

# Topology and differentiability of Labyrinths in the disc and annulus

by Gilbert Levitt and Harold Rosenberg

Introduction. The study of differential equations in the plane which are locally of the form  $\frac{dy}{dx} = F(x,y)$ , gives rise to labyrinths. They are limit sets of bounded solutions to this equation. This is made precise in [Ro], where the singularities considered are thorns and tripods. In part I of this paper, we shall extend the results of [Ro] to differential equations with  $n$ -prong singularities, in the disc and annulus. For the disc, the story is not essentially different from the previous case. However, for the annulus, the study is quite different and more complicated. In both cases, we obtain a topological structure theorem for solutions of the equation.

In part IV we discuss the differentiability of foliations with labyrinths. We prove there are no  $C^2$  labyrinths in the disc or annulus, where differentiability is in the sense of local models at singularities. Thus the Poincaré Bendixson theorem is true for  $C^2$  differential equations of this type.

The second author wishes to acknowledge that an unpublished manuscript of M. Herman was useful in the construction of the example in remark 4, page 31 of [Ro]. In his manuscript, an example is constructed of a  $C^\infty$  homeomorphism of the circle, with an exceptional minimal set and exactly one critical point. This construction is published in [Ha]. The techniques in [Gu] are in the same spirit. We also wish to thank Remi Langevin for drawing the figures in [Ro].

## I. Preliminaries

We consider foliations of compact surfaces, whose singularities are thorns and saddles with any number  $p > 3$  of prongs.

We define an arational foliation as in [Ro]:  $F$  is arational if  $F$  has no interior compact leaf,  $F$  is transverse to  $\partial M$ , and no separatrices join two singularities.

We define the spreading of a leaf as in [Ro] p. 3, but we allow the spreading of a singular leaf (see figure 1); this operation replaces a  $p$ -prong saddle by a  $(p + 1)$ -prong saddle and a thorn. If  $x$  and  $y$  are points on the same leaf, we denote by  $(x,y)$  the leaf joining  $x$  to  $y$ . We also use this notation when one of the points is a singularity and the leaf of the other point is a separatrix of the singularity.

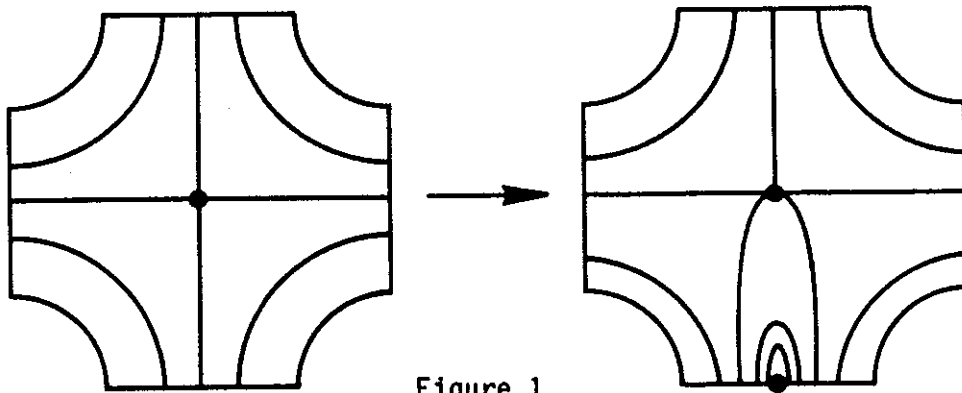


Figure 1

A standard foliation of the half-disc  $D^+$  is defined as in [Ro] p. 4, by consecutively spreading a finite number of leaves in the foliation of  $D^+$  by concentric circles, but we allow the top point of  $D^+$  to be a saddle (see figure 2).

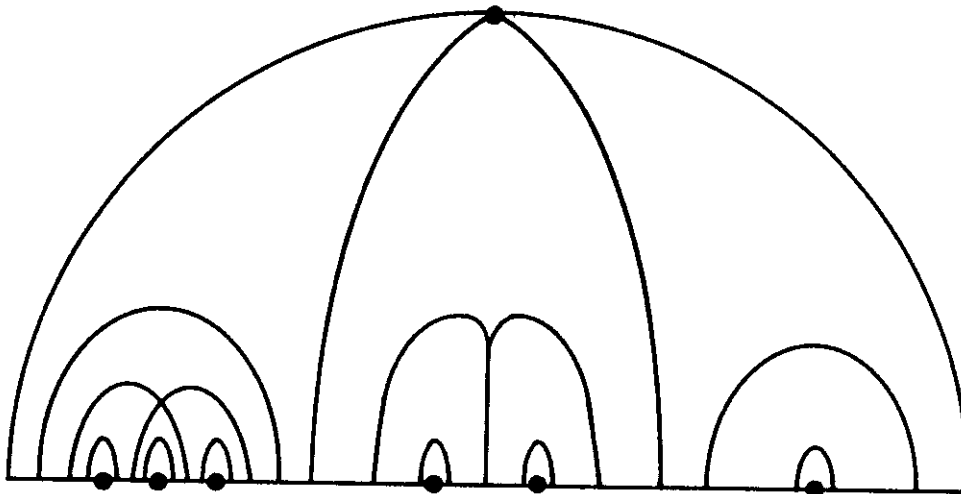


Figure 2

A labyrinth over a segment  $J$  (resp. a simple closed curve  $C$ ), and its standard extension to the disc  $D^2$  (resp. the annulus  $A$ ), are defined as in [Ro] pp. 4-5.

For future reference, we note the following useful fact: in an arational foliation, every leaf is cut by a transverse simple closed curve, and every infinite leaf is cut infinitely often by some transverse curve (compare [Ro], p. 76). Also note that the extension lemma ([Ro], lemma 2.2, p. 9) remains valid in our situation.

## II. Arational foliations of the disc.

Just as in [Ro], we first prove:

Theorem 1. Let  $F$  be an arational foliation of  $D^2$  such that no leaf joins a point of  $\partial D^2$  to a thorn. Then  $F$  is the standard extension to  $D^2$  of a labyrinth on a segment  $J$ .

1) The first step in the proof is to construct an arc  $J$  passing through all the thorns and transverse to  $F$  (except at the thorns). The proof we give is simpler than the one in [Ro], pp. 10-13.

Lemma 1. Let  $F$  be an arational foliation of a compact surface  $M$ . Any two points  $x$  and  $y$  in  $M - \text{Sing } F$  can be joined by an immersed transverse arc.

In fact we can join  $x$  and  $y$  by an embedded arc, but we do not need this. Using lemma 1, we construct an immersed  $J$  passing through all the thorns. We then replace it by an embedded arc, as in [Ro] pp. 13-15.

### Proof of lemma 1.

Fix  $x$ , and consider the subset  $V$  of  $M - \text{Sing } F$  consisting of all the points that can be joined to  $x$  by a transverse (immersed) arc. This subset is

obviously nonempty and open. We claim that it is also closed (and therefore contains  $y$ ).

Let  $z$  be a point in the frontier of  $V$ , and  $C$  a transverse closed curve meeting the leaf of  $z$ . For  $z' \in V$  close enough to  $z$ , we can extend to  $z$  any transverse arc joining  $x$  to  $z'$  (see figure 3). Thus  $z \in V$ .

