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Abstract

This paper considers invariant manifolds of global trajectories of retarded Functional Differential Equations in $\mathbb{R}^n$. The persistence, smoothness and stability of such manifolds where the flow is given by an Ordinary Differential Equation (ODE) in $\mathbb{R}^n$ is studied for small perturbations of ODEs. The novelty of the present approach lies in the use of the dynamics of the flow on the manifolds, instead of their attractivity properties.

I. Introduction

The study of invariant sets consisting of global trajectories has an important role in the geometric theory of Functional Differential Equations (FDE). Here, we are concerned with retarded FDEs and use as phase space the Banach space $C = C([-r, 0]; \mathbb{R}^n)$ of continuous functions from the interval $[-r, 0]$ into $\mathbb{R}^n$ $(r > 0)$, taken with the usual uniform norm. This paper studies invariant sets where the flow of the FDE is given by an Ordinary Differential Equation (ODE) in $\mathbb{R}^n$, in a sense to be made precise in the text. Conditions for these invariant sets to be differentiable manifolds are given, when the FDE is a small perturbation of an ODE in $\mathbb{R}^n$. The order of smoothness of these manifolds is also studied.

An ODE in $\mathbb{R}^n$ can be considered, in a natural way, as an FDE with phase space C. If the ODE is defined by a continuously differentiable vector field with bounded derivative on $\mathbb{R}^n$, then all solutions $x(t)$ of the ODE are global, i.e., defined for all $t \in \mathbb{R}$, and the flow in the phase space C is given by the evolution of restrictions to intervals of length r taken along a solution of the ODE. In particular, through each point of C which is such a segment of length r of a solution of the ODE, there exists a global trajectory of the flow in C. For other points of C, the flow cannot be continued backwards in time, but proceeding forward r units of time the flow leads to a point in C which is a segment of a solution of the ODE, in the preceding sense. This shows that the set of all segments of length r of solutions of the ODE is an invariant C^1-manifold which attracts strongly. The trajectories starting in the manifold are global and remain in the manifold for all time, and the trajectories starting outside the manifold will lie on the manifold after r units of time. From the general theory of hyperbolic invariant manifolds, it is to be expected that an invariant C^1-manifold will persist, under small perturbations of the retarded type. This was noted by Kurzweil [2] for the case where the functions giving the ODE and the perturbations are bounded and differentiable and the derivatives satisfy a "modulus of continuity" condition.

The results in another paper of Kurzweil [3] apply to the case when the unperturbed equation is not an ODE, but the set of initial data for global solutions is a C^1-manifold and the rates of the flow in directions normal to the manifold are higher than in tangential directions.

The results of Kurzweil mentioned above were obtained by a contraction mapping argument applied to functions defined on the manifold of initial data of global solutions of the unperturbed equation. The mapping to which the contraction mapping principle is applied is defined in such a way that its fixed point is a function with range equal to the set of initial data for global solutions of the perturbed equation.

In the present paper, the existence and the smoothness of manifolds where the flow of retarded equations is given by an ODE in $\mathbb{R}^n$ is established by determining directly the ODE giving the flow in such manifolds. This is accomplished using a nonlinear variation of constants formula. The novelty of the present approach lies in the use of the dynamics of the flow on the manifolds, instead of their attractivity properties.

2. Preliminaries

The equations considered are retarded FDEs of the form

$$\dot{x}(t) = F[x(t)] + g(x_t) \quad , \tag{1_g}$$

where F is a continuously differentiable function from $\mathbb{R}^n$ into $\mathbb{R}^n$ which has a bounded and globally lipschitzian first derivative, and g is a globally lipschitzian function from C into $\mathbb{R}^n$ which is taken to be a small perturbation of (1_0) in a sense that will be clear later on. As usual, given a function x of real variable and with values in $\mathbb{R}^n$, x_t denotes the function from $[-r, 0]$ into $\mathbb{R}^n$ which satisfies $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$. The solution of (1_g) in C which satisfies $x_0 = \phi$ is denoted by $x(\phi, g)$ and its value at time t is denoted by $x(t; \phi, g)$. The subset of C consisting of initial data for global solutions of (1_g) (i.e., solutions defined for all $t \in (-\infty, +\infty)$) is denoted by A_g. It is an invariant set under (1_g).

