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AVERAGED MOTION OF CHARGED PARTICLES IN A CURVED STRIP

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Abstract. This paper is concerned with the motion of electrically charged particles in "curved" infinite strip. The equations of motion are $\Delta\varphi = -P$, $\partial^2\psi/\partial t^2 = -\nabla\varphi(\psi, t)$, and $P = P_0(\psi^{-1}) J(\psi^{-1})$, with initial and boundary conditions, where φ is the electric potential, ψ is the particle trajectory, P is the charge distribution, and J is the Jacobian. The lower boundary Γ_0 of the strip is grounded. Therefore, once a particle reaches Γ_0, it loses its charge and becomes immobile. We prove existence and uniqueness of solutions for small time. For nearly axially symmetric initial data, we also show that the solution can be extended until the time when "almost all" particles have migrated to Γ_0. A numerical simulation based on our model is implemented. The results indicate that particles tend to accumulate less around the convex parts of Γ_0 and more around the concave parts.

Key Words. Charged particles, two-phase materials, homogenization, dynamical system, conservation law.

AMS Subject Classification. Primary 70F99, 78A35. Secondly 35A05, 35B60, 45G15

1. The Model. We consider a simplified model of motion of electrically charged spherical particles, with uniform charge and mass, in a "curved" infinite strip in R^n :

$$\Omega = \{x = (x', x_n) \in R^n : \ g(x') < x_n < 1\}.$$

Let $P(x,t)$ denote the mass (or charge) distribution of the particles at a point x in Ω and time $t \geq 0$, and $\varphi(x,t)$ the electric potential. By Maxwell's equation,

$$\Delta\varphi = -P \quad \text{in} \quad \Omega. \tag{1.1}$$

A voltage difference M is maintained between the upper boundary $\Gamma_\infty = \{x_n = 1\}$ and the lower boundary

$$\Gamma_0 = \{(x', x_n) : \ x_n = g(x'), \ x' \in R^{n-1}\};$$

thus,

$$\varphi = M \quad \text{on} \quad \Gamma_\infty \tag{1.2}$$

and

$$\varphi = 0 \quad \text{on} \quad \Gamma_0. \tag{1.3}$$

The particles may collide, but collisions are assumed to be "soft" in the sense that nothing mechanically happens when two particles move into the same spot. We also neglect gravity. Then the only force acting on the particles is the electric field $-\nabla\varphi$. If we denote by $\psi(x,t)$ the particle trajectory then, by Newton's law,

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$$(1.4) \qquad \frac{d^2\psi(x,t)}{dt^2} = -\nabla\varphi(\psi(x,t),t).$$

By conservation of mass we also have

$$(1.5) \qquad P(x,t) = P_0\left(\psi^{-1}(x,t)\right) J\left(\psi^{-1}(x,t)\right), \; P(x,0) = P_0(x),$$

where $P_0(x)$ is the initial distribution, J is the Jacobian and ψ^{-1} is the inverse of the mapping $x \mapsto \psi(x,t)$ that we require to be 1-1. Finally, we prescribe the initial conditions

$$(1.6) \qquad \psi(x,0) = x, \;\; \psi_t(x,0) = \psi^0(x).$$

The model (1.1)–(1.6) was developed and studied by the authors [3] in case Ω is the entire space R^n. It was proved that a unique solution exists for a small time interval $0 \le t \le T$, but, in general, not for all time (since $P(x,t)$ may blow up in finite time). For a class of initial data, however, the solution was proved to exist for all time t [3].

We note that if we introduce the Eulerian variables $z = \psi(x,t)$, $\vec{v}(z,t) = \psi_t(x,t)$ then (1.4) becomes

$$(1.7) \qquad \vec{v}_t(z,t) + (\vec{v}\nabla_z)\vec{v}(z,t) = -\nabla_x\varphi.$$

Differentiating in t the relation

$$J(\psi(x,t))\,P(z,t) = P_0(x)$$

(which is another way of writing (1.5)) and using the well-known formula

$$\frac{\partial}{\partial t} J(\psi(x,t)) = (\nabla\cdot\vec{v})\,J(\psi(x,t)),$$

we obtain

$$(1.8) \qquad P_t + \nabla\cdot(P\vec{v}) = 0,$$

which is the standard form of conservation of mass. Although equations (1.7), (1.8) together with (1.1) look similar to the Euler-Poisson system [5], it is difficult to work with this form of the model, because we cannot very well account for the particles that leave the domain Ω. We shall therefore stick to the Lagrangian variable x and the formulation (1.1)–(1.6).

The geometry of the domain Ω and the boundary conditions (1.2), (1.3) are motivated by a problem in electrostatic spray painting [1] [2, Chap. 4]. The surface of the sprayer is located at $\Gamma_\infty = \{x_n = 1\}$ and the workpiece (which is being painted) is Γ_0. The workpiece is grounded ($\varphi = 0$) whereas a large potential M is maintained at Γ_∞. The electrostatically charged paint particles stream through Ω from Γ_∞ toward Γ_0 and, as they reach the workpiece Γ_0, they stick to it to form a paint layer. In the present model we assume that, once reaching Γ_0, the particles lose their electrostatic charge. We also assume that the thickness of the layer formed by the particles that accumulate along the boundary Γ_0 is small compared to the thickness of the strip Ω, so that the domain Ω may be assumed to be fixed in time. Finally, and most importantly, we consider here only the submodel whereby no new particles are injected from Γ_∞ into Ω at times $t > 0$, i.e., all the charged particles are already in Ω at time $t = 0$.

In Sections 2 and 3 we shall establish local existence and uniqueness for the system (1.1)–(1.6) by an adaptation of the method in [3]. In Section 4 we shall consider the case of axially symmetric data:

$$g(x') = 0,\ P_0(x) = P_0(x_n),\ \psi^0(x) = (0, bx_n), \tag{1.9}$$

and establish global solution, i.e., the solution exists for $0 < t < T$, where T is such that all the charged particles have migrated to Γ_0 by the time T.

In Section 5 we shall consider a perturbation problem with the initial data

$$g(x') = \varepsilon h(x'),\ P_0(x) = p_0,\ \psi^0(x) = 0,$$

where p_0 is a positive constant and ε is a small parameter. Using the method of Section 3 and deriving some *a priori* estimates, we show that there is a unique solution for $0 \le t \le T_\varepsilon$ where $T_\varepsilon \to T_0$ as $\varepsilon \to 0$ and T_0 is such that all the charged particles for the problem with $\varepsilon = 0$ have reached Γ_0. In Section 6 we consider for simplicity the case n=2 and establish the asymptotic expansion

$$\psi_\varepsilon = \psi_0 + \varepsilon\psi_1 + O\left(\varepsilon^2\right),\ f_\varepsilon = f_0 + \varepsilon f_1 + O\left(\varepsilon^2\right),\ \varphi_\varepsilon = \varphi_0 + \varepsilon\varphi_1 + O\left(\varepsilon^2\right), \tag{1.10}$$

where (φ_0, ψ_0, f_0) is the solution of an axially symmetric system and (φ_1, ψ_1, f_1) is the solution of the linearized problem about (φ_0, ψ_0, f_0).

We are interested in the thickness $W_\varepsilon(x')$ of the layer of the particles that have accumulated at a point (x', x_n) of the workpiece Γ_0 during the time interval $0 \le t \le T_\varepsilon$. Using (1.10), we can write

$$W_\varepsilon(x') = W_0 + \varepsilon W_1(x') + O\left(\varepsilon^2\right) \tag{1.11}$$

where W_0 is a constant. In Section 7 we compute numerically the solution of linearized problem and, in particular, the term $W_1(x')$. The numerical results explore the relation of $W_1(x')$ to the geometry of the workpiece.

2. Local existence. We anticipate that the particles will move downwards toward the workpiece Γ_0. Denote by Γ_t the surface consisting of all points x in Ω such that the trajectory $s \mapsto \psi(x, s)$ hits Γ_0 at time t, and write

$$\Gamma_t = \{(x', f(x', t)) : x' \in R^{n-1}\},$$

i.e., $f(x', t)$ is the function such that its graph is Γ_t. Note that $g(x') \le f(x', t) \le 1$ and $f(x', 0) = g(x')$. We assume that as soon as the trajectory $s \mapsto \psi(x, s)$ hits Γ_0, it becomes inactive (i.e., immobile and with zero charge). Therefore we need to consider the function $\psi(x,t)$ only for x in the closure of the domain

$$\Omega_t = \{(x', x_n) : \ f(x',t) < x_n < 1\}.$$

Setting

$$\Omega_t^\psi = \psi(\Omega_t, t) = \{\psi(x,t) : \ x \in \Omega_t\},$$

we note that the lower boundary of Ω_t^ψ is Γ_0. Denote by $\mathcal{X}_K$ the characteristic function of a set K. We can now reformulate problem (1.1)–(1.6) more precisely: Find functions $\varphi(x,t)$, $P(x,t)$, $\psi(x,t)$ and $f(x',t)$ satisfying

$$\Delta\varphi(x,t) = -P(x,t)\,\mathcal{X}_{\Omega_t^\psi} \ \text{ in } \ \Omega, \tag{2.1}$$

$$\psi_{tt}(x,t) = -\nabla\varphi(\psi(x,t),t) \ \text{ if } \ f(x',t) < x_n < 1, \tag{2.2}$$

$$P(x,t) = P_0\left(\psi^{-1}(x,t)\right) J\left(\psi^{-1}(x,t)\right) \ \text{ in } \ \Omega_t^\psi, \tag{2.3}$$

$$\psi_n(x', f(x',t), t) = g(\psi_1(x', f(x',t), t), \dots, \psi_{n-1}(x', f(x',t), t)), \tag{2.4}$$

$$g(x') \le f(x',t) \le 1 \tag{2.5}$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_n)$, together with the boundary conditions

$$\varphi = 0 \ \text{ on } \ \Gamma_0, \ \ \varphi = M \ \text{ on } \ \Gamma_\infty, \ \ \varphi \text{ is bounded in } \Omega, \tag{2.6}$$

and the initial conditions

$$\psi(x,0) = x, \ \psi_t(x,0) = \psi^0(x), \ f(x',0) = g(x'). \tag{2.7}$$

In (2.1)–(2.7) we have implicitly assumed that $\psi(x,t)$ is defined only in the closure of Ω_t, ψ^{-1} and P in (2.3) are defined in Ω_t^ψ, P is understood to be zero in $\Omega\backslash\Omega_t^\psi$ in (2.1), and

$$\Omega_t^\psi = \psi(\Omega_t, t) \subset \Omega, \ \psi(\cdot,t) : \ \Omega_t \to \Omega_t^\psi \ \text{ is invertible.} \tag{2.8}$$

We shall further require that

$$f_t(x',t) > 0. \tag{2.9}$$

This condition is motivated by the physical assumptions that the particles in Ω move downward and that no new particles are injected from Γ_∞ at time $t > 0$.

