Superintegrable Systems in Darboux spaces

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Abstract

Almost all research on superintegrable potentials concerns spaces of constant curvature. In this paper we find by exhaustive calculation, all superintegrable potentials in the four Darboux spaces of revolution that have at least two integrals of motion quadratic in the momenta, in addition to the Hamiltonian. These are two-dimensional spaces of nonconstant curvature. It turns out that all of these potentials are equivalent to superintegrable potentials in complex Euclidean 2-space or on the complex 2-sphere, via "coupling constant metamorphosis" (or equivalently, via Stäckel multiplier transformations). We present tables of the results.

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1 Introduction

In a previous article [1] we have studied superintegrability in a two-dimensional space of nonconstant curvature, in particular one of the so called Darboux spaces, given by Koenigs [2]. In this article we study the remaining three spaces of nonconstant curvature from the point of view of superintegrability. This involves the addition of a potential to each of the spaces given by Koenigs. We recall that classical superintegrability relating to a Hamiltonian $H(x_1, \ldots, x_n, p_1, \ldots, p_n) = H(x, p)$ implies the existence of $2n-1$ globally defined constants of the motion. For the purposes of this article we restrict this definition to require that there exist $2n-1$ globally defined functionally independent constants of the motion X_i, $i = 1, \ldots, 2n-1$ that are quadratic in the canonical momenta p_i. This clearly implies the relations

$$\{H, X_\ell\} = \sum_{i=1}^{n} \left(\frac{\partial X_\ell}{\partial x_i} \frac{\partial H}{\partial p_i} - \frac{\partial X_\ell}{\partial p_i} \frac{\partial H}{\partial x_i} \right) = 0 \,, \qquad i = 1, \ldots, 2n-1 \,.$$

The concepts of integrability and superintegrability also have their analogue in quantum mechanics. A superintegrable quantum mechanical system is described by $2n-1$ (independent) quantum observables $\hat{H} = \hat{X}_1, \hat{X}_2, \ldots, \hat{X}_{2n-1}$ that satisfy the commutation relations

$$[\hat{H}, \hat{X}_i] = \hat{H}\hat{X}_i - \hat{X}_i\hat{H} = 0, \qquad i = 1, \ldots, 2n-1.$$

The analogue of quadratic superintegrability in this case is that each of the quantum observables is a second order partial differential operator. Systematic studies of superintegrable systems have been conducted in spaces of constant curvature in two dimensions [3, 4, 5, 6, 7].

In this article we solve the following problem. Given a Riemannian space in two dimensions with infinitesimal distance $ds^2 = \sum_{i,j=1}^{2} g_{ij}(u) du^i du^j$, and $u = (u^1, u^2)$, the classical Hamiltonian has the form

$$H = \sum_{i,j=1}^{2} g^{ij} p_i p_j + V(u)$$

and the corresponding Schrödinger equation is

$$\hat{H}\Psi = \frac{1}{\sqrt{g}} \partial_{u^i} \left(\sqrt{g} g^{ik} \partial_{u^k} \Psi \right) + V(u)\Psi = E\Psi$$

where $\sqrt{g} = \det(g_{ij})$. Koenigs found all free Hamiltonians $H = \sum g^{ij} p_i p_j$ admitting at least two extra functionally independent constants of the motion of the form

$$\Lambda = \sum_{i,j=1}^{2} a^{ij}(u) p_i p_j, \qquad a^{ij} = a^{ji}.$$

He obtained a number of families of solutions; in particular, spaces that admitted three extra quadratic constants. There must then be a functional relation between these and, furthermore, in each case there is a Killing vector, i.e., a function $\mu = \sum_{i=1}^{2} a^i(u) p_i$ that satisfies $\{H,\mu\}=0$. One of the three quadratic constants is a square of the Killing vector μ.

The problem we solve here is supplemental to that of Koenigs: Suppose we have a Hamiltonian $H=\sum g^{ij}p_ip_j+V(u)$ that admits a Killing vector. We determine the *potentials* that correspond to superintegrability, i.e., potentials such that we can find at least two extra functionally independent quadratic constants of the form

$$\Lambda=\sum_{i,j=1}^{2} a^{ij}(u)p_ip_j+\lambda(u)\,.$$

A necessary condition that this be possible is that the Riemannian space be one of the four listed by Koenigs:

1. $ds^2=(x+y)dxdy$

2. $ds^2=\left(\frac{a}{(x-y)^2}+b\right)dxdy$

3. $ds^2=\left(ae^{-(x+y)/2}+be^{-x-y}\right)dxdy$

4. $ds^2=\frac{a\left(e^{(x-y)/2}+e^{(y-x)/2}\right)+b}{\left(e^{(x-y)/2}-e^{(y-x)/2}\right)^2}dxdy.$

The first of these spaces, type one, or D_1, has been treated in detail in an earlier paper [1]. Here we treat the remaining three Darboux spaces in a similar and unified way. Sections 2, 3, and 4 are devoted to the spaces D_2, D_3, and D_4, respectively. In each space we follow the same pattern.

1. We first consider a classical free particle system and give the free Hamiltonian H_0, the Killing vector K and the two Killing tensors X_1 and X_2 in a space with a conformally Euclidean metric (real or complex). We choose coordinates u and v in which the first order constant is $K=p_v$, hence u is an ignorable variable, not appearing in the metric or in the Hamiltonian.

2. We present an embedding of the two-dimensional Darboux space into a three-dimensional flat space.

3. We present a polynomial relation between the four integrals of motion H, K, X_1 and X_2, and also the polynomial algebra generated by these integrals.

4. We consider the quantum mechanics of a free particle in the corresponding Darboux space, i.e., write the corresponding Hamiltonian and integrals of motion as linear operators. We then establish that the relations between these operators are the same as those between the classical quantities.

5. We use the fact that the Killing vector K generates a one-dimensional Lie transformation group to classify all integrals of motion

$$\lambda=aX_1+bX_2+cK^2 \tag{1.1}$$

into conjugation classes. Each class gives rise to a coordinate system in which the Hamilton-Jacobi and Schrödinger equations allow the separation of variables. We construct these separable coordinate systems explicitly and solve the corresponding separated equations (classical and quantum).

6. By construction, the free classical and quantum systems in Darboux spaces are all quadratically superintegrable: they have 3 functionally independent integrals of motion. We introduce potentials that do not destroy this superintegrability. Thus we present systematically all superintegrable classical and quantum systems of the form
$$H = H_0 + V(u,v) \tag{1.2}$$
where H_0 is the free Hamiltonian in the space D_2, D_3, or D_4. To obtain this result we make use of the fact that to be quadratically superintegrable, a Hamiltonian in a Darboux space must allow the separation of variables in at least two coordinate systems.

A separate section, Section 5, is devoted to the relation between superintegrable systems in Darboux spaces and two-dimensional spaces of constant curvature.

2 Darboux spaces of type two

2.1 The free particle and separating coordinate systems

If we allow rescaling of the variables x and y, as well as the Hamiltonian H then we can always take H to be of the form

$$H_0 = \frac{(x-y)^2}{(x-y)^2-1}\ p_x p_y\,. \tag{2.1}$$

In the coordinates $x = \frac{1}{2}(v+iu)$, $y = \frac{1}{2}(v-iu)$ this Hamiltonian becomes

$$H_0 = \frac{u^2(p_u^2+p_v^2)}{u^2+1}\,.$$

Associated with the Hamiltonian are three integrals of the free motion

$$K = p_v, \qquad X_1 = \frac{2v(p_v^2-u^2p_u^2)}{u^2+1} + 2up_up_v, \quad X_2 = \frac{(v^2-u^4)p_v^2+u^2(1-v^2)p_u^2}{u^2+1} + 2uvp_up_v.$$

These three integrals satisfy the following polynomial algebra relations:

$$\{K, X_1\} = 2(K^2-H_0), \qquad \{K, X_2\} = X_1, \qquad \{X_1, X_2\} = 4KX_2. \tag{2.2}$$

They are functionally dependent via the relation

$$X_1^2 - 4K^2X_2 + 4H_0X_2 - 4H_0^2 = 0\,. \tag{2.3}$$

The corresponding problem in quantum mechanics can be obtained via the usual quantisation rules and symmetrisation:

$$\hat{H}_0 = \frac{u^2}{u^2+1}(\partial_u^2 + \partial_v^2), \quad \hat{K} = \partial_v, \quad \hat{X}_1 = \frac{2v}{(u^2+1)}(\partial_v^2 - u^2\partial_u^2) + 2u\partial_u\partial_v + \partial_v,$$

$$\hat{X}_2 = \frac{1}{u^2+1}\Big((v^2-u^4)\partial_v^2 + u^2(1-v^2)\partial_u^2\Big) + 2uv\partial_u\partial_v + u\partial_u + v\partial_v - \frac{1}{4},$$

where the constant in the last expression is taken for convenience. The commutation relations are identical with those of the corresponding classical algebra:

$$[\hat{K}, \hat{X}_1] = 2(\hat{K}^2 - \hat{H}_0) \qquad [\hat{K}, \hat{X}_2] = \hat{X}_1, \qquad [\hat{X}_1, \hat{X}_2] = 2\{\hat{K}, \hat{X}_2\}.$$

Here $\{\hat{K}, \hat{X}_2\} = \frac{1}{2}(\hat{K}\hat{X}_2 + \hat{X}_2\hat{K})$. The operator relation (that exists in analogy with the functional relation in the classical case) is

$$\hat{X}_1^2 - 2\{\hat{K}^2, \hat{X}_2\} + 4\hat{H}_0\hat{X}_2 - 4\hat{H}_0^2 - \hat{H}_0 + 4\hat{K}^2 = 0.$$

The line element $ds^2 = (du^2 + dv^2)(u^2+1)/u^2$ can be realised as a two-dimensional surfaced embedded in three dimensions by

$$X = \frac{v\sqrt{u^2+1}}{u}, \quad Y - T = \frac{\sqrt{u^2+1}}{u}, \quad Y + T = -\frac{(2u^4 + 5u^2 + 8v^2)\sqrt{u^2+1}}{8u} - \frac{3}{8}\text{arcsinh} u\,,$$

in which case,

$$ds^2 = dX^2 + dY^2 - dT^2 = \frac{u^2+1}{u^2}(du^2 + dv^2)\,.$$

We wish to determine all the essentially different separable coordinate systems for the free classical or quantum particle. In order to do this we need to consider a general quadratic constant of the form $\lambda = aX_1 + bX_2 + cK^2$. Under the adjoint action of $\exp(\alpha K)$, X_1 and X_2 transform according to

$$X_1 \to X_1 + 2\alpha(K^2 - H_0), \qquad X_2 \to X_2 + \alpha X_1 + \alpha^2(K^2 - H_0).$$

¿From these transformation formulae we see that if $b \neq 0$ we can always take λ in the form $\lambda = X_2 + \beta K^2$. If $b = 0$ then there are two representatives possible: X_1 or K^2. We have the following cases:

$$X_2 + \beta K^2\,, \qquad X_1\,, \qquad K^2\,. \tag{2.4}$$

We now demonstrate the explicit coordinates for each of these representatives using methods of our previous article[1].

2.1.1 Coordinates associated with $X_2 + \beta K^2$

If we choose $\beta = b^2$, $b \neq 0$ suitable coordinates ω, φ are

$$u = b\cosh\omega\cos\varphi, \quad v = b\sinh\omega\sin\varphi\,, \tag{2.5}$$

the standard form of elliptical coordinates in the plane. The classical Hamiltonian has the form

$$H_0 = \frac{p_\omega^2 + p_\varphi^2}{\sec^2\varphi - \mathrm{sech}^2\omega + b^2\left(\cosh^2\omega - \cos^2\varphi\right)}\,.$$

The corresponding quadratic constant, expressed in these coordinates is

$$X_2 + b^2K^2 = \frac{\left(\sec^2\varphi + b^2\sin^2\varphi\right)p_\omega^2 + \left(\mathrm{sech}^2\omega - b^2\sinh^2\omega\right)p_\varphi^2}{\left(\sec^2\varphi - \mathrm{sech}^2\omega\right) + b^2\left(\cosh^2\omega - \cos^2\varphi\right)}\,.$$

The Hamilton-Jacobi equation is

$$\frac{\left(\frac{\partial S}{\partial \omega}\right)^2 + \left(\frac{\partial S}{\partial \varphi}\right)^2}{\sec^2\varphi - \mathrm{sech}^2\omega + b^2\left(\cosh^2\omega - \cos^2\varphi\right)} = E\,,$$

with solutions of the form

$$S(\omega,\varphi) = \frac{b\sqrt{E}}{2}\left(\int \frac{1}{\Omega}\sqrt{\frac{(\Omega+\beta_1)(\Omega+\beta_2)}{\Omega-1}}\,d\Omega + \int \frac{1}{\Phi}\sqrt{\frac{(\beta_1-\Phi)(\Phi-\beta_2)}{1-\Phi}}\,d\Phi\right)\,.$$

where $\beta_1+\beta_2 = -\lambda/Eb^2$, $\beta_1\beta_2 = -1/b^2$, $\Phi = \cos^2\varphi$, $\Omega = \cosh^2\omega$. The corresponding Schrödinger equation

$$\frac{\left(\partial_\varphi^2 + \partial_\omega^2\right)\Psi}{\sec^2\varphi - \mathrm{sech}^2\omega + b^2\left(\cosh^2\omega - \cos^2\varphi\right)} = E\Psi$$

has solutions of the form

$$\Psi = \sqrt{\cos\varphi\cosh\omega}\; S_n^{m(j)}\left(i\sinh\omega, -\frac{1}{4}Eb\right)Ps_n^m\left(\sin\varphi, -\frac{1}{4}Eb\right)\,, \qquad j=1,2$$

where $S_n^{m(j)}(z,\kappa)$ and $Ps_n^m(t,\kappa)$ are spheroidal functions [8] and $E = m^2 - \frac{1}{4}$.

2.1.2 Coordinates associated with X_2

Here we use polar coordinates:

$$u = r\cos\theta\,, \qquad v = r\sin\theta\,. \tag{2.6}$$

The classical Hamiltonian has the form

$$H_0 = \frac{r^2p_r^2 + p_\theta^2}{r^2 + \sec^2\theta}$$

and the corresponding quadratic constant is

$$X_2 = \frac{r^2\sec^2\theta\,p_r^2 - p_\theta^2}{r^2 + \sec^2\theta}\,.$$

The Hamilton-Jacobi equation in these coordinates is

$$\frac{r^2\left(\frac{\partial S}{\partial r}\right)^2+\left(\frac{\partial S}{\partial \theta}\right)^2}{r^2+\sec^2\theta}=E\,,$$

with solution

$$\begin{aligned} S(r,\theta) &= \sqrt{Er^2+\lambda}-\sqrt{\lambda}\,\operatorname{arctanh}\sqrt{\frac{Er^2+\lambda}{\lambda}} \\ &\quad -\sqrt{\lambda}\,\log\left(\sqrt{\lambda}\sin\theta+\sqrt{E-\lambda\cos^2\theta}\right) \\ &\quad +\frac{1}{2}\sqrt{E}\,\operatorname{arccosh}\left(\frac{(E+\lambda)\cos^2\theta-2E}{(E-\lambda)\cos^2\theta}\right)\,. \end{aligned}$$

The corresponding Schrödinger equation is

$$\frac{r^2\frac{\partial^2\Psi}{\partial r^2}+\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)}{r^2+\sec^2\theta}=E\Psi\,,$$

and has solutions of the form

$$\Psi=\sqrt{r\sin\theta}\;C_{\ell+\frac{1}{2}}\left(\sqrt{-E}\;r\right)P_\ell^m(\cos\theta)\,,\qquad E=m^2-\frac{1}{4}\,,$$

where $C_\nu(z)$ is a Bessel function and $P_\ell^n(\cos\theta)$ is an associated Legendre polynomial [8].