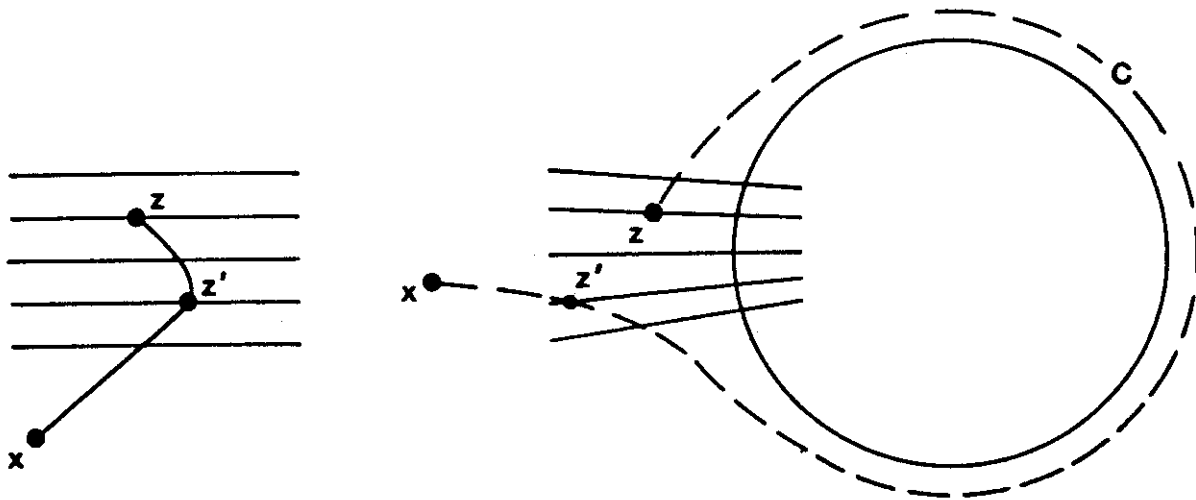


Figure 3

2) Let  $J$  be as above. Any half-leaf that does not go to a singularity goes to  $J$  or  $\delta D$ . If a separatrix from a saddle  $s$  goes to  $\delta D$  (without meeting  $J$ ), all the other separatrices of  $s$  go to  $J$ . No regular leaf goes from  $\delta D$  to  $\delta D$  without meeting  $J$ .

Same proof as in [Ro] (proof of 2.4, p. 16), using the fact that a  $p$ -prong saddle is a singularity of positive index  $(p-2)/2$ .

3) Let  $x_1, \dots, x_n$  be the points of  $\delta D$  whose leaves go to saddles  $y_1, \dots, y_n$  (before intersecting  $J$ ). Note that there is at least one such point: consider a segment of a leaf going from  $\delta D$  to  $J$ , and use the extension lemma ([Ro], p. 9) in order to lift it until reaching a singularity; since no leaf joins a point of  $\delta D$  to a thorn, this singularity is a saddle.

The end of the proof of theorem 1 is as in [Ro], pp. 16-20, using for each  $y_i$  the two separatrices adjacent to the separatrix  $(y_i, x_i)$ .

Theorem 2. Let  $F$  be an arational foliation of  $D^2$ . Then  $F$  is obtained by a finite number of consecutive spreadings from either:

- a) the trivial arational foliation of  $D^2$  with exactly two thorns, or
- b) the standard extension of a labyrinth  $L$  on a segment  $J$ .

The spreadings are done on (possibly singular) leaves going to  $\delta D^2$ .

Proof of Theorem 2.

The proof is by induction on the number  $q$  of leaves joining a thorn to  $\delta D^2$ . We know that the theorem is true for  $q = 0$ .

Now suppose that a leaf joins a thorn  $z$  to a point  $x \in \delta D^2$ . Apply the extension lemma to leaves near  $(z, x)$ . If  $F$  is not the trivial foliation with two thorns, one arrives at a saddle  $s$  (see figure 4).

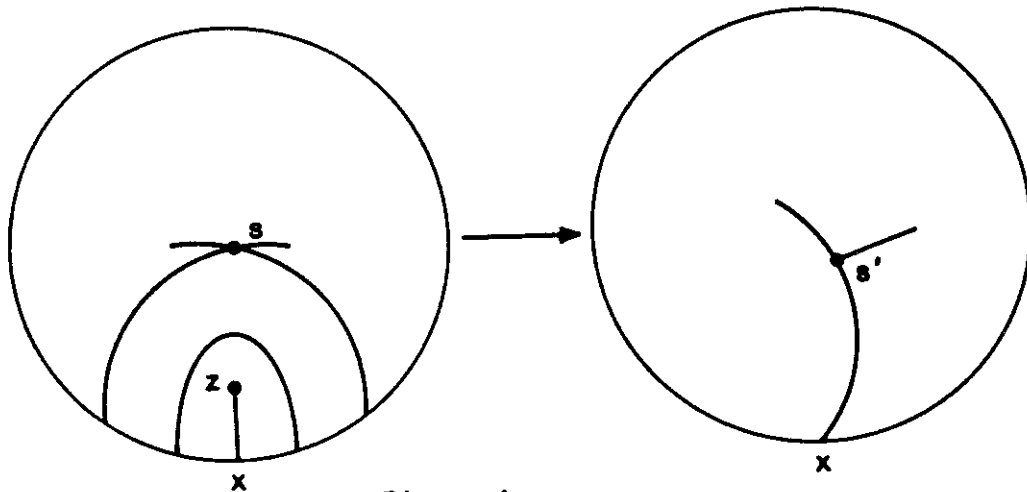


Figure 4

Modify the foliation as in figure 4, replacing  $s$  by a saddle  $s'$  with one less prong (a regular point if  $s$  is a tripod). The result now follows by applying the induction hypothesis to the new foliation  $F'$ ; one passes from  $F'$  to  $F$  by spreading the leaf  $(s', x)$ .

### III. Arational foliations of the annulus

For foliations with  $p$ -prong saddles, it turns out that the situation on the annulus  $A$  is much more complicated than on the disc. We start with an example of an arational foliation of  $A$  which cannot be obtained by spreading the product foliations or the standard extension of a labyrinth on a simple closed curve  $C$ .

Say that a singularity  $s$  (saddle or thorn) is joined to the boundary if at least one separatrix of  $s$  reaches  $\partial M$ .

#### Example 1

Suppose  $G$  is an arational foliation of  $D^2$  such that no thorn and exactly one saddle  $s$  is joined to  $\partial D^2$ . Consider a punctured annulus (or pair of pants) with the foliation pictured in figure 5, and fill in  $D$  with  $D^2$  equipped with  $G$ , in such a way that  $x$  is identified with a point of  $\partial D^2$  whose leaf goes to  $s$ .