To prove existence and uniqueness we introduce the space $C^{m+\alpha}$ of functions for which the first m derivatives are α-Hölder continuous; the norm will be denoted by $\|\cdot\|_{C^{m+\alpha}}$. For any function $\varphi(x,t)$ defined in $\Omega \times [0,T]$, we shall briefly write $\|\varphi\|_{C^{m+\alpha}}$ instead of $\|\varphi(\cdot,t)\|_{C^{m+\alpha}(\Omega)}$, and $\nabla\varphi$ for the spatial gradient. Throughout the paper, we make the following assumptions:

$$g \in C^{2+\alpha}\left(R^{n-1}\right), \ g \le 1 - 2\delta_0 \ (\delta_0 > 0), \ P_0 \in C^\alpha\left(\bar{\Omega}\right), \ P_0 \ge 0, \tag{2.10}$$

$$(2.11) \qquad \psi^0 \in C^{1+\alpha}\left(\bar{\Omega}\right), \ \psi^0_n(x) \le 0, \ \vec{n}(y) \cdot \psi^0(x) \le 0 \text{ for } x \in \ \Omega, \ y \in \Gamma_0,$$

where $\vec{n}$ is the inward normal vector to Γ_0. In this and the next section we also assume, for simplicity, that

$$(2.12) \qquad \psi^0 = 0 \text{ and } P_0 = 0 \ \text{ for } |x| \text{ large.}$$

It is sometimes inconvenient to work directly with the function $\psi(x,t)$ because the domain Ω_t (of its spatial variable x) varies with time t. We therefore introduce a new function Ψ, defined in $\Omega \times [0,T]$, by

$$(2.13) \qquad \Psi(x', x_n, t) = \psi(x', z_n, t), \ \text{ or } \ \psi(x', x_n, t) = \Psi(x', w_n, t),$$

where

$$(2.14) \qquad z_n = f(x',t) + \frac{1 - f(x',t)}{1 - g(x')}(x_n - g(x')),$$

$$(2.15) \qquad w_n = g(x') + \frac{1 - g(x')}{1 - f(x',t)}(x_n - f(x',t)).$$

Note that

$$(2.16) \qquad \nabla\Psi = \nabla\psi \cdot \begin{bmatrix} I_{n-1} & 0 \\ \nabla_{x'} z_n & \frac{1-f}{1-g} \end{bmatrix}, \ \nabla\psi = \nabla\Psi \cdot \begin{bmatrix} I_{n-1} & 0 \\ \nabla_{x'} w_n & \frac{1-g}{1-f} \end{bmatrix},$$

and

$$(2.17) \qquad \Psi_t = \psi_t + D_t z_n D_n \psi, \ \ \psi_t = \Psi_t + D_t w_n D_n \Psi,$$

where D_i means differentiating with respect to the ith spatial variable, D_t is the derivative in t, and I_m the identity matrix in R^m. A solution of (2.1)–(2.7) for $0 \le t \le T$ is said to be classical if (i) f and ∇f are continuous in $R^{n-1} \times [0,T]$, (ii) P is continuous in $\bigcup_{0 \le t \le T}\left(\overline{\Omega_t^{\psi}} \times \{t\}\right)$, (iii) ψ, $\nabla\psi$, ψ_{tt} are continuous in $\bigcup_{0 \le t \le T}\left(\overline{\Omega_t} \times \{t\}\right)$, (iv) φ is continuous in $\bar{\Omega} \times [0,T]$, and $\Delta\varphi$ is bounded in $\bar{\Omega} \times [0,T]$. In the sequel, all solutions are understood to be classical solutions. We further require that, for any $0 \le t \le T$,

$$(2.18) \qquad f(\cdot,t), \ \Psi(\cdot,t) \in C^{1+\alpha} \ \text{ and } \ f_t(\cdot,t), \ \Psi_t(\cdot,t) \in C^{1+\alpha}$$

in their respective domains. We shall often denote a solution simply by (ψ, f), or (Ψ, f).

THEOREM 2.1. *If (2.10)–(2.12) hold and M is large enough (depending on the initial data) then there exists a unique solution of (2.1)–(2.7) for $0 \le t \le T$, for some $T > 0$.*

The proof is given in the next section.

3. Proof of Theorem 2.1. For any $\eta \geq 1$, $T > 0$, denote by $K(\eta, T)$ the set of all functions (Ψ, f) which satisfy the following conditions:

(i) $\Psi(x,t)$ is defined in $\bar{\Omega} \times [0,T]$ with values in $\bar{\Omega}$, and $f(x',t)$ is defined in $R^{n-1} \times [0,T]$ with values in R^1;

(ii) For any $0 \leq t \leq T$, (2.18) holds and

$$\|\Psi(\cdot,t) - Id(\cdot)\|_{C^{1+\alpha}}, \ \|f(\cdot,t)\|_{C^{1+\alpha}}, \ \|f_t(\cdot,t)\|_{C^\alpha}, \ \|\Psi_t(\cdot,t)\|_{C^\alpha} \leq \eta,$$

$$\|f - g\|_{L^\infty} \leq \delta_0, \ \|\nabla\Psi - I_n\|_{L^\infty} \leq \frac{1}{4\eta},$$

where $Id(\cdot)$ denotes the identity mapping;

(iii) For any $0 \leq t \leq T$, $x = (x', x_n) \in R^n$,

$$1 \geq f(x',t) \geq g(x'), \ f_t(x',t) > 0, \ f(x',0) = g(x'),$$

$$\Psi(x,0) = x, \ \Psi_t(x,0) = \psi^0(x) - \begin{pmatrix} 0 \\ D_t z_n(x,0) \end{pmatrix},$$

where z_n is defined in (2.14), and

$$\Psi_n(x', g(x'), t) = g(\Psi_1(x', g(x'), t), \ldots, \Psi_{n-1}(x', g(x'), t)).$$

For any $(\Psi, f) \in K(\eta, T)$, we shall define a mapping $\mathcal{A}(\Psi, f) = (\tilde{\Psi}, \tilde{f})$ in such a way that (Ψ, f) is a solution of (2.1)–(2.7) if and only if it is a fixed point of $\mathcal{A}$. We construct the mapping $\mathcal{A}$ in several steps:

Step 1. Given (Ψ, f), define ψ by (2.13), P by (2.3) and φ by (2.1), (2.6).

Step 2. Set $\varphi = \varphi_1 + M\varphi_2$, where φ_1, φ_2 are bounded functions satisfying

$$\Delta\varphi_1 = -P\mathcal{X}_{\Omega_t^\psi} \text{ in } \Omega, \tag{3.1}$$

$$\varphi_1 = 0 \text{ on } \Gamma_0 \cup \Gamma_\infty, \tag{3.2}$$

and

$$\Delta\varphi_2 = 0 \text{ in } \Omega, \tag{3.3}$$

$$\varphi_2 = 0 \text{ on } \Gamma_0, \ \varphi_2 = 1 \text{ on } \Gamma_\infty. \tag{3.4}$$

From condition (ii) and (2.16), $\|\nabla\psi - I_n\|_{L^\infty} \leq (4\eta)^{-1}$; hence $|J(\psi^{-1})|$ is uniformly bounded. By standard estimates for elliptic equations (more precisely, L^p-estimates and the Sobolev embedding), we can obtain the bound $|\nabla\varphi_1| \leq C_1$. Using the maximum principle, we readily find that

$$\frac{\partial}{\partial x_n}\varphi_2 \geq C_2 > 0 \text{ in } \Omega, \ \frac{\partial}{\partial \vec{n}}\varphi_2 \geq C_2 \text{ on } \Gamma_0,$$

where the constant C_2 depends only on $\|g\|_{C^{2+\alpha}}$. It follows that

$$\frac{\partial}{\partial x_n}\varphi \geq C_3 \text{ in } \Omega, \quad \frac{\partial}{\partial \vec{n}}\varphi \geq C_3 > 0 \text{ on } \Gamma_0, \tag{3.5}$$

provided that M is large enough. By the method in [3] we can show that φ_1 is a $C^{2+\alpha}$ function and therefore so is φ. We extend φ to the whole space R^n by a function $\hat{\varphi}$ such that

$$\|\hat{\varphi}\|_{C^2(R^n)} \leq C \|\varphi\|_{C^2(\Omega)},$$

where C depends only on g.

<u>*Step 3*</u>. Define $\tilde{\psi}(x,t)$, for any $x \in \Omega$, as the unique solution of

$$\tilde{\psi}(x,t) = x + t\psi^0(x) - \int_0^t \int_0^s \nabla\hat{\varphi}\left(\tilde{\psi}(x,\tau),\tau\right) d\tau ds; \tag{3.6}$$

the method of successive iterations shows that indeed there exists a unique solution to (3.6). Set

$$F(x', x_n, t) = \tilde{\psi}_n(x,t) - g\left(\tilde{\psi}_1(x,t), ..., \tilde{\psi}_{n-1}(x,t)\right). \tag{3.7}$$

Then

$$\begin{aligned} D_n F(x,t) &= 1 + tD_n\psi_n^0 - \int_0^t \int_0^s D_n\left(D_n\hat{\varphi}\left(\tilde{\psi}(x,\tau),\tau\right)\right) d\tau ds \\ &\quad -\nabla_{x'}g \cdot \left[tD_n\left(\psi^0\right)' - \int_0^t \int_0^s D_n\left(\nabla_{x'}\hat{\varphi}\left(\tilde{\psi}(x,\tau),\tau\right)\right) d\tau ds \right], \end{aligned}$$

where $(\psi^0)' = \left(\psi_1^0, ..., \psi_{n-1}^0\right)$. Hence, for small $T > 0$ (dependent on η),

$$D_n F(x,t) \geq \frac{1}{2} \text{ if } 0 \leq t \leq T. \tag{3.8}$$

From the implicit function theorem it then follows that there exists a unique function $x_n = \tilde{f}(x',t)$ of

$$F\left(x', \tilde{f}(x,t), t\right) \equiv 0. \tag{3.9}$$

We now use (2.13) to define $\tilde{\Psi}$ corresponding to $\tilde{\psi}$, $\tilde{f}$, and then define $\mathcal{A}(\Psi, f) = \left(\tilde{\Psi}, \tilde{f}\right)$.

<u>*Step 4*</u>. We claim that

(a) $g(x') \leq \tilde{f}(x',t) \leq 1$, $\tilde{f}_t(x',t) > 0$, and $\tilde{f}(x',0) = g(x')$;
(b) $\tilde{\psi}(x,t) \in \Omega$ for $\tilde{f}(x',t) < x_n < 1$;
(c) $\left(\tilde{\psi}, \tilde{f}\right)$ (hence $\left(\tilde{\Psi}, \tilde{f}\right)$) is independent of the particular extension $\hat{\varphi}$ of φ.

Proof. (a) From (3.7) and (3.9) at $t = 0$, we have $\tilde{f}(x',0) = g(x')$. Next we differentiate (3.7) in t to obtain

$$F_t(x,t) = D_t\tilde{\psi}_n - \nabla_{x'}g \cdot D_t\tilde{\psi}' = \sqrt{1+|\nabla_{x'}g|^2}\vec{n} \cdot D_t\tilde{\psi}(x,t), \tag{3.10}$$

where $\vec{n}$ is the inward normal to Γ_0 at $\left(\tilde{\psi}'(x,t), g\left(\tilde{\psi}'(x,t)\right)\right)$. Differentiating (3.6) in t we also have, for T small,

$$\vec{n} \cdot D_t\tilde{\psi}(x,t) = \vec{n} \cdot \psi^0(x) - \int_0^t \vec{n} \cdot \nabla\hat{\varphi}\left(\tilde{\psi}(x,s), s\right) ds < 0 \text{ if } t \leq T,$$

where we used (3.5) and assumption (2.11); consequently, $F_t(x', f(x',t), t) < 0$. Since, by (3.9),

$$(D_nF)\tilde{f}_t(x',t) + F_t(x', f(x',t), t) = 0, \tag{3.11}$$

it follows (using also (3.8)) that $\tilde{f}_t > 0$. Finally, since $\tilde{f}(x',0) = g(x') \leq 1 - 2\delta_0$, we obtain $g(x') \leq \tilde{f}(x',t) \leq 1 - \delta_0$ if t is small.

(b) Since $F\left(x', \tilde{f}(x',t), t\right) \equiv 0$ and $F(x', x_n, t)$ is strictly monotone increasing in x_n, it follows that $F(x', x_n, , t) > 0$ if $x_n > \tilde{f}(x',t)$. Hence, by (3.7),

$$\tilde{\psi}_n(x,t) > g\left(\tilde{\psi}'\right) \text{ if } x_n > \tilde{f}(x',t).$$

To complete the proof of (b), it remains to show that $\tilde{\psi}_n(x,t) \leq 1$ in the set $\tilde{f}(x',t) < x_n < 1$. Suppose this is not true. Then we can find a point (x^0, t^0) such that $\tilde{f}(x^{0\prime}, t^0) \leq x_n^0 < 1$, $\tilde{\psi}_n(x^0,t^0) = 1$ and $\tilde{\psi}_n(x^0,t) \leq 1$ for $t \leq t^0$. From (3.6),

$$\begin{aligned} 1 &= \tilde{\psi}_n\left(x^0,t^0\right) = x_n^0 + t\psi_n^0\left(x^0\right) - \int_0^{t^0}\int_0^s D_n\hat{\varphi}\left(\tilde{\psi}\left(x^0,\tau\right),\tau\right) d\tau ds \\ &= x_n^0 + t\psi_n^0\left(x^0\right) - \int_0^{t^0}\int_0^s D_n\varphi\left(\tilde{\psi}\left(x^0,\tau\right),\tau\right) d\tau ds \leq x_n^0 < 1 \quad \text{(by (3.5))}, \end{aligned}$$

a contradiction.