2.1.3 Coordinates associated with X_1

A suitable choice of coordinates is

$$u=\xi\eta,\quad v=\frac{1}{2}(\xi^2-\eta^2)\,. \tag{2.7}$$

The classical Hamiltonian in these coordinates has the form

$$H_0=\frac{p_\xi^2+p_\eta^2}{\xi^2+\eta^2+\frac{1}{\xi^2}+\frac{1}{\eta^2}}.$$

The corresponding quadratic constant is

$$X_1=\frac{\left(\eta^2+\frac{1}{\eta^2}\right)p_\xi^2-\left(\xi^2+\frac{1}{\xi^2}\right)p_\eta^2}{\xi^2+\eta^2+\frac{1}{\xi^2}+\frac{1}{\eta^2}}\,.$$

The Hamilton-Jacobi equation has the form

$$\frac{\left(\frac{\partial S}{\partial\xi}\right)^2+\left(\frac{\partial S}{\partial\eta}\right)^2}{\xi^2+\eta^2+\frac{1}{\xi^2}+\frac{1}{\eta^2}}=E\,,$$

which has solution

$$
\begin{aligned}
S(\xi,\eta) \;\; = \;\; & -\frac{\sqrt{E\xi^4+E-\lambda\xi^2}}{\xi^2} - \frac{\lambda}{2\sqrt{E}}\ \operatorname{arctanh}\left(\frac{\lambda\xi^2-2E}{2\sqrt{E}\sqrt{E\xi^4+E-\lambda\xi^2}}\right) \\
& + \sqrt{E}\ \log\left(\sqrt{E}(2E\xi^2-\lambda)+2E\sqrt{E\xi^4+E-\lambda\xi^2}\right) \\
& - \frac{\sqrt{E\eta^4+E+\lambda\eta^2}}{\eta^2} - \frac{\lambda}{2\sqrt{E}}\ \operatorname{arctanh}\left(\frac{\lambda\eta^2+2E}{2\sqrt{E}\sqrt{E\xi^4+E+\lambda\xi^2}}\right) \\
& + \sqrt{E}\ \log\left(\sqrt{E}(2E\xi^2+\lambda)+2E\sqrt{E\xi^4+E+\lambda\xi^2}\right) .
\end{aligned}
$$

The corresponding Schrödinger equation is

$$
\frac{\partial_\xi^2\Psi+\partial_\eta^2\Psi}{\xi^2+\eta^2+\frac{1}{\xi^2}+\frac{1}{\eta^2}} = E\Psi \,.
$$

Typical solutions are

$$
\Psi = \frac{1}{\sqrt{\xi\eta}}\ M_{\chi,\mu}(\sqrt{E}\xi^2)M_{-\chi,\mu}(\sqrt{E}\eta^2)
$$

where $M_{\chi,\mu}(z)$ is a Whittaker function [9] and $E = 4\mu^2 - \frac{1}{4}$.

2.1.4 Coordinates associated with K^2

The representative K^2 has associated with it the coordinates u and v, in which the ignorable variable has a fundamental role to play. The Hamiltonian and constant associated with this separation have already been given. The Hamilton-Jacobi equation has the form

$$
\frac{u^2}{u^2+1}\left(\left(\frac{\partial S}{\partial u}\right)^2+\left(\frac{\partial S}{\partial v}\right)^2\right) = E\,,
$$

which has solution, with separation constant c,

$$
S(u,v) = \sqrt{u^2(E-c^2)+E} - \sqrt{E}\ \operatorname{arctanh}\sqrt{\frac{u^2(E-c^2)+E}{E}} + cv\,.
$$

The corresponding Schrödinger equation has the form

$$
\frac{u^2}{u^2+1}\left(\partial_u^2\Psi+\partial_v^2\Psi\right) = E\Psi.
$$

Typical solutions are

$$
\Psi = \sqrt{u}\ C_\nu\left(\sqrt{m^2-E}\ u\right)e^{mv}
$$

where $E = \nu^2 - \frac{1}{4}$.

It is no surprise that the Hamiltonian is separable in elliptic, parabolic and polar coordinates, since, if we write the classical equation $H = E$ in u, v coordinates we obtain

$$
p_u^2 + p_v^2 - E\left(\frac{1}{u^2}+1\right) = 0.
$$

This equation is essentially the same form as a flat space superintegrable system with Cartesian coordinates u, v and potential α/u^2, viz

$$p_u^2 + p_v^2 + \frac{\alpha}{u^2} - E = 0.$$

It is known to be solvable via the separation of variables Ansatz in elliptic, Cartesian, polar and parabolic coordinates. This correspondence between flat space superintegrable systems and their curved analogues is essentially the way all the curved superintegrable systems can be obtained and is discussed in more detail in section 5.

2.2 Superintegrability for Darboux spaces of type two.

In this section we address the problem of superintegrability for the Hamiltonian

$$H_0 = \frac{u^2(p_u^2 + p_v^2)}{u^2+1}\,. \tag{2.8}$$

This is done in exactly the same manner as it was for the Darboux space of type 1 in a previous paper [1]. The free space Hamiltonian is given and we compute the possible potentials that correspond to superintegrability. There are four possibilities:

[A] $H = \dfrac{u^2}{u^2+1}\left(p_u^2 + p_v^2 + a_1\left(\dfrac{1}{4}u^2 + v^2\right) + a_2 v + \dfrac{a_3}{u^2}\right).$

A basis for the additional constants of the motion is

$$\begin{aligned} R_1 &= X_1 + \frac{a_1}{2}v\left(u^2 + \frac{u^2+4v^2}{u^2+1}\right) + \frac{a_2}{2}\left(u^2 + \frac{4v^2}{u^2+1}\right) - \frac{2a_3 v}{u^2+1}\,, \\ R_2 &= K^2 + a_1 v^2 + a_2 v\,. \end{aligned}$$

These, along with $R = \{R_1, R_2\}$, form a quadratic algebra

$$\{R, R_1\} = -\frac{1}{2}\frac{\partial R^2}{\partial R_2}\,, \qquad \{R, R_2\} = \frac{1}{2}\frac{\partial R^2}{\partial R_1} \tag{2.9}$$

that is determined by the identity

$$\begin{aligned} R^2 = {} & 16R_2^3 - 4a_1 R_1^2 - 32HR_2^2 - 8a_2 R_1 R_2 \\ & + 8a_2 H R_1 + 16(H^2 + a_1 H - a_1 a_3)R_2 + 4a_2^2 H - 4a_2^2 a_3\,. \end{aligned}$$

The classical equation of motion $H - E = 0$ is

$$p_u^2 + p_v^2 + a_1\left(\frac{1}{4}u^2 + v^2\right) + a_2 v + \frac{a_3 - E}{u^2} - E = 0.$$

The basic form of this equation is a superintegrable system in flat space, but with rearranged constants, which is solvable via separation of variables in Cartesian and parabolic coordinates.

This accords with the fact that the leading part of a quadratic constant for this Hamiltonian will be an element of the orbits represented by X_1 and K^2. So this Hamiltonian also separates in the 'parabolic' coordinates ξ, η (2.7) and in these coordinates takes the form

$$H = \frac{p_\xi^2 + p_\eta^2 + \frac{1}{4}a_1\left(\xi^6+\eta^6\right) + \frac{1}{2}a_2\left(\xi^4-\eta^4\right) + a_3\left(\frac{1}{\eta^2}+\frac{1}{\xi^2}\right)}{\xi^2+\eta^2+\frac{1}{\xi^2}+\frac{1}{\eta^2}} .$$

Adding the same potential and coordinate functions to the quantum Hamiltonian $\hat{H}_0$ and its corresponding commuting operators $\hat{X}_1$ and $\hat{K}^2$, we obtain the operators

$$\begin{aligned}
\hat{H} &= \hat{H}_0 + \frac{u^2}{u^2+1}\left(a_1\left(\frac{1}{4}u^2+v^2\right)+a_2 v+\frac{a_3}{u^2}\right) \\
\hat{R}_1 &= \hat{X}_1 + \frac{a_1}{2}v\left(u^2+\frac{u^2+4v^2}{u^2+1}\right)+\frac{a_2}{2}\left(u^2+\frac{4v^2}{u^2+1}\right)-\frac{2a_3 v}{u^2+1}, \\
\hat{R}_2 &= \hat{K}^2 + a_1 v^2 + a_2 v .
\end{aligned}$$

$\hat{R}_1$ and $\hat{R}_2$ commute with $\hat{H}$ and along with $\hat{R}=[\hat{R}_1,\hat{R}_2]$, obey the corresponding quantum quadratic algebra relations

$$\begin{aligned}
[\hat{R},\hat{R}_1] &= -24\hat{R}_2^2 + 4a_2\hat{R}_1 + 32\hat{H}\hat{R}_2 - 8\hat{H}^2 - 8a_1\hat{H} + 6a_1 + 8a_1a_3 , \\
[\hat{R},\hat{R}_2] &= -4a_1\hat{R}_1 - 4a_2\hat{R}_2 + 4a_2\hat{H}
\end{aligned}$$

and the operator identity

$$\begin{aligned}
\hat{R}^2 = {} & 16\hat{R}_2^3 - 4a_1\hat{R}_1^2 - 32\hat{H}\hat{R}_2^2 - 4a_2\{\hat{R}_1,\hat{R}_2\} \\
& + 8a_2\hat{H}\hat{R}_1 + 16\hat{H}^2\hat{R}_2 + 16a_1\hat{H}\hat{R}_2 - 4a_1(4a_3-11)\hat{R}_2 \\
& + 4\left(a_2^2+8a_1\right)\hat{H} - 4b_2^2\left(a_3+\tfrac{3}{4}\right) .
\end{aligned}$$

[B] $H = \dfrac{u^2}{u^2+1}\left(p_u^2+p_v^2+b_1\left(u^2+v^2\right)+\dfrac{b_2}{u^2}+\dfrac{b_3}{v^2}\right).$

The additional constants of the motion have the form

$$R_1 = X_2 + \frac{u^2+v^2}{u^2+1}\left(b_1(u^2+v^2)-b_2-b_3\frac{u^2}{v^2}\right), \qquad R_2 = K^2 + b_1 v^2 + \frac{b_3}{v^2} .$$

The corresponding quadratic algebra relations can be determined, using (2.9), from the identity

$$\begin{aligned}
R^2 = {} & 16R_1R_2^2 - 16b_1R_1^2 - 16HR_1R_2 + 32b_1(H-b_2-b_3)R_1 + 16(H+b_3-b_2)HR_2 \\
& - 16(b_1+b_3)H^2 + 32b_1(b_2-b_3)H - 16b_1(b_2-b_3)^2 .
\end{aligned}$$

The equation of motion $H-E=0$ becomes

$$p_u^2+p_v^2+b_1(u^2+v^2)+\frac{(b_2-E)}{u^2}+\frac{b_3}{v^2}-E=0 .$$

This is a superintegrable system in flat space, but with rearranged constants, which is solvable via separation of variables in Cartesian, polar and elliptic coordinates. Again, this agrees with the observation that for this Hamiltonian we have quadratic constants with leading parts K^2, X_2 and $X_2+\beta K^2$. In the latter two coordinate systems, the Hamiltonian takes the forms:

(i) Elliptical coordinates (2.5)

$$H=\frac{p_\omega^2+p_\varphi^2+\frac{1}{4}b_1b^2(\sinh^2 2\omega+\sin^2 2\varphi)+b_2(\sec^2\varphi-\mathrm{sech}^2\omega)+b_3(\mathrm{cosec}^2\varphi+\mathrm{cosech}^2\omega)}{\sec^2\varphi-\mathrm{sech}^2\omega+b^2\left(\cosh^2\omega-\cos^2\varphi\right)}\,.$$

(ii) Polar coordinates (2.6)

$$H=\frac{r^2p_r^2+p_\theta^2+b_1r^4+b_2\sec^2\theta+b_3\mathrm{cosec}^2\theta}{r^2+\sec^2\theta}\,.$$

The corresponding quantum algebra relations are

$$\begin{aligned}
[\hat R,\hat R_1] &= -8\{\hat R_1,\hat R_2\}+8\hat H\hat R_1+12\hat R_2-8\hat H^2+8(b_2-b_3-\tfrac{3}{4})\hat H\,,\\
[\hat R,\hat R_2] &= 8\hat R_2^2-16b_1\hat R_1-8\hat H\hat R_2+16b_1\hat H-16b_1(b_2+b_3+\tfrac{3}{4})\,,\\
\hat R^2 &= 8\{\hat R_1,\hat R_2^2\}-8\hat H\{\hat R_1,\hat R_2\}+16\hat H^2\hat R_2-16b_1\hat R_1^2-76\hat R_2^2\\
&\quad+32b_1\hat H\hat R_1-8b_1(4(b_3+b_2)+3)\hat R_1+16(b_3-b_2+\tfrac{19}{4})\hat H\hat R_2\\
&\quad-16(b_1+b_3+\tfrac{3}{4})\hat H^2-8b_1(4(b_3-b_2)+3)\hat H+b_1\Big(36+48b_3-(4(b_3-b_2)+3)^2\Big)
\end{aligned}$$

[C] $$H=\frac{p_\xi^2+p_\eta^2+c_1+\frac{c_2}{\xi^2}+\frac{c_3}{\eta^2}}{\xi^2+\eta^2+\frac{1}{\xi^2}+\frac{1}{\eta^2}}\,.$$

The additional constants of the motion are

$$\begin{aligned}
R_1 &= X_1+\frac{c_1\xi^2(\eta^4+1)+c_2(\eta^4+1)-c_3(\xi^4+1)}{(\xi^2\eta^2+1)(\xi^2+\eta^2)}\,,\\
R_2 &= X_2+\frac{c_1(\xi^2+\eta^2)-c_2(\eta^4-1)-c_3(\xi^4-1)}{4(\xi^2\eta^2+1)}\,.
\end{aligned}$$

The corresponding Poisson algebra can be determined from the identity

$$\begin{aligned}
R^2 &= 4R_1^2R_2-(c_2+c_3)R_1^2+16HR_2^2-4c_1R_1R_2+2c_1c_3R_1-16H^2R_2\\
&\quad+4(c_2+c_3)H^2+(c_1^2-4c_2c_3)H-c_1^2c_3
\end{aligned}$$

The Hamiltonian can be written in separable form for the following coordinate systems.

(i) Displaced Elliptic coordinates $\xi=b'\cosh\omega'\cos\varphi'$, $\eta=b'\sinh\omega'\sin\varphi'$.

$$H=\frac{p_{\omega'}^2+p_{\varphi'}^2+c_1b'^2(\cosh^2\omega'-\cos^2\varphi')+c_2(\sec^2\varphi'-\mathrm{sech}^2\omega')+c_3(\mathrm{cosec}^2\varphi'+\mathrm{cosech}^2\omega')}{b'^4(\cosh^4\omega'-\cos^4\varphi'-\cosh^2\omega'+\cos^2\varphi')+\sec^2\varphi'+\mathrm{cosec}^2\varphi'+\mathrm{cosech}^2\omega'-\mathrm{sech}^2\omega'}\,.$$

These coordinates are not those given in (2.5) and are related to u and v by

$$u=\frac{1}{4}b'^2\sinh 2\omega'\sin 2\varphi'\,,\qquad v=\frac{1}{4}b'^2\left(\cosh 2\omega'\cos 2\varphi'+1\right)\,.$$

(ii) Polar coordinates $\xi = r'\cos\theta'$, $\eta = r'\sin\theta'$.

$$H = \frac{r'^2 p_{r'}^2 + p_{\theta'}^2 + c_1 r'^2 + c_2 \mathrm{cosec}^2\theta' + c_3\sec^2\theta'}{r'^4 + \sec^2\theta' + \mathrm{cosec}^2\theta'}\,.$$

These coordinates are not those given in (2.6) and are related to u and v by

$$u = \frac{1}{2}r'^2\sin 2\theta'\,, \qquad v = \frac{1}{2}r'^2\cos 2\theta'\,.$$

The corresponding quantum algebra relations are

$$\begin{aligned}
[\hat R, \hat R_1] &= -2\hat R_1^2 - 2c_1\hat R_1 - 16\hat H\hat R_2 + 8\hat H^2 - 6\hat H \\
[\hat R, \hat R_2] &= 2\{\hat R_1, \hat R_2\} - (c_2+c_3)\hat R_1 - 2c_1\hat R_2 + c_1c_3 \\
\hat R^2 &= 2\{\hat R_1^2, \hat R_2\} + 16\hat H\hat R_2^2 - (c_2+c_3+4)\hat R_1^2 - 2c_1\{\hat R_1, \hat R_2\} + 2c_1(c_3+2)\hat R_1 \\
&\quad - 16\hat H^2\hat R_2 + 12\hat H\hat R_2 + 4(c_2+c_3)\hat H^2 + (c_1^2 - 4c_2c_3 - 3(c_2+c_3))\hat H - \tfrac{1}{4}(3+4c_3)c_1^2
\end{aligned}$$

The equation of motion $H - E = 0$ is

$$p_\xi^2 + p_\eta^2 + c_1 - E(\xi^2+\eta^2) + \frac{(c_2-E)}{\xi^2} + \frac{(c_3-E)}{\eta^2} = 0.$$

This is a superintegrable system in flat space, but with rearranged constants, which is solvable via separation of variables in Cartesian, polar and elliptic coordinates.