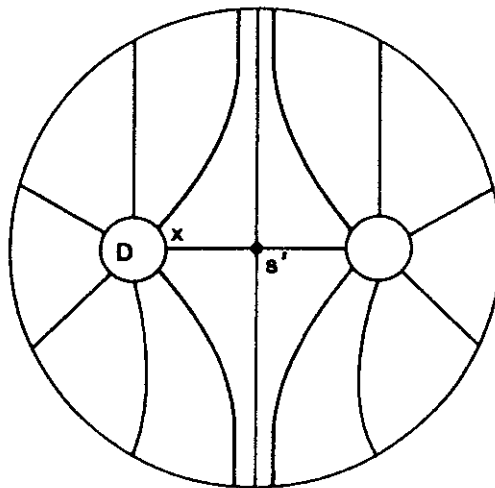


Figure 5

After collapsing to a point  $s''$  the saddle connection  $(s, x, s')$ , we get an arational foliation  $F$  of  $A$  with the property that exactly one singularity (namely  $s''$ ) is joined to  $\partial A$ . Such a foliation cannot be the standard extension of a labyrinth, since in the extension of a labyrinth there are at least two saddles joined to  $\partial A$  (one on each side of  $C$ ).

Example 2

Here is another example. Suppose  $G$  is an arational foliation of  $D^2$  such that no thorn and exactly one saddle  $s_1$  (resp. exactly two saddles  $s_1$  and  $s_2$ ) is (are) joined to  $\delta D^2$ . Glue  $D^2$  to a punctured annulus equipped with the foliation pictured in figure 6, in such a way that  $x$  is identified with a point of  $\delta D^2$  whose leaf goes to  $s_1$ , and  $y$  to a point of  $\delta D^2$  whose leaf is regular (resp. whose leaf goes to  $s_2$ ).

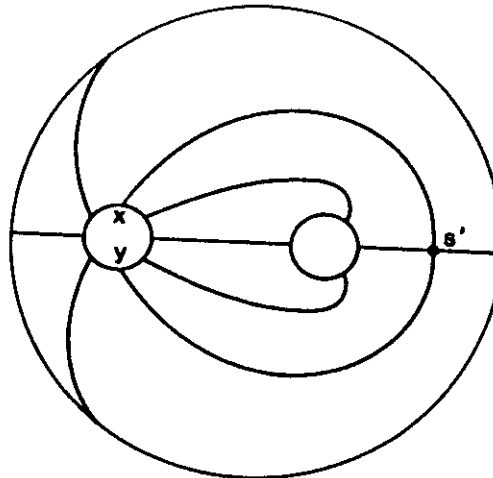


Figure 6

Then collapse to a point the saddle connection(s)  $(s_1, x, s')$  (resp.  $(s_1, x, s')$  and  $(s_2, y, s')$ ). Just as above, the resulting foliation on  $A$  is not the standard extension of a labyrinth.

Theorem 3. Let  $F$  be an arational foliation of the annulus  $A$ . Then  $F$  is obtained by a finite number of spreadings of either

- the product foliation, or
- the standard extension of a labyrinth on a simple closed curve  $C$ , or
- a foliation obtained from  $D^2$  as above (examples 1 and 2).

The rest of the section is devoted to the proof. Note that we need only consider the case when no thorn is joined to  $\delta A$  : the extension to the general case is as above (proof of theorem 2).

We first remark that, if  $F$  is not the product foliation, each component  $C_i$  of  $\delta A$  contains a point whose leaf goes to a saddle: apply the extension lemma to a segment of leaf joining a point  $x \in C_i$  to another point of  $\delta A$  or to a point belonging to a transverse curve meeting the leaf of  $x$  infinitely often.

If no singularity is joined to  $\delta A$ , then  $F$  is the product foliation. Now assume that exactly one singularity  $s$  is joined to  $\delta A$ , and consider two separatrices  $(s,a)$  and  $(s,b)$  joining  $s$  to different components of  $\delta A$ .

First suppose these two separatrices are adjacent. Then another separatrix, adjacent to  $(s,a)$  or  $(s,b)$ , must also go to  $\delta A$  (see figure 7i), and  $F$  is obtained as in example 1 (note that the dotted curve on figure 7ii is

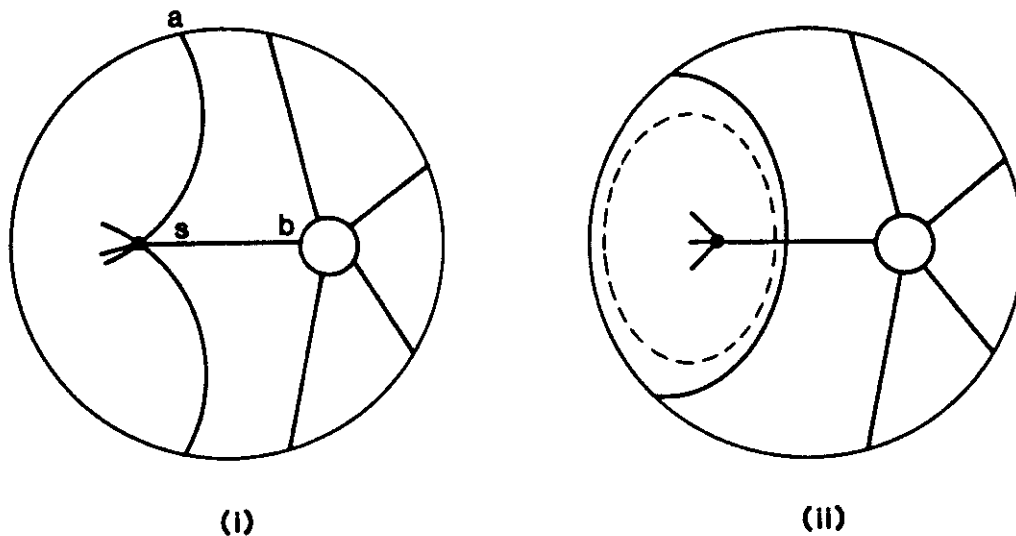


Figure 7

Now suppose  $(s,a)$  and  $(s,b)$  are not adjacent (see figure 8). Then  $F$  is obtained as in example 2; the foliation  $G$  has one (resp. two) saddle(s) joined to  $\delta D^2$  if there is (resp. there is not) a separatrix adjacent to both  $(s,a)$  and  $(s,b)$ .



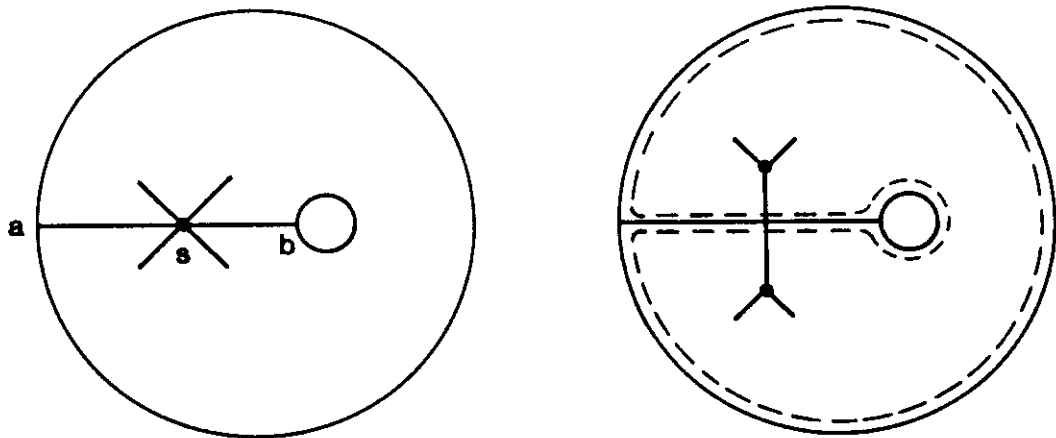


Figure 8

From now on we assume that there are two distinct saddles  $s_1$  and  $s_2$  joined to points  $m_1$  and  $m_2$  of  $\delta A$ . We will show that  $F$  is the standard extension of a labyrinth.