(c) By (b), $\tilde{\psi}(x,t) \in \Omega$ if $\tilde{f}(x',t) \leq x_n \leq 1$, and therefore $\hat{\varphi}\left(\tilde{\psi}(x,t),t\right) = \varphi\left(\tilde{\psi}(x,t),t\right)$. Hence for any $x \in \Omega$, $\tilde{\psi}(x,t)$ is a solution of (2.2) for $t < \min\left(\tilde{t}(x), T\right)$, where $\tilde{t}(x)$ is the largest number less that T such that $\tilde{f}\left(x', \tilde{t}(x)\right) \leq x_n$. Suppose that $\check{\varphi}$ is another extension of φ. Then by the above procedure we can define $\check{\psi}, \check{f}$ and $\check{F}$ corresponding to $\check{\varphi}$. By (b), we know that $\check{\psi}(x,t)$ solves (2.2) for $t < \min\left(\check{t}(x), T\right)$. By uniqueness of the solution to (2.2) (for fixed x), we then obtain $\tilde{\psi}(x,t) = \check{\psi}(x,t)$ for $t < \min\left(\tilde{t}(x), \check{t}(x)\right)$. It follows (from (3.7)) that $F(x', x_n, t) = \check{F}(x', x_n, t)$ for $t \leq \min\left(\tilde{t}(x), \check{t}(x)\right)$ and, since the solution to $F(x', x_n, t) = 0$ is unique, $\tilde{f}(x',t) = \check{f}(x',t)$. We thus conclude that $\tilde{\psi}$ is independent of the extension of φ.

Step 5. To prove that $\mathcal{A}$ maps $K(\eta, T)$ into itself (for small T and large η), we need to estimate the $C^{1+\alpha}$-norms of $\tilde{\psi}$ and $\tilde{f}$. Using the method in [3], we can show that

$$\left\|\tilde{\psi}(\cdot,t) - Id(\cdot)\right\|_{C^{1+\alpha}} + \left\|\tilde{\psi}_t(\cdot,t)\right\|_{C^{1+\alpha}} \leq \left\|\psi^0\right\|_{C^{1+\alpha}} + tC(\eta,T) \tag{3.12}$$

for $0 \leq t \leq T$, where $C(\eta, T)$ depends on the initial data and on η, T and is an increasing function in T. By (3.10), (3.11) and $C^{1+\alpha}$ estimates of $\tilde{\psi}_t$ in (3.12), we also have

$$\left\|\tilde{f}_t(\cdot,t)\right\|_{C^\alpha} \leq C + tC(\eta,T),$$

where the constant C depends only on the initial data g, P_0 and ψ^0. Since also $\psi(x,0) = 0$, if we choose η large enough and T sufficient small, then $\left(\tilde{\Psi}, \tilde{f}\right) \in K(\eta, T)$.

Having constructed the mapping $\mathcal{A}$, we now define successively

$$\left(\Psi^{(m+1)}, f^{(m+1)}\right) = \mathcal{A}\left(\Psi^{(m)}, f^{(m)}\right),$$

and let

$$\delta^{(m)}(t) = \sup_{x\in\Omega}\left|\Psi^{(m+1)}(x,t) - \Psi^{(m)}(x,t)\right|$$

$$+ \sup_{x'\in R^{n-1}}\left|f^{(m+1)}(x',t) - f^{(m)}(x',t)\right|.$$

As in [3], we can show that

$$\delta^{(m)}(t) \leq C^m t^m \left|\log t\right|^m. \tag{3.13}$$

Therefore, for $0 \leq t \leq T$ (T small), the sequence $\Psi^{(m)}(x,t)$ is uniformly convergent as $m \to \infty$. It follows that the mapping $\mathcal{A}$ admits a fixed point. As in [3], we can also establish uniqueness by slightly modifying the proof of the estimates (3.13). Clearly, a fixed point of $\mathcal{A}$ is a solution to (2.1)–(2.7). Conversely, any solution to (2.1)–(2.7) must belong to $K(\eta, T)$ for some η and T, and is a fixed point of $\mathcal{A}$. This concludes the proof of Theorem 2.1.

Note that in the above proof, M was taken large enough so that (3.5) holds. The choice of M depends only on $\|P\|_{L^\infty}$. Since, for small time, $\|P\|_{L^\infty}$ is bounded by $C\|P_0\|_{L^\infty}$, M actually depends only on $\|P_0\|_{L^\infty}$.

4. Axially symmetric solution. The results in the Sections 2 and 3 can be extended to the cases where P_0 and ψ^0 do not have compact supports. We shall be interested in the global solutions, that is, the solutions that exist until time T when all the particles have migrated to Γ_0, i.e., $P(x,T) = 0$. From [3] we know that, in general, a global solution does not exist . Therefore we first restrict our attention to the axially symmetric data, i.e.,

$$P_0(x) = P_0(x_n),\; g(x') = 0 \tag{4.1}$$

and

$$\psi^0(x) = (0, ..., 0, b_n x_n), \quad b_n \leq 0; \tag{4.2}$$

in the next section we shall consider a small perturbation of such data.

Set

$$\Omega_0 = \{x \in R^n : \ 0 < x_n < 1\}.$$

Since in (4.1) and (4.2) all data are axially symmetric with respect to x_n-axis, we expect that the solution will also be axially symmetric. Thus, we seek a solution in Ω_0 of the form

$$\begin{cases} \varphi(x,t) = \varphi_1(x_n,t) + Mx_n, \\ \psi_i(x,t) = x_i \ \text{ for } \ 1 \leq i \leq n-1, \\ \psi_n(x,t) = \psi_n(x_n,t), \\ f(x',t) = f(t) \end{cases} \tag{4.3}$$

with $P = P(x_n, t)$. From equation (2.1) and the boundary conditions in (2.6),

$$\begin{cases} \Delta\varphi_1 = -P\mathcal{X}_{\Omega_t^\psi} \ \text{ in } \ \Omega_0 \\ \varphi_1 = 0 \ \text{ on } \ \Gamma_0 \cup \Gamma_\infty. \end{cases} \tag{4.4}$$

Hence, for $x_n \leq \psi_n(1,t)$,

$$\begin{aligned} D_n\varphi_1(x_n,t) &= A(t) - \int_0^{x_n} P(\xi,t)\, d\xi \\ &= A(t) - \int_0^{x_n} P_0\left(\psi_n^{-1}(\xi,t)\right) D_n\left(\psi_n^{-1}(\xi,t)\right) d\xi \\ &= A(t) - \int_{f(t)}^{\psi_n^{-1}(x_n,t)} P_0(\xi)\, d\xi, \end{aligned}$$

where we have used the fact that $\psi_n(f(t),t) = 0$. It follows that

$$D_n\varphi_1(x_n,t) = \begin{cases} A(t) - \int_{f(t)}^{\psi_n^{-1}(x_n,t)} P_0(\xi,t)\, d\xi & \text{for } 0 \leq x_n \leq \psi_n(1,t), \\ A(t) - \int_{f(t)}^{1} P_0(\xi,t)\, d\xi & \text{for } \psi_n(1,t) \leq x_n \leq 1. \end{cases} \tag{4.5}$$

By (2.2)

$$\frac{d^2\psi_n(x_n t)}{dt^2} = -D_n\varphi(\psi_n,t) = -D_n\varphi_1(\psi_n,t) - M.$$

Therefore, as long as the trajectory lies in Ω_0,

$$\psi_n(x_n,t) = (1+b_nt)\,x_n - \frac{t^2}{2}M - \int_0^t\int_0^s \left(A(\tau) - \int_{f(\tau)}^{x_n} P_0(\xi)\, d\xi \right) d\tau ds. \tag{4.6}$$

To determines $A(t)$ we integrate both sides of (4.5) and use the boundary conditions in (4.4):

$$\int_0^{\psi_n(1,t)} \left(A(t) - \int_{f(t)}^{\psi_n^{-1}(x_n,t)} P_0(\xi,t)\, d\xi \right) dx_n + \int_{\psi_n(1,t)}^{1} \left(A(t) - \int_{f(t)}^{1} P_0(\xi,t)\, d\xi \right) dx_n = 0,$$

or

$$A(t) - (1-\psi_n(1,t)) \int_{f(t)}^{1} P_0(\xi,t)\, d\xi = \int_0^{\psi_n(1,t)} \int_{f(t)}^{\psi_n^{-1}(x_n,t)} P_0(\xi,t)\, d\xi dx_n$$

$$= \int_{f(t)}^{1} \left(\int_{f(t)}^{z} P_0(\xi)\, d\xi \right) D_n\psi_n(z,t)\, dz \ \text{ (by substituting } z = \psi_n^{-1}(x_n,t)\text{)}. \tag{4.7}$$

From (4.6) we have, by differentiation,

$$D_n\psi_n(x_n,t) = (1+b_nt) + \frac{t^2}{2}P_0(x_n). \tag{4.8}$$

Substituting this into the right-hand side of (4.7), we find the following expression for $A(t)$:

$$\begin{aligned} A(t) &= (1-\psi_n(1,t)) \int_{f(t)}^{1} P_0(\xi)\, d\xi \\ &\quad + \int_{f(t)}^{1} \left(\int_{f(t)}^{z} P_0(\xi)\, d\xi \right) \left(1 + b_nt + \frac{t^2}{2}P_0(z) \right) dz. \end{aligned} \tag{4.9}$$

To compute $\psi_n(1,t)$ we integrating (4.8) (with respect to x_n) and use the fact that $\psi_n(f(t),t) = 0$:

$$\psi_n(1,t) = (1+b_nt)(1-f(t)) + \frac{t^2}{2}\int_{f(t)}^{1} P_0(\xi)\, d\xi. \tag{4.10}$$

Finally, using the relation $\psi_n(f(t),t) = 0$ in (4.6), we arrive at the following formula for f:

$$f(t) = \frac{Mt^2}{2(1+b_nt)} + \frac{1}{1+b_nt}\int_0^t\int_0^s \left(A(\tau) - \int_{f(\tau)}^{f(t)} P_0(\xi)\, d\xi \right) d\tau ds. \tag{4.11}$$

Theorem 4.1. *Assume that $b_n \le 0$, $P_0 \ge 0$ and P_0 is continuous. Then there is a constant $M_0 > 0$ (depending on b_n and P_0) such that for any $M \ge M_0$ there exists a unique solution to the system (4.9)–(4.11) for $0 \le t \le T(M, b_n)$, and*

$$0 \le f(t) \le 1,\ f'(t) > 0 \ \text{ for } \ 0 \le t < T(M, b_n),$$

$$f(t) \nearrow 1 \ \text{ as } \ t \nearrow T(M, b_n);$$

moreover,

$$\frac{\partial T}{\partial M} \le 0, \qquad \frac{\partial T}{\partial b_n} \ge 0. \tag{4.12}$$

Proof. Consider first this case where $D_n P_0$ is Lipschitz continuous. By extension, we may assume that P_0 is defined for all $x \in R^1$. Multiplying (4.11) by $(1 + b_n t)$ and differentiating in t, we get

$$\begin{aligned} &\left[1 + b_n t + \frac{t^2}{2} P_0(f(t))\right] f'(t) \\ &= Mt - b_n f(t) + \int_0^t \left(A(\tau) - \int_{f(\tau)}^{f(t)} P_0(\xi)\, d\xi \right) d\tau. \end{aligned} \tag{4.13}$$

Differentiating (4.13) once again, we obtain

$$\begin{aligned} &\left[1 + b_n t + \frac{t^2}{2} P_0(f(t))\right] f''(t) \\ &= M + A(t) - \left[2b_n + 2tP_0(f(t)) + \frac{t^2}{2} P_0'(f(t)) f'(t)\right] f'(t). \end{aligned} \tag{4.14}$$

From (4.9) and (4.10), one sees that the right-hand side of (4.14) is Lipschitz continuous in $f(t)$, $f'(t)$ and t. Therefore, by standard ODE theory, equation (4.14) together with the initial conditions $f(0) = f'(0) = 0$ has a unique solution as long as $1 + b_n t > 0$.