Definition

Given a retarded FDE with phase space C and an integer $k \geqslant 0$, a manifold $B \subset C$ is said to be a manifold where the flow is given by a C^k ODE in $\mathbb{R}^n$, if there exists a function $F : \mathbb{R}^n \to \mathbb{R}^n$ of class C^k such that the ODE $\dot{x} = F(x)$ has unique solutions for each arbitrary initial condition $x(0) = a \in \mathbb{R}^n$, all solutions of this ODE are also solutions of the given FDE with orbits lying in B, and vice versa, the solutions of the given FDE through initial data $\phi \in B$ at $t = 0$ are also solutions of the ODE for $t \geqslant -r$.

Remark

The preceding definition implies that such manifolds are homeomorphic (if $k = 0$) or diffeomorphic (if $k \geqslant 1$) to $\mathbb{R}^n$. Thus, they are C^0 (if $k = 0$) or C^1 (if $k \geqslant 1$) manifolds and are n-dimensional. These manifolds are invariant under the given FDE and, in particular, they are included in the set of initial data for global solutions of the FDE. Besides, the mapping $\pi : \phi \to \phi(0)$, from B into $\mathbb{R}^n$, must be one-to-one and onto, and $f(\phi) = F(\phi(0))$ for $\phi \in B$, and $F(a) = f(\pi^{-1} a)$ for $a \in \mathbb{R}^n$.

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The maximal solution of an ODE $\dot{x} = F(x)$ having unique solutions for all initial conditions $x(0) = a \in \mathbb{R}^n$, is denoted by $t \to \xi(t; a, F)$. The derivatives of solutions relative to the initial data are denoted by

$$H_F(t, a) = \frac{\partial \xi}{\partial a}(t; a, F) \quad . \tag{2}$$

It was mentioned in the Introduction that a nonlinear variation of constants formula is used in the proof of existence of the invariant manifolds for the perturbed equations. Although this formula may be known to the reader, it is stated and proved here for completeness.

Lemma 1

Consider the differential equation

$$\dot{x} = F(x) + g(t) \tag{3}$$

where F is a lipschitzian and continuously differentiable function defined in $\mathbb{R}^n$ and g is a continuous function of real variable and values in $\mathbb{R}^n$. A differentiable function x from an interval $I \subset \mathbb{R}$ containing the origin into $\mathbb{R}^n$, and satisfying $x(0) = a$, is a solution of (3) if and only if it satisfies the following (nonlinear) variation of constants formula, for $t \in I$:

$$x(t) = \xi(t; a, F) + \int_0^t H_F\big(t - s, x(s)\big) g(s) ds \quad . \tag{4}$$

Proof

Let $x: I \to \mathbb{R}^n$ be a differentiable function with $x(0) = a$. It is clear that we can write

$$x(t) = \xi(t; a, F) + \int_0^t \frac{d}{ds}\Big[\xi\big(t - s; x(s), F\big)\Big] ds \quad . \tag{5}$$

Now,

$$\frac{d}{ds}\Big[\xi\big(t - s; x(s), F\big)\Big] = -\frac{\partial \xi}{\partial t}\big(t - s; x(s), F\big) + \frac{\partial \xi}{\partial x}\big(t - s; x(s), F\big)\dot{x}(s) \quad .$$

We note that, by uniqueness of solutions,

$$\xi(t; x_0, F) = \xi\Big(t + h; \xi(-h; x_0, F), F\Big)$$

and, taking derivatives relative to h at $h = 0$, we get

$$0 = \frac{\partial \xi}{\partial t}(t; x_0, F) - \frac{\partial \xi}{\partial x}(t; x_0, F) F(x_0) \ .$$

Consequently, since $\frac{\partial \xi}{\partial x}(t - s; x(s), F) = H_F(t - s, x(s))$, we have

$$\frac{d}{ds}\Big[\xi\big(t - s; x(s), F\big)\Big] = H_F\big(t - s, x(s)\big)\Big[\dot{x}(s) - F\big(x(s)\big)\Big] \quad .$$

Noting that $\dot{x}(s) - F\big(x(s)\big) = g(s)$, we see that formula (5) becomes (4). Q.E.D.