[D] $H = \dfrac{u^2(p_u^2 + p_v^2 + d)}{u^2+1}$.

The additional constants of the motion are

$$R_1 = X_1 + \frac{2dv}{u^2+1}, \qquad R_2 = X_2 + \frac{d(u^2+v^2)}{u^2+1}, \qquad K = p_v.$$

The corresponding Poisson algebra relations are

$$\{K, R_1\} = 2K^2 - 2H + 2d\,, \qquad \{K, R_2\} = R_1\,, \qquad \{R_1, R_2\} = -4KR_2\,.$$

The functional relation between these constants is

$$R_1^2 - 4K^2R_2 + 4(H-d)R_2 - 4H^2 + 4dH = 0\,.$$

The Hamiltonian can be written in separable form for all the possible types of separable coordinates we have discussed viz.

(i) Elliptic coordinates (2.5).

$$H = \frac{p_\omega^2 + p_\varphi^2 + b^2 d(\cosh^2\omega - \cos^2\varphi)}{b^2(\cosh^2\omega - \cos^2\varphi) + \sec^2\varphi - \mathrm{sech}^2\omega}\,.$$

(ii) Polar coordinates (2.6).

$$H = \frac{r^2 p_r^2 + p_\theta^2 + dr^2}{r^2 + \sec^2\theta}\,.$$

(iii) Parabolic coordinates (2.7).

$$H = \frac{p_\xi^2 + p_\eta^2 + d(\xi^2+\eta^2)}{\xi^2+\eta^2+\frac{1}{\xi^2}+\frac{1}{\eta^2}}\,.$$

The corresponding quantum algebra relations have the form

$$[\hat{K},\hat{R}_1] = 2\hat{K}^2 - 2\hat{H} + 2d, \qquad [\hat{K},\hat{R}_2] = \hat{R}_1, \qquad [\hat{R}_1,\hat{R}_2] = 2\{\hat{K},\hat{R}_2\}.$$

The operator identity satisfied by the defining operators of the quantum algebra is

$$\hat{R}_1^2 - 2\{\hat{K}^2,\hat{R}_2\} + 4\hat{H}\hat{R}_2 - 4d\hat{R}_2 + 4\hat{K}^2 - 4\hat{H}^2 + (4d-1)\hat{H} = 0\,.$$

The equation of motion $H - E = 0$ is

$$p_u^2 + p_v^2 + d - E - \frac{E}{u^2} = 0.$$

This is a superintegrable system in flat space, but with rearranged constants, which is solvable via separation of variables in Cartesian, polar, elliptic and parabolic coordinates.

3 Darboux spaces of type three.

3.1 The free particle and separating coordinate systems

With rescaling and translation of the variables x and y the Hamiltonian H has the form

$$H_0 = \frac{e^{(x+y)/2}}{1+e^{-(x+y)/2}}\,p_x p_y. \tag{3.1}$$

In coordinates $x = u - iv$, $y = u + iv$ we can write this Hamiltonian in positive definite form

$$H_0 = \frac{1}{4}\frac{e^{2u}(p_u^2+p_v^2)}{e^u+1}.$$

Associated with the Hamiltonian are three integrals of the free motion

$$K = p_v\,, \qquad X_1 = \frac{1}{4}\frac{e^{2u}}{e^u+1}\cos v\, p_u^2 - \frac{1}{4}\frac{e^u(e^u+2)}{e^u+1}\cos v\, p_v^2 + \frac{1}{2}e^u \sin v\, p_u p_v\,,$$

$$X_2 = \frac{1}{4}\frac{e^{2u}}{e^u+1}\sin v\, p_u^2 - \frac{1}{4}\frac{e^u(e^u+2)}{e^u+1}\sin v\, p_v^2 - \frac{1}{2}e^u \cos v\, p_u p_v\,.$$

The integrals satisfy the polynomial algebra relations

$$\{K, X_1\} = -X_2, \qquad \{K, X_2\} = X_1, \qquad \{X_1, X_2\} = K H_0.$$

They are functionally dependent via the relation

$$X_1^2 + X_2^2 - H_0^2 - H_0 K^2 = 0.$$

The corresponding problem in quantum mechanics can readily be obtained via the usual quantisation rules and symmetrisation.

$$\hat{H}_0 = \frac{1}{4}\frac{e^{2u}}{e^u+1}\left(\partial_u^2 + \partial_v^2\right), \qquad \hat{K} = \partial_v\,,$$

$$\begin{aligned}
\hat{X}_1 &= \frac{1}{4}\frac{e^{2u}}{e^u+1}\cos v\partial_u^2 - \frac{1}{4}\frac{e^u(e^u+2)}{e^u+1}\cos v\partial_v^2 + \frac{1}{2}e^u\sin v\partial_u\partial_v + \frac{1}{4}e^u\cos v\partial_u + \frac{1}{4}e^u\sin v\partial_v\,,\\
\hat{X}_2 &= \frac{1}{4}\frac{e^{2u}}{e^u+1}\sin v\partial_u^2 - \frac{1}{4}\frac{e^u(e^u+2)}{e^u+1}\sin v\partial_v^2 - \frac{1}{2}e^u\cos v\partial_u\partial_v + \frac{1}{4}e^u\sin v\partial_u - \frac{1}{4}e^u\cos v\partial_v\,.
\end{aligned}$$

The commutator algebra obtained has the same form as the Poisson algebra, and the identity relating the operators is

$$\hat{X}_1^2 + \hat{X}_2^2 - \hat{H}_0^2 - \hat{H}_0\hat{K}^2 + \frac{1}{4}\hat{H}_0 = 0\,.$$

The line element $ds^2 = (e^{-u} + e^{-2u})(du^2 + dv^2)$ can be realised as a two-dimensional surface embedded in three dimensions by

$$X = v\sqrt{e^{-u} + e^{-2u}}\,, \qquad Y - T = \sqrt{e^{-u} + e^{-2u}}$$

$$Y + T = (1 - v^2)\sqrt{e^{-u} + e^{-2u}} + \log\left(1 + 2e^{-u} + 2\sqrt{e^{-u} + e^{-2u}}\right) + \frac{1}{2}\arctan\left(2\sqrt{e^{-u} + e^{-2u}}\right),$$

in which case,

$$ds^2 = dX^2 + dY^2 - dT^2 = (e^{-u} + e^{-2u})(du^2 + dv^2)\,.$$

Just as we have done in other cases, we wish to determine all the essentially different separable coordinate systems for the free classical or quantum particle. To do this we need to consider a general quadratic constant of the form $\lambda = aX_1 + bX_2 + cK^2$. Under the adjoint action of $\exp(\alpha K)$, X_1 and X_2 transform according to

$$X_1 \to \cos\alpha\ X_1 - \sin\alpha\ X_2, \qquad X_2 \to \sin\alpha\ X_1 + \cos\alpha\ X_2.$$

¿From this transformation law we see that λ can take five different forms:

$$K^2, \qquad X_1, \qquad X_1 + \gamma K^2, \qquad X_1 + iX_2, \qquad X_1 + iX_2 - K^2. \tag{3.2}$$

We now demonstrate the explicit coordinates in the case of each of these representatives.

3.1.1 Coordinates associated with K^2

These are the coordinates associated with the ignorable coordinate v and the Hamiltonian has already been given in the u, v coordinates. The Hamilton-Jacobi equation is

$$\frac{1}{4}\frac{e^{2u}}{e^u+1}\left(\left(\frac{\partial S}{\partial u}\right)^2+\left(\frac{\partial S}{\partial v}\right)^2\right)=E\,,$$

with solutions

$$\begin{aligned} S(u,v) &= -\frac{\sqrt{4E(1+e^u)-c^2e^{2u}}}{ce^u}-\frac{\sqrt{E}}{c}\mathrm{arctanh}\left(\frac{\sqrt{E}(e^u+2)}{\sqrt{4E(1+e^u)-c^2e^{2u}}}\right) \\ &\quad + i\log\left(i(c^2e^u-2E)+c\sqrt{4E(1+e^u)-c^2e^{2u}}\right)+cv\,. \end{aligned}$$

The corresponding Schrödinger equation is

$$\frac{1}{4}\frac{e^{2u}}{e^u+1}(\partial_u^2+\partial_v^2)\Psi=E\Psi,$$

with solutions of the form

$$\Psi=e^{-u/2}M_{-1/\sqrt{-E},\,\pm m}(4\sqrt{-E}\,e^{-u})e^{imv}.$$

3.1.2 Coordinates associated with X_1

For the second representative in (3.2), a suitable choice of variables is

$$\xi=2e^{-u/2}\cos\frac{v}{2}\,,\qquad \eta=2e^{-u/2}\sin\frac{v}{2}\,. \tag{3.3}$$

In terms of these coordinates the classical Hamiltonian has the form

$$H_0=\frac{p_\xi^2+p_\eta^2}{4+\xi^2+\eta^2}$$

and the corresponding quadratic constant is

$$X_1=\frac{(2+\eta^2)p_\xi^2-(2+\xi^2)p_\eta^2}{2(4+\xi^2+\eta^2)}\,.$$

In ξ, η coordinates the classical Hamilton-Jacobi equation is

$$\frac{\left(\frac{\partial S}{\partial\xi}\right)^2+\left(\frac{\partial S}{\partial\eta}\right)^2}{4+\xi^2+\eta^2}=E\,,$$

which has the solution

$$\begin{aligned} S &= \frac{1}{2}\xi\sqrt{E\xi^2+2E-\lambda}+\left(\frac{2E-\lambda}{2\sqrt{E}}\right)\log\left(E+\sqrt{E\xi^2+2E-\lambda}\right) \\ &\quad +\frac{1}{2}\eta\sqrt{E\eta^2+2E+\lambda}+\left(\frac{2E+\lambda}{2\sqrt{E}}\right)\log\left(E+\sqrt{E\eta^2+2E+\lambda}\right)\,. \end{aligned}$$

The Schrödinger equation is

$$\frac{(\partial_\xi^2 + \partial_\eta^2)\Psi}{4 + \xi^2 + \eta^2} = E\Psi\,,$$

which has typical solutions

$$\Psi = D_{(\lambda - 2E)/\sqrt{4E}}\Big(\pm(4E)^{1/4}\xi\Big) D_{-(\lambda+E)/\sqrt{4E}}\Big(\pm(4E)^{1/4}\eta\Big)\,,$$

in terms of parabolic cylinder functions $D_\nu(z)$ [8, 9].

3.1.3 Coordinates associated with $X_1 + \gamma K^2$

For the third case it is convenient to take the representative as $b^2X_1 + 2K^2$. Here we identify coordinates via

$$\xi = b\cosh\omega\cos\varphi\,, \qquad \eta = b\sinh\omega\sin\varphi\,. \tag{3.4}$$

The classical Hamiltonian has the form

$$H_0 = \frac{p_\omega^2 + p_\varphi^2}{2b^2(\cosh 2\omega - \cos 2\varphi) + \frac{1}{4}b^4(\cosh^2 2\omega - \cos^2 2\varphi)}$$

and the corresponding quadratic constant in these coordinates is

$$b^2X_1 + 2K^2 = \frac{(8\cos 2\varphi - b^2\sin 2\varphi)p_\omega^2 + (8\cosh 2\omega + b^2\sinh 2\omega)p_\varphi^2}{8b^2(\cosh 2\omega - \cos 2\varphi) + b^4(\cosh^2 2\omega - \cos^2 2\varphi)}\,.$$

In the φ, ω coordinates the classical Hamilton-Jacobi equation has the form

$$\frac{\left(\dfrac{\partial S}{\partial\omega}\right)^2 + \left(\dfrac{\partial S}{\partial\varphi}\right)^2}{2b^2(\cosh 2\omega - \cos 2\varphi) + \frac{1}{4}b^4(\cosh^2 2\omega - \cos^2 2\varphi)} = E\,,$$

and has the solution

$$S(\omega,\varphi) = \frac{1}{4}b^2\sqrt{E}\left(\int\sqrt{\frac{(\Omega-\alpha_1)(\Omega-\alpha_2)}{\Omega^2-1}}\,d\Omega + \int\sqrt{\frac{(\beta_1-\Phi)(\Phi-\beta_2)}{1-\Phi^2}}\,d\Phi\right)$$

where $\alpha_1 + \alpha_2 = -\beta_1 - \beta_2 = -8/b^2$, $\alpha_1\alpha_2 = -\beta_1\beta_2 - 4\lambda/Eb^2$, $\Omega = \cosh 2\omega$, $\Phi = \cos 2\varphi$. The corresponding Schrödinger equation

$$\frac{\partial_\omega^2\Psi + \partial_\varphi^2\Psi}{2b^2(\cosh 2\omega - \cos 2\varphi) + \frac{1}{4}b^4(\cosh^2 2\omega - \cos^2 2\varphi)} = E\Psi\,,$$

separates with $\Psi = \Phi(\varphi)\Omega(\omega)$ in the equations

$$\begin{aligned}
\Big(\partial_\varphi^2 + 2b^2E\cos 2\varphi + \tfrac{1}{8}b^4E\cos 4\varphi + \lambda + \tfrac{1}{8}b^4E\Big)\Phi &= 0\,,\\
\Big(\partial_\omega^2 - 2b^2E\cos 2\omega - \tfrac{1}{8}b^4E\cos 4\omega - \lambda - \tfrac{1}{8}b^4E\Big)\Omega &= 0\,,
\end{aligned}$$

which has typical solutions

$$\begin{aligned}\Psi_1 &= \mathrm{gc}_m\left(\varphi, b\sqrt{-E}, 2b\sqrt{-E}\right)\mathrm{gc}_m\left(i\omega, b\sqrt{-E}, 2b\sqrt{-E}\right)\\ \Psi_2 &= \mathrm{gs}_m\left(\varphi, b\sqrt{-E}, 2b\sqrt{-E}\right)\mathrm{gs}_m\left(i\omega, b\sqrt{-E}, 2b\sqrt{-E}\right)\end{aligned}$$

with corresponding separation constant given by $\lambda_m = \mu_m b^2(1+b^2)E/8$. The functions appearing here are even and odd Whittaker Hill functions [10].