Consider first the case that $b_n < 0$. If we substitute (4.10) into (4.9) we find, after some manipulations, that for $t \ge \tau$,

$$\begin{aligned} A(\tau) - \int_{f(\tau)}^{f(t)} P_0(\xi)\, d\xi &= -\frac{\tau^2}{2} \int_{f(\tau)}^{1} P_0(\xi) \int_{\xi}^{1} P_0(\eta)\, d\eta\, d\xi \\ &\quad + \int_{f(t)}^{1} P_0(\xi)\, d\xi - (1 + b_n \tau) \int_{f(\tau)}^{1} \int_{\xi}^{1} P_0(\eta)\, d\eta d\xi. \end{aligned} \tag{4.15}$$

It follows that

$$M + A(\tau) - \int_{f(\tau)}^{f(t)} P_0(\xi)\, d\xi \ge \gamma$$

for any $\gamma > 0$, if

$$M \geq \frac{\tau^2}{2}\left(\int_0^1 P_0(\xi)\, d\xi\right)^2 + \int_0^1 P_0(\xi)\, d\xi + \gamma. \tag{4.16}$$

We choose

$$M_0 = \frac{T_0^2}{2}\left(\int_0^1 P_0(\xi)\, d\xi\right)^2 + \int_0^1 P_0(\xi)\, d\xi + \gamma, \quad T_0 = -\frac{1}{2b_n} \tag{4.17}$$

where γ is a positive constant that will be determined later on. Then, for any $M \geq M_0$, (4.16) holds for $t < T_0$. Hence from (4.13),

$$f'(t) > 0 \ \text{ for } \ 0 \leq t \leq T_0,$$

and from (4.11),

$$f(t) \geq \frac{\gamma t^2}{(1 + b_n t)} \ \text{ for } \ 0 \leq t \leq T_0$$

as long as $f(t) \leq 1$. It follows that if

$$\gamma \geq \frac{1 + b_n T_0}{T_0^2} = 2b_n^2\,,$$

then, for any $M \geq M_0$, $f(t) \nearrow 1$ as $t \nearrow T(M, b_n)$, for some $T(M, b_n) \leq T_0$. The solution $f(t)$ of (4.14) determines the unique solution of (4.9)-(4.11).

In case $b_n = 0$, we choose

$$T_0 = \left(\frac{1}{2}\int_0^1 P_0(\xi)\, d\xi\right)^{-\frac{1}{2}}, \quad \gamma \geq \int_0^1 P_0(\xi)\, d\xi,$$

so that

$$M_0 = 3\int_0^1 P_0(\xi)\, d\xi. \tag{4.18}$$

As before, if $M \geq M_0$ there exists a $T(M, b_n) \leq T_0$ such that $f'(t) > 0$ for $t < T(M, b_n)$ and

$$f(t) \nearrow 1 \ \text{ as } \ t \nearrow T(M, b_n).$$

From now on we fix

$$\gamma = \max\left(2b_n^2, \int_0^1 P_0(\xi)\, d\xi\right) \tag{4.19}$$

and take M_0 to be the maximum of the two number M_0 in (4.17) and (4.18). Then, if $M \geq M_0$ the assertions of the theorem follow.

We now drop the assumption that P_0' is Lipschitz continuous function. Let P_m be a sequence of smooth functions that converges uniformly to P_0. For each m, there exists a solution $f_m(t)$ for $t \leq T_0$. From (4.13), we know that f_m and f_m' is uniformly bounded. Let $A_m(t)$ denote the function $A(t)$ corresponding to f_m. It is easily seen that

$$|A_l(t) - A_m(t)| \leq C\,|f_l(t) - f_m(t)| + C\,\|P_l - P_m\|_{L^\infty},$$

and, from (4.11),

$$|f_l(t) - f_m(t)| \leq Ct^2 \left(\sup_{\tau \leq t} |f_l(\tau) - f_m(\tau)| + \|P_l - P_m\|_{L^\infty} \right).$$

Hence for small t, $f_m(t)$ converges uniformly to a function $f(t)$; by (4.13), $f_m'(t)$ also converges uniformly to $f'(t)$. A step-by-step argument now shows that $f_m(t) \to f(t)$ for all t as long as $f(t)$ remains less than 1. Finally, the assertions in (4.12) follow by substituting $t = T(M, b_n)$ in (4.11) and differentiating with respect to M and b_n.

REMARK 4.1. If P_0 is piecewise continuous then we can extend Theorem 4.1 by establishing existence and uniqueness of a solution f in interval (t_i, t_{i+1}) where $f(t_i)$ and $f(t_{i+1})$ are two adjacent discontinuities of P_0. One can show that f is Lipschitz continuous; furthermore, $f(t)$ is continuously differentiable and $f'(t) > 0$ if $t \neq t_i$.

We next show that system (4.9)–(4.11) is equivalent to system (2.1)–(2.7) with the data (4.1) and (4.2).

THEOREM 4.2. *Let $(A(t), f(t))$ be a solution of (4.9)–(4.11) established in Theorem 4.1. Then the functions in (4.3) with φ_1 and ψ_n defined by (4.5) and (4.6) form the unique solution of (2.1)–(2.7).*

Proof. Actually the only thing that remains to be proved is that

$$0 \leq \psi_n(x_n, t) \leq 1 \text{ if } f(t_n) \leq x_n \leq 1,\ 0 \leq t \leq T \quad (T = T(M, b_n)). \tag{4.20}$$

To prove it, notice that by (4.10), (4.13) and (4.19), $D_t\psi_n(1,t) \leq 0$. Since $\psi_n(1,0) = 1$ (by (4.10)), we conclude that

$$\psi_n(1,t) \leq 1 \text{ for } t \leq T.$$

Since by (4.8) $D_n\psi_n(x_n,t) > 0$, if follows that $\psi_n(x_n,t) \leq 1$. Next $\psi_n(f(t), t) = 0$, and since $D_n\psi_n(x_n,t) > 0$, $\psi_n(x_n,t) \geq 0$ if $x_n \geq f(t)$.

EXAMPLE 4.1. Let

$$b_n = 0,\ P_0(x_n) = p_0\mathcal{X}_{(0,1)}(x_n),\ p_0 > 0. \tag{4.21}$$

Then

$$\psi_n(x_n, t) = \left(1 + \frac{p_0 t^2}{2}\right)(x_n - f(t)), \tag{4.22}$$

where $f(t)$ solves

$$\left(1+\frac{p_0t^2}{2}\right)f''(t)+2p_0tf'(t)=A(t)+M,\quad f(0)=f'(0)=0, \tag{4.23}$$

and

$$A(t)=\frac{p_0}{2}(1-f(t))\left[\left(1+\frac{p_0t^2}{2}\right)f(t)+1-\frac{p_0t^2}{2}\right]. \tag{4.24}$$

Note that (4.23) is equivalent to

$$\left(1+\frac{p_0t^2}{2}\right)f(t)=\frac{Mt^2}{2}+\int_0^t\int_0^s(A(\tau)+p_0f(\tau))\,d\tau ds. \tag{4.25}$$

5. A perturbed system. In the previous section we established existence of a global solution if the domain and the initial data are axially symmetric. We now consider system (2.1)–(2.9) in the case where

$$g(x')=\varepsilon h(x'),\ \psi_t(x,0)=0,\ P_0(x)=p_0 \tag{5.1}$$

where ε is small, $p_0>0$ is a constant, and $h(x')$ is a $C^{2+\alpha}$ function. For simplicity, we assume that $\varepsilon>0$, $h(x)\geq 0$. By slightly modifying the proof of Theorem 2.1, we can show that there exists a unique solution $(\psi_\varepsilon,f_\varepsilon,\varphi_\varepsilon)$, for $0\leq t\leq T$ for some $T>0$ which is independent of ε. We shall use the following notations:

$$\begin{aligned}
\Omega_\varepsilon&=\{(x',x_n):\ \varepsilon h(x')<x_n<1,\ x'\in R^{n-1}\},\\
\Omega_{\varepsilon t}&=\left\{(x',x_n):\ f_\varepsilon(x')<x_n<1,\ x'\in R^{n-1}\right\},\\
\Gamma_\varepsilon&=\{(x',x_n):\ x_n=\varepsilon h(x'),\ x'\in R^n\},\\
\Gamma_{\varepsilon t}&=\{(x',x_n):\ x_n=f_\varepsilon(x'),\ x'\in R^n\};
\end{aligned}$$

the set $\Omega_{\varepsilon t}^{\psi_\varepsilon}$ is defined similarly. In this section Ω_0 and Γ_0 are understood as Ω_ε and Γ_ε respectively, for $\varepsilon=0$. To establish global existence, we need the following lemmas.

LEMMA 5.1. *Let $0<a<1$, $g_1(x')\in C^{2+\alpha}(R^{n-1})$, $g_2(x')\in C^{1+\alpha}(R^{n-1})$, and $b(x')\in C^{2+\alpha}(R^{n-1})$. For any ε small such that $a+\|\varepsilon g_2\|_{L^\infty}<1$, set*

$$\begin{aligned}
G_\varepsilon&=\{(x',x_n):\ \varepsilon g_1(x')<x_n<a+\varepsilon g_2(x')\},\\
G_1&=\{(x',x_n):\ \varepsilon g_1(x')<x_n<1\},\\
G_0&=\{(x',x_n):\ 0<x_n<a\}.
\end{aligned}$$

Suppose u_ε is a bounded solution of

$$\Delta u_\varepsilon=q\mathcal{X}_{G_\varepsilon}\ \text{ in } G_1,\ u_\varepsilon=0\ \text{ on } x_n=1,\ u_\varepsilon=b(x')\ \text{ on } x_n=\varepsilon g_1(x'), \tag{5.2}$$

where $q\in C^\alpha(G_1)$. Then there exists a positive constant ε_0 (depending on g_1,g_2 and a) such that $\|\varepsilon_0g_1\|_{C^{2+\alpha}}+\|\varepsilon_0g_2\|_{C^{1+\alpha}}\leq 1$, and for $|\varepsilon|\leq\varepsilon_0$

$$\|u_\varepsilon\|_{C^{2+\alpha}(G_\varepsilon)}\leq\frac{C}{a^{2+\alpha}}\left(\|q\|_{C^\alpha}+\|b\|_{C^{2+\alpha}}\right), \tag{5.3}$$

where C is a constant independent of a, b, g_i, q and ε. Moreover, if $g_1 \geq 0,\ g_2 \leq 0$, and u_0 is the solution corresponding to $\varepsilon = 0$, then

$$\|u_\varepsilon - u_0\|_{C^{2+\alpha}(G_\varepsilon)} \leq \frac{\varepsilon C}{a^{2+\alpha}} \left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right), \tag{5.4}$$

for $0 \leq \varepsilon \leq \varepsilon_0$.

Note that in (5.4) the assumptions $\varepsilon \geq 0, g_1 \geq 0,\ g_2 \leq 0$ are made only for the sake of simplicity so that $G_\varepsilon \subseteq G_0$. Without these assumptions, however, (5.4) is still true provided u_0 is "properly" extended to G_ε.

Proof. Consider first the case $a = 1/2$. Choose ε_0 such that $|\varepsilon_0 g_i(x')| \leq 1/8$. For any $x_0' \in R^{n-1}$, we take a cut-off function $r(x')$ that equals 1 if $|x' - x_0'| \leq 1/4$ and 0 if $|x' - x_0'| \geq 1/2$. Denote by $B_\rho = B_\rho(x_0')$ the ball in R^n of radius ρ centered at $(x_0', \varepsilon g_2(x_0'))$. Set

$$w(x) = \omega_2 \iint\limits_{G_\varepsilon} \frac{q(\xi)\, r(\xi)}{|x - \xi|^{n-2}}\, d\xi,$$

where the constant ω_2 is chosen so that

$$\Delta w = rq\mathcal{X}_{G_\varepsilon} \ \text{ in } \ G_1.$$

By [3, Lemma 4.2],

$$\|w\|_{C^2\left(G_\varepsilon \cap B_{1/4}\right)} + \|w\|_{C^{2+\alpha}\left(G_\varepsilon \cap B_{1/4}\right)} \leq C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right). \tag{5.5}$$

The function $v_\varepsilon = u_\varepsilon - w$ satisfies

$$\Delta v_\varepsilon = 0 \quad \text{in} \quad G_1 \cap B_{1/4}.$$

By the maximum principle, $\|u_\varepsilon\|_{L^\infty} \leq C\left(\|q\|_{L^\infty} + \|b\|_{L^\infty}\right)$, and the same estimate then holds for v_ε in $G_\varepsilon \cap B_{1/4}$ (by (5.5)). It follows (by the interior Schauder estimates) that the $C^{2+\alpha}$ norm of v_ε in $G_1 \cap B_{1/8}$ is bounded by $C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right)$. Therefore,

$$\|u_\varepsilon\|_{C^{2+\alpha}\left(G_\varepsilon \cap B_{1/8}\right)} \leq C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right). \tag{5.6}$$

By the Schauder estimates, the same bound holds in any subsets of G_ε that satisfy $dist\left(x, \{x_n = \varepsilon g_2(x')\}\right) \geq 1/16$, and thus the assertion (5.3) is valid for $a = 1/2$.