3. Existence

In this section, equation (1_g) is considered with $F:\mathbb{R}^n \to \mathbb{R}^n$ being a C^1 function with first derivative bounded and globally lipschitzian. As discussed in the Introduction, the subset of C consisting of segments of solutions of the ODE $\dot{x} = F(x)$ is a C^1-manifold where the flow is given by a C^1 ODE in $\mathbb{R}^n$.

Theorem 2

Let $F:\mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and have first derivatives bounded and globally lipschitzian, and let $B_0 = \{\xi(\cdot\,; a, F) \in C : a \in \mathbb{R}^n\}$. If $g: C \to \mathbb{R}^n$ is uniformly lipschitzian on a neighborhood of B_0, $V = \{\phi \in C: \inf_{\psi \in B_0} |\phi - \psi| \leq \delta\}$ for some $\delta > 0$, with $\sup_{\phi \in B_0} |g(\phi)|$ and the Lipschitz constant of g on V both sufficiently small, then there exists a C^0-manifold B_g where the flow of

$$\dot{x}(t) = F[x(t)] + g(x_t) \tag{6}$$

is given by a C^0 ODE in $\mathbb{R}^n$, $\dot{x} = (F+K_g)(x)$, such that the function $h_g(\theta, a) = \xi(\theta; a, F+K_g) - \xi(\theta; a, F)$ is continuous, bounded, and globally lipschitzian in the second argument. Furthermore, B_g is the unique subset of V with these properties, and both B_g and the associated ODE depend continuously on g, in the sense that B_g approaches B_0 in the Hausdorff metric and the vector field, defining the ODE which gives the flow on B_g, approaches F in the uniform metric of the space of continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$, as $\sup_{\phi \in B_0} |g(\phi)|$ converges to zero.

Proof

The proof consists in the construction of a function $K_g: \mathbb{R}^n \to \mathbb{R}^n$, so that the elements of B_g are segments of solutions of the ODE

$$\dot{x}(t) = [F+K_g](x(t)) \quad . \tag{7}$$

The solutions of this ODE must also satisfy the given FDE (6). It follows that K_g must satisfy

$$K_g\left[\xi(t;b,\, F+K_g)\right] \;=\; g\left[\xi(t+\cdot\,;b,\, F+K_g)\right] \quad . \tag{8}$$

On the other hand, we expect to relate B_0 to B_g through a mapping $h_g : [-r,0] \times \mathbb{R}^n \to \mathbb{R}$ in such a way that $\phi \in B_0$ is transformed into a point of B_g by the mapping $\phi \mapsto \phi + h_g(\cdot\,,\phi(0))$. Thus, we want to have

$$\xi(\theta;a,\, F+K_g) \;=\; \xi(\theta;a,F) + h_g(\theta,a) \quad , \qquad \theta \in [-r,0] \tag{9}$$

and then, also

$$K_g(b) \;=\; g\left[\xi(\cdot\,;b,F) + h_g(\cdot\,,b)\right] \quad . \tag{10}$$

In order to make this construction precise, we need to establish the existence of the function h_g with the preceding properties. Equation (9) and the nonlinear variation of constants formula for equation (6), imply that h_g must be a fixed point of the mapping G_g given by

$$G_g(h)(\theta,a) \;=\; \int_0^\theta H_F\left[\theta - s,\, \xi(s;a,F) + h(s,a)\right] g\Big[\xi\big(\cdot\,;\xi(s;a,F) + h(s,a),F\big) + h\big(\cdot\,;\xi(s;a,F) + h(s,a)\big)\Big]\, ds \;. \tag{11}$$