The hard part of the proof is to find a non-contractible simple closed curve  $C$  passing through all the thorns and transverse to  $F$  (except at the thorns). Once we have  $C$ , the end of the proof is as in [Ro], p. 24.

Just as above (first step of the proof of theorem 1), we can construct an embedded segment  $J$  transverse to  $F$  and passing through all the thorns. In fact, we also need  $J$  to be disjoint from  $(s_1, m_1)$  and  $(s_2, m_2)$ . The proof that we can find such a  $J$  will be postponed until the end of this section.

From now on, we will say that a separatrix goes to  $\delta A$  only when it does so without meeting  $J$ .

As in [Ro], top of p. 22, it is easy to see that every half-leaf which does not go to a singularity reaches  $J$  or  $\delta A$ . Also note that, if  $(s, a)$  and  $(s, b)$  are separatrices going to the same component of  $\delta A$  (without meeting  $J$ ), then the foliation is "trivial" in the annulus bounded by  $(s, a)$  and  $(s, b)$  (see figure 9), and every separatrix between  $(s, a)$  and  $(s, b)$  goes to  $J$ .

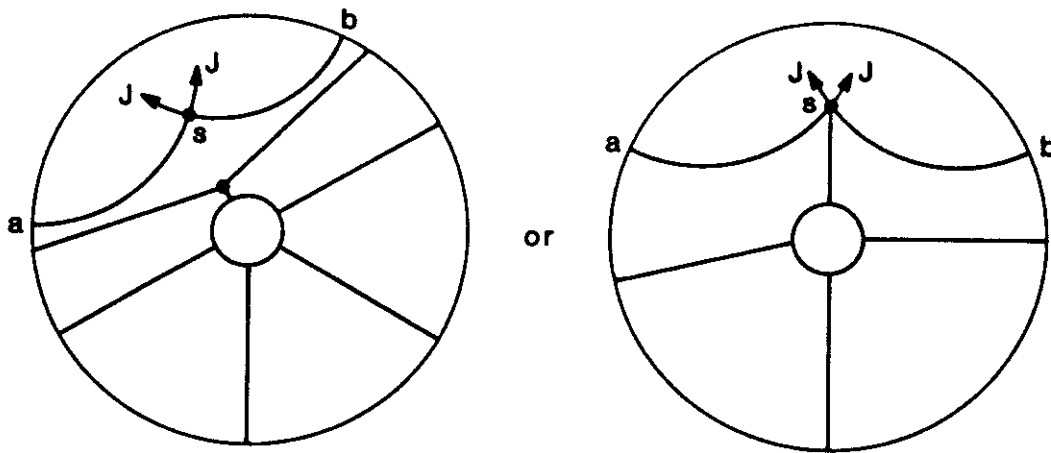


Figure 9

We claim that there is a saddle with a separatrix going to  $J$  and an adjacent separatrix going to  $\delta A$  ; the proof we are about to give is simpler than the proof of step 2 in [Ro], pp. 22-23.

In  $M - \text{Sing } F$  , consider the union of all leaves meeting  $\delta A$  , and the union of all leaves meeting  $J$  . These two sets are open, and we know that they cover  $M - \text{Sing } F$  . It follows that there is a point  $m \in \delta A$  whose leaf goes to a point  $n \in J$  . Apply the extension lemma to the segment  $(m,n)$  in order to obtain the desired saddle (keeping in mind that no thorn is joined to  $\delta A$  ) .

Now we distinguish two cases:

- 1) there is a saddle  $s$  with some separatrix  $(s,m)$  going to  $\delta A$  and both adjacent separatrices going to  $J$  .

There are five subcases a to e , according to how these separatrices reach  $J$  . For clarity,  $J$  and  $(s,m)$  will be in the same position on all pictures. The two points where the separatrices reach  $J$  are called A and B.

First consider subcase a (see figure 10). We show how to construct C on figures 10 and 11. The part of C outside of a neighborhood of J is pictured on figure 10 (dotted line); the part of C near J is pictured on figure 11. Note that for Euler characteristic reasons there has to be an upward thorn

t between A and B . At other thorns (not represented on figure 11), C passes as in figure 12.

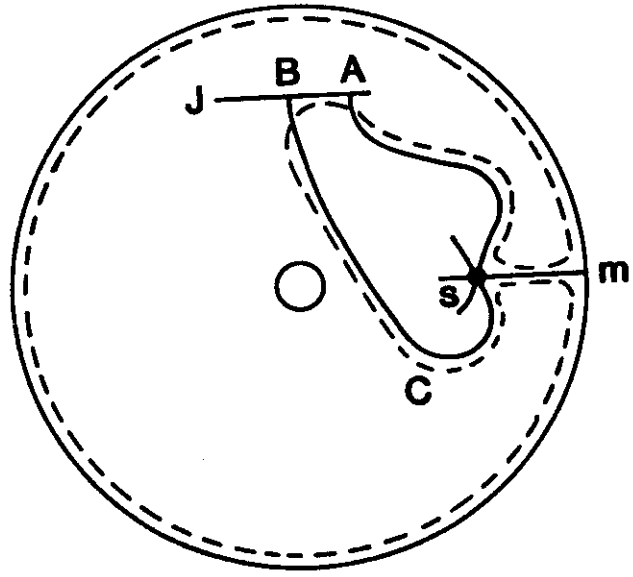


Figure 10 (case a)