To prove (5.4), we first show that

$$\|u_\varepsilon - u_0\|_{L^\infty(G_\varepsilon)} \leq \varepsilon C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right). \tag{5.7}$$

Introduce a change of variables

$$\xi' = x', \quad \xi_n = \frac{x_n - \varepsilon g_1(x')}{1 - \varepsilon g_1(x')}, \quad \tilde{u}_\varepsilon(\xi) = u_\varepsilon(x). \tag{5.8}$$

Then $\tilde{u}_\varepsilon$ is defined in Ω_0 and satisfies

$$\begin{aligned} &\Delta \tilde{u}_\varepsilon = \varepsilon\theta + \tilde{q}\mathcal{X}_{\tilde{G}_\varepsilon} \text{ in } \Omega_0, \\ &\tilde{u}_\varepsilon = 0 \text{ on } x_n = 1, \\ &\tilde{u}_\varepsilon(x', 0) = b(x'), \end{aligned}$$

where $\tilde{G}_\varepsilon$ is the image of G_ε by the mapping (5.8), θ is a bounded function (by (5.6)), and $\tilde{q}(\xi) \equiv q(x)$ satisfies $|\tilde{q} - q| \leq C\varepsilon$. It is now easy to deduce (5.7) by L^p estimates for the Poisson equation.

Next, for any fixed x'_0, we choose a smooth function $r(x')$ such that $r(x') = 1$ if $|x' - x'_0| \leq 2$ and $r(x') = 0$ if $|x' - x'_0| \geq 3$, and set $\hat{u}_\varepsilon = ru_\varepsilon$, $\hat{u}_0 = ru_0$. Then

$$\Delta \hat{u}_\varepsilon = q_\varepsilon + rq\mathcal{X}_{G_\varepsilon} \text{ in } \Omega_0, \quad \Delta \hat{u}_0 = q_0 + rq\mathcal{X}_{G_0} \text{ in } \Omega_0,$$

where

$$q_\varepsilon = \tilde{u}_\varepsilon \Delta r + 2\nabla \tilde{u}_\varepsilon \cdot \nabla r, \quad q_0 = u_0 \Delta r + 2\nabla u_0 \cdot \nabla r.$$

It is easily seen that

$$\|q_\varepsilon - q_0\|_{C^\alpha(G_\varepsilon)} \leq C \|u_\varepsilon - u_0\|_{C^{1+\alpha}(G_\varepsilon)}. \tag{5.9}$$

We can express both $\hat{u}_\varepsilon$ and $\hat{u}_0$ as Newtonian potentials in the bounded domains plus boundary integrals. By applying the methods in [3, section 12] (to estimate the Newtonian potentials) and [4, Section 3] (to estimate the boundary integrals), we can derive the bound

$$\|\hat{u}_\varepsilon - \hat{u}_0\|_{C^{2+\alpha}(G_\varepsilon \cap B_2)} \leq \varepsilon C \left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right) + C \|q_\varepsilon - q_0\|_{C^\alpha(G_\varepsilon)}. \tag{5.10}$$

There is actually a difference between the treatments in [3] and in (5.10). In [3], we break the domain of integration G_ε into two parts. One part is a ball tangent to the boundary whereas the measure of the remaining part is less than εC. Here we take, instead of a ball, the intersection of G_ε with a half space that is tangent to ∂G_ε at a boundary point; then the remaining region is contained in a strip of width less than εC. The rest of the estimates are essentially same.

From (5.9) and (5.10) it follows that

$$\|u_\varepsilon - u_0\|_{C^{2+\alpha}(G_\varepsilon \cap B_1)} \leq \varepsilon C \left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right) + C \|u_\varepsilon - u_0\|_{C^{1+\alpha}(G_\varepsilon)}.$$

Using partition of unity, we get

$$\|u_\varepsilon - u_0\|_{C^{2+\alpha}(G_\varepsilon)} \leq \varepsilon C \left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right) + C \|u_\varepsilon - u_0\|_{C^{1+\alpha}(G_\varepsilon)},$$

and then, by interpolation and (5.7), the proof of (5.4) (in case $a = 1/2$) follows.

For general $a > 0$, we use the scaling $w_\varepsilon(x) = u_\varepsilon(2ax)$. Then

$$\Delta w_\varepsilon = 4a^2 \tilde{q}\mathcal{X}_{\tilde{G}_\varepsilon} \text{ in } G_1/2a,$$

$$w_\varepsilon = 0 \text{ on } x_n = \frac{1}{2a}, \quad w_\varepsilon = \tilde{b} \text{ on } x_n = \frac{\varepsilon \tilde{g}_1}{2a},$$

where $\tilde{b}(x') = b(2ax')$, $\tilde{q}(x) = q(2ax)$, $\tilde{g}_1(x') = g_1(2ax')$, $\tilde{G}_\varepsilon = G_\varepsilon/2a$. By the maximum principle,

$$\|w_\varepsilon\|_{L^\infty} = \|u_\varepsilon\|_{L^\infty} \le \|b\|_{L^\infty} + C\|q\|_{L^\infty}.$$

From the proof of the case $a = 1/2$, we see that the estimates on the C^α-norm of the first two derivatives remain true if we replace the domain G_1 by G_1/ρ for $\rho \le 1$, and the constants C do not depend on ρ. Hence (5.3) and (5.4) hold for w_ε. We thus obtain

$$\|w_\varepsilon\|_{C^{2+\alpha}(\tilde{G}_\varepsilon)} \le C\left(a^2\|\tilde{q}\|_{C^\alpha} + \left\|\tilde{b}\right\|_{C^{2+\alpha}} + \|q\|_{L^\infty}\right),$$

$$\|w_\varepsilon - w_0\|_{C^{2+\alpha}(\tilde{G}_\varepsilon)} \le \varepsilon C\left(a^2\|\tilde{q}\|_{C^\alpha} + \left\|\tilde{b}\right\|_{C^{2+\alpha}} + \|q\|_{L^\infty}\right),$$

if $|\varepsilon g_1|, |\varepsilon g_2| \le a/8$ (so that Ω_ε is connected and of class $C^{1+\alpha}$), where $w_0(x) = u_0(2ax)$. Estimates (5.3) and (5.4) now follow by scaling back.

Let (ψ_0, f_0, φ_0) be the global solution corresponding to $\varepsilon = 0$ for $0 \le t \le T_0$, constructed in Example 4.1, with $f_0(T_0) = 1$; the function ψ_0 is understood to be defined for all x in R^n by extending its n-th component, as defined in (4.22), to all $x_n \in R$.

Lemma 5.2. *Suppose the solution $(\psi_\varepsilon, f_\varepsilon, \varphi_\varepsilon)$ of (2.1)–(2.9) exists for $0 \le t \le T$, and $1 - f_\varepsilon(x', t) \ge \delta$ for some $\delta > 0$. Then there exists an $\varepsilon_0 > 0$ and a constant C such that if*

$$\psi_\varepsilon = \psi_0 + \varepsilon\psi_\varepsilon^{(1)}, \quad f_\varepsilon = f_0 + \varepsilon f_\varepsilon^{(1)} \tag{5.11}$$

for $\varepsilon \le \varepsilon_0$, then

$$\left\|\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} + \left\|D_t\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} + \left\|f_\varepsilon^{(1)}\right\|_{C^{1+\alpha}} \le C, \tag{5.12}$$

for all $0 \le t \le T$.

Proof. By (5.11), we have, for $f_\varepsilon(x', t) < x_n < 1$,

$$\varepsilon\frac{d^2\psi_\varepsilon^{(1)}}{dt^2} = -\nabla\varphi_\varepsilon(\psi_\varepsilon, t) + \nabla\varphi_0(\psi_0, t). \tag{5.13}$$

We can also write

$$\begin{aligned} \Delta\varphi_\varepsilon &= -p_0 J\left(\psi_0^{-1}\right)\mathcal{X}_{\Omega_{\varepsilon t}}^{\psi_\varepsilon} + \theta_\varepsilon \mathcal{X}_{\Omega_{\varepsilon t}}^{\psi_\varepsilon}, \\ \Delta\varphi_0 &= -p_0 J\left(\psi_0^{-1}\right)\mathcal{X}_{\Omega_{0t}}^{\psi_0}, \end{aligned}$$

where $\theta_\varepsilon = p_0\left(J\left(\psi_0^{-1}\right) - J\left(\psi_\varepsilon^{-1}\right)\right)$. By (5.11),

$$\|\theta_\varepsilon\|_{C^\alpha} \le \varepsilon C\left\|\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}. \tag{5.14}$$

Set $\varphi_\varepsilon = u_\varepsilon + v_\varepsilon$, where u_ε satisfies

$$\Delta u_\varepsilon = -p_0 J\left(\psi_0^{-1}\right)\mathcal{X}_{\Omega_{\varepsilon t}}^{\psi_\varepsilon}$$

and the boundary condition (2.6). By the estimate (5.4) of Lemma 5.1 (with $g_1 = h$, $a = \psi_{0n}(1,t)$, $g_2(x') = \psi_n^{(1)}(x',1,t)$),

$$\|\nabla u_\varepsilon(\psi_\varepsilon,t) - \nabla\varphi_0(\psi_0,t)\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} \le \varepsilon C\left\|\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}, \tag{5.15}$$

where C depends only on δ. On the other hand, v_ε satisfies

$$\Delta v_\varepsilon = \theta_\varepsilon \mathcal{X}_{\Omega_{\varepsilon t}}^{\psi_\varepsilon}$$

with zero boundary conditions. Again by Lemma 5.1 and (5.14),

$$\|\nabla v_\varepsilon(\psi_\varepsilon,t)\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} \le C\|\theta_\varepsilon\|_{C^\alpha} \le \varepsilon C\left\|\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}. \tag{5.16}$$

It follows from (5.13), (5.15) and (5.16) that

$$\left\|\frac{d^2\psi_\varepsilon^{(1)}}{dt^2}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} \le C\left\|\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})},$$

so that

$$\left\|\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} + \left\|D_t\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} \le C. \tag{5.17}$$

Next, in view of (5.1), (2.4) becomes

$$\psi_{\varepsilon,n}(x', f_\varepsilon, t) = \varepsilon h\left(\psi_\varepsilon'(x', f_\varepsilon, t)\right).$$

Hence, by (5.11) and (4.22),

$$-\varepsilon\left(1 + \frac{p_0 t^2}{2}\right) f_\varepsilon^{(1)} = \varepsilon h\left(\psi_\varepsilon'(x', f_\varepsilon, t)\right) - \varepsilon\psi_{\varepsilon,n}^{(1)}(x', f_\varepsilon, t),$$

so that

$$\left\|f_\varepsilon^{(1)}\right\|_{C^{1+\alpha}} \le C\left\|\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} + C.$$

Combining this with (5.17), the lemma follows.

We can now establish "global" existence for the perturbed system.