3.1.4 Coordinates associated with $X_1 + iX_2$

In the case of a system specified by the fourth representative there are, in fact, no separable coordinates. However, in the coordinates

$$x = \xi + i\eta\,, \qquad y = \frac{1}{2}\left(\xi - i\eta\right)^2\,,$$

the classical Hamiltonian takes the form

$$H_0 = \frac{2p_x p_y}{2y^{-1/2} + x}$$

and the corresponding constant is

$$X_1 + iX_2 = 2yH_0 - p_x^2\,.$$

The solution of the Hamilton-Jacobi equation

$$\frac{2\frac{\partial S}{\partial x}\frac{\partial S}{\partial y}}{2y^{-1/2} + x} = E$$

is

$$S = x\sqrt{Ey - \lambda} + \sqrt{E}\log\left(\sqrt{E}\,y - \frac{\lambda}{2\sqrt{E}} + \sqrt{Ey^2 - \lambda y}\right)\,.$$

The corresponding Schrödinger equation is

$$\frac{2\partial_x\partial_y\Psi}{2y^{-1/2} + x} = E\Psi$$

which has solutions

$$\Psi = \frac{\left(2E^{3/2}y - E^{1/2} + 2E\sqrt{Ey^2 - \lambda y}\right)^{\sqrt{E}}\, e^{x\sqrt{Ey-\lambda}}}{\sqrt{Ey - \lambda}}\,.$$

3.1.5 Coordinates associated with $X_1 + iX_2 - K^2$

In the case of a system specified by the fifth representative an appropriate choice of coordinates is

$$\xi = \frac{\mu-\nu}{2\sqrt{\mu\nu}} + \sqrt{\mu\nu}\,, \qquad \eta = i\left(\frac{\mu-\nu}{2\sqrt{\mu\nu}} - \sqrt{\mu\nu}\right)\,. \tag{3.5}$$

The corresponding classical Hamiltonian has the form

$$H_0 = \frac{\mu^2 p_\mu^2 - \nu^2 p_\nu^2}{(\mu+\nu)(2+\mu-\nu)},$$

and the quadratic constant is

$$X_1 + iX_2 - K^2 = \frac{\nu^2(\mu+2)\mu p_\nu^2 - \mu^2(\nu-2)\nu p_\mu^2}{(\mu+\nu)(2+\mu-\nu)}\,.$$

In the μ, ν coordinate system the classical Hamilton-Jacobi equation has the form

$$\frac{\mu^2\left(\dfrac{\partial S}{\partial \mu}\right)^2 - \nu^2\left(\dfrac{\partial S}{\partial \nu}\right)^2}{(\mu+\nu)(2+\mu-\nu)} = E\,,$$

which has the solution

$$\begin{aligned}
S(\mu,\nu) \;=\;& \sqrt{E\mu^2+2E\mu+\lambda} + \sqrt{E}\ \log\left(\sqrt{E}(1+\mu) + \sqrt{E\mu^2+2E\mu+\lambda}\right)\\
& - \sqrt{\lambda}\ \mathrm{arctanh}\left(\frac{\lambda+E\mu}{\sqrt{\lambda}\sqrt{E\mu^2+2E\mu+\lambda}}\right)\\
& + \sqrt{E\nu^2-2E\nu+\lambda} + \sqrt{E}\ \log\left(\sqrt{E}(1-\nu) + \sqrt{E\nu^2-2E\nu+\lambda}\right)\\
& - \sqrt{\lambda}\ \mathrm{arctanh}\left(\frac{\lambda-E\nu}{\sqrt{\lambda}\sqrt{E\nu^2-2E\nu+\lambda}}\right)\,.
\end{aligned}$$

The Schrödinger equation

$$\frac{\mu\partial_\mu\left(\mu\partial_\mu\Psi\right) - \nu\partial_\nu\left(\nu\partial_\nu\Psi\right)}{(\mu+\nu)(2+\mu-\nu)} = E\Psi\,,$$

separates with $\Psi = A(\mu)B(\nu)$ into the equations

$$\Big(\mu\partial_\mu(\mu\partial_\mu) - E\mu^2 - 2E\mu - \rho^2\Big)A(\mu) = 0\,, \qquad \Big(\nu\partial_\nu(\nu\partial_\nu) - E\nu^2 - 2E\nu - \rho^2\Big)B(\nu) = 0\,,$$

and has solutions, in terms of the Whittaker function $M_{\lambda,\chi}$, of the form [9]

$$\frac{1}{\sqrt{\mu\nu}} M_{\sqrt{E},\rho}\left(2\sqrt{E}\ \mu\right) M_{-\sqrt{E},\rho}\left(2\sqrt{E}\ \nu\right)\,.$$

If we write the classical equation $H = E$ in ξ, η coordinates we obtain

$$p_\xi^2 + p_\eta^2 - E(4+\xi^2+\eta^2) = 0.$$

This is in the form of a flat space superintegrable system which can be solved by separation of variables in Cartesian, polar, hyperbolic and elliptic coordinates.

3.2 Superintegrability for Darboux spaces of type three.

In this section we address the problem of superintegrability for the Hamiltonian

$$H = \frac{1}{4}\frac{e^{2u}(p_u^2+p_v^2)}{e^u+1}. \tag{3.6}$$

We arrive at three possibilities: [**A**], [**B**], [**C**].

$$[\mathbf{A}]\ \ H = \frac{p_\xi^2+p_\eta^2+a_1\xi+a_2\eta+a_3}{4+\xi^2+\eta^2}$$

The additional constants have the form

$$\begin{aligned} R_1 &= X_1 + \frac{2a_1\xi(2+\eta^2)-2a_2\eta(2+\xi^2)+a_3(\eta^2-\xi^2)}{4(4+\xi^2+\eta^2)}\,,\\ R_2 &= X_2 + \frac{a_1\eta(\eta^2-\xi^2+4)+a_2\xi(\xi^2-\eta^2+4)-2a_3\xi\eta}{4(4+\xi^2+\eta^2)}\,. \end{aligned}$$

The corresponding quadratic algebra can be determined from the identity

$$\begin{aligned} R^2 &= HR_1^2+HR_2^2+\tfrac{1}{8}(a_2^2-a_1^2)R_1-\tfrac{1}{4}a_1a_2R_2\\ &\quad -H^3+\tfrac{1}{2}a_3H^2+\tfrac{1}{16}(2a_2^2+2a_1^2-a_3^2)H-\tfrac{1}{32}a_3(a_1^2+a_2^2)\,. \end{aligned}$$

This Hamiltonian separates in a family of coordinate systems obtained by translating the given separable system via $\xi \to \xi + a,\ \ \eta \to \eta - a$. The corresponding quantum algebra relations are

$$\begin{aligned} [\hat R,\hat R_1] &= -\hat H\hat R_2+\tfrac{1}{8}a_1a_2\,, \qquad [\hat R,\hat R_2] = \hat H\hat R_1+\tfrac{1}{16}(a_2^2-a_1^2)\,,\\ \hat R^2 &= \hat H\hat R_1^2+\hat H\hat R_2^2+\tfrac{1}{8}(a_2^2-a_1^2)\hat R_1-\tfrac{1}{4}a_1a_2\hat R_2\\ &\quad -\hat H^3+\tfrac{1}{2}(a_3+\tfrac{1}{2})\hat H^2+\tfrac{1}{16}(2a_1^2+2a_2^2-a_3^2)\hat H-\tfrac{1}{32}a_3(a_1^2+a_2^2) \end{aligned}$$

As in the case of free motion, the equation $H = E$ becomes

$$p_\xi^2+p_\eta^2+a_1\xi+a_2\eta+a_3-E(4+\xi^2+\eta^2)=0\,.$$

Again, this is a superintegrable system in flat space but with rearranged constants.

$$[\mathbf{B}]\ \ H = \frac{p_\xi^2+p_\eta^2+\frac{b_1}{\xi^2}+\frac{b_2}{\eta^2}+b_3}{4+\xi^2+\eta^2}\,.$$

The additional constants are

$$R_1 = X_1 + \frac{2b_1\eta^2(\eta^2+2)-2b_2\xi^2(\xi^2+2)+b_3(\eta^2-\xi^2)}{4(4+\xi^2+\eta^2)}\,, \tag{3.7}$$

$$R_2 = K^2+\frac{b_1\eta^2}{4\xi^2}+\frac{b_2\xi^2}{4\eta^2}\,. \tag{3.8}$$

The corresponding quadratic algebra relations are determined by

$$
\begin{aligned}
R^2 \;&=\; -4R_1^2R_2-(b_1+b_2)R_1^2+4HR_2^2+2(b_1-b_2)HR_1+\tfrac{1}{2}b_3(b_2-b_1)R_1\\
&\quad +4H^2R_2-2b_3HR_2+\tfrac{1}{4}b_3^2R_2-(b_1+b_2)H^2\\
&\quad +\left(\tfrac{1}{2}b_3(b_1+b_2)-b_1b_2\right)H-\tfrac{1}{16}b_3^2(b_1+b_2).
\end{aligned}
$$

This Hamiltonian separates in all the separable coordinate systems given in Section 2.1. The Hamiltonian has the explicit forms

(i) In u, v coordinates,

$$
H=\frac{e^{2u}\left(p_u^2+p_v^2+\frac{1}{4}b_1\sec^2\frac{v}{2}+\frac{1}{4}b_2\mathrm{cosec}^2\frac{v}{2}+b_3e^{-u}\right)}{4(e^u+1)}.
$$

(ii) In the elliptical coordinates (3.4),

$$
H=\frac{p_\omega^2+p_\varphi^2+b_1\left(\sec^2\varphi-\mathrm{sech}^2\omega\right)+b_2\left(\mathrm{cosec}^2\varphi+\mathrm{cosech}^2\omega\right)+b_3b^2\left(\cosh^2\omega-\cos^2\varphi\right)}{2b^2(\cosh 2\omega-\cos 2\varphi)+\frac{1}{4}b^4(\cosh^2 2\omega-\cos^2 2\varphi)}.
$$

The corresponding quantum algebra relations have the form

$$
\begin{aligned}
[\hat R,\hat R_1] \;&=\; 2\hat R_1^2-4\hat H\hat R_2-2\hat H^2+\left(b_3+\tfrac{1}{2}\right)\hat H-\tfrac{1}{8}b_3^2\\
[\hat R,\hat R_2] \;&=\; -2\{\hat R_1,\hat R_2\}-(b_1+b_2+1)\hat R_1+(b_1-b_2)\hat H+\tfrac{1}{4}(b_2-b_1)b_3\,,\\
\hat R^2 \;&=\; -2\{\hat R_1^2,\hat R_2\}-(b_1+b_2+5)\hat R_1^2+4\hat H\hat R_2^2\\
&\quad +2(b_1-b_2)\hat H\hat R_1+b_3(b_2-b_1)\hat R_1+4\hat H^2\hat R_2-(2b_3-1)\hat H\hat R_2+\tfrac{1}{4}b_3^2\hat R_2\\
&\quad -(b_1+b_2-2)\hat H^2+\left(\tfrac{1}{2}(b_3+\tfrac{3}{2})(b_1+b_2)-b_3-b_1b_2-\tfrac{1}{2}\right)H-\tfrac{1}{16}b_3^2(b_1+b_2-2).
\end{aligned}
$$

As in the case of free motion, we observe that equation $H=E$ becomes

$$
p_\xi^2+p_\eta^2+\frac{b_1}{\xi^2}+\frac{b_2}{\eta^2}+b_3-E(4+\xi^2+\eta^2)=0.
$$

This is a superintegrable system in flat space, with rearranged constants, that separates variables in Cartesian, polar and elliptic coordinates.

[C] $$H=\frac{\mu^2p_\mu^2-\nu^2p_\nu^2+c_1(\mu+\nu)+c_2\frac{\mu+\nu}{\mu\nu}+c_3\frac{\mu^2-\nu^2}{\mu^2\nu^2}}{(\mu+\nu)(2+\mu-\nu)}.$$

The additional constants of the motion have the form

$$
R_1=X_1+iX_2-\frac{c_1\mu^2\nu^2+c_2\mu\nu+2c_3(1+\mu-\nu)}{\mu\nu(2+\mu-\nu)}\qquad R_2=K^2-c_2\frac{\mu-\nu}{\mu\nu}-c_3\frac{(\mu-\nu)^2}{\mu^2\nu^2}.
$$

The corresponding quadratic Poisson algebra relations can be determined from

$$
\begin{aligned}
R^2 \;&=\; -4R_2R_1^2+8c_2HR_1-4c_1c_2R_1+16c_3HR_2\\
&\quad +16c_3H^2+4(c_2^2-4c_1c_3)H+4c_1^2c_3\,.
\end{aligned}
$$

The quantum algebra relations are

$$[\hat{R},\hat{R}_1]=2\hat{R}_1^2-8c_3\hat{H}\,,\qquad [\hat{R},\hat{R}_2]=-2\{\hat{R}_1,\hat{R}_2\}-\hat{R}_1+4c_2\hat{H}-2c_1c_2\,,$$

$$\hat{R}^2=-2\{\hat{R}_1^2,\hat{R}_2\}+8c_2\hat{H}\hat{R}_1+16c_3\hat{H}\hat{R}_2-5\hat{R}_1^2-4c_1c_2\hat{R}_1 +16c_3\hat{H}^2+4(c_3+c_2^2-4c_1c_3)\hat{H}+4c_1^2c_3\,.$$

As in the case of free motion, equation $H=E$ becomes

$$p_\xi^2+p_\eta^2+2c_1+\frac{8c_2}{(\xi+i\eta)^2}+\frac{16c_3(\xi-i\eta)}{(\xi+i\eta)^3}-E(4+\xi^2+\eta^2)=0,$$

a superintegrable system in flat space with rearranged constants, that separates variables in polar and hyperbolic coordinates.

[D] $H=\dfrac{\mu^2p_\mu^2-\nu^2p_\nu^2+d_1\mu+d_2\nu+d_3(\mu^2+\nu^2)}{(\mu+\nu)(2+\mu-\nu)}.$

The additional constants of the motion have the form

$$R_1 = X_1+iX_2-K^2-\frac{\mu\nu\Big(d_1(\nu-2)+d_2(\mu+2)+2d_3(\nu-\mu+\mu\nu)\Big)}{(\mu+\nu)(2+\mu-\nu)}\,,$$

$$R_2 = X_1-iX_2-\frac{(\mu-\nu)\Big((\mu-\nu)(d_1\mu+d_2\nu)-2d_3(\mu^2+\nu^2+\mu\nu(2+\mu-\nu))\Big)}{4\mu\nu(\mu+\nu)(2+\mu-\nu)}\,.$$

The corresponding quadratic Poisson algebra can be determined from

$$R^2 = 4R_1R_2^2-4HR_1R_2+d_3^2R_1-4H^2R_2+2(d_1+d_2)HR_2-d_1d_2R_2 +4H^3-2(d_1+d_2)H^2+\frac{1}{4}\Big((d_1+d_2)^2+d_3(d_2-d_1)\Big)H-d_3(d_1^2-d_2^2).$$

This classical system also separates in elliptical coordinates obtained by choosing new variables defined by the roots of the characteristic equation of R_1+R_2, that is, the elliptical coordinates (3.4) with $b=2i$. In these variables the Hamiltonian has the form

$$H=\frac{p_\omega^2+p_\varphi^2+2(d_1+d_2)(\cos 2\varphi-\cosh 2\omega)+2(d_1-d_2)(2i\sin 2\varphi+\sinh 2\omega)+2d_3(\sinh 4\omega+2i\sin 4\phi)}{8(\cos 2\varphi-\cosh 2\omega)+4(\cosh^2 2\omega-\cos^2 2\varphi)}\,.$$

The corresponding quantum algebra relations are

$$[\hat{R},\hat{R}_1] = -2\{\hat{R}_1,\hat{R}_2\}+2\hat{H}\hat{R}_1+\hat{R}_2+2\hat{H}^2-\left(d_1+d_2+\tfrac{1}{2}\right)\hat{H}+\tfrac{1}{2}d_1d_2$$

$$[\hat{R},\hat{R}_2] = 2\hat{R}_2^2-2\hat{H}\hat{R}_2+\tfrac{1}{2}d_3^2$$

$$\hat{R}^2 = 2\{\hat{R}_1,\hat{R}_2^2\}-5\hat{R}_2^2-2\hat{H}\{\hat{R}_1,\hat{R}_2\}+d_3^2\hat{R}_1-4\hat{H}^2\hat{R}_2+(2d_1+2d_2+5)\hat{H}\hat{R}_2-d_1d_2\hat{R}_2 +4\hat{H}^3-(2d_1+2d_2+1)\hat{H}^2+\left(\tfrac{1}{4}(d_1+d_2)^2+d_3(d_2-d_1)\right)\hat{H}-\tfrac{1}{4}d_3(d_3-d_1^2+d_2^2)$$

As in the case of free motion we observe that equation $H=E$ becomes

$$p_\xi^2+p_\eta^2+d_1+d_2-4d_3+\frac{(d_2-d_1)(\xi-i\eta)}{\sqrt{(\xi-i\eta)^2+4}}+\frac{8d_3(\xi+i\eta)}{\sqrt{(\xi-i\eta)^2+4}\Big(\xi-i\eta+\sqrt{(\xi-i\eta)^2+4}\Big)^2}$$

$$= (E - d_3)(4 + \xi^2 + \eta^2)\,,$$

a superintegrable system in flat space with rearranged constants that separates variables in elliptic and hyperbolic coordinates.