To prove the existence of fixed points of this mapping, we apply the contraction mapping principle in a set

$$S(D,\Delta) = \Big\{ h : [-r,0] \times \mathbb{R}^n,\;\; h \text{ is continuous},\;\; h(0,a) = 0,\;\; |h(\theta,a)| \leqslant D \text{ and } |h(\theta,a) - h(\theta,\bar{a})| \leqslant \Delta |a - \bar{a}|, \text{ for all } a,\bar{a} \in \mathbb{R}^n,\; \theta \in [-r,0] \Big\} \;. \tag{12}$$

This set is a closed subset of the Banach space of continuous functions from $[-r,0] \times \mathbb{R}^n$ into $\mathbb{R}^n$, taken with the uniform norm $\|h\| = \sup\{|h(\theta,a)| : \theta \in [-r,0],\; a \in \mathbb{R}^n\}$. We need, therefore, to prove that G_g maps $S(D,\Delta)$ into itself and is a contraction on this set, in the sense that there exists $0 < \lambda < 1$ such that

$$\|G_g(h) - G_g(\bar{h})\| \leqslant \lambda\|h-\bar{h}\| , \qquad \text{for } h, \bar{h} \in S(D, \Delta) ,$$

uniformly in g, for $\sup_{\phi \in B_0} |g(\phi)|$ and the Lipschitz constant of g both sufficiently small. This is a straightforward computation based on formula (11) and the following inequalities, where $\alpha = \sup\{|F'(a)| : a \in \mathbb{R}^n\}$ and $L(F')$ is the Lipschitz constant for F':

$$|\xi(t; a, F) - \xi(t; \bar{a}, F)| \leqslant e^{\alpha|t|}|a - \bar{a}| \tag{13}$$

$$|H_F(t, a)| \leqslant e^{\alpha|t|} \tag{14}$$

$$|H_F(t, a) - H_F(t, \bar{a})| \leqslant \begin{cases} \dfrac{L(F')}{\alpha} (e^{2\alpha|t|} - e^{\alpha|t|}) |a - \bar{a}| , & \text{if } \alpha \neq 0 \\ 0 , & \text{if } \alpha = 0 \end{cases} . \tag{15}$$

The proof of these inequalities is based on noticing that $\xi(t; a, F) - \xi(t; \bar{a}, F)$ is a solution of the differential inequality $|\dot{w}| \leqslant \alpha|w|$, $H_F(t, a)$ is the solution of the linear variational equation $\dot{w} = F'(\xi(t; a, F))w$ with initial condition $w(0) = I$, and $H_F(t, a) - H_F(t, \bar{a})$ is the solution of the equation

$$\dot{w} = F'(\xi(t; a, F))w + \left[F'(\xi(t; a, F)) - F'(\xi(t; \bar{a}, F))\right] H_F(t, \bar{a})$$

with the initial condition $w(0) = 0$.

If we denote by $L(g)$ the Lipschitz constant of g on V and take $D \leqslant \delta$, we get from formula (11) and inequalities (13) - (15), for every $h, \bar{h} \in S(D, \Delta)$, $a, \bar{a} \in \mathbb{R}^n$, $\theta \in [-r, 0]$, in the case when $\alpha \neq 0$

$$|G_g(h)(\theta, a)| \leqslant \frac{e^{\alpha r} - 1}{\alpha} \left[\sup_{\phi \in B_0} |g(\phi)| + D\,L(g)\right] \tag{16a}$$

$$\left|G_g(h)(\theta,a) - G_g(h)(\theta,\bar{a})\right| \leqslant \left\{ \frac{L(F')}{2\alpha^2}\left[2e^{\alpha r}(e^{\alpha r} - \alpha r - 1) + (e^{\alpha r} - 1)^2\Delta\right] \times \left[\sup_{\phi\in B_0} |g(\phi)| + D(g)\right] + \frac{1}{\alpha}\left(\alpha r\, e^{\alpha r} + (e^{\alpha r} - 1)\Delta\right)(e^{\alpha r} + \Delta)\, L(g) \quad |a - \bar{a}|\right\} \tag{17a}$$