THEOREM 5.1. *For any $\delta > 0$, there exists an $\varepsilon_0(\delta) > 0$ such that for $\varepsilon \le \varepsilon_0(\delta)$, there exists a unique solution $(\psi_\varepsilon, f_\varepsilon, \varphi_\varepsilon)$ of (2.1)–(2.9) with (5.1) for $0 \le t \le T_0 - \delta$.*

Proof. By Theorem 2.1, there exists a unique solution for $0 \le t \le T$, T is small and independent of ε. To extend the solution beyond $t = T$, we need to verify the following conditions:

$$D_t\psi_{\varepsilon,n}(x,T) \le 0 \text{ in } \Omega_{\varepsilon T}, \quad \vec{n}\cdot D_t\psi_\varepsilon(x,T) \le 0 \text{ on } \Gamma_0, \tag{5.18}$$

$$\inf\left\{\left|1-f_\varepsilon(x',T)\right|:\ x'\in R^{n-1}\right\}\geq\omega>0, \tag{5.19}$$

$$D_n\psi_{\varepsilon,n}(x,T)-\varepsilon\nabla_{x'}h(\psi_\varepsilon')\cdot D_n\psi_\varepsilon'(x,T)\geq\gamma>0; \tag{5.20}$$

notice that (5.18) ensures that $D_t f_\varepsilon>0$; (5.20) guarantees that we can solve for $f_\varepsilon(x',T+t)$ from $F=0$, and (5.19) implies that $\Omega_{\varepsilon T}$ is connected and is of class $C^{1+\alpha}$. Inequality (5.18) holds if M is large enough, independently of ε. Set

$$2\omega_\delta=1-f_0(T_0-\delta)>0.$$

By continuity, (5.19) holds for $\omega=\omega_\delta$ if T small (depending only on the initial data, since $D_t f_\varepsilon$ is bounded by a constant depending on the initial data.) Next, since by (5.11),

$$D_n\psi_{\varepsilon,n}=\left(1+\frac{p_0t^2}{2}\right)+\varepsilon D_n\psi_{\varepsilon,n}^{(1)},$$

(5.20) holds for $\gamma=1/2$ if ε is small. We can now apply the proof of Theorem 2.1 with slight modifications (since $\psi_\varepsilon(x,T)\neq x$ in general) to extend the solution to $0\leq t\leq T+\Delta T$ where ΔT is a positive number which depends only on ω,γ, and the $C^{1+\alpha}$-norms of the data at $t=T$. If $T+\Delta T<T_0-\delta$, then,

$$1-f_0(T+\Delta T)\geq1-f_0(T_0-\delta)=2\omega_\delta$$

since f_0 is monotone increasing. By Lemma 5.2, $f_\varepsilon^{(1)}$ and $\psi_\varepsilon^{(1)}$ are bounded, uniformly with respect to ε. Hence for ε small, (5.18)–(5.20) hold with $\omega=\omega_\delta,\gamma=1/2$, at $t=T+\Delta T$. By applying Theorem 2.1 once again, we can extend the solution up to $0\leq t\leq T+2\Delta T$, if ε is small enough. Repeating the above procedure a finite number of times, we obtain a solution for $0\leq t\leq T_0-\delta$. Note that at each step, we may need to decrease the size of ε. However, the length of the time interval is fixed, until we reach $T_0-\delta$.

6. Asymptotic expansion. In this section, we consider system (2.1)–(2.9) with data (5.1), and seek a more precise asymptotic expansion then (5.11), namely

$$f_\varepsilon(x',t) = f_0(t)+\varepsilon f^{(1)}(x',t)+\varepsilon^2f_\varepsilon^{(2)}, \tag{6.1}$$

$$\psi_\varepsilon(x,t) = \psi_0(x,t)+\varepsilon\psi^{(1)}(x,t)+\varepsilon^2\psi_\varepsilon^{(2)}, \tag{6.2}$$

$$\varphi_\varepsilon(x,t) = \varphi_0(x,t)+\varepsilon\varphi^{(1)}(x,t)+\varepsilon^2\varphi_\varepsilon^{(2)}. \tag{6.3}$$

For simplicity we shall work in R^2, instead of R^n, and denote the point in R^2 by (x,y). By elementary calculation,

$$J\left(\psi_\varepsilon^{-1}(x,t)\right)=J\left(\psi_0^{-1}(x,t)\right)+\varepsilon H(x,y,t)+O\left(\varepsilon^2\right), \tag{6.4}$$

and

$$\nabla\varphi_\varepsilon\left(\psi_\varepsilon\left(x,y,t\right),t\right)=\nabla\varphi_0\left(\psi_0\left(x,y,t\right),t\right)$$

$$+\varepsilon\begin{pmatrix} D_x\varphi^{(1)}\left(\psi_0\left(x,t\right),t\right) \\ -p_0Q\left(t\right)\psi_2^{(1)}\left(x,y,t\right)+D_y\varphi^{(1)}\left(\psi_0\left(x,t\right),t\right)\end{pmatrix}+O\left(\varepsilon^2\right),$$

where

$$H(x,y,t)=-Q\left(t\right)D_x\psi_1^{(1)}\left(\psi_0^{-1}\left(x,y,t\right),t\right)-Q\left(t\right)^2D_y\psi_2^{(1)}\left(\psi_0^{-1}\left(x,y,t\right),t\right), \tag{6.5}$$

and

$$Q\left(t\right)=\left(1+\frac{p_0t^2}{2}\right)^{-1}.$$

Substituting these expressions in the system (2.1)–(2.7) and comparing the coefficients of the ε^0 and ε^1 terms, we find that

$$\frac{d^2\psi_1^{(1)}\left(x,y,t\right)}{dt^2}=-D_x\varphi^{(1)}\left(\psi_0,t\right), \tag{6.6}$$

$$\frac{d^2\psi_2^{(1)}\left(x,y,t\right)}{dt^2}=p_0Q\left(t\right)\psi_2^{(1)}\left(x,y,t\right)-D_y\varphi^{(1)}\left(\psi_0,t\right), \tag{6.7}$$

$$\psi^{(1)}\left(x,y,0\right)=\psi_t^{(1)}\left(x,y,0\right)=0, \tag{6.8}$$

$$\Delta\varphi^{(1)}\left(x,y,t\right)=\begin{cases}-p_0H\left(x,y,t\right) & \text{for } 0<y<\psi_{0,2}\left(1,t\right), \\ 0 & \text{for } \psi_{0,2}\left(1,t\right)<y<1,\end{cases} \tag{6.9}$$

$$\varphi^{(1)}\left(x,y,t\right)=-\left(A\left(t\right)+M\right)h\left(x\right) \text{ on } \Gamma_\varepsilon, \tag{6.10}$$

$$\varphi^{(1)}\left(x,y,t\right)=0 \text{ on } \Gamma_1. \tag{6.11}$$

Further, differentiating the relation,

$$\psi_{\varepsilon,2}\left(x,f_\varepsilon\left(x,t\right),t\right)=\varepsilon h\left(x\right)\psi_{\varepsilon,1}\left(x,f_\varepsilon\left(x,t\right),t\right)$$

with respect to ε and taking $\varepsilon=0$, we get

$$D_y\psi_{0,2}\left(f_0\left(t\right),t\right)f^{(1)}\left(x,t\right)+\psi^{(1)}\left(x,f_0\left(t\right),t\right)=h\left(x\right).$$

Since, by (4.22), $D_y\psi_{0,2}\left(y,t\right)=Q\left(t\right)^{-1}$, we conclude that

$$f^{(1)}\left(x,t\right)=\left(h\left(x\right)-\psi^{(1)}\left(x,f_0\left(t\right),t\right)\right)Q\left(t\right). \tag{6.12}$$

We have thus formally obtained a linearized system (6.5)–(6.12). We shall prove that it has a unique solution and that the asymptotic formulas (6.1)–(6.3) hold with $f_\varepsilon^{(2)},\psi_\varepsilon^{(2)},\varphi_\varepsilon^{(2)}$ bounded. Note that since $f^{(1)}$ can be expressed in terms of $\psi^{(1)}$, we only need to consider (6.5)–(6.11).

THEOREM 6.1. *Let (ψ_0, f_0) be the solution (4.22)–(4.25) for $0 \le t \le T_0$ such that $f_0(T_0) = 1$. Then there exists a unique solution $\left(\psi^{(1)}, f^{(1)}\right)$ of (6.5)–(6.11) for $0 \le t < T_0$. Furthermore, $\psi^{(1)}$ and $D_t\psi^{(1)}$ can be continuously extended to $\Omega_0 \times [0, T_0)$, and $\psi^{(1)}(\cdot,\cdot,t)$, $D_t\psi^{(1)}(\cdot,\cdot,t) \in C^{1+\alpha}(\Omega_0)$.*

Proof. We first fix a $T_1 < T_0$ and derive *a priori* estimates. Clearly, $\psi_{0,2}(1,t) = Q(t)^{-1}(1 - f_0(t)) \ge \delta > 0$ if $0 \le t \le T_1$. Hence the region

$$\Omega_t^{\psi_0} = \{(x,y):\ 0 < y < \psi_{0,2}(1,t)\}$$

contains an infinite horizontal strip and then by Lemma 5.1,

$$\left\|\varphi^{(1)}\right\|_{C^{2+\alpha}\left(\Omega_t^{\psi_0}\right)} \le C\left(\|H\|_{C^{\alpha}\left(\Omega_t^{\psi_0}\right)} + 1\right),$$

where C depends on T_1 (or δ), $\psi_{0,2}(1,t)$, $A(t)$ and M. From (6.5),

$$\|H\|_{C^{\alpha}\left(\Omega_t^{\psi_0}\right)} \le C\left\|\nabla\psi^{(1)}\right\|_{C^{\alpha}(\Omega_t)}.$$

It follows that

$$\left\|\varphi^{(1)}(\psi_0,t)\right\|_{C^{2+\alpha}(\Omega_t)} \le C\left(\left\|\nabla\psi^{(1)}\right\|_{C^{\alpha}(\Omega_t)} + 1\right).$$

Differentiating (6.6) and (6.7) in the spatial variables and using the above estimate, we obtain, by Gronwall's inequality,

$$\left\|\nabla\psi^{(1)}\right\|_{C^{\alpha}(\Omega_t)} + \left\|\nabla\psi_t^{(1)}\right\|_{C^{\alpha}(\Omega_t)} \le C, \tag{6.13}$$

where C depends on T_1, $\psi_{0,2}(1,t)$, $A(t)$ and M.

We now proceed to prove existence for a small time interval $0 \le t \le T_2$. For any $\psi^{(1)}(\cdot,\cdot,t) \in C^{1+\alpha}(\Omega_t)$, $0 \le t \le T_2 \le T_1$ such that $\psi^{(1)}$ and $D_t\psi^{(1)}$ are continuous in (x,y,t), we define H by (6.5) and $\varphi^{(1)}$ by (6.9)–(6.11). Substituting $\psi^{(1)}$ and $\varphi^{(1)}$ into the right-hand sides of (6.6) and (6.7), we solve the left-hand sides by integration, using the initial condition (6.8); we denote the solution by $\hat\psi^{(1)} = \left(\hat\psi_1^{(1)}, \hat\psi_2^{(1)}\right)$. By Lemma 5.1 (and the derivation of (6.13)), it is easily seen that the mapping $\psi^{(1)} \mapsto \hat\psi^{(1)}$ is contraction mapping if T_2 is small. Therefore there exists a fixed point, which is then a solution of the linearized problem (6.5)–(6.11) for $0 \le t \le T_2$.

Using (6.13), we can further extend the solution to $0 \le t \le T_2 + \beta$ with β depending only on the constant C on the right-hand side of (6.13). By a step-by-step argument, the solution can be extended up to $t = T_1$. Uniqueness can be proved in the same way as in Theorem 2.1. Since T_1 can be chosen arbitrarily close to T_0, the theorem follows.

REMARK 6.1. By applying Lemma 5.1 to φ_x, we deduce that if $q \in C^{1+\alpha}$, $b \in C^{3+\alpha}$, then $\varphi \in C^{3+\alpha}$

In Theorem 5.3 we have established existence of a uniqueness solution $(\psi_\varepsilon, f_\varepsilon, \varphi_\varepsilon)$ for (2.1)–(2.9) for $0 \le t \le T_\varepsilon$, where $T_\varepsilon \le T_0$, $T_\varepsilon \to T_0$ as $\varepsilon \to 0$. We wish to justify that the formal expansions (6.1)–(6.3). It will be convenient to correspond Ψ_ε, Ψ_0 and $\Psi^{(1)}$ to ψ_ε, ψ_0 and $\psi^{(1)}$ as in (2.13).