[E] $H = \dfrac{p_\xi^2 + p_\eta^2 + c}{4 + \xi^2 + \eta^2}.$

The additional constants of the motion are

$$R_1 = X_1 + \frac{c}{4}\frac{\eta^2 - \xi^2}{4 + \xi^2 + \eta^2}\,, \qquad R_2 = X_2 - \frac{c}{2}\frac{\xi\eta}{4 + \xi^2 + \eta^2}\,,$$

and K. The corresponding Poisson algebra relations have the form

$$\{K, R_1\} = -R_2, \qquad \{K, R_2\} = R_1\,, \qquad \{R_1, R_2\} = HK\,,$$

and the functional relation between these constants is

$$R_1^2 + R_2^2 - HK^2 - H^2 + \frac{c}{2}H - \frac{c^2}{16} = 0\,.$$

This Hamiltonian separates in all of the four types of separable coordinate systems available, and the corresponding expressions for the Hamiltonian can be deduced from [2] by taking $b_3 = c$, $b_1 = b_2 = 0$.

The quantum algebra relations are

$$[\hat{K}, \hat{R}_1] = -\hat{R}_2\,, \qquad [\hat{K}, \hat{R}_2] = \hat{R}_1\,, \qquad [\hat{R}_1, \hat{R}_2] = \hat{H}\hat{K}\,,$$

and the associated operator identity is

$$\hat{R}_1^2 + \hat{R}_2^2 - \hat{H}\hat{K}^2 - \hat{H}^2 + \left(\frac{c}{2} + \frac{1}{4}\right)\hat{H} - \frac{c^2}{16} = 0\,.$$

4 Darboux spaces of type four

4.1 The free particle and separating coordinate systems

With rescaling of the variables x and y, the Hamiltonian H can be taken in the form

$$H_0 = \frac{(e^{x-y} - e^{y-x})^2}{e^{x-y} + e^{y-x} + a}\; p_x p_y. \tag{4.1}$$

In coordinates $x = v + iu$, $y = v - iu$, we can write the Hamiltonian as

$$H_0 = -\frac{\sin^2 2u (p_u^2 + p_v^2)}{2\cos 2u + a}\,.$$

It admits constants of the motion

$$K = p_v \quad X_1 = e^{2v}\left(-H_0 + \cos 2u\; p_u^2 + \sin 2u\; p_u p_v\right), \quad X_2 = e^{-2v}\left(-H_0 + \cos 2u\; p_v^2 - \sin 2u\; p_u p_v\right).$$

These integrals satisfy the polynomial algebra relations

$$\{K, X_1\} = 2X_1, \qquad \{K, X_2\} = -2X_2, \qquad \{X_1, X_2\} = -8K^3 - 4aKH_0.$$

They are functionally dependent via the relation

$$X_1X_2 - K^4 - aK^2H_0 - H_0^2 = 0.$$

The corresponding quantum operators are

$$\hat{H}_0 = \frac{-\sin^2 2u}{2\cos 2u + a}\left(\partial_u^2 + \partial_v^2\right), \qquad \hat{X}_1 = e^{2v}\left(-\hat{H}_0 + \cos 2u(\partial_v^2 + \partial_v) + \sin 2u(\partial_u\partial_v + \partial_u)\right),$$

$$\hat{K} = \partial_v, \qquad \hat{X}_2 = e^{-2v}\left(-\hat{H}_0 + \cos 2u(\partial_v^2 - \partial_v) - \sin 2u(\partial_u\partial_v - \partial_u)\right).$$

Their algebra is determined by the relations

$$[\hat{K}, \hat{X}_1] = 2\hat{X}_1, \qquad [\hat{K}, \hat{X}_2] = -2\hat{X}_2, \qquad [\hat{X}_1, \hat{X}_2] = -8\hat{K}^3 - 4a\hat{K}\hat{H}_0 - 4\hat{K},$$

and the operator identity is

$$\frac{1}{2}\{\hat{X}_1, \hat{X}_2\} - \hat{K}^4 - a\hat{H}_0\hat{K}^2 - 5\hat{K}^2 - \hat{H}_0^2 - a\hat{H}_0 = 0.$$

The line element $ds^2 = (2\cos u + a)(du^2 + dv^2)/\sin^2 2u$ can be realized as a two-dimensional surface embedded in $E(2,1)$ by (assuming $a > 2$)

$$X = \sqrt{a + 2\cos 2u}v, \qquad Y - T = \sqrt{a + 2\cos 2u},$$

$$Y+T = \frac{(a-2)}{\sqrt{2(a+2)}}[\Pi(\chi, \sqrt{\frac{a-2}{a+2}}\frac{2}{(r_1+1)}, p) + \Pi(\chi, \sqrt{\frac{a-2}{a+2}}\frac{2}{(r_2+1)}, p)] - \sqrt{a + 2\cos 2u}v^2,$$

where

$$\sin\chi = \sqrt{\frac{(a+2)(\cos 2u + 1)}{2(a + 2\cos 2u)}}, \qquad p = \frac{2}{\sqrt{a+2}},$$

and Π is an elliptic integral of the third kind [8]. Then $ds^2 = dX^2 + dY^2 - dT^2$.

Just as we have done in other cases, we wish to determine all the essentially different separable coordinate systems for the free classical or quantum particle. To do this we need to consider a general quadratic constant of the form $\lambda = aX_1 + bX_2 + cK^2$. Under the adjoint action of $\exp(\alpha K)$, X_1 and X_2 transform according to

$$X_1 \to \exp(-2\alpha)X_1, \qquad X_2 \to \exp(2\alpha)X_2.$$

If we regard two such quadratic expressions as equivalent if they are related by a combination of group motions and the discrete transformation observed above, then the equivalence classes of these expressions can be chosen to have the following representatives:

$$K^2, \qquad X_2, \qquad \gamma X_2 + K^2, \qquad X_1 + X_2 + \gamma K^2. \tag{4.2}$$

In the last of these are three cases to distinguish: $\gamma = 0$, $\gamma = 2$ and $\gamma \neq 0, 2$. The various separable systems involved can now be computed.

4.1.1 Coordinates associated with K^2

These are the coordinates associated with the ignorable coordinate v and the Hamiltonian has already been given in the u, v coordinates. The Hamilton-Jacobi equation is

$$-\frac{\sin^2 2u}{2\cos 2u + a}\left(\left(\frac{\partial S}{\partial v}\right)^2 + \left(\frac{\partial S}{\partial u}\right)^2\right) = E\,.$$

It has typical solutions

$$\begin{aligned} S(u,v) &= -i\log\left(i(c^2\cos 2u - E) + c\sqrt{E(a+2\cos 2u) + c^2\sin^2 2u}\right) \\ &\quad + \frac{1}{2c}\sqrt{E(a+2)}\,\operatorname{arctanh}\left(\frac{E(a+1) + c^2 + (E - c^2)\cos 2u}{\sqrt{E(a+2)(E(a+2\cos 2u) + c^2\sin^2 2u)}}\right) \\ &\quad + \frac{1}{2c}\sqrt{E(a-2)}\,\operatorname{arctanh}\left(\frac{E(a-1) + c^2 + (E + c^2)\cos 2u}{\sqrt{E(a-2)(E(a+2\cos 2u) + c^2\sin^2 2u)}}\right) + cv\,. \end{aligned}$$

The corresponding Schrödinger equation is

$$\frac{\sin^2 2u}{2\cos 2u + a}\left(\frac{\partial^2\Psi}{\partial v^2} + \frac{\partial^2\Psi}{\partial u^2}\right) = E\Psi\,,$$

which has the solution

$$\Psi = {}_2F_1\left(\frac{1}{2}(\lambda - \epsilon_+ - \epsilon_-), \frac{1}{2}(\lambda + \epsilon_+ + \epsilon_-), \epsilon_+ + \frac{1}{2}, \sin^2 u\right) e^{\lambda v}\,,$$

where

$$\epsilon_\pm = \frac{1}{2} + \frac{1}{2}\sqrt{1 - (a \pm 2)E}\,, \tag{4.3}$$

and ${}_2F_1$ is a Gaussian hypergeometric function [11].

4.1.2 Coordinates associated with X_2

If we choose new coordinates

$$x = \log\left(\frac{1}{2}(\mu - i\nu)\right)\,, \qquad y = \log\left(\frac{1}{2}(\mu + i\nu)\right)\,, \tag{4.4}$$

then the Hamiltonian takes the rational form

$$H_0 = -\frac{4\mu^2\nu^2(p_\mu^2 + p_\nu^2)}{(a+2)\mu^2 + (a-2)\nu^2}\,.$$

In this case the corresponding choice of coordinates has already been given, and the quadratic constant in these coordinates is

$$X_2 = \frac{4(a+2)\mu^2 p_\mu^2 - 4(a-2)\nu^2 p_\nu^2}{(a+2)\mu^2 + (a-2)\nu^2}.$$

The Hamilton-Jacobi equation

$$-\frac{4\mu^2\nu^2\left(\left(\frac{\partial S}{\partial\mu}\right)^2+\left(\frac{\partial S}{\partial\nu}\right)^2\right)}{(a+2)\mu^2+(a-2)\nu^2}=E$$

has solution

$$\begin{aligned}S(\mu,\nu) &= i\sqrt{(a-2)E+\lambda\mu^2}-i\sqrt{(a-2)E}\ \operatorname{arctanh}\sqrt{\frac{(a-2)E+\lambda\mu^2}{(a-2)E}}\\ &\quad+\sqrt{(a+2)E-\lambda\nu^2}-\sqrt{(a+2)E}\ \operatorname{arctanh}\sqrt{\frac{(a+2)E-\lambda\nu^2}{(a+2)E}}\end{aligned}$$

The corresponding Schrödinger equation has Bessel function solutions of the form

$$\Psi=\sqrt{\mu\nu}\ C_{\frac12\sqrt{1-E(a-2)}}\left(\tfrac12\sqrt{\lambda}\ \mu\right)\ C_{\frac12\sqrt{1-E(a+2)}}\left(\tfrac12\sqrt{\lambda}\ i\nu\right).$$

4.1.3 Coordinates associated with γX_2+K^2

In the case of the third representative the transformation

$$\mu=c\cosh\omega\cos\varphi\,,\qquad \nu=c\sinh\omega\sin\varphi \tag{4.5}$$

gives the classical Hamiltonian

$$H=\frac{4(p_\omega^2+p_\varphi^2)}{(a-2)(\operatorname{sech}^2\omega-\sec^2\varphi)-(a+2)(\operatorname{cosech}^2\omega+\operatorname{cosec}^2\varphi)}.$$

The classical constant associated with this coordinate system is

$$\begin{aligned}&-\frac{c^2}{4}X_2+K^2\\ &= \frac{((a-2)\sec^2\varphi+(a+2)\operatorname{cosec}^2\varphi)\,p_\omega^2+\big((a-2)\operatorname{sech}^2\omega-(a+2)\operatorname{cosech}^2\omega\big)\,p_\varphi^2}{(a-2)(\operatorname{sech}^2\omega-\sec^2\varphi)-(a+2)(\operatorname{cosech}^2\omega+\operatorname{cosec}^2\varphi)}.\end{aligned}$$

The Hamilton-Jacobi equation in these coordinates is

$$\frac{4(p_\omega^2+p_\varphi^2)}{(a-2)(\operatorname{sech}^2\omega-\sec^2\varphi)-(a+2)(\operatorname{cosech}^2\omega+\operatorname{cosec}^2\varphi)}=E$$

and has solutions of the form

$$\begin{aligned}S(\omega,\varphi) &= \frac14\sqrt{\lambda}\log\left(\sqrt{\lambda}\ (\lambda\cos 2\varphi+2aE)+\lambda\sqrt{8E+4aE\cos 2\varphi-\lambda\sin^2 2\varphi}\right)\\ &\quad-\frac14\sqrt{(a+2)E}\ \operatorname{arctanh}\left(\frac{2Ea+8E-\lambda+\cos 2\varphi(\lambda+2Ea)}{2\sqrt{(a+2)E(8E+4aE\cos 2\varphi-\lambda\sin^2 2\varphi)}}\right)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4}\sqrt{(a-2)E}\ \operatorname{arctanh}\left(\frac{-2Ea+8E-\lambda+\cos 2\varphi(-\lambda+2Ea)}{2\sqrt{(a-2)E(8E+4aE\cos 2\varphi-\lambda\sin^2 2\varphi)}}\right)\\
&+\frac{1}{4}\sqrt{\lambda}\log\left(\sqrt{\lambda}\ (\lambda\cosh 2\omega-4E)+\lambda\sqrt{\lambda\sinh^2 2\omega-8E\cosh 2\omega-4aE}\right)\\
&+\sqrt{a+2}\ \arctan\left(\frac{4E+\lambda+4aE+\cosh 2\omega(4E-\lambda)}{2\sqrt{(a+2)E(\lambda\sinh^2\omega-8E\cosh 2\omega-4aE)}}\right)\\
&+\sqrt{a-2}\ \arctan\left(\frac{-4E+\lambda+4aE+\cosh 2\omega(4E+\lambda)}{2\sqrt{(a-2)E(\lambda\sinh^2\omega-8E\cosh 2\omega-4aE)}}\right).
\end{aligned}
$$

The Schrödinger equation has the form

$$
\frac{4\left(\dfrac{\partial^2\Psi}{\partial\omega^2}+\dfrac{\partial^2\Psi}{\partial\varphi^2}\right)}{(a-2)\left(\mathrm{sech}^2\omega-\sec^2\varphi\right)-(a+2)\left(\mathrm{cosech}^2\omega+\mathrm{cosec}^2\varphi\right)}=E\Psi
$$

with corresponding solutions

$$
\begin{aligned}
\Psi=(\sin\varphi\sinh\omega)^{\epsilon_-}\,(\cos\varphi\cosh\omega)^{\epsilon_+}\,{}_2F_1\left(\frac{\epsilon_++\epsilon_--\lambda}{2},\frac{\epsilon_++\epsilon_-+\lambda}{2},\epsilon_-+\frac{1}{2},\sin^2\varphi\right)\\
\times\,{}_2F_1\left(\frac{\epsilon_++\epsilon_--\lambda}{2},\frac{\epsilon_++\epsilon_-+\lambda}{2},\epsilon_-+\frac{1}{2},-\sinh^2\omega\right)
\end{aligned}
$$

with $\epsilon_\pm$ defined by (4.3).

4.1.4 Coordinates associated with $X_1+X_2+\gamma K^2$

For the coordinates corresponding to the fourth representative we make the transformation $u=\arctan(\exp\alpha)$, $v=\beta/2$, so our Hamiltonian has the form

$$
H=-4\frac{p_\alpha^2+\mathrm{sech}^2\alpha p_\beta^2}{a-2\tanh\alpha}\,. \tag{4.6}
$$

This can be realised in terms of projective coordinates on a two-dimensional complex sphere via $s_1=\cosh\alpha\cosh\beta$, $s_2=i\cosh\alpha\sinh\beta$, $s_3=i\sinh\alpha$ where $s_1^2+s_2^2+s_3^2=1$. The Hamiltonian can be written as

$$
H=4\frac{J_1^2+J_2^2+J_3^2}{\frac{2is_3}{\sqrt{s_1^2+s_2^2}}+a}\,.
$$

These two ways of realising the classical Hamiltonian are useful in determining the various possible separable coordinate systems.