$$\left\|G_g(h) - G_g(\bar{h})\right\| \leqslant \left\{ \frac{L(F')}{2\alpha^2}(e^{\alpha r} - 1)^2\left[\sup_{\phi\in B_0} |g(\phi)| + D\, L(g)\right] + \frac{e^{\alpha r} - 1}{\alpha}\left[e^{\alpha r} + 1 + \Delta\right] L(g)\right\} \left\|h - \bar{h}\right\| \quad , \tag{18a}$$

and in the case when $\alpha = 0$

$$\left|G_g(h)(\theta,a)\right| \leqslant r\left[\sup_{\phi\in B_0} |g(\phi)| + D\, L(g)\right] \tag{16b}$$

$$\left|G_g(h)(\theta,a) - G_g(h)(\theta,\bar{a})\right| \leqslant r(1+\Delta)^2 L(g)\, |a - \bar{a}| \tag{17b}$$

$$\left\|G_g(h) - G_g(\bar{h})\right\| \leqslant r(2+\Delta)\, L(g) \left\|h - \bar{h}\right\| \quad . \tag{18b}$$

It is clear from these inequalities that G_g is a contraction in $S(D,\Delta)$, provided $\sup_{\phi\in B_0} |g(\phi)|$ and $L(g)$ are sufficiently small. The contraction mapping principle guarantees that there exists a unique fixed point h_g of the mapping G_g in the set $S(D,\Delta)$, provided that $\sup_{\phi\in B_0} |g(\phi)|$ and the Lipschitz constant of g on the set V are sufficiently small.

We can now define $K_g : \mathbb{R}^n \to \mathbb{R}^n$ by equation (10) and consider, as suggested before, the set

$$B_g = \left\{\xi(\cdot\,; a, F+K_g) \in C : a \in \mathbb{R}^n\right\} . \tag{19}$$

Let $y(\theta) = \xi(\theta; a, F) + h_g(\theta, a)$, $\theta \in [-r, 0]$. It is clear from formula (11), that $h_g(\theta, a)$ is differentiable in θ. Consequently, $y(\theta)$ is differentiable also. From formula (11), we see that the equation $h_g = G_g(h_g)$ can now be written as

$$y(\theta) = \xi(\theta; a, F) + \int_0^\theta H_F\big(\theta - s, y(s)\big) K_g\big(y(s)\big) ds \ ,$$

which is the nonlinear variation of constants formula for

$$\dot{x} = F(x) + K_g(x) \ .$$

It follows that equation (9) holds, and

$$\begin{aligned} \dot{\xi}(t; a, F+K_g) &= F\big[\xi(t; a, F+K_g)\big] + K_g\big[\xi(t; a, F+K_g)\big] \\ &= F\big[\xi(t; a, F+K_g)\big] + g\big[\xi(\cdot\,; \xi(t; a, F+K_g), F+K_g)\big] \\ &= F\big[\xi(t; a, F+K_g)\big] + g\big[\xi(t+\cdot\,; a, F+K_g)\big] \ . \end{aligned}$$

Therefore $\xi(t; a, F+K_g)$ is a global solution of (6), and the set B_g is invariant under (6). Consequently B_g is a manifold where the flow of (6) is given by the C^0 ODE $\dot{x} = (F+K_g)(x)$.

The continuous dependence of B_g and $F+K_g$ on g, in the sense given in the statement of the theorem, follows from the continuous dependence of the fixed point h_g on g. Let $0 < \lambda < 1$ denote the contraction constant, uniform in g for $\sup_{\phi \in B_0} |g(\phi)|$ and $L(g)$ sufficiently small. Then

$$\|h_g\| = \|G_g(h_g)\| = \|G_g(h_g) - G_g(0)\| + \|G_g(0)\| \leqslant \lambda \|h_g\| + \|G_g(0)\|$$

and

$$G_g(0) = \int_0^\theta H_F\big(\theta - s, \xi(s; a, F)\big)\, g\big(\xi(s+\cdot\,; a, F)\big) ds \ .$$

Consequently,

$$\|h_g\| \leqslant (1+\lambda)^{-1}\|G_g(0)\| \qquad \text{and} \qquad \|G_g(0)\| \leqslant \frac{e^{\alpha r}}{\alpha} \sup_{\phi \in B_0} |g(\phi)| \quad .$$

These inequalities show that h_g approaches zero as $\sup_{\phi \in B_0} |g(\phi)|$ converges to zero, or that h_g depends continuously on g, in the sense of the statement.