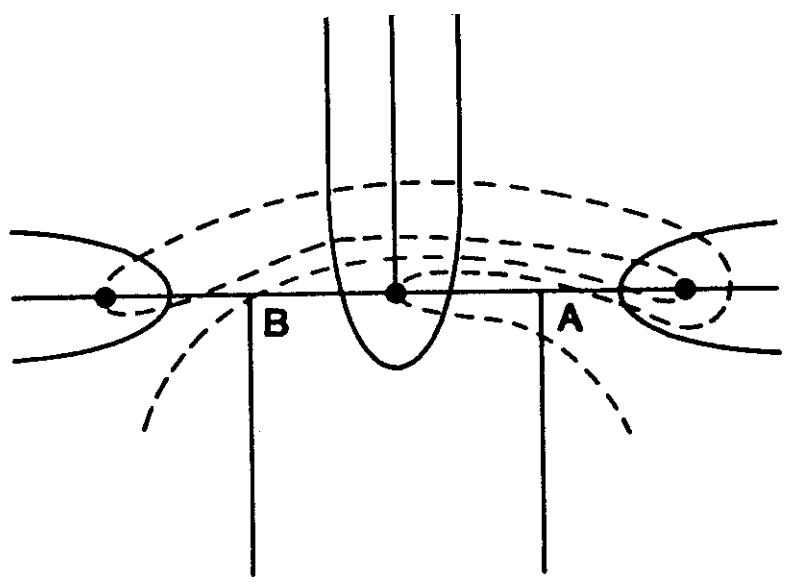


Figure 11

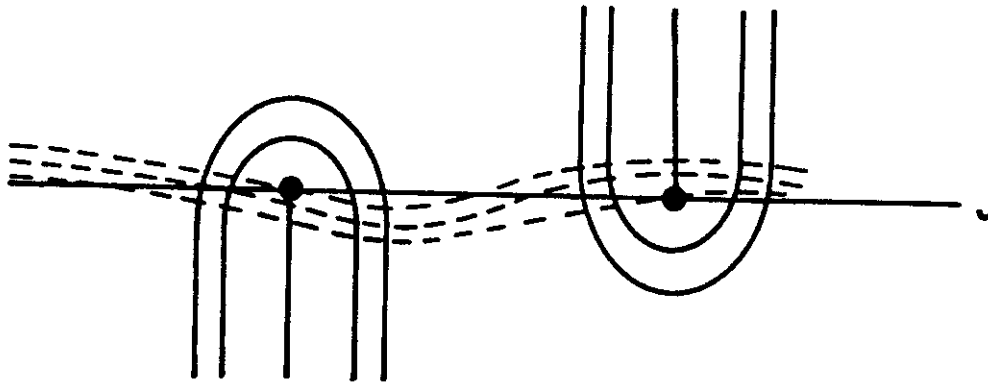


Figure 12

Cases b and c are dealt with similarly (see figures 13 and 14).

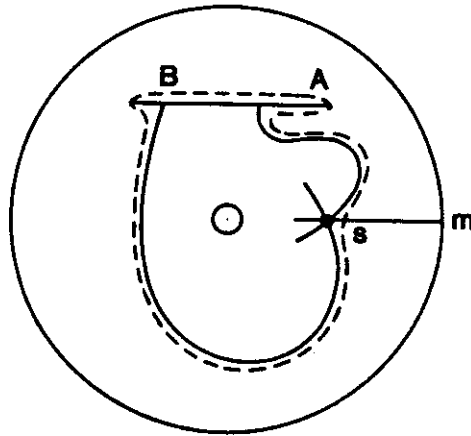


Figure 13 (case b)

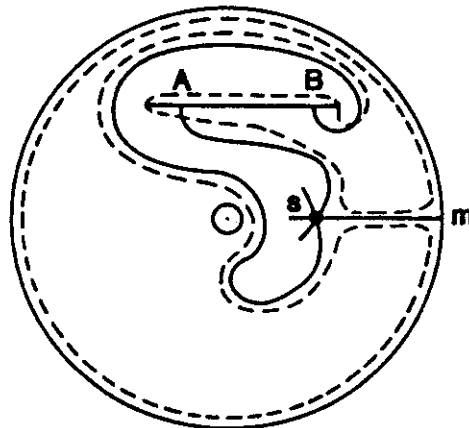


Figure 14 (case c)

Now we consider case d , when the separatrices reach J on opposite sides (there are four possibilities, of which we represent only two on figure 15; the other two are dealt with similarly).

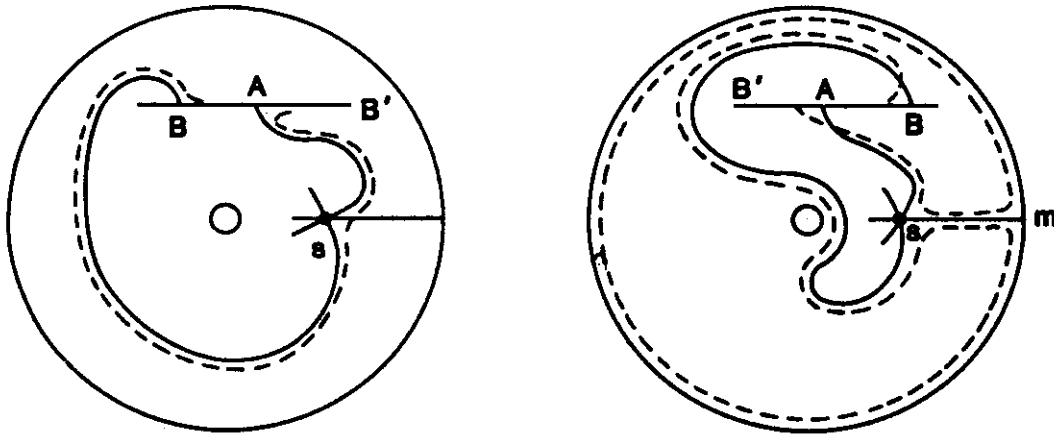


Figure 15 (case d)

The curve C is constructed outside of a neighborhood of J as before (see figure 15). To construct it near J , note that the separatrix of the thorn B' meets J between B and B' ; hence there has to be a thorn t between A and B or a downward thorn t' between A and B . This thorn t or t' is used as a turnaround point for C (see figure 16).

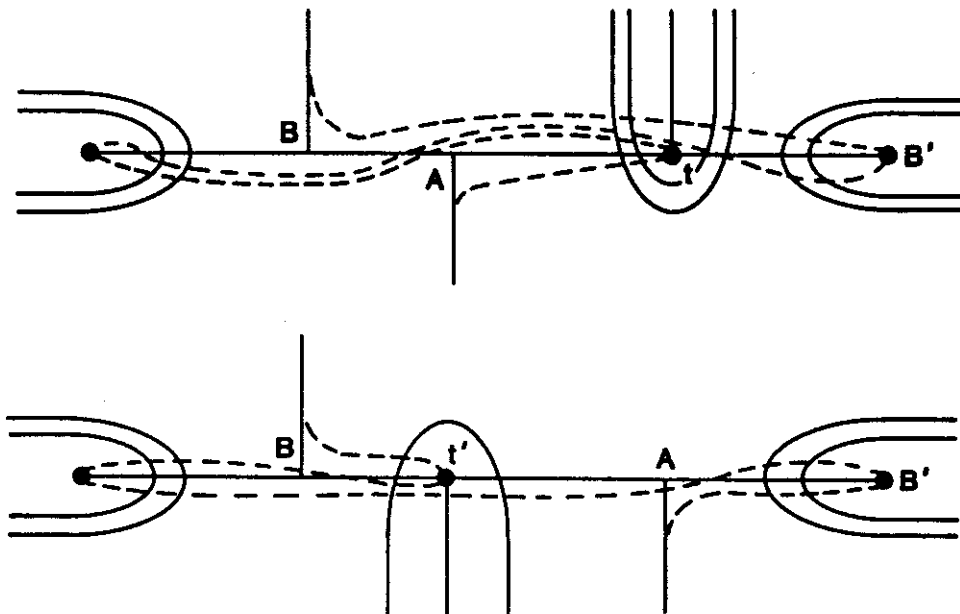


Figure 16

The remaining case e is pictured in figure 17. First assume that  $(s,m)$  is the only separatrix of  $s$  that goes to  $\delta A$ . Then there are two adjacent separatrices of  $s$  going to  $J$  in homotopically different ways (see figure 18), and it is easy to construct  $C$  in a neighborhood of  $J$  union these separatrices.