We consider the most general case first, i.e., $\gamma\neq 0,2$. We make use of the transformation equations

$$
\sinh\alpha=i\frac{XY+1}{2\sqrt{XY}}\,,
$$

$$\tanh\beta = \frac{2\sqrt{(A_+X+A_-)(A_-X+A_+)(A_+Y+A_-)(A_-Y+A_+)}}{(A_+^2+A_-^2)(XY+1)+2A_+A_-(X+Y)},$$

applied to (4.6) to give classical Hamiltonian in the form

$$H = -16XY\frac{X(A_+X+A_-)(A_-X+A_+)p_X^2 - Y(A_+Y+A_-)(A_-Y+A_+)p_Y^2}{A_+A_-(X-Y)\Big((a+2)XY-a+2\Big)}.$$

The corresponding classical constant associated with this coordinate system is

$$\begin{aligned} & X_1+X_2+2\frac{A_+^2+A_-^2}{A_+^2-A_-^2}K^2 \\ = \;& 16\frac{(A_+X+A_-)(A_-X+A_+)\Big(a(A_+Y+A_-)(A_-Y+A_+)-2A_+A_-(Y^2-1)\Big)X^2p_X^2}{A_+A_-(A_+^2-A_-^2)(X-Y)\Big((a+2)XY-a+2\Big)} \\ & -16\frac{(A_+Y+A_-)(A_-Y+A_+)\Big(a(A_+X+A_-)(A_-X+A_+)-2A_+A_-(X^2-1)\Big)Y^2p_Y^2}{A_+A_-(A_+^2-A_-^2)(X-Y)\Big((a+2)XY-a+2\Big)}. \end{aligned}$$

The Hamilton-Jacobi equation has the form

$$-16XY\frac{X(A_+X+A_-)(A_-X+A_+)\left(\frac{\partial S}{\partial X}\right)^2+Y(A_+Y+A_-)(A_-Y+A_+)\left(\frac{\partial S}{\partial Y}\right)^2}{A_+A_-(X-Y)\Big((a+2)XY-a+2\Big)}$$

and solutions

$$S(X,Y) = \frac{1}{\sqrt{A_+A_-}}\left(\lambda_X\int\frac{1}{X}\sqrt{\frac{a_X-X}{(b-X)(c-X)}}\,dX+\lambda_Y\int\frac{1}{Y}\sqrt{\frac{a_Y-Y}{(b-Y)(c-Y)}}\,dY\right)$$

where $\lambda_X-\lambda_Y = -(a+2)EA_+A_-/16$, $a_X = (a-2)E\lambda_X/16$, $a_Y = (a-2)E\lambda_Y/16$, $b=-A_+/A_-$, $c=-A_-/A_+$.

A further change of coordinates

$$X = -\frac{1}{k}\mathrm{sn}^2(\alpha'+iK',k)\,,\qquad Y = -\frac{1}{k}\mathrm{sn}^2(\beta'+iK',k)\,,\qquad k=\frac{A_+}{A_-}$$

is convenient for writing the Schrödinger equation

$$\frac{16\left(\frac{\partial^2\Psi}{\partial\alpha'^2}+\frac{\partial^2\Psi}{\partial\beta'^2}\right)}{(a+2)k^4\left(\mathrm{sn}^2(\alpha',k)-\mathrm{sn}^2(\beta',k)\right)+k^2(a-2)} = E\Psi\,.$$

The separated equations are versions of Lame's equation [12]. Indeed if we look for solutions of the form $\Psi = A(\alpha')B(\beta')$ then

$$\begin{aligned} \frac{\partial^2A(\alpha')}{\partial\alpha'^2}+\left(-\frac{1}{16}k^4E(a+2)\mathrm{sn}^2(\alpha',k)-\lambda_1\right)A(\alpha') = 0\,, \\ \frac{\partial^2B(\beta')}{\partial\beta'^2}+\left(-\frac{1}{16}k^4E(a+2)\mathrm{sn}^2(\beta',k)-\lambda_2\right)B(\beta') = 0 \end{aligned}$$

where $\lambda_1 - \lambda_2 = -E(a-2)k^2/16$. Solutions of these separation equations can be represented as Riemann P functions [13] of the form

$$P(z) = \begin{pmatrix} 0 & 0 & k^{-2} & \infty & \\ 0 & 0 & 0 & \frac{1}{4}\left(1 - \frac{1}{2}\sqrt{4 + k^2 E(a+2)}\right) & \mathrm{sn}^2(z,k) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4}\left(1 + \frac{1}{2}\sqrt{4 + k^2 E(a+2)}\right) & \end{pmatrix}$$

for $z = \alpha', \beta'$.

The case $\gamma = 0$ can easily be deduced by putting $A_+ = iA_-$, as can be seen from the expression for the associated classical constant.

If $\gamma = 2$ then a convenient choice of coordinates is

$$x = \log\Big(\tan(\varphi' - i\omega')\Big)\,, \qquad y = \log\Big(\tan(\varphi' + i\omega')\Big)\,. \tag{4.7}$$

The corresponding classical Hamiltonian has the form

$$H = -\frac{p_{\varphi'}^2 + p_{\omega'}^2}{\dfrac{a+2}{\sinh^2 2\omega'} + \dfrac{a-2}{\sin^2 2\varphi'}}\,.$$

The classical constant is

$$X_1 + X_2 + 2K^2 = aH + \frac{(a+2)\sin^2 2\varphi' p_{\varphi'}^2 - (a-2)\sinh^2 2\omega' p_{\omega'}^2}{(a+2)\sin^2 2\varphi' + (a-2)\sinh^2 2\omega'}$$

The Hamilton-Jacobi equation in these coordinates is

$$-\frac{\left(\dfrac{\partial S}{\partial \varphi'}\right)^2 + \left(\dfrac{\partial S}{\partial \omega'}\right)^2}{\dfrac{a+2}{\sinh^2 2\omega'} + \dfrac{a-2}{\sin^2 2\varphi'}} = E$$

which has solutions

$$\begin{aligned} S(\varphi', \omega') \;=\; & \frac{i}{2}\sqrt{\lambda}\,\arctan\sqrt{\frac{(a-2)E}{\lambda}\;\sec^2 2\varphi' + \tan^2 2\varphi'} \\ & - \frac{i}{2}\sqrt{(a-2)E}\,\operatorname{arctanh}\sqrt{\sec^2 2\varphi' + \frac{\lambda}{(a-2)E}\tan^2 2\varphi'} \\ & + \frac{i}{2}\sqrt{\lambda}\,\arctan\sqrt{\frac{(a+2)E}{\lambda}\;\mathrm{sech}^2 2\omega' - \tanh^2 2\omega'} \\ & - \frac{i}{2}\sqrt{(a+2)E}\,\operatorname{arctanh}\sqrt{\mathrm{sech}^2 2\omega' - \frac{\lambda}{(a+2)E}\tanh^2 2\omega'} \end{aligned}$$

The corresponding Schrödinger equation is

$$-\frac{\dfrac{\partial^2 \Psi}{\partial \varphi'^2} + \dfrac{\partial^2 \Psi}{\partial \omega'^2}}{\dfrac{a+2}{\sinh^2 2\omega'} + \dfrac{a-2}{\sin^2 2\varphi'}} = E\Psi\,,$$

which has solutions of the form

$$\Psi = \sqrt{\sin 2\varphi' \sinh 2\omega'}\; P_\nu^{\frac{1}{2}\sqrt{1-(a-2)E}}(\cos 2\varphi') P_\nu^{\frac{1}{2}\sqrt{1-(a+2)E}}(\cosh 2\omega')$$

where $P_\nu^\mu(z)$ is a solution of Legendre's equation.

This completes the list of possible coordinate systems which are inequivalent and separable for this particular Hamiltonian. We notice in particular that the equation $H - E = 0$ can be written in the equivalent forms

$$\mu^2(p_\mu^2 + p_\nu^2) + \frac{1}{4}E\left(a - 2 + (a+2)\frac{\mu^2}{\nu^2}\right) = 0, \qquad J_1^2 + J_2^2 + J_3^2 - E\left(\frac{2is_3}{\sqrt{s_1^2 + s_2^2}} + a\right) = 0\,,$$

both superintegrable systems on the complex two-sphere, the first of which is written in horospherical coordinates.

4.2 Superintegrability for Darboux spaces of type four.

There are various possibilities for the potential in this case: [**A**], [**B**], [**C**], [**D**].

$$[\mathbf{A}]\;\; H = -\frac{4\mu^2\nu^2}{(a+2)\mu^2 + (a-2)\nu^2}\left(p_\mu^2 + p_\nu^2 + a_1 + a_2\left(\frac{1}{\mu^2} + \frac{1}{\nu^2}\right) + a_3(\mu^2 + \nu^2)\right).$$

The additional constants of the motion have the form

$$\begin{aligned} R_1 &= K^2 + a_1(\mu^2 + \nu^2) + a_3(\mu^2 + \nu^2)^2\,, \\ R_2 &= X_2 + \frac{2a_1\Big((a+2)\mu^2 - (a-2)\nu^2\Big) + 16a_2 + 4a_3\Big((a+2)\mu^4 - (a-2)\nu^4\Big)}{(a+2)\mu^2 + (a-2)\nu^2}\,. \end{aligned}$$

The corresponding quadratic algebra relations are determined by

$$\begin{aligned} R^2 &= 16R_1R_2^2 - 256a_3R_1^2 - 64a_1R_1R_2 - 256aa_3HR_1 - 1024a_2a_3R_1 + 64a_1HR_2 \\ &\quad - 256a_3H^2 - 64a_1(a+2)H - 256a_1^2a_2\,. \end{aligned}$$

This Hamiltonian admits a separation of variables in coordinates corresponding to the equivalence first, second and third classes of Section 4.1. For the second this is covered by the choice of coordinates μ, ν.

(i) For coordinates corresponding to the first equivalence class, we obtain the Hamiltonian in the form

$$H = -\frac{\sin^2 2u\,(p_u^2 + p_v^2 + 4a_1e^{2v} + 4a_2\mathrm{cosec}^2 2u + 4a_3e^{4v})}{2\cos 2u + a}\,.$$

(ii) For coordinates corresponding to the third representative (4.5) the Hamiltonian takes form

$$\begin{aligned} H &= \frac{4(p_\omega^2 + p_\varphi^2) + 4a_1c^2(\cosh^2\omega - \cos^2\varphi)}{(a-2)\left(\mathrm{sech}^2\omega - \sec^2\varphi\right) - (a+2)\left(\mathrm{cosech}^2\omega + \mathrm{cosec}^2\varphi\right)} \\ &\quad + \frac{16a_2\left(\mathrm{cosech}^2 2\omega + \mathrm{cosec}^2 2\varphi\right) + a_3c^4(\sinh^2 2\omega + \sin^2 2\varphi)}{(a-2)\left(\mathrm{sech}^2\omega - \sec^2\varphi\right) - (a+2)\left(\mathrm{cosech}^2\omega + \mathrm{cosec}^2\varphi\right)} \end{aligned}$$

The quantum algebra relations are

$$
\begin{aligned}
[\hat{R},\hat{R}_1] &= -8\{\hat{R}_1,\hat{R}_2\}-16\hat{R}_2-32a_1\hat{H}\,,\\
[\hat{R},\hat{R}_2] &= 8\hat{R}_2^2-256a_3\hat{R}_1-128aa_3\hat{H}-32(a_1^2+4a_3+16a_2a_3)
\end{aligned}
$$

together with the operator relation

$$
\begin{aligned}
\hat{R}^2 &= 8\{\hat{R}_1,\hat{R}_2^2\}-256a_3\hat{R}_1^2-80\hat{R}_2^2-256aa_3\hat{H}\hat{R}_1-64(16a_2a_3+a_1^2+4a_3)\hat{R}_1\\
&\quad +64a_1\hat{H}\hat{R}_2-256a_3\hat{H}^2+64a(4a_3-a_1^2)\hat{H}+128(a1^2+4a_3+8a_2a_3-2a_1^2a_2).
\end{aligned}
$$

As in the case of free motion we observe that the equation $H=E$ is

$$
p_\mu^2+p_\nu^2+a_1+\frac{a_2-\frac{1}{4}(a-2)E}{\mu^2}+\frac{a_2-\frac{1}{4}(a+2)E}{\nu^2}+a_3(\mu^2+\nu^2)=0\,,
$$

a superintegrable system in flat space with rearranged constants, that separates in elliptic and hyperbolic coordinates.

$$
\textbf{[B]}\ \ H=-\frac{\sin^2 2u\left(p_v^2+p_u^2+\dfrac{b_2}{\sinh^2 v}+\dfrac{b_3}{\cosh^2 v}\right)+b_1}{2\cos 2u+a}
$$

The additional constants are

$$
\begin{aligned}
R_1 &= X_1+X_2\\
&\quad +\frac{2b_1\cosh 2v+(b_2+b_3)(4-a^2)+(\cos 4u+2a\cos 2u+3)\left(\dfrac{b_2}{\sinh^2 v}-\dfrac{b_3}{\cosh^2 v}\right)}{2\cos 2u+a}\\
R_2 &= K^2+\frac{b_2}{\sinh^2 v}+\frac{b_3}{\cosh^2 v}\,.
\end{aligned}
$$

The quadratic algebra is given by

$$
\begin{aligned}
R^2 &= 16R_1^2R_2-64R_2^3-64aHR_2^2+64(2b_3-2b_2-b_1)R_2^2+32a(b_2+b_3)R_1R_2\\
&\quad -64H^2R_2+64(b_2+b_3)HR_1+128a(b_3-b_2)HR_2\\
&\quad -16\Big((4-a^2)(b_2+b_3)^2+8b_1(b_2-b_3)\Big)R_2+128(b_3-b_2)H^2-64b_1(b_2+b_3)^2\,.
\end{aligned}
$$

This Hamiltonian admits a separation of variables in coordinate systems corresponding to the first and fourth equivalence classes of (4.2). The defining expressions have already been given in terms of coordinates for the first. For the fourth, we distinguish two cases.