Theorem 6.2. *Let $(\psi_\varepsilon, f_\varepsilon, \varphi_\varepsilon)$ be a solution of system (2.1)–(2.9) with data (5.1) in $0 \le t \le T_\varepsilon$. Then there exists a constant C such that for $0 \le t \le T_\varepsilon$*

$$\left\| f_\varepsilon - \left(f_0 + \varepsilon f^{(1)}\right)\right\|_{C^{1+\alpha}} \le C\varepsilon^2, \tag{6.14}$$

$$\left\| \Psi_\varepsilon - \left(\Psi_0 + \varepsilon \Psi^{(1)}\right)\right\|_{C^{1+\alpha}} \le C\varepsilon^2. \tag{6.15}$$

Proof. For simplicity, we assume that both ψ_ε and $\psi^{(1)}$ are defined in $\Omega_{\varepsilon t}^{\psi_\varepsilon}$. Then (6.15) is equivalent to

$$\left\| \psi_\varepsilon - \left(\psi_0 + \varepsilon \psi^{(1)}\right)\right\|_{C^{1+\alpha}\left(\Omega_{\varepsilon t}^{\psi_\varepsilon}\right)} \le C\varepsilon^2. \tag{6.16}$$

In $\Omega_{\varepsilon t}$, $\psi_\varepsilon^{(2)}$ satisfies

$$\varepsilon^2 \frac{d^2 \psi_{\varepsilon,1}^{(2)}}{dt^2} = -D_x \varphi_\varepsilon\left(\psi_\varepsilon\right) + D_x \varphi_0\left(\psi_0\right) + \varepsilon D_x \varphi^{(1)}\left(\psi_0, t\right), \tag{6.17}$$

$$\varepsilon^2 \frac{d^2 \psi_{\varepsilon,2}^{(2)}}{dt^2} = -D_y \varphi_\varepsilon\left(\psi_\varepsilon\right) + D_y \varphi_0\left(\psi_0\right) - p_0 Q\left(t\right) \psi_2^{(1)}\left(x, y, t\right) + D_y \varphi^{(1)}\left(\psi_0, t\right). \tag{6.18}$$

Substituting (6.3) into (6.17) and (6.18), we obtain

$$\varepsilon^2 \frac{d^2 \psi_{\varepsilon,1}^{(2)}}{dt^2} = -\varepsilon^2 D_x \varphi_\varepsilon^{(2)}\left(\psi_\varepsilon\right) + I_1, \tag{6.19}$$

$$\varepsilon^2 \frac{d^2 \psi_{\varepsilon,2}^{(2)}}{dt^2} = -\varepsilon^2 D_y \varphi_\varepsilon^{(2)}\left(\psi_\varepsilon\right) + I_2, \tag{6.20}$$

where

$$\begin{aligned}
I_1 &= D_x \varphi_0\left(\psi_0\right) - D_x \varphi_0\left(\psi_\varepsilon\right) + \varepsilon D_x \varphi^{(1)}\left(\psi_0\right) - \varepsilon D_x \varphi^{(1)}\left(\psi_\varepsilon\right), \\
I_2 &= D_y \varphi_0\left(\psi_0\right) - D_y \varphi_0\left(\psi_\varepsilon\right) - \varepsilon p_0 Q \psi_2^{(1)} + \varepsilon D_y \varphi^{(1)}\left(\psi_0\right) - \varepsilon D_y \varphi^{(1)}\left(\psi_\varepsilon\right).
\end{aligned}$$

Since $\varphi_0\left(x, y, t\right) = \varphi_0\left(y, t\right)$,

$$I_1 = \varepsilon D_x \varphi^{(1)}\left(\psi_0\right) - \varepsilon D_x \varphi^{(1)}\left(\psi_\varepsilon\right).$$

By (6.2),

$$\left\|\nabla I_1\left(\cdot, t\right)\right\|_{L^\infty(\Omega_{\varepsilon t})} \le C\varepsilon^2 \left(\left\|\nabla \psi_\varepsilon^{(2)}\left(\cdot, t\right)\right\|_{L^\infty(\Omega_{\varepsilon t})} + 1\right),$$

$$\left\|\nabla I_1\left(\cdot, t\right)\right\|_{C^\alpha(\Omega_{\varepsilon t})} \le C\varepsilon^2 \left(\left\|\psi_\varepsilon^{(2)}\left(\cdot, t\right)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}^{1+\alpha} + 1\right),$$

where C depends also on $C^{1+\mu}$-norm of ψ_ε. Analogously, since $D_y^2\varphi_0(\psi_0) = -p_0 Q$ and consequently

$$D_y\varphi_0(\psi_0) - D_y\varphi_0(\psi_\varepsilon) - \varepsilon p_0 Q\psi_2^{(1)} = -\varepsilon^2 p_0 Q\psi_\varepsilon^{(2)},$$

we obtain (by using the regularity of φ_0 in Remark 6.1)

$$\begin{aligned} \|\nabla I_2(\cdot,t)\|_{L^\infty(\Omega_{\varepsilon t})} &\leq C\varepsilon^2\left(\left\|\nabla\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{L^\infty(\Omega_{\varepsilon t})}+1\right), \\ \|\nabla I_2(\cdot,t)\|_{C^\alpha(\Omega_{\varepsilon t})} &\leq C\varepsilon^2\left(\left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}+1\right). \end{aligned}$$

Using these estimates in (6.19) and (6.20), we find that

$$\begin{aligned} &\left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} \\ \text{(6.21)}\qquad &\leq C + C\int_0^t\int_0^s\left(\left\|\varphi_\varepsilon^{(2)}(\psi_\varepsilon,\tau)\right\|_{C^{2+\alpha}(\Omega_{\varepsilon s})}+\left\|\psi_\varepsilon^{(2)}(\cdot,\tau)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon s})}\right)d\tau ds. \end{aligned}$$

We next estimate $\varphi_\varepsilon^{(2)}$. By (6.2),

$$\left\|J\left(\psi_\varepsilon^{-1}\right) - J\left(\psi_0^{-1}\right) - \varepsilon H\right\|_{C^\alpha} \leq C\varepsilon^2\left(\left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon s})}+1\right), \tag{6.22}$$

where C depends on $\|\psi_\varepsilon(\cdot,t)\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}$. Substituting (6.3) into (2.1) and using (6.9), one sees that $\varphi_\varepsilon^{(2)}$ solves

$$\varepsilon^2\Delta\varphi_\varepsilon^{(2)} = J\left(\psi_\varepsilon^{-1}\right)\mathcal{X}_{\Omega_{\varepsilon t}^{\psi_\varepsilon}} - J\left(\psi_0^{-1}\right)\mathcal{X}_{\Omega_{0t}^{\psi_0}} - \varepsilon H\mathcal{X}_{\Omega_{0t}^{\psi_0}}.$$

Using (6.22) and Lemma 5.1, we then obtain

$$\left\|\varphi_\varepsilon^{(2)}(\psi_\varepsilon)\right\|_{C^{2+\alpha}(\Omega_{\varepsilon s})} \leq C\left(\left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon s})}+1\right). \tag{6.23}$$

Substituting (6.23) into (6.21) and applying Gronwall's inequality, we find that

$$\left\|\varphi_\varepsilon^{(2)}(\psi_\varepsilon)\right\|_{C^{2+\alpha}(\Omega_{\varepsilon s})} + \left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon s})} \leq C, \tag{6.24}$$

which implies, in particular, the assertion (6.16). Expanding the relation

$$\psi_{\varepsilon,2}(x,f_\varepsilon) = \varepsilon h(\psi_{\varepsilon,1}(x,f_\varepsilon)),$$

in ε, and using the equations

$$\begin{aligned} &\psi_{0,2}(x,f_0) = \varepsilon h(\psi_{0,1}(x,f_0)), \\ &(D_y\psi_{0,2})f_1 + \psi_2^{(1)}(x,f_0) = h(\psi_{0,1}(x,f_0)), \end{aligned}$$

we can express $f_\varepsilon^{(2)}$ in terms of the derivatives of $\psi_\varepsilon^{(2)}$ and, by the estimate for $\psi_\varepsilon^{(2)}$ in (6.24),

$$\left\|f_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}} \leq C,$$

which is the assertion (6.14).

7. Numerical results. In this section, we take $n = 2$ and compute the thickness W_ε of the layer of the particles that have accumulated along the workpiece Γ_ε in time T, $T < T_0$. We are interested in the relation between the local profile of W_ε and the geometry of Γ_ε.

The density along the boundary Γ_ε is $P(x, \varepsilon h(x), t)$. By (2.3) and (6.4), the thickness $W_\varepsilon(x)$ at $(x, \varepsilon h(x)) \in \Gamma_\varepsilon$ can be written as

$$W_\varepsilon(x) = \int_1^T P(x, \varepsilon h(x), t)\, dt = W_0 + \varepsilon W_1(x) + O\left(\varepsilon^2\right) \tag{7.1}$$

where

$$W_0 = \int_1^T p_0 Q(t)\, dt = const, \quad W_1(x) = \int_1^T p_0 H(x, 0, t)\, dt. \tag{7.2}$$

We want to compute $W_1(x)$.

Differentiating (6.6) and (6.7) in x and y respectively and introducing the functions

$$u(x,y,t) = D_x \psi_1^{(1)}(x,y,t), \quad v(x,y,t) = D_y \psi_2^{(1)}(x,y,t),$$

we obtain the system

$$\frac{d^2 u(x,y,t)}{dt^2} = -\varphi_{xx}^{(1)}(x, \psi_{0,2}(y,t), t), \tag{7.3}$$

$$\frac{d^2 v(x,y,t)}{dt^2} = p_0 Q(t)\, v - Q(t)^{-1} \varphi_{yy}^{(1)}(x, \psi_{0,2}(y,t), t), \tag{7.4}$$

with the initial data

$$u(x,y,0) = v(x,y,0) = u_t(x,y,0) = v_t(x,y,0) = 0, \tag{7.5}$$

where $\varphi^{(1)}$ satisfies (6.9). By (6.5)

$$H(x,y,t) = -Q(t)\, u\left(\psi_0^{-1}(x,y,t), t\right) - Q(t)^2\, v\left(\psi_0^{-1}(x,y,t), t\right). \tag{7.6}$$

To simplify the computation of $W_1(x)$ we introduce the function

$$G(x,y,t) = -Q(t)^{-2} H(\psi_0(x,y,t), t). \tag{7.7}$$

Substituting H form (7.6) into (7.7), differentiating (7.7) in t, and then using (6.9), (7.3), (7.4), we get

$$\frac{d^2 G(x,y,t)}{dt^2} = 2 p_0 t u_t(x,y,t). \tag{7.8}$$

and, by integration in t,

$$\begin{aligned} G_t &= p_0 \int_0^t u_t(x,y,t)\, dt \\ &= p_0 \left(t^2 u_t(x,y,t) + \int_0^t s^2 D_x^2 \varphi^{(1)}(\psi_0, s)\, ds \right). \end{aligned}$$

Integrating once more and using (7.3), (7.4), we obtain, for $f_0(t) < y < 1$,

$$
\begin{aligned}
G(x,y,t) &= \frac{p_0}{3}\int_0^t u_t(x,y,t)\,dt + p_0\int_0^t\int_0^\tau s^2 D_x^2\varphi^{(1)}(\psi_0,s)\,dsd\tau \\
&= \frac{p_0 t^3}{3}u_t - \frac{p_0}{3}\int_0^t s^3 u_{tt}(x,y,s)\,ds + p_0\int_0^t\int_0^\tau s^2 D_x^2\varphi^{(1)}(\psi_0,s)\,dsd\tau \\
&= -\frac{p_0 t^3}{3}\int_0^t\left(-\frac{p_0 t^3}{3} - \frac{2p_0 s^3}{3} + p_0 t s^2\right) D_x^2\varphi^{(1)}(\psi_0,s)\,ds
\end{aligned}
\tag{7.9}
$$

where we have used the identity

$$\int_0^t\int_0^\tau s^2 k(s)\,dsd\tau = t\int_0^t s^2 k(s)\,ds - \int_0^t s^3 k(s)\,ds.$$

Since $G(x,y,t)$ is defined only for $f_0(t) < y < 1$, it is more convenient to work with another function $\hat{G}$, defined by

$$\hat{G}(x,y,t) = G(x,(1-f_0(t))y + f(t),t),\ 0 < y < 1.$$

Equation (7.9) is equivalent to

$$
\begin{aligned}
\hat{G}(x,y,t) &= \int_0^t D_x^2\varphi^{(1)}\left(x,\frac{(1-f_0(t))y + f_0(t) - f_0(s)}{Q(s)},s\right) \\
&\quad\times\left(-\frac{p_0 t^3}{3} - \frac{2p_0 s^3}{3} + p_0 t s^2\right) ds.
\end{aligned}
\tag{7.10}
$$

Equation (7.10) is supplemented by (6.9)–6.11), where

$$H(x,y,t) = -Q(t)^2\,\hat{G}\left(x,\frac{Q(t)\,y}{1-f_0(t)},t\right). \tag{7.11}$$