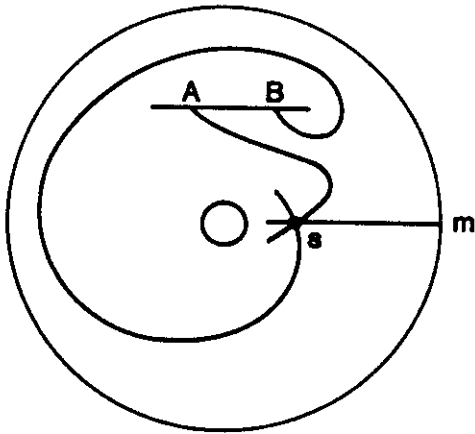


Figure 17 (case e)

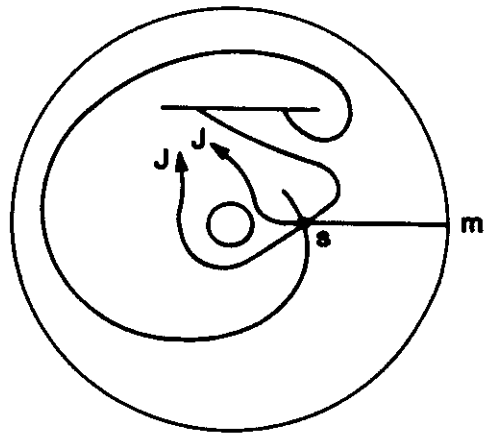


Figure 18

Finally, if there is another separatrix  $(s,n)$  going to  $\delta A$ , we recall that there is another saddle  $s'$  which is joined to  $\delta A$  in  $A - J$  (because  $J$  is disjoint from  $(s_1, m_1)$  and  $(s_2, m_2)$ ). Only one separatrix of  $s'$  can reach  $\delta A$  (in  $A - J$ ), and for the adjacent separatrices we are reduced to one of cases a, c or d.

2) for every separatrix going to  $\delta A$ , at least one of the two adjacent separatrices also goes to  $\delta A$ .

Recall that there is a saddle  $s$  with a separatrix going to  $J$  and an adjacent separatrix going to  $\delta A$ , and that there are at least two saddles

joined to  $\delta A$  in  $A - J$ . This easily implies that we are in one of two situations represented on figure 19 ( $s$  and  $s'$  may be tripods).

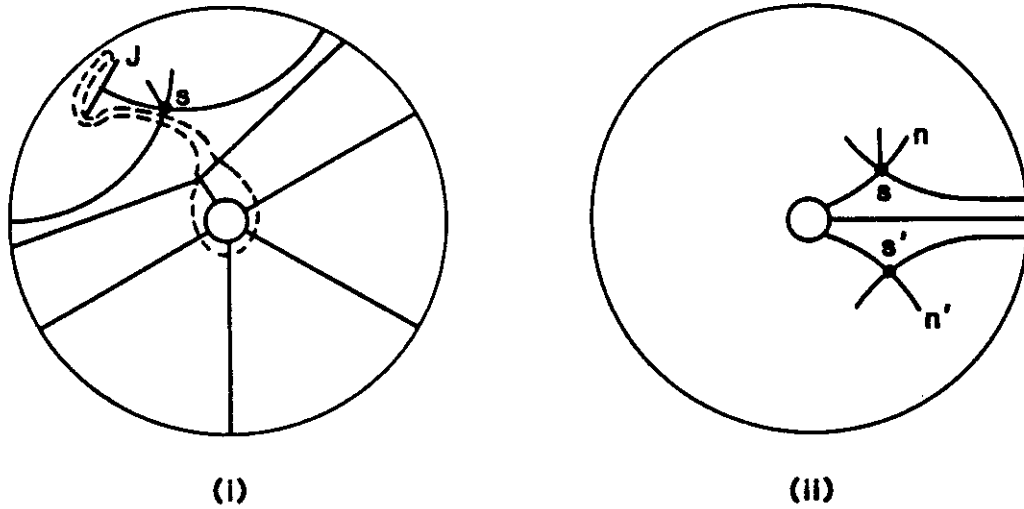


Figure 19

The picture shows what to do in the first situation. In the second, the separatrices  $(s,n)$  and  $(s',n')$  meet  $J$ , and we can apply the same analysis as in case 1. In subcases a to d, the construction of  $C$  is as above. In subcase e, one has to use the separatrix of  $s'$  situated the furthest to the left (see figure 20); this separatrix is  $(s',n')$  if  $s'$  is a tripod.

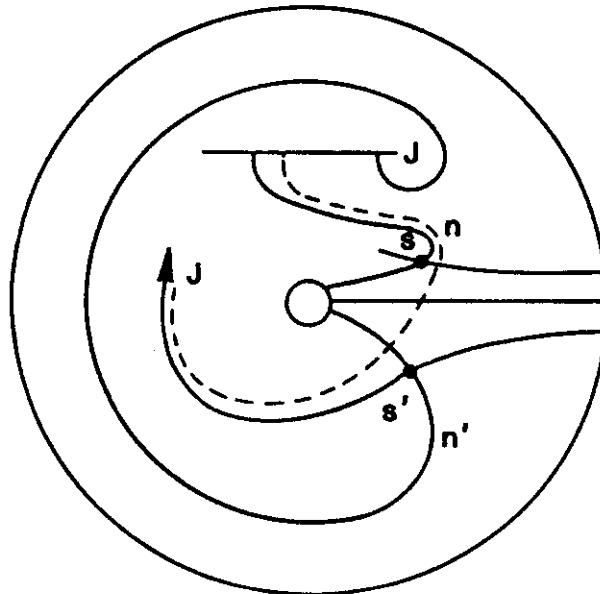


Figure 20

To complete the proof of theorem 3, we now have to show that we had the right to assume  $J$  disjoint from  $(s_1, m_1)$  and  $(s_2, m_2)$ . In order to do that, we can apply the arguments used above (first step of proof of prop. 1), provided we know that any leaf different from  $(s_1, m_1)$  and  $(s_2, m_2)$  meets some (immersed) closed transverse curve disjoint from  $(s_1, m_1)$  and  $(s_2, m_2)$ . We shall now prove this fact, under the simplifying assumption that no regular leaf goes from  $\delta A$  to  $\delta A$ .

Consider in  $N = M - [\text{Sing } F \cup (s_1, m_1) \cup (s_2, m_2)]$  the union  $W$  of all leaves meeting some transverse curve  $C$  with the following two properties:  $C$  is disjoint from  $(s_1, m_1)$  and  $(s_2, m_2)$ , and  $C$  meets at least one thorn separatrix. We will show that  $W = N$ .

The set  $W$  is nonempty, open, and its frontier in  $N$  consists of separatrices of finite length (since a leaf in the frontier cannot meet a transverse curve contained in  $N$ ). Note that each component of  $M - \text{Sing } F$  - (separatrices of finite length) contains at least one thorn (this is where we use our simplifying assumption) and therefore is contained in  $W$ . It follows that a separatrix  $\alpha$  in the frontier of  $W$  is adherent to  $W$  on both sides, hence contained in  $W$  (see figure 21).