(i) $\gamma\neq 2$

$$
\begin{aligned}
H &= 16XY\,\frac{X(A_-X-A_+)(A_+X-A_-)p_X^2+Y(A_-Y+A_+)(A_+Y+A_-)p_Y^2}{A_+A_-(X-Y)(a-2-(a+2)XY)}\\
&\quad +\frac{b_1(XY+1)+\dfrac{4b_2(A_-^2-A_+^2)XY}{(A_+Y+A_-)(A_+X+A_-)}+\dfrac{4b_3(A_-^2-A_+^2)XY}{(A_-Y+A_+)(A_-X+A_+)}}{a-2-(a+2)XY}
\end{aligned}
$$

(ii) $\gamma = 2$

$$H = -\frac{p_{\varphi'}^2 + p_{\omega'}^2 + b_1\left(\frac{1}{\sinh^2 2\omega'} + \frac{1}{\sin^2 2\varphi'}\right) + \frac{4b_2}{\cos^2 2\varphi'} + \frac{4b_3}{\cosh^2 2\omega'}}{\frac{a+2}{\sinh^2 2\omega'} + \frac{a-2}{\sin^2 2\varphi'}}$$

The corresponding quantum algebra relations are

$$\begin{aligned}
[\hat{R}, \hat{R}_1] &= -8\hat{R}_1^2 + 96\hat{R}_2^2 + 64a\hat{H}\hat{R}_2 - 16a(b_2+b_3)\hat{R}_1 + 64(2b_2 - 2b_3 + b_1 + 3)\hat{R}_2 \\
&\quad + 32\hat{H}^2 + 32a(2b_2 - 2b_3 + 1)\hat{H} \\
&\quad + 64b_1(b_2 - b_3) - 8(a^2-4)(b_2+b_3)^2 + 32(b_1 + 2b_2 - 2b_3)\,, \\
[\hat{R}, \hat{R}_2] &= 8\{\hat{R}_1, \hat{R}_2\} + 16a(b_2+b_3)\hat{R}_2 - 16\hat{R}_1 + 32(b_2+b_3)\hat{H} - 16a(b_2+b_3)\,, \\
R^2 &= -64\hat{R}_2^3 + 8\{\hat{R}_1^2, \hat{R}_2\} - 64a\hat{H}\hat{R}_2^2 - 64\hat{H}^2\hat{R}_2 - 80\hat{R}_1^2 - 64(2b_2 - 2b_3 + b_1 + 7)\hat{R}_2^2 \\
&\quad + 16a(b_2+b_3)\{\hat{R}_1, \hat{R}_2\} + 64(b_2+b_2)\hat{H}\hat{R}_1 + 64a(2b_3 - 2b_2 - 1)\hat{H}\hat{R}_2 \\
&\quad - 160a(b_2+b_3)\hat{R}_1 + 16\Big((a^2-4)(b_2+b_3)^2 + 8(b_1+1)(b_3-b_2) - 4b_1 + 32\Big)\hat{R}_2 \\
&\quad + 128(b_3 - b_2 + 1)\hat{H}^2 + 128a(b_2 - b_3 + 1)\hat{H} \\
&\quad + (b_2+b_3)^2(128 - 80a^2 - 64b_1) - 128(b_1+2)(b_3 - b_2 - 1) - 256.
\end{aligned}$$

As in the case of free motion, equation $H - E = 0$ is

$$J_1^2 + J_2^2 + J_3^2 + \frac{2b_1}{\sqrt{s_1^2+s_2^2}\left(s_1 + \sqrt{s_1^2+s_2^2}\right)} + \frac{2b_2}{\sqrt{s_1^2+s_2^2}\left(s_1 - \sqrt{s_1^2+s_2^2}\right)} + b_3$$

$$-E\left(\frac{2is_3}{\sqrt{s_1^2+s_2^2}} + a\right) = 0\,,$$

a superintegrable system on the complex sphere that separates variables in spherical, elliptic and degenerate elliptic type 1 coordinates.

$$\textbf{[C]}\quad H = -\frac{p_{\varphi'}^2 + p_{\omega'}^2 + \frac{c_1}{\cos^2\varphi'} + \frac{c_2}{\cosh^2\omega'} + c_3\left(\frac{1}{\sin^2\varphi'} - \frac{1}{\sinh^2\omega'}\right)}{\frac{a+2}{\sinh^2 2\omega'} + \frac{a-2}{\sin^2 2\varphi'}}\,.$$

These are coordinates associated with $\gamma = 2$ in the fourth representative from (4.2). The constants of the motion associated with this Hamiltonian are

$$\begin{aligned}
R_1 &= X_1 + X_2 + 2K^2 + aH \\
&\quad + \frac{\frac{a+2}{\sinh^2 2\omega'}\left(\frac{c_3}{\sin^2\varphi'} + \frac{c_1}{\cos^2\varphi'}\right) + \frac{a-2}{\sin^2 2\varphi'}\left(\frac{c_3}{\sinh^2\omega'} - \frac{c_2}{\cosh^2\omega'}\right)}{\frac{a+2}{\sinh^2 2\omega'} + \frac{a-2}{\sin^2\omega'}}\,.
\end{aligned}$$

$$
\begin{aligned}
R_2 &= X_1 - X_2 + \frac{1}{\dfrac{a+2}{\sinh^2 2\omega'} + \dfrac{a-2}{\sin^2 \omega'}} \\
&\times \left[\frac{a+2}{\sinh^2 2\omega'} \left(c_1 \cosh 2\omega' \tan^2 \varphi' - c_2 \cos 2\varphi' - \frac{c_3 \Big(2\cos^2 \varphi' (\sinh^2 \omega' - \sin^2 \varphi') \Big) + 1}{\sin^2 \varphi'} \right) \right. \\
&\left. + \frac{a-2}{\sin^2 2\varphi'} \left(c_2 \cos 2\varphi' \tanh^2 \omega' + c_1 \cosh 2\omega' - \frac{c_3 \Big(2\cosh^2 \omega' (\sinh^2 \omega' - \sin^2 \varphi) + 1 \Big)}{\sinh^2 \omega'} \right) \right] .
\end{aligned}
$$

They satisfy the quadratic algebra determined by the identity

$$
\begin{aligned}
R^2 &= 16R_1^3 - 16R_1R_2^2 - 32aHR_1^2 + 32(c_2 - c_1)R_1^2 + 16(a^2-4)H^2R_1 \\
&+ 32((a+2)c_1 - (a-2)c_2 + 4c_3)HR_1 \\
&- 16(2c_3^2 - c_1^2 - c_2^2 + 6c_3(c_1+c_2) + 4c_1c_2)R_1 - 32(c_2-c_3)(c_1-c_3)R_2 \\
&- 16((a+2)(c_1-c_3)^2 + (a-2)(c_2-c_3)^2)H - 32(c_1-c_2)(3c_3^2 - c_1c_2 - c_3(c_1+c_2))
\end{aligned}
$$

The Hamiltonian admits a separation of variables in a number of coordinates systems corresponding to various combinations of the operators R_1 and R_2. We exhibit the various possibilities.

(i) For the constant $R_1 - R_2$, the associated separable coordinates are those corresponding to the third representative in (4.2) with $\gamma = 1$. In these coordinates, the Hamiltonian is

$$
H = \frac{4(p_\omega^2 + p_\varphi^2) + \dfrac{c_1+c_2+2c_3}{2\sinh^2 2\omega} - \dfrac{(c_1+c_2)\cosh 2\omega}{2\sinh^2 2\omega} + \dfrac{c_3 \cos 2\varphi}{\sin^2 2\varphi}}{(a-2)\left(\dfrac{1}{\cosh^2 \omega} - \dfrac{1}{\cos^2 \varphi} \right) - (a+2)\left(\dfrac{1}{\sinh^2 \omega} + \dfrac{1}{\sin^2 \varphi} \right)} .
$$

(ii) In coordinates corresponding to rotations of the fourth representative in (4.2) with $\gamma \neq 0, 2$, that is, $B_+^2 X_1 + (B_+^2 - B_-^2)X_2 + (2B_+^2 - B_-^2)K^2$, the corresponding Hamiltonian has the form

$$
\begin{aligned}
H = 16 &\left[-X(B_\mp + X)(B_\pm + X)p_X^2 + Y(B_\mp + Y)(B_\pm + Y)p_Y^2 + \frac{c_1}{4}\left(\frac{1}{Y} - \frac{1}{X} \right) \right. \\
&\left. + \frac{c_2}{4}(X - Y) + \frac{c_3}{4}(B_\mp^2 - B_\pm^2)\left(\frac{1}{1+B_\mp Y} - \frac{1}{1+B_\mp X} + \frac{1}{1+B_\pm Y} - \frac{1}{1+B_\pm X} \right) \right] \Big/ \\
&\left[(B_\mp^2 - B_\pm^2)\left(\frac{a-2}{1+B_\pm X} - \frac{a-2}{1+B_\pm Y} + \frac{a+2}{1+B_\mp Y} - \frac{a+2}{1+B_\mp X} \right) \right. \\
&\left. + \left(\frac{a-2}{X} - \frac{a-2}{Y} + (a+2)(X-Y) \right) \right] .
\end{aligned}
$$

Here, $B_\pm = B_+/B_-$ and $B_\mp = B_-/B_+$. The Hamiltonian associated with R_2 can be obtained from this last case by taking $B_- = \sqrt{2}B_+$.

The quantum algebra relations are

$$\begin{aligned}[\hat{R},\hat{R}_1] &= 8\{\hat{R}_1,\hat{R}_2\}+16\hat{R}_2+16(c_1-c_3)(c_2-c_3)\,,\\ [\hat{R},\hat{R}_2] &= 24\hat{R}_1^2-8\hat{R}_2^2-32a\hat{H}\hat{R}_1+8(a^2-4)\hat{H}^2+32(c_1-c_2-\tfrac{3}{2})\hat{R}_1\\ &\quad +16((a+2)c_1-(a-2)c_2+a+64c_3)\hat{H}\\ &\quad +8c_1^2+8c_2^2-16c_3^2-32c_1c_2-48c_3(c_1+c_2)+16(c_1-c_2)\,.\end{aligned}$$

The operator identity is

$$\begin{aligned}\hat{R}^2 &= 16\hat{R}_1^3-8\{\hat{R}_1,\hat{R}_2^2\}+32(c_2-c_1-\tfrac{7}{2})\hat{R}_1^2-80\hat{R}_2^2+16(a^2-4)\hat{H}^2\hat{R}_1\\ &\quad +32\Big((a+2)c_1-(a-2)c_2+4c_3+a\Big)\hat{H}\hat{R}_1+16\Big(c_1^2+c_2^2-2c_3^2-6c_3(c_1+c_2)\\ &\quad -4c_1c_2+2(c_1-c_2)-8\Big)\hat{R}_1-32(c_2-c_3)(c_1-c_3)\hat{R}_2+16(a^2-4)\hat{H}^2\\ &\quad -16\Big((a+2)((c_1-c_3)^2-2c_1)+(a-2)((c_2-c_3)^2+2c_2)-8c_3-4a\Big)\hat{H}\\ &\quad -32(c_1-c_2)(3c_3^2-c_1c_2-c_3(c_1+c_2))\\ &\quad +32(c_1^2+c_2^2-4c_3(c_1+c_2)-2c_1c_2+2c_1-2c_2).\end{aligned}$$

As in the case of free motion, the equation $H=E$ is

$$J_1^2+J_2^2+J_3^2-\frac{i(c_1+c_2+2c_3)s_1}{4\sqrt{s_2^2+s_3^2}}+\frac{i(c_1-c_2)(s_1+is_2-s_3)}{4\sqrt{2}\sqrt{(s_1+is_2)(s_3-is_2)}}$$

$$+\frac{(2c_3-c_1-c_2)(s_1+is_2+s_3)}{\sqrt{(s_1+is_2)(s_3+is_2)}}+\frac{i(c_1-c_2)}{4\sqrt{2}}-E\left(a+\frac{2is_1}{\sqrt{s_2^2+s_3^2}}\right)=0\,.$$

which is a superintegrable system on the complex sphere, with rearranged constants, that separates variables in elliptic and degenerate elliptic coordinates of type 1.

[D] $H=-\dfrac{4\mu^2\nu^2\left[p_\mu^2+p_\nu^2+d\left(\dfrac{1}{\mu^2}+\dfrac{1}{\nu^2}\right)\right]}{(a+2)\mu^2+(a-2)\nu^2}.$

This Hamiltonian admits three classical constants of the motion

$$R_1=X_1+\frac{d(\mu^2+\nu^2)^2}{(a+2)\mu^2+(a-2)\nu^2}\,,\quad R_2=X_2+\frac{16d}{(a+2)\mu^2+(a-2)\nu^2}\,,\quad K=\mu p_\mu+\nu p_\nu\,.$$

The Poisson quadratic algebra satisfies the relations

$$\{K,R_1\}=2R_1,\qquad \{K,R_2\}=-2R_2,\qquad \{R_1,R_2\}=-8K^3-4aKH-16dK.$$

These three extra constants are related via the identity

$$-R_1R_2+K^4+aHK^2+4dK^2+H^2=0\,.$$

This Hamiltonian admits a separation of variables in all the coordinate systems that are possible. We need only give the expressions in terms of the fourth representatives. In the coordinate system associated with the fourth representative and for which $\gamma \neq 2$ the Hamiltonian can be written as

$$\begin{aligned} H &= 16XY\,\frac{X(A_+X - A_-)(A_-X - A_+)p_X^2 - Y(A_+Y + A_-)(A_-Y + A_+)p_Y^2}{(X-Y)(a-2-XY(a+2))A_+A_-} \\ &\quad - \frac{4dA_+A_-\,(X^2Y + Y + XY^2 + X)}{(X-Y)(-a+2+XY(a+2))A_+A_-}\,, \end{aligned}$$

and for the case $\gamma = 2$ this Hamiltonian has the form

$$H = \frac{p_{\varphi'}^2 + p_{\omega'}^2 + d\left(\dfrac{1}{\sinh^2 2\omega'} + \dfrac{1}{\sin^2 2\varphi'}\right)}{\dfrac{a+2}{\sinh^2 2\omega'} + \dfrac{a-2}{\sin^2 2\varphi'}}\,.$$

The corresponding quadratic algebra relations are

$$[\hat{K}, \hat{R}_1] = 2\hat{R}_1, \qquad [\hat{K}, \hat{R}_2] = -2\hat{R}_2, \qquad [\hat{R}_1, \hat{R}_2] = -8\hat{K}^3 - 4a\hat{H}\hat{K} - 16d\hat{K} - 4\hat{K},$$

subject to the operator identity

$$-\frac{1}{2}\{\hat{R}_1, \hat{R}_2\} + \hat{H}^2 + a\hat{H}\hat{K}^2 + \hat{K}^4 + a\hat{H} + (5+4d)\hat{K}^2 + 4d = 0\,.$$

This completes the analysis of the superintegrable potentials associated with the four metrics of Darboux.

5 Relationship to constant curvature superintegrable potentials

In sections 2–4 we have found, by means of exhaustive calculation, all superintegrable potentials in the Darboux spaces of revolution having two or more quadratic integrals. Once these are expressed in suitable coordinates, it is clear that each is simply a multiple of one of the superintegrable potentials on the complex Euclidean plane or two-sphere, that have been enumerated in [7], though that was by no means evident in advance.

In each case we can start with a Hamiltonian of the form

$$H = H_0 + \alpha V_0\,, \tag{5.1}$$

where V_0 is a function of the coordinates x and y, and α is a constant. Dividing the Hamilton-Jacobi equation, $H = E$, throughout by V_0 and rearranging gives a new Hamilton-Jacobi equation in which the roles of the energy E and parameter α have been exchanged.

$$H' = \frac{H_0}{V_0} - \frac{E}{V_0} = -\alpha\,. \tag{5.2}$$

Clearly, the integrability and separability of one system guarantees that of the other. It is this relationship between the harmonic oscillator potential written in Cartesian coordinates and the Coulomb potential in parabolic coordinates that has been discovered by many authors. Transformations of this type relating integrable systems were described in a more general context by Hietarinta *et al* in [14] and called *coupling constant metamorphosis*. See also [15] where the *Stäckel transform* and its close connection with variable separation was emphasized.

The preservation of integrability under such a transformation can be demonstrated explicitly by noting that if $\{H_0, L_0\} = 0$ and

$$H = H_0 + \alpha V_0 \quad \text{and} \quad L = L_0 + \alpha \ell_0 \tag{5.3}$$

are in involution, i.e., $\{H, L\} = 0$, then so are

$$H' = \frac{H_0}{V_0} \quad \text{and} \quad L' = L_0 - \ell_0 H' \,. \tag{5.4}$$

Any identities involving integrals associated with (5.1), give rise to corresponding identities involving integrals associated with (5.2) and are obtained by the replacements

$$\alpha \to -H' \qquad \text{and} \qquad H \to 0\,. \tag{5.5}$$

5.1 Generating the Darboux spaces of revolution by coupling constant metamorphosis

Taking each of the degenerate potentials from [7], that is, the potentials with Hamiltonians having one first order and two quadratic integrals and performing a coupling constant metamorphosis we arrive at a Hamiltonian having one first order K and two quadratic constants, X_1 and X_2. These must be free Hamiltonians either on one of the four Darboux spaces of revolution or one of the constant curvature spaces, $E_2(\mathbb{C})$ or $S_2(\mathbb{C})$. After comparing the Hamiltonians so generated, it can been seen that this approach generates all of the Darboux spaces of revolution.

Knowing the Poisson algebra for each Hamiltonian involved and how coupling constant metamorphosis modifies this algebra, we can determine which Hamiltonian has been generated, even if it appears in unfamiliar coordinates. Note that some transformations reproduce the free Hamiltonian on $E_2(\mathbb{C})$ or $S_{2,\mathbb{C}}$, and some Darboux spaces can be generated from two distinct constant curvature potentials.