Q. E. D.

Remark

From the inequalities (16) - (18), it is possible to obtain explicit estimates on how large $\sup_{\phi \in B_0} |g(\phi)|$ and the Lipschitz constant of g on the neighborhood V of B_0 can be for applicability of the preceding proof. It is easy to verify that, in the case $\alpha \neq 0$, these conditions reduce to

$$\left[\sup_{\phi \in B_0} |g(\phi)| + D\,L(g)\right] < \frac{\alpha}{e^{\alpha r} - 1}\, D$$

and

$$\frac{L(F')}{2\alpha^2}\left[\frac{2e^{\alpha r}}{\Delta}(e^{\alpha r} - \alpha r - 1) + (e^{\alpha r} - 1)^2\right]\left[\sup_{\phi \in B_0} |g(\phi)| + D\,L(g)\right]$$

$$+ \frac{1}{\alpha}\left[\alpha r \frac{e^{\alpha r}}{\Delta} + (e^{\alpha r} - 1)\right](e^{\alpha r} + \Delta)\, L(g) < 1 \quad ,$$

and in the case $\alpha = 0$,

$$\sup_{\phi \in B_0} |g(\phi)| + D\,L(g) \leqslant D/r$$

and

$$L(g) < \frac{\Delta}{r(1+\Delta)^2} \quad ,$$

where $0 < D \leqslant \delta$ and Δ is an arbitrarily fixed positive number. In the case $\alpha = 0$, these inequalities amount to

$$L(g) < 1/4r$$

$$\sup_{\phi \in B_0} |g(\phi)| \leqslant \delta\big(1/r - L(g)\big) \quad ,$$

where δ is the size of V as in the theorem: $V = \left\{\phi \in C : \inf_{\psi \in B_0} |\phi - \psi| \leqslant \delta\right\}$.

Based on these observations, it is possible to prove the following result valid for perturbations of equations which may include retarded terms.

Corollary 3

Let $f : C \to \mathbb{R}^n$ be a globally lipschitzian function and let B_0 be a manifold where the flow of

$$\dot{x}(t) = f(x_t)$$

is given by a C^1 ODE in $\mathbb{R}^n$, $\dot{x}(t) = F[x(t)]$, with F having bounded and globally lipschitzian first derivative. If $\alpha = \sup\left\{|F'(a)| : a \in \mathbb{R}^n\right\}$ and the Lipschitz constant of f, denoted by L(f), satisfies

$$L(f) < \begin{cases} \dfrac{\alpha}{e^{\alpha r}(\sqrt{\alpha r} + \sqrt{e^{\alpha r} - 1})^2} - \alpha \quad , & \text{if } \alpha \neq 0 \\ 1/4r - \alpha \quad , & \text{if } \alpha = 0 \quad , \end{cases}$$

and if $g : C \to \mathbb{R}^n$ is uniformly lipschitzian on a neighborhood of B_0, $V = \left\{\phi \in C : \inf_{\psi \in B_0} |\phi - \psi| \leqslant \delta\right\}$ for some $\delta > 0$, with $\sup_{\phi \in B_0} |g(\phi)|$ and the Lipschitz constant of g on V both sufficiently small, then there exists a C^0-manifold B_g where the flow of the perturbed equation

$$\dot{x} = f(x_t) + g(x_t)$$

is given by a C^0 ODE. Furthermore, B_g and the associated ODE depend continuously on g and they are unique, in the sense of Theorem 2.