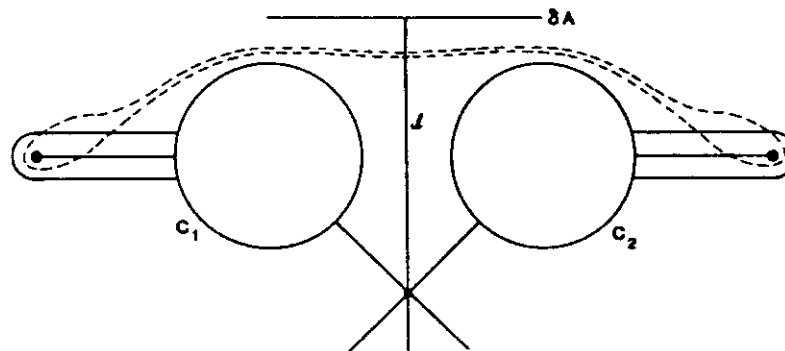


Figure 21

#### IV. Differentiability

There is no obvious definition of differentiability for a singular foliation of a surface  $M$ .



One possibility is to say that  $F$  is of class  $C^r$  if the restriction of  $F$  to  $M - \text{Sing } F$  is defined by a  $C^r$  line field.

With this definition, Schwartz's theorem [Sc] implies that in a singular  $C^2$  foliation every compact invariant set contains a compact leaf or a singularity. Conversely, Gutierrez's theorem [Gu] makes it likely that, if every compact invariant set contains a compact leaf or a singularity, then  $F$  is topologically conjugate to a  $C^\infty$  foliation (this is proved in [Gu] for orientable foliations, and in [Ro] pp. 31-32 for a special family of labyrinths).

We get another natural definition by imposing local models near the singularities. More precisely, we shall say from now on that  $F$  is of class  $C^r$  if there exists a  $C^r$ -atlas  $\{\phi_\alpha: U_\alpha \rightarrow D_\alpha\}$ , where each  $\phi_\alpha$  carries the chart domain  $U_\alpha$  foliated by  $F$  to a subset  $D_\alpha$  of the complex plane foliated by the level sets of the function  $|\text{Re} \cdot z^{k/2}|$  ( $k$  is a positive integer, depending on  $\alpha$ ; if  $k$  is odd, the absolute value sign removes the ambiguity in the definition of  $z^{k/2}$ ; if  $0 \in D_\alpha$ , then  $U_\alpha$  contains a thorn if  $k = 1$ , a  $k$ -prong saddle if  $k > 3$ ).

This notion of differentiability imposes surprisingly strong restrictions on the global behavior of  $F$ :

Theorem 4. Let  $F$  be an arational foliation of the disc or annulus. If  $F$  is of class  $C^2$ , then every regular leaf is compact (and goes from boundary to boundary).

We refer the reader to [Le] for a more general discussion of differentiability and corollaries of theorem 4.

Proof of theorem 4.

The proof is a generalization of a well-known argument of Denjoy ([De], [VK]). First we note that it suffices to show that regular leaves meeting  $\partial M$  are compact: if this is true, then the union of compact regular leaves and

separatrices reaching  $\delta M$  is nonempty and open in  $M - \text{Sing } F$ , but by the extension lemma it is also closed.

We assume that there is a point  $m \in \delta M$  whose leaf  $\ell$  is regular and does not return to  $\delta M$ , and we argue towards a contradiction. We orient  $\ell$ , starting from  $m$ .

Using a transverse curve  $C$  meeting  $\ell$  infinitely often, we construct a transverse segment  $H$ , with endpoints  $a$  and  $b$  on  $\delta M$ , which meets  $\ell$  infinitely often (see figure 22i). From now on we shall picture  $H$  as horizontal (see figure 22ii); it separates  $M$  into two regions (top and bottom).

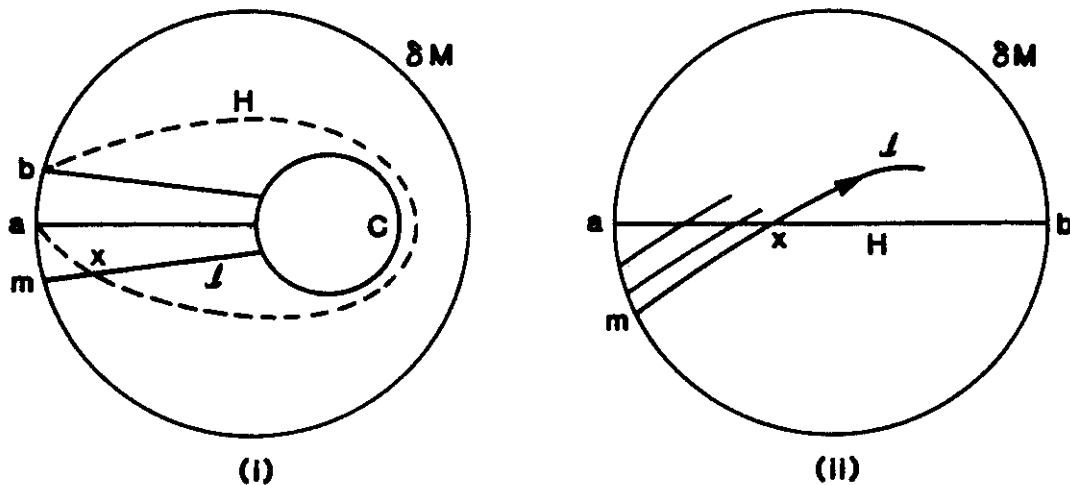


Figure 22

Consider the limit set of  $\ell$  (equal to  $\bar{\ell} - \ell$ ). It is compact and meets  $H$ . Let  $t$  be the point of  $(\bar{\ell} - \ell) \cap H$  furthest to the left.

Lemma 4.1.  $\ell$  meets the interval  $[a, t) \subset H$  finitely many times; in other words,  $\ell$  accumulates on  $t$  only from the right.

This lemma is a special case of a general fact, valid for any arational foliation: if an open transverse interval does not meet the limit set of a half-leaf  $\ell$ , then it meets  $\ell$  finitely many times.

Proof of lemma 4.1.

Consider the points of  $[a,t)$  where a separatrix first meets  $[a,t)$ , for instance  $a'$  and  $a''$  on figure 23. There are finitely many of them, and we call  $a'$  the furthest to the right (see figure 23);  $a' = a$  if  $[a,t)$  meets no separatrix. We claim that  $\ell$  meets  $(a',t)$  at most once.