For each Hamiltonian we have four linearly independent constants of the motion. These, however, cannot be functionally independent and there is always a polynomial identity in K, X_1, X_2 and H that is of fourth order in the momenta. We can use this identity to classify the possible Hamiltonians. Up to freedoms in choosing X_1 and X_2, scalings of K and coupling constant metamorphosis, we find that there are 5 classes of identities that involve all of the constants. The correspondences between these identities, degenerate superintegrable potentials from [7] and the Darboux spaces of revolution are summarised in Table 1. Note that because we allow coupling constant metamorphosis, H has the same status as parameters in the the potential and the coefficients A and B appearing in the representative identities may be functions of H. The labels in bold (e.g. **E3**, **S3**,...) refer to [7]. Those Hamiltonians in Table 1 on the complex two-sphere, that is, **S3**, **S5** and **S6**, are represented with three coordinates s_1, s_2 and s_3 constrained by

$s_1^2+s_2^2+s_3^2=1$ and $J_1=s_2p_{s_3}-s_3p_{s_2}$, $J_2=s_3p_{s_1}-s_1p_{s_3}$ and $J_3=s_1p_{s_2}-s_2p_{s_1}$. The potentials **E12**, **E14**, **E4** and **E13** are functions of $x-iy$ and hence division of $p_x^2+p_y^2$ by these potentials reproduces the flat space Hamiltonian.

For example, starting from the algebraic identity for constants associated with the Hamiltonian and integrals

$$H=p_x^2+p_y^2+\frac{\alpha}{x^2}+\alpha \qquad (\mathbf{E6})\,, \tag{5.6}$$

$$X_1=(xp_y-yp_x)p_x-\frac{\alpha y}{x^2}\,, \qquad X_2=(xp_y-yp_x)^2+\frac{\alpha y^2}{x^2}\,, \qquad K=p_y\,,$$

that is [7],

$$X_1^2+K^2X_2-(H-\alpha)X_2+\alpha K^2=0\,,$$

we find that applying the transformation (5.4) gives

$$H'=\frac{p_x^2+p_y^2}{\frac{1}{x^2}+1}\,, \qquad X_1'=(xp_y-yp_x)p_x-\frac{y}{x^2}\frac{p_x^2+p_y^2}{\frac{1}{x^2}+1}=\frac{y(p_y^2-x^2p_x^2)}{x^2+1}+xp_xp_y\,,$$

$$X_2'=(xp_y-yp_x)^2+\frac{y^2}{x^2}\frac{p_x^2+p_y^2}{\frac{1}{x^2}+1}=\frac{(x^2+x^4-y^2)p_y^2+x^2y^2p_x^2}{x^2+1}-2xyp_xp_y\,, \qquad K'=K\,,$$

and using (5.5),

$$X_1'^2+K'^2X_2'-H'X_2'-H'K'^2=0\,.$$

Then

$$X_1''=2X_1'\,, \qquad X_2''=-X_2'+H'\,, \qquad H''=H'\,, \qquad K''=K'\,,$$

gives

$$X_1''^2-4K''^2X_2''+4H''X_2''-4H''^2=0\,,$$

the identity (2.3) associated with the Darboux space of type two (2.1).

5.2 Generating superintegrable potentials on Darboux spaces

The H_0 in equation (5.3) may itself contain potential terms and if these are chosen so that H is superintegrable, then so will be H'.

For example, taking the superintegrable Hamiltonian on the complex two-sphere **S1** [7],

$$H=J_1^2+J_2^2+J_3^2+\frac{\alpha}{(s_1-is_2)^2}+\frac{\beta s_3}{(s_1-is_2)^3}+\frac{\gamma(1-4s_3^2)}{(s_1-is_2)^4}+\delta$$

and dividing though by $(s_1-is_2)^{-2}-1$ gives, after a change of coordinates, the superintegrable potential [**A**] in a Darboux space of type 2. The same Hamiltonian can be generated by dividing **E2** throughout by $x^{-2}+1$.

Each potential in Table 1 is compatible with the addition of further terms while maintaining superintegrability, and in using the method demonstrated above, all superintegrable Hamiltonians found in Sections 2–4 can be generated. The correspondences are given below.

Table 1: Correspondences between constant curvature superintegrable potentials and Hamiltonians for Darboux spaces of revolution.

Degenerate superintegrable potential on $E_2(\mathbb{C})$ or $S_2(\mathbb{C})$	Hamiltonian for Darboux space of revolution	Representative identity
E5: $4x$	D_1: $\dfrac{p_u^2+p_v^2}{4u}$	$X_1^2+AX_2+K^4+B=0$
E6: $\dfrac{1}{x^2}+1$ **S5**: $\dfrac{1}{(s_1-is_2)^2}-1$	D_2: $\dfrac{u^2(p_u^2+p_v^2)}{u^2+1}$	$X_1^2+K^2X_2+AX_2+B=0$
E12: $\dfrac{\alpha(x-iy)}{\sqrt{(x-iy)^2+c^2}}+\beta$ **E14**: $\dfrac{\alpha}{\sqrt{x-iy}}+\beta$	$E_2(\mathbb{C})$	$X_1^2+K^2X_2+A=0$
E3: x^2+y^2+4 **E18**: $\dfrac{2}{\sqrt{x^2+y^2}}+1$	D_3: $\dfrac{p_u^2+p_v^2}{4+u^2+v^2}$	$X_1X_2+AK^2+B=0$
S3: $\dfrac{a+2}{s_3^2}-a+2$ **S6**: $\dfrac{2is_3}{\sqrt{s_1^2+s_2^2}}+a$	D_4: $\dfrac{p_u^2+p_v^2}{\frac{a+2}{u^2}+\frac{a-2}{v^2}}$	$X_1X_2+K^4+AK^2+B=0$
E4: $\alpha(x-iy)+\beta$ **E13**: $\dfrac{\alpha}{\sqrt{x-iy}}+\beta$	$E_2(\mathbb{C})$	$K^2X_1+AX_2+B=0$

5.2.1 Darboux spaces of type 1

The potential **E5**, $V_0 = 4x$, appears in each of

$$\mathbf{E2} \quad : \quad \alpha(4x^2+y^2)+\beta x+\frac{\gamma}{x^2}+\delta$$
$$\mathbf{E3'} \quad : \quad \alpha(x^2+y^2)+\beta x+\gamma y+\delta$$
$$\mathbf{E9} \quad : \quad \frac{\alpha}{\sqrt{x-iy}}+\beta x+\frac{\gamma(2x-iy)}{\sqrt{x-iy}}+\delta\,.$$

The potential labelled **E3′** is a translation of **E3** . Adding these potentials to $H_0 = p_x^2+p_y^2$ and dividing by $4x$ produces the two real non-degenerate potentials found in [1] and an additional complex one given in this paper. (The details of the quadratic algebra and defining operators for the Hamiltonian derived from **E9** can be computed using (5.2)).

5.2.2 Darboux spaces of type 2

The potentials **E6** and **S5** appear in each of following.

$$\mathbf{E1}\ \ [\mathbf{B}] \quad : \quad \alpha(x^2+y^2)+\frac{\beta}{x^2}+\frac{\gamma}{y^2}+\delta$$
$$\mathbf{E2}\ \ [\mathbf{A}] \quad : \quad \alpha(4x^2+y^2)+\beta x+\frac{\gamma}{y^2}+\delta$$
$$\mathbf{E16}\ \ [\mathbf{C}] \quad : \quad \frac{1}{\sqrt{x^2+y^2}}\left(\alpha+\frac{\beta}{x+\sqrt{x^2+y^2}}+\frac{\gamma}{x-\sqrt{x^2+y^2}}\right)+\delta$$
$$\mathbf{S1}\ \ [\mathbf{A}] \quad : \quad \frac{\alpha}{(s_1-is_2)^2}+\frac{\beta s_3}{(s_1-is_2)^3}+\frac{\gamma(1-4z^2)}{(x-iy)^4}+\delta$$
$$\mathbf{S2}\ \ [\mathbf{B}] \quad : \quad \frac{\alpha}{s_3^2}+\frac{\beta}{(s_1-is_2)^2}+\frac{\gamma(s_1+is_2)}{(s_1-is_2)^3}+\delta$$
$$\mathbf{S4}\ \ [\mathbf{C}] \quad : \quad \frac{\alpha}{(s_1-is_2)^2}+\frac{\beta s_3}{\sqrt{s_1^2+s_2^2}}+\frac{\gamma}{(s_1-is_2)\sqrt{s_1^2+s_2^2}}+\delta$$

The superintegrable system generated after dividing by $x^{-2}+1$ or $(s_1-is_2)^{-2}-1$ as appropriate is indicated by label the [**A**], [**B**] or [**C**]. The apparent over abundance of superintegrable potentials generated in this way for D_2 is resolved by noting that the same potential can appear in more than one coordinate system.

5.2.3 Darboux spaces of type 3

The potentials **E3** and **E18** appear in each of

$$\mathbf{E1}\ \ [\mathbf{B}] \quad : \quad \alpha(x^2+y^2)+\frac{\beta}{x^2}+\frac{\gamma}{y^2}+\delta$$
$$\mathbf{E3'}\ \ [\mathbf{A}] \quad : \quad \alpha(x^2+y^2)+\beta x+\gamma y+\delta$$
$$\mathbf{E7}\ \ [\mathbf{D}] \quad : \quad \frac{\alpha(x-iy)}{\sqrt{(x-iy)^2-c^2}}+\frac{\beta(x+iy)}{\sqrt{(x-iy)^2-c^2}\left((x-iy)+\sqrt{(x-iy)^2-c^2}\right)^2}$$

$$+\gamma(x^2+y^2)+\delta$$

E8 [C] : $$\frac{\alpha(x+iy)}{(x-iy)^3}+\frac{\beta}{(x-iy)^2}+\gamma(x^2+y^2)+\delta$$

E16 [B] : $$\frac{1}{\sqrt{x^2+y^2}}\left(\alpha+\frac{\beta}{x+\sqrt{x^2+y^2}}+\frac{\gamma}{x-\sqrt{x^2+y^2}}\right)+\delta$$

E17 [C] : $$\frac{\alpha}{\sqrt{x^2+y^2}}+\frac{\beta}{(x+iy)^2}+\frac{\gamma}{(x+iy)\sqrt{x^2+y^2}}+\delta$$

E19 [D] : $$\frac{\alpha(x-iy)}{\sqrt{(x-iy)^2-4}}+\frac{\beta}{\sqrt{(x+iy)(x-iy+2)}}$$
$$+\frac{\gamma}{\sqrt{(x+iy)(x-iy-2)}}+\delta$$

E20 [A] : $$\frac{1}{\sqrt{x^2+y^2}}\left(\alpha+\beta\sqrt{x+\sqrt{x^2+y^2}}+\gamma\sqrt{x-\sqrt{x^2+y^2}}\right)+\delta$$

As before, once the possibility of changes of coordinates in taken into account, the above list produces only those superintegrable potentials found in section 3.2.

5.2.4 Darboux spaces of type 4

The potentials **S3** and **S6** appear in each of

S2 [A] : $$\frac{\alpha}{s_3^2}+\frac{\beta}{(s_1-is_2)^2}+\frac{\gamma(s_1+is_2)}{(s_1-is_2)^3}+\delta$$

S4 [A] : $$\frac{\alpha}{(s_1-is_2)^2}+\frac{\beta s_3}{\sqrt{s_1^2+s_2^2}}+\frac{\gamma}{(s_1-is_2)\sqrt{s_1^2+s_2^2}}+\delta$$

S7 [B,C] : $$\frac{\alpha s_1}{\sqrt{s_2^2+s_3^2}}+\frac{\beta s_2}{s_3^2\sqrt{s_2^2+s_3^2}}+\frac{\gamma}{s_3^2}+\delta$$

S8 [C] : $$\frac{\alpha s_1}{\sqrt{s_2^2+s_3^2}}+\frac{\beta(s_1+is_2-s_3)}{\sqrt{(s_1+is_2)(s_3-is_2)}}+\frac{\gamma(s_1+is_2+s_3)}{\sqrt{(s_1+is_2)(s_3+is_2)}}+\delta$$

S9 [B] : $$\frac{\alpha}{s_1^2}+\frac{\beta}{s_2^2}+\frac{\gamma}{s_3^2}+\delta$$

As before, once the possibility of changes of coordinates in taken into account, the above list produces only those superintegrable potentials found in section 4.2.

6 Conclusion

In this paper we have discussed in some detail three of the four Darboux spaces of revolution that have at least two integrals of classical motion quadratic in the momenta in addition to the Hamiltonian. In each case we have also presented an exhaustive list of potentials for each of these spaces which when added to the Hamiltonians of these spaces preserve this property i.e. that there are still two extra integrals of the classical motion. These are the superintegrable systems associated with the systems of Darboux. The property of extra integrals also extends easily to the case of the corresponding quantum systems. For each of these systems we have calculated the corresponding quadratic algebra relations and shown that in each case the Hamiltonians that we obtain arise from constant curvature systems via a coupling constant transformation. We have also discussed the solutions of the corresponding classical and quantum problems in each of the inequivalent coordinate systems and have also given some of the embeddings of these spaces in three dimensions. In the last section we have shown how the free Hamiltonians of Darboux are related to the superintegrable Hamiltonians on spaces of constant curvature via coupling constant transformations. We also list how the corresponding superintegrable systems of spaces of constant curvature are related in this way to the superintegrable systems that we have found. This classification is comprehensive and complete.

Let us very briefly review the current status of superintegrability in two-dimensional spaces. Most of the published work [3, 4, 5, 6, 7] concerns quadratic superintegrability for classical, or quantum Hamiltonians of the form kinetic energy plus a scalar potential. Once a specific space is chosen, superintegrable systems in the space can be classified under the action of the corresponding isometry group. Systems in the same class are not only mathematically equivalent, but also have the same physical properties. In classical mechanics they will have the same trajectories and the trajectories will be periodic, if they are bounded. Similarly, in quantum mechanics superintegrable systems in the same class will have the same energy levels and eigenspaces.

Quadratically superintegrable systems exist in spaces of constant curvature and also in Darboux spaces. A Darboux space is defined by the fact that it allows one Killing vector and two (irreducible) Killing tensors. This article completes the task of classifying all quadratically superintegrable systems in all of the above spaces.

The results are quite rich. Indeed, in the real Euclidean space E_2, we have four $E(2)$ classes of superintegrable systems [3, 4]. They are physically quite diverse. One is an isotropic harmonic oscillator with additional terms, called **E1** above in Section 5. A second is an anisotropic harmonic oscillator with additional terms (called **E2** above). The third and fourth are Kepler (or Coulomb) systems with two different types of additional terms, respectively. In complex Euclidean space $E_2(\mathbb{C})$, or correspondingly in the pseudo-Euclidean space $E(1,1)$, one obtains 6 more classes [5].

Two classes of superintegrable systems exist on the real sphere S_2, four more on the complex sphere $S_2(\mathbb{C})$ [6]. On the real Darboux spaces $D_1, \ldots, D_4$ we have obtained 3, 4, 4, and 4 classes of systems, respectively. One more for the complex space $D_3(\mathbb{C})$.

¿From the mathematical point of view the situation is much more unified. As was stressed above, superintegrable systems that may correspond to quite different physical situations may be related by coupling constant metamorphosis. Once we allow this type of equivalence, many fewer equivalence classes exist. For instance, in real Euclidean space we only have two classes, because the Kepler potentials with additional terms are equivalent to isotropic harmonic oscillators (in one case with the additional terms). All superintegrable systems in Darboux spaces are related by coupling constant metamorphosis to systems in spaces of constant curvature. For D_1, D_2 and D_3 this is always flat space, complex or real. Two of the systems in D_4 are related to systems in real Euclidean space.

The other two are related to systems on a complex sphere. The relation is of course not unique and depends on the choice of coordinates (see Section 5).

A typical feature of quadratic superintegrability for scalar potentials is that quantum and classical superintegrable potentials coincide. They allow separation of variables in at least two coordinate systems in the Schrödinger and Hamilton-Jacobi equation, respectively.

Superintegrability involving third order integrals of motion has also been considered [16, 17]. There the situation is quite different. Multiseparability is lost. More interestingly, quantum superintegrable systems exist (in real Euclidean space) that have no classical analog (in the classical limit they reduce to free motion).

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