We can now express W_1 in the form

$$W_1(x) = -p_0\int_1^T Q(s)^2\,\hat{G}(x,0,s)\,ds.$$

The assumption $P_0(x) = p_0$ means that initially we have a uniform distribution of particles everywhere in the domain. In the problem of spray painting, initially there are no paint particles in the domain and, for a certain period of time, say $0 \le t \le t_0$, a cloud of particles with uniform distribution is injected from Γ_∞ at a uniform rate. In order to represent this situation more closely, we should replace the assumption $P_0(x) \equiv p_0$ by the assumption

$$P_0(x,y) = \begin{cases} 0 & \text{if } 0 \le y < a \\ p_0 & \text{if } a \le y \le 1, \end{cases} \tag{7.12}$$

for some $0 < a < 1$. Note that although we do not include here continuous injection of particles across Γ_∞ fort $t > 0$, the assumption (7.12) is an approximation to the spray painting model with t_0 proportional to a. The system (7.10), (7.11) and (6.9)–6.11) then becomes

$$\hat{G}(x,y,t) = P_0^2\left((1-f_0(t))\,y + f_0(t)\right)\int_0^t \left(-\frac{p_0 t^3}{3} - \frac{2p_0 s^3}{3} + p_0 t s^2\right) \times D_x^2\varphi^{(1)}\left(x, \psi_{02}\left((1-f_0(t))\,y + f_0(t), s\right), s\right) ds, \tag{7.13}$$

$$\Delta\varphi^{(1)}(x,y,t) = \begin{cases} Q\left(\psi_{02}^{-1}, t\right)\hat{G}\left(x, \frac{\psi_{02}^{-1}(y,t)-f_0}{1-f_0(t)}, t\right) & \text{if } 0 < y < \psi_{02}(1,t) \\ 0 & \text{if } \psi_{02}(1,t) < y < 1, \end{cases} \tag{7.14}$$

$$\varphi^{(1)} = 0 \text{ on } \Gamma_1,\ \varphi^{(1)} = -(A(t)+M)h(x) \text{ on } \Gamma_0, \tag{7.15}$$

where

$$Q(y,t) = J\left(\psi_0(x,y,t)\right)^{-1}, \tag{7.16}$$

$A(t), f_0(t), \varphi_0$,and ψ_0 are the solutions of (4.5), (4.8)–(4.11) corresponding to (7.12). The ε order term of the thickness is then

$$W_1(x) = -\int_1^T Q\left(f_0(s), s\right)^2 \hat{G}(x,0,s)\, ds + W_{10}, \tag{7.17}$$

where

$$W_{10} = \int_1^T \left[Q_y\left(f_0(s),s\right)Q\left(f_0(s),s\right)P_0\left(f_0\right) + Q\left(f_0(s),s\right)^2 P_{0y}\left(f_0\right)\right] ds$$

is a constant.

The system (7.12)–(7.16) has a unique solution (the proof is similar to that for (7.10), (7.11) and (6.9)–6.11)), although do not establish here the global existence and the asymptotic estimates (as in Section 5 & 6) for the initial data (7.12).

NUMERICAL COMPUTATION. We now describe the numerical results for the system (7.12)–(7.16). It is based on solving the Poisson equation (7.14) by the finite element method and the integral equation (7.13) by the Euler forward method. We choose $M = 50$, $p_0 = 10$, $a = 0.7$, and then solve ODE (4.11) to get $T_0 = 0.202$. The computational results show that the scheme is stable if the partition (in both spatial and time directions) is sufficiently fine. Figures 1-3 illustrate the results of the painting thickness (modulo an additive constant) for three different workpieces with the uniform space and time meshes $dx = 10^{-2}$ and $dt = T_0/100$, respectively; the small figures within Figures 1-3 are the rescaled profiles of the corresponding painting thickness.

CONCLUSION. *The accumulation of paint is smaller around the convex points of the workpiece and larger around the concave points.*

This result is in agreement with the numerical results obtained by Ellwood and Braslaw [1]; their method is based on a discrete model of a finite number of elastic particles.

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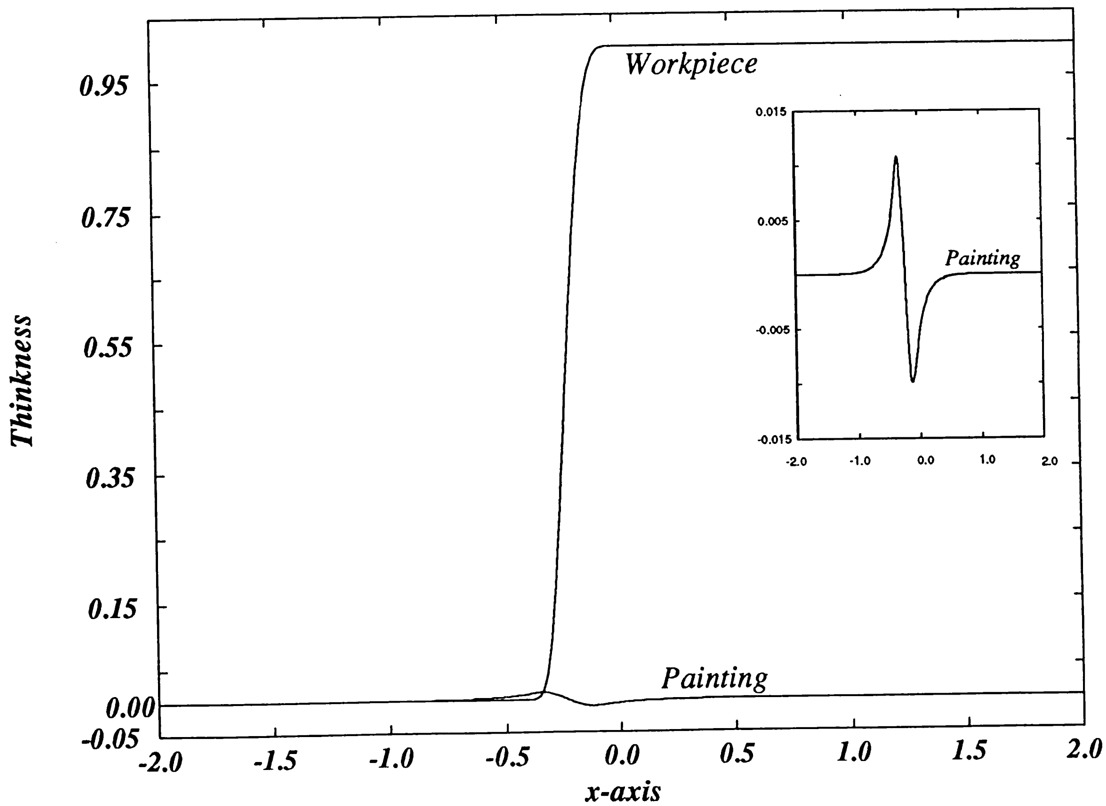

Figure 1.

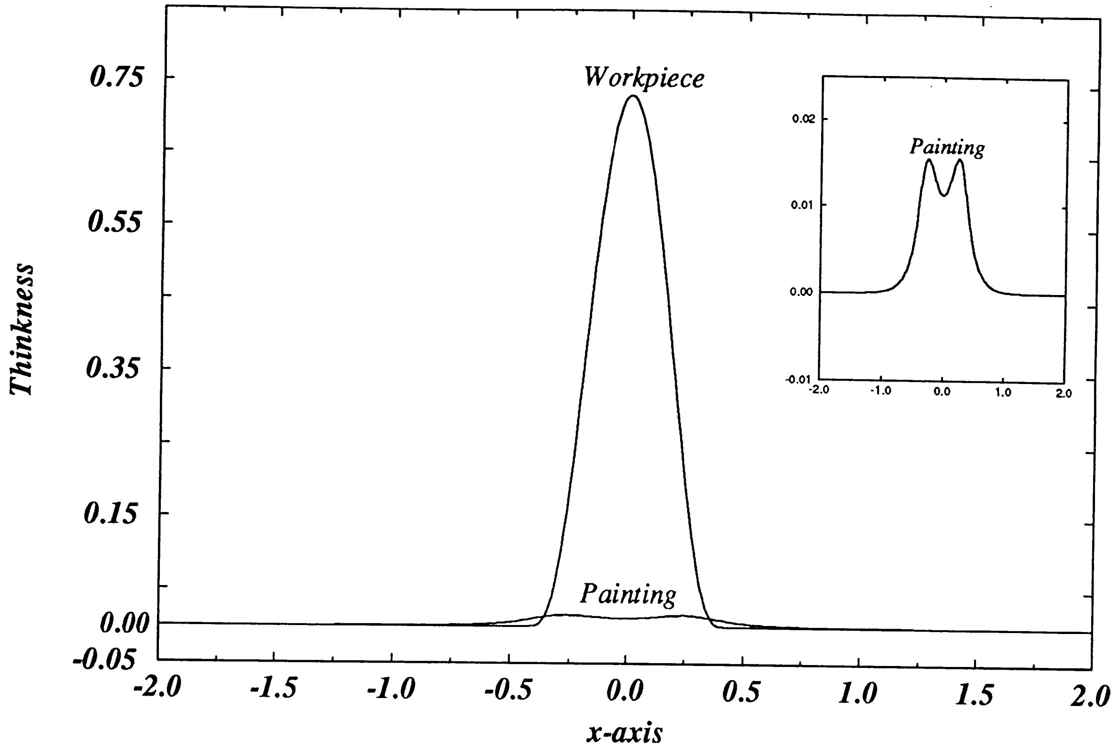

Figure 2.

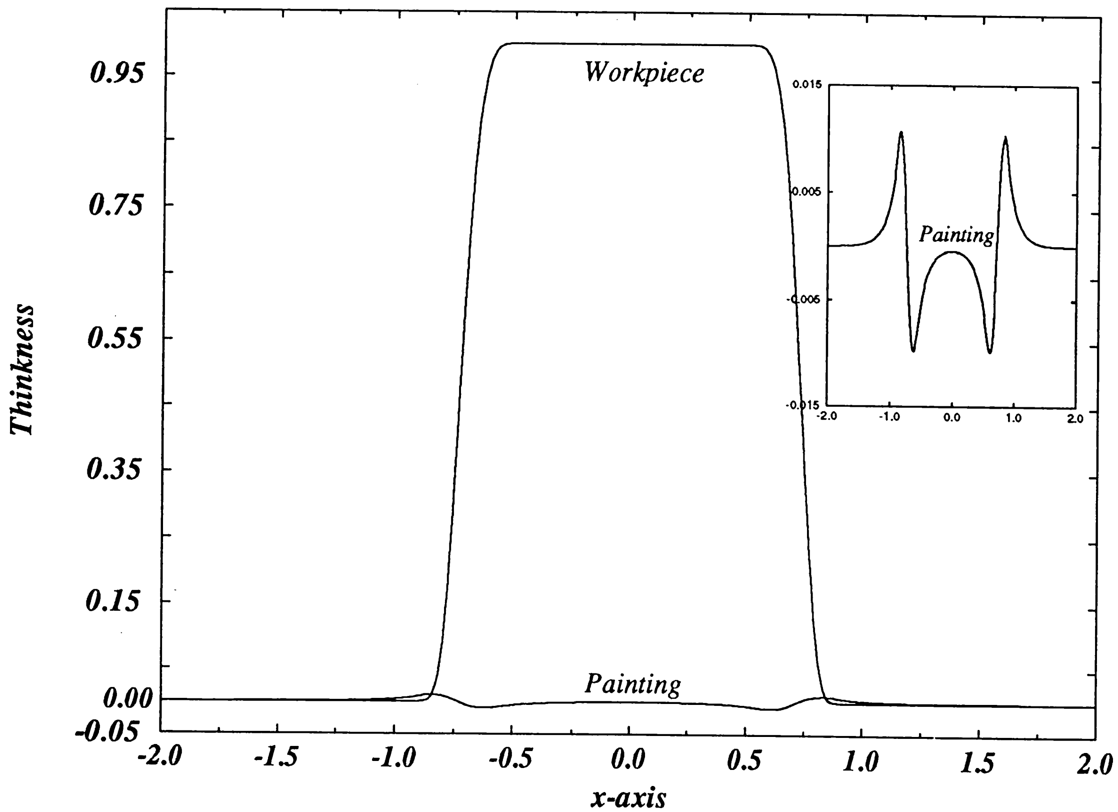

Figure 3.

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