb{C})$.

#### 2.4.4.1 Siegel-type parabolic case

For $\Delta - \Delta_B^P = \{\alpha_n\}$, the derived group of ${}^LM$ is of type $A_{n-1}$. The corresponding $H$ is

$$H = \mathrm{diag}(n-1, n-3, \ldots, 1-n, 1-n, 3-n, \ldots, n-1)$$

We identify ${}^L\mathfrak{n}$ with $n \times n$ matrices skew-symmetric spanned by all $\{\alpha'_{ij}\}$ and $\{\alpha'_i\}_{i=1}^{n-1}$. We have

$${}^L\mathfrak{n} = r_1 \simeq V_{2(n-2)} \oplus V_{2(n-4)} \oplus \cdots \oplus \begin{cases} V_0 & n \text{ even} \\ V_2 & n \text{ odd} \end{cases}$$

We have simple poles at

$$s = n-1, n-3, n-5, \ldots, \begin{cases} 1 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

For $\Delta - \Delta_B^P = \{\alpha_n\}$, we have ${}^L\mathfrak{n} = r_1$ spanned by all $\{\alpha'_{ij}\}$ and $\{\alpha_{i,n-1}\}_{i=1}^{n-1}$. Therefore the result is the same as the above case, namely, simple poles at

$$s = n-1, n-3, n-5, \ldots, \begin{cases} 1 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

See remark 20.

#### 2.4.4.2 Non-Siegel parabolic subgroups

For $\Delta - \Delta_B^P = \{\alpha_a\}$ for $1 \leq a \leq n-2$, the derived group of ${}^L M$ is of type $A_{a-1} \times D_b$ with $a+b=n$ and $b \geq 2$. The corresponding $H$ is

$$H = (a-1, \dots, 1-a, 2(b-1), \dots, 2, 0, 0, -2, \dots, 2(1-b), a-1, \dots, 1-a)$$

using fact 19. We have ${}^L\mathfrak{n} = r_1 \oplus r_2$ where

$$r_2 \simeq V_{2(a-2)} \oplus V_{2(a-4)} \oplus \cdots \oplus \begin{cases} V_0 & a \text{ even} \\ V_2 & a \text{ odd} \end{cases}$$

This contributes simple poles at

$$s = \frac{a-1}{2}, \frac{a-3}{2}, \dots, \begin{cases} \frac{1}{2} & a \text{ even} \\ 1 & a \text{ odd} \end{cases}$$

We parametrize $r_1$ by two $a \times 2b$ matrices with the corresponding $H$-eigenvalues

$$\begin{bmatrix} a-2b+1 & \cdots & a-1 & a-1 & \cdots & a+2b-3 \\ a-2b-1 & & a-3 & a-3 & & a+2b-5 \\ \vdots & & \vdots & \vdots & & \vdots \\ 3-a-2b & \cdots & 1-a & 1-a & \cdots & -a+2b-1 \end{bmatrix}$$

The presence of two consecutive zeros in $H$ means we get an "extra" $V_{a-1}$. Writing it separately,

$$r_1 \simeq V_{a-1} \oplus \bigoplus_{k=a+2b-3-2(\min\{a,2b-1\}-1)}^{a+2b-3} V_k \quad k \text{ increments of } 2$$

This contributes simple poles at

$$k = \frac{a+2b-1}{2}, \dots, \frac{a+2b-1}{2} - (\min\{a, 2b-1\} - 1)\,; \quad \frac{a+1}{2}$$

We have poles at

$$\frac{a+1}{2}, \frac{a-1}{2}, \dots, \frac{a+2b-1}{2} - (\min\{a, 2b-1) - 1)$$

with double poles when

$$\frac{a+1}{2} \geq \frac{a+2b-1}{2} - (\min\{a, 2b-1) - 1) \iff \min\{a, 2b-1\} \geq b$$

at

$$\frac{a+1}{2}, \dots, \frac{a+2b+1}{2} - \min\{a, 2b-1\}.$$

## 2.5 Computations for exceptional Chevalley groups

We use the standard numbering of roots for the exceptional groups as in Bourbaki [3]. The ordering of roots by height is available in Springer [41] (with a different numbering).