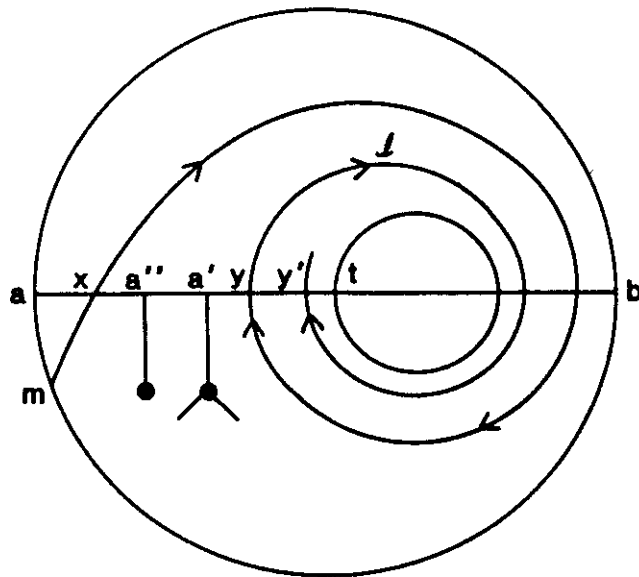


Figure 23

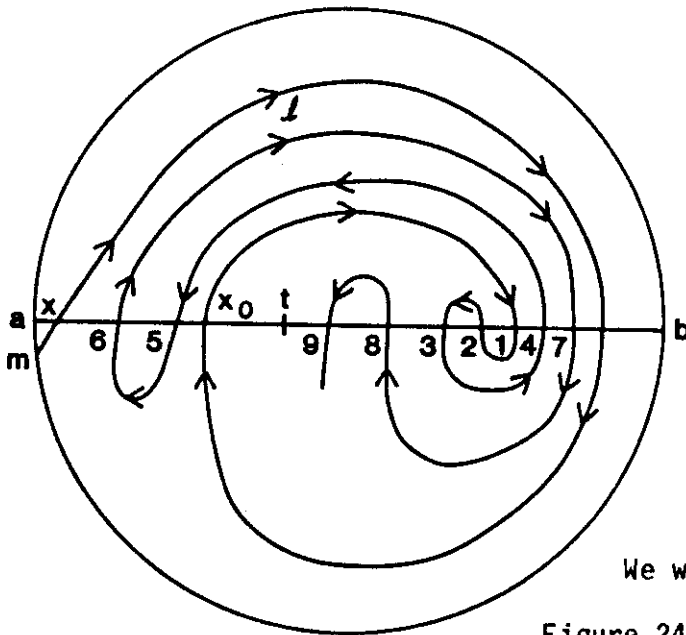
Suppose on the contrary that  $y$  and  $y'$  are successive intersections of  $\ell$  with  $(a',t)$ . First note that  $\ell$  goes in the same direction (up or down) at  $y$  and  $y'$ : if not, the extension lemma would show that some separatrix meets  $[a,t)$  for the first time between  $y$  and  $y'$ .

We now apply the extension lemma to the segment of  $\ell$  between  $y$  and  $y'$ , and conclude that either the leaf through  $t$  meets  $H$  on the left of  $t$ , or  $\ell$  spirals towards a compact leaf or a union of singularities and separatrices. Both conclusions are absurd. This proves 4.1.

Let  $x_0$  be the point of  $\ell \cap [a,t)$  furthest to the right ( $x_0$  exists because  $\ell \cap [a,t)$  is finite and contains  $x$ , see figure 22i or 23).

Following  $\lambda$  from  $x_0$  on, we call  $x_1, \dots, x_n, \dots$  its successive intersections with  $H$ .

Consider the integers  $p$  such that the interval  $[x_0, x_p] \cap H$  contains  $t$  but contains no  $x_n$ ,  $0 < n < p$  (see figure 24, where  $p = 1, 2, 3, 8, 9, \text{etc.}$ ). These integers form an infinite sequence whose terms are alternatively odd and even, and we fix an odd  $p = 2q + 1$ . Note that the points  $x_i$  ( $1 < i < p - 1$ ) with odd and even subscripts alternate on  $H$ .



We write  $1, 2, \dots$  instead of  $x_1, x_2, \dots$

Figure 24

We shall reach a contradiction, for  $p$  large enough, by considering on  $H$  a partially defined second return map. Assume that at  $x_0$  the leaf  $\lambda$  is going up (as on figure 24); then we define the downward second return map  $f$ , by saying that  $f(y)$  is the first point where the half-leaf leaving  $y$  downward returns to  $H$  from the top (for instance, on figure 24,  $f(x_0) = x$ ,  $f(x_1) = x_3$ ,  $f(x_2) = x_0$ ;  $f(x)$  is not defined). If  $\lambda$  is going down at  $x_0$ , we consider the upward return map instead.

The domain of definition  $U$  of  $f$  is a finite nonempty collection of open intervals (at whose endpoints the half-leaf reaches a singularity before meeting  $H$  twice). The map  $f$  is an orientation-preserving  $C^2$ -diffeomorphism between

$U$  and  $f(U)$ . Our definition of differentiability implies furthermore that the logarithm of the derivative of  $f$  has bounded variation on  $U$ : there is a number  $V$  such that, for any set  $y_0 < y_1 < \dots < y_{2h-1}$  on  $U$ , we have

$$\left| \sum_{i=0}^{2h-1} (-1)^i \log Df(x_i) \right| < V.$$

Let  $I \Subset \mathcal{M}$  be an open interval containing  $m$  and meeting no separatrix. Leaves starting at points of  $I$  stay close to  $\ell$ , and for every  $n$  they cut out an open interval  $I_n \Subset H$  around  $x_n$ . The intervals  $I_n$  are disjoint, and their length  $|I_n|$  tends to 0 as  $n$  goes to infinity.

We have  $I_p = f^q(I_1)$  and  $I_0 = f^q(I_{p-1})$ , so we can find points  $y \in I_1$  and  $z \in I_{p-1}$  such that  $|I_p| = Df^q(y) \cdot |I_1|$  and  $|I_0| = Df^q(z) \cdot |I_{p-1}|$ . Using the chain rule for derivatives, we obtain

$$\begin{aligned} \log(|I_p| \cdot |I_{p-1}| / |I_0| \cdot |I_1|) &= \log Df(y) + \log Df(f(y)) + \dots + \log Df(f^{q-1}(y)) \\ &\quad - \log Df(z) - \log Df(f(z)) - \dots - \log Df(f^{q-1}(z)). \end{aligned}$$

The points  $y, f(y), \dots, f^{q-1}(y)$  belong to the intervals  $I_1, I_3, \dots, I_{p-2}$ ; the points  $z, f(z), \dots, f^{q-1}(z)$  belong to  $I_{p-1}, \dots, I_4, I_2$ . The alternating property pointed out earlier then implies that the absolute value of  $\log(|I_p| \cdot |I_{p-1}| / |I_0| \cdot |I_1|)$  is less than  $V$ , a contradiction for  $p$  large enough.

UER de Mathematique  
 Universite de Paris VII  
 2 Place Jussieu  
 75251 Paris Cedex 05  
 FRANCE

References

- [De] A. DENJOY, Sur les courbes definies par les equations differentielles a la surface du tore, J. Math. Pures et Appl. 11 (1932), 333-375.
- [Gu] C. GUTIERREZ, Smoothing continuous flows and the converse of Denjoy-Schwartz theorem, An. Ac. Bras. de Cien. 51 (1979), 581-589.
- [Le] G. LEVITT, La decomposition dynamique et la differentiability des feuilletages des surfaces, version revisee, preprint.
- [Ro] H. ROSENBERG, Labyrinths in the disc and surfaces, Ann. of Math. 117 (1983), 1-33.
- [Sc] A. SCHWARTZ, A generalization of the Poincare-Bendixson theorem to closed two-dimensional manifolds, Amer. Jour. of Math. 85 (1963), 453-458.
- [VK] E. VAN KAMPEN, The topological transformations of a simple closed curve into itself, Amer. J. Math. 57 (1935), 142-152.
- [Ha] G.R. HALL, A  $C^\infty$  Denjoy counterexample, Erg. Th. and Dynam. Syst., Vol. 1 (1981), 261-273.