Proof

The perturbed equation can be written as

$$\dot{x}(t) = F[x(t)] + \left\{ f(x_t) - F[x(t)] + g(x_t) \right\} .$$

Taking into consideration Theorem 2 and the Remark preceding the present corollary, it follows that it is sufficient to show that $D, \Delta > 0$ can be chosen so that the estimates for applicability of the proof of Theorem 2, as given in the Remark, are satisfied for the function $\bar{g} : C \to \mathbb{R}^n$ given by $\bar{g}(\phi) = f(\phi) - F[\phi(0)]$. Since the flow of $\dot{x} = f(x_t)$ on B_0 is given by the ODE $\dot{x} = F[x(t)]$, we have necessarily $f(\phi) = F[\phi(0)]$, for all $\phi \in B_0$. Consequently, $\sup_{\phi \in B_0} [\bar{g}(\phi)] = 0$ and, since D can be chosen arbitrarily small, the conditions given in the Remark preceding the present corollary, when applied to the function $\bar{g}$, reduce to

$$L(\bar{g}) < 1/4r \quad , \qquad \text{if } \alpha = 0$$

and

$$L(\bar{g}) < \min \left\{ \frac{\alpha}{e^{\alpha r} - 1} \; , \; \frac{\alpha}{(e^{\alpha r} + \Delta)\left(e^{\alpha r} - 1 + \alpha r \dfrac{e^{\alpha r}}{\Delta}\right)} \right\} , \quad \text{if } \alpha \neq 0 .$$

The second term inside the curly brackets is always smaller than the first and its maximum value is achieved when $\Delta = e^{\alpha r}\left[\alpha r/(e^{\alpha r} - 1)\right]^{1/2}$. Using this value in the preceding estimate for $L(\bar{g})$, we obtain

$$L(\bar{g}) < \frac{\alpha}{e^{\alpha r}\left(\sqrt{\alpha r} + \sqrt{e^{\alpha r} - 1}\right)^2} \quad , \qquad \text{if } \alpha \neq 0 .$$

Since $L(\bar{g}) \leqslant L(f) + \alpha$, these conditions on $L(\bar{g})$ are guaranteed by the conditions assumed for $L(f)$ in the hypothesis of the corollary. Q.E.D.

The following example illustrates the application of the preceding result.

Example 1

Let us consider

$$\dot{x}(t) = G[x(t)] - G[x(t-r)]$$

where $G: \mathbb{R}^n \to \mathbb{R}^n$ is a globally lipschitzian function with Lipschitz constant $L(G)$, and $r > 0$. The manifold $B_0 \subset C([-r, 0]; \mathbb{R}^n)$ consisting of the constant functions defined on $[-r, 0]$, is a manifold where the flow is given by the ODE $\dot{x} = F(x) = 0$. Therefore, $\alpha = \sup\{|F'(a)| : a \in \mathbb{R}\} = 0$. The right-hand side of the equation is given by the function $f: C \to \mathbb{R}^n$ satisfying $f(\phi) = G[\phi(0)] - G[\phi(-r)]$. Consequently, f is globally lipschitzian and has Lipschitz constant $L(f) = 2L(G)$. Thus, the preceding corollary can be applied whenever $r < 1/(8L(G))$ to show that for small perturbations of the given equation there exists a unique manifold with the properties of Theorem 2, where the flow is given by a C^0 ODE in $\mathbb{R}^n$.

4. Smoothness

The following result gives conditions for smoothness of the invariant manifold B_g and the associated ODE, whose existence was established in the previous section.

Theorem 4

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^{k+1} $(k \geqslant 1)$ function with bounded derivatives up to order $k+1$, and let $B_0 = \{\xi(\cdot\,; a, F) : a \in \mathbb{R}^n\}$. If $g : C \to \mathbb{R}^n$ is a C^k function having bounded derivatives up to order k on a neighborhood of B_0, $V = \{\phi \in C : \inf_{\psi \in B_0} |\phi - \psi| \leqslant \delta\}$ for some $\delta > 0$, with $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$ both sufficiently small, then there exists a C^k-manifold B_g where the flow of

$$\dot{x}(t) = F[x(t)] + g(x_t) \tag{20}$$

is given by a C^k ODE. Furthermore, B_g is the unique subset of V with the properties stated in Theorem 2.