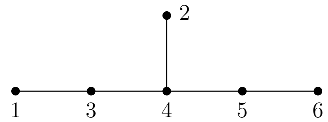
The Chevalley groups $E_6, E_7$, and $E_8$ are self dual and under $G \to {}^L G$ there is no relabelling of vertices in the corresponding map between Dynkin diagrams. This is not true for $G_2$ and $F_4$.

We write $(\mu; m)$ to indicate that the Eisenstein series has a pole of order $m$ at $s = \mu$. If the pole is simple, we only write $\mu$. The results of this section have been obtained using computers by Segal and Halawi [17].

### 2.5.1 Type $E_6$

The Dynkin diagram with standard numbering is

$E_6$:




#### 2.5.1.1 Poles for $P_1, P_6$

$$r_1 \simeq V_4 \oplus V_{10}$$

$$s = 3, 6$$

#### 2.5.1.2 Poles for $P_3, P_5$

$$r_1 \simeq V_1 \oplus V_3 \oplus V_5 \oplus V_7$$

$$r_2 \simeq V_4$$

$$s = \left(\frac{3}{2}; 2\right), \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$$

#### 2.5.1.3 Poles for $P_4$

$$r_1 \simeq V_1^{\oplus 2} \oplus V_3^{\oplus 2} \oplus V_5$$

$$r_2 \simeq V_0 \oplus V_2 \oplus V_4$$

$$r_3 \simeq V_1$$

$$s = \left(\frac{1}{2}; 2\right), 1, \left(\frac{3}{2}; 3\right), \left(\frac{5}{2}; 2\right), \frac{7}{2}$$

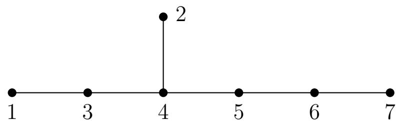
#### 2.5.1.4 Poles for $P_2$

$$r_1 \simeq V_3 \oplus V_5 \oplus V_9$$
$$r_2 \simeq V_0$$
$$s = \frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{11}{2}$$

### 2.5.2 Type $E_7$

The Dynkin diagram with standard numbering is

$E_7$:




#### 2.5.2.1 $P_1$

$$r_1 \simeq V_5 \oplus V_9 \oplus V_{15}$$
$$r_2 \simeq V_0$$
$$s = \frac{1}{2}, \frac{7}{2}, \frac{11}{2}, \frac{17}{2}$$

#### 2.5.2.2 $P_2$

$$r_1 \simeq V_0 \oplus V_4 \oplus V_6 \oplus V_8 \oplus V_{12}$$
$$r_2 \simeq V_6$$
$$s = 1, 2, 3, 4, 5, 7$$

#### 2.5.2.3 $P_3$

$$r_1 \simeq V_1 \oplus V_3 \oplus V_5 \oplus V_7 \oplus V_9$$
$$r_2 \simeq V_0 \oplus V_4 \oplus V_8$$
$$r_3 \simeq V_1$$
$$s = \left(\frac{1}{2}; 2\right), \left(\frac{3}{2}; 2\right), \left(\frac{5}{2}; 2\right), \frac{7}{2}, \frac{9}{2}, \frac{11}{2}$$