Proof

Under the current hypothesis, it results from Theorem 2 that there exists a unique manifold B_g with the properties in that theorem, where the flow of (20) is given by a C^0 ODE, $\dot{x} = (F+K_g)(x)$, and K_g, B_g are defined by expressions (10) and (19), in terms of the fixed point h_g of the mapping G_g defined by (11). Consequently, it only remains to prove that $h_g(\theta, a)$ is of class C^k as a function of the variable a. This will be first established for $k = 1$ and $k = 2$. For higher k we use induction. This procedure is similar to the one used by Fenichel [1] for the smoothness of invariant hyperbolic manifolds for flows.

Using formulas (9) - (11) and abbreviating

$$z(t, a) = \xi(t; a, F+K_g) \quad , \qquad t \in \mathbb{R} \tag{21}$$

so that

$$z(\theta, a) = \xi(\theta; a, F) + h_g(\theta, a) \quad , \qquad \theta \in [-r, 0] \quad , \tag{22}$$

we can write

$$h_g(\theta, a) = \int_0^\theta H_F\big(\theta - s,\ z(s,a)\big)\, g\Big[\xi\big(\cdot\,; z(s,a), F\big) + h_g\big(\cdot,\ z(s,a)\big)\Big]\, ds\ . \qquad (23)$$

Let us assume that the hypothesis of the theorem holds for $k = 1$. If the derivative $\partial h_g/\partial a$ exists, it must satisfy the equation obtained by formally differentiating both sides of (23) relative to a. In particular, $\partial h_g/\partial a$ must be a fixed point of the mapping T_g defined for $n \times n$ matrix-valued functions $v(\theta, a)$ and for $b \in \mathbb{R}^n$ by

$$\Big[T_g(v)b\Big](\theta, a) = \int_0^\theta \Big\{D_2H_F\big(\theta - s,\ z(s,a)\big)\Big[H_F(s,a)b + v(s,a)b\Big]\, g\Big[z(s+\cdot\,,\ a)\Big]$$

$$+ H_F\big(\theta - s, z(s,a)\big)\, g'\Big[z(s+\cdot\,, a)\Big]\Big[H_F\big(\cdot\,,\ z(s,a)\big) + v\big(\cdot\,,\ z(s,a)\big)\Big] \qquad (24)$$

$$\times \Big[H_F(s,a)b + v(s,a)b\Big]\Big\}\, ds\ ,$$

where D_2H_F denotes the derivative of H_F relative to its second argument. Let us consider the sequence of $n \times n$ matrix-valued functions v_i defined inductively by

$$v_1 = 0 \qquad \text{and} \qquad v_{i+1} = T_g(v_i)\ . \qquad (25)$$

We shall see that $\{v_i\}$ is a Cauchy sequence in the space of $n \times n$ matrix-valued continuous functions defined in $[-r, 0] \times \mathbb{R}^n$, taken with the uniform operator norm

$$\|v\| = \sup\Big\{|v(\theta, a)b| : \theta \in [-r, 0],\ a, b \in \mathbb{R}^n,\ |b| = 1\Big\}\ .$$

Following this, one can show that the limit of the sequence $\{v_i\}$ is, indeed, the derivative $\partial h_g/\partial a$.

In order to prove that $\{v_i\}$ is a Cauchy sequence, we use the following inequalities, where $\alpha = \sup\Big\{|F'(a)| : a \in \mathbb{R}^n\Big\}$ and $\beta = \sup\Big\{|F''(a)| : a \in \mathbb{R}^n\Big\}$:

$$\Big|H_F(t, a)\Big| \leqslant e^{\alpha|t|} \qquad (26)$$

$$\left|D_2H_F(t,a)\right| \leqslant \begin{cases} \dfrac{\beta}{\alpha}\, e^{2\alpha|t