#### 2.5.2.4 $P_4$

$$r_1 \simeq V_0 \oplus V_2^{\oplus 2} \oplus V_4^{\oplus 2} \oplus V_6$$

$$r_2 \simeq V_2^{\oplus 2} \oplus V_4 \oplus V_6$$

$$r_3 \simeq V_2 \oplus V_4$$

$$r_4 \simeq V_2$$

$$s = \frac{1}{2}, \frac{2}{3}, (1;4), \frac{3}{2}, (2;3), (3;2), 4$$

#### 2.5.2.5 $P_5$

$$r_1 \simeq V_0 \oplus V_2 \oplus V_4^{\oplus 2} \oplus V_6 \oplus V_8$$

$$r_2 \simeq V_2 \oplus V_4 \oplus V_6$$

$$r_3 \simeq V_4$$

$$s = (1;3), \frac{3}{2}, (2;2), (3;2), 4, 5$$

#### 2.5.2.6 $P_6$

$$r_1 \simeq V_3 \oplus V_5 \oplus V_9 \oplus V_{11}$$

$$r_2 \simeq V_0 \oplus V_8$$

$$s = \frac{1}{2}, \left(\frac{5}{2}; 2\right), \frac{7}{2}, \frac{11}{2}, \frac{13}{2}$$

#### 2.5.2.7 $P_7$

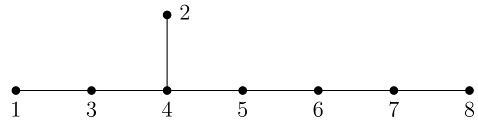
$$r_1 \simeq V_0 \oplus V_8 \oplus V_{16}$$

$$s = \frac{1}{2}, 5, 9$$

### 2.5.3 Type $E_8$

The Dynkin diagram with standard numbering is

$E_8$:



#### 2.5.3.1 $P_1$

$$r_1 \simeq V_3 \oplus V_9 \oplus V_{11} \oplus V_{15} \oplus V_{21}$$
$$r_2 \simeq V_0 \oplus V_{12}$$
$$s = \frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{11}{2}, \frac{13}{2}, \frac{17}{2}, \frac{23}{2}$$

#### 2.5.3.2 $P_2$

$$r_1 \simeq V_3 \oplus V_5 \oplus V_7 \oplus V_9 \oplus V_{11} \oplus V_{15}$$
$$r_2 \simeq V_0 \oplus V_4 \oplus V_8 \oplus V_{12}$$
$$r_3 \simeq V_7$$
$$s = \frac{1}{2}, \left(\frac{3}{2}; 2\right), \left(\frac{5}{2}; 2\right), \left(\frac{7}{2}; 2\right), \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{17}{2}$$

#### 2.5.3.3 $P_3$

$$r_1 \simeq V_1 \oplus V_3 \oplus V_5 \oplus V_7 \oplus V_9 \oplus V_{11}$$
$$r_2 \simeq V_0 \oplus V_4 \oplus V_6 \oplus V_8 \oplus V_{12}$$
$$r_3 \simeq V_5 \oplus V_7$$
$$r_4 \simeq V_6$$
$$s = \frac{1}{2}, 1, \frac{7}{6}, \left(\frac{3}{2}; 3\right), 2, \left(\frac{5}{2}; 2\right), \left(\frac{7}{2}; 2\right), \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$$

#### 2.5.3.4 $P_4$

$$r_1 \simeq V_1 \oplus V_3^{\oplus 2} \oplus V_5^{\oplus 2} \oplus V_7$$

$$r_2 \simeq V_0 \oplus V_2 \oplus V_4^{\oplus 2} \oplus V_6 \oplus V_8$$

$$r_3 \simeq V_1 \oplus V_3 \oplus V_5 \oplus V_7$$

$$r_4 \simeq V_2 \oplus V_4 \oplus V_6$$

$$r_5 \simeq V_1 \oplus V_3$$
$$r_6 \simeq V_4$$
$$s = \frac{3}{10}, \left(\frac{1}{2}; 5\right), \frac{3}{4}, \frac{5}{6}, (1; 2), \frac{7}{6}, \left(\frac{3}{2}; 4\right), 2, \left(\frac{5}{2}; 3\right), \left(\frac{7}{2}; 2\right), \frac{9}{2}$$

#### 2.5.3.5 $P_5$

$$r_1 \simeq V_1 \oplus V_3^2 \oplus V_5^2 \oplus V_7 \oplus V_9$$

$$r_2 \simeq V_0 \oplus V_2 \oplus V_4^2 \oplus V_6 \oplus V_8$$

$$r_3 \simeq V_1 \oplus V_3 \oplus V_5 \oplus V_7$$

$$r_4 \simeq V_2 \oplus V_6$$

$$r_5 \simeq V_3$$

$$s = \left(\frac{1}{2};4\right), \frac{5}{6}, (1;2), \frac{7}{6}, \left(\frac{3}{2};4\right), 2, \left(\frac{5}{2};3\right), \left(\frac{7}{2};2\right), \frac{9}{2}, \frac{11}{2}$$

#### 2.5.3.6 $P_6$

$$r_1 \simeq V_2 \oplus V_4 \oplus V_6 \oplus V_8 \oplus V_{10} \oplus V_{12}$$

$$r_2 \simeq V_2 \oplus V_6 \oplus V_8 \oplus V_{10}$$

$$r_3 \simeq V_4 \oplus V_{10}$$

$$r_4 \simeq V_2$$

$$s = \frac{1}{2}, (1;2), (2;3), \frac{5}{2}, (3;2), 4, 5, 6, 7$$

#### 2.5.3.7 $P_7$

$$r_1 \simeq V_1 \oplus V_7 \oplus V_9 \oplus V_{15} \oplus V_{17}$$

$$r_2 \simeq V_0 \oplus V_8 \oplus V_{16}$$

$$r_3 \simeq V_1$$

$$s = \left(\frac{1}{2};2\right), \frac{3}{2}, \frac{5}{2}, \left(\frac{9}{2};2\right), \frac{11}{2}, \frac{17}{2}, \frac{19}{2}$$

#### 2.5.3.8 $P_8$

$$r_1 \simeq V \oplus V_{17} \oplus V_{27}$$

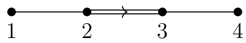
$$r_2 \simeq V_0$$

$$s = \frac{1}{2}, \frac{11}{2}, \frac{19}{2}, \frac{29}{2}$$

### 2.5.4 Type $F_4$

The Dynkin diagram with standard numbering is

$F_4$:




Recall that there is a relabelling of roots under $G \to {}^LG$: $\alpha_1^\vee \rightsquigarrow \alpha_4$, $\alpha_2^\vee \rightsquigarrow \alpha_3$, $\alpha_3^\vee \rightsquigarrow \alpha_2$, and $\alpha_4^\vee \rightsquigarrow \alpha_1$. If $P_i = N_i \rtimes M_i$ corresponds to $\alpha_i$, then

$$ {}^L\mathfrak{n}_1 \simeq \mathfrak{n}_4 \quad {}^L\mathfrak{n}_2 \simeq \mathfrak{n}_3, \quad {}^L\mathfrak{n}_3 \simeq \mathfrak{n}_2, \quad \text{and} \quad {}^L\mathfrak{n}_4 \simeq \mathfrak{n}_1 $$

under the action of the corresponding $\mathfrak{sl}_2\mathbb{C}$.

#### 2.5.4.1 $P_1$

$$ r_1 \simeq V_0 \oplus V_6 $$
$$ r_2 \simeq V_6 $$
$$ s = 1, 2, 4 $$

#### 2.5.4.2 $P_2$

$$ r_1 \simeq V_1 \oplus V_3 $$
$$ r_2 \simeq V_0 \oplus V_2 \oplus V_4 $$
$$ r_3 \simeq V_1, \quad r_4 = V_2 $$
$$ s = \left(\frac{1}{2}; 3\right), 1, \left(\frac{3}{2}; 2\right), \frac{5}{2} $$

#### 2.5.4.3 $P_3$

$$ r_1 = V_1 \oplus V_3 \oplus V_5 $$
$$ r_2 = V_0 \oplus V_4 $$
$$ r_3 = V_1 $$
$$ s = \left(\frac{1}{2}; 2\right), \left(\frac{3}{2}; 2\right), \frac{5}{2}, \frac{7}{2} $$

#### 2.5.4.4 $P_4$

$$ r_1 = V_3 \oplus V_9 $$
$$ r_2 = V_0 $$
$$ s = \frac{1}{2}, \frac{5}{2}, \frac{11}{2} $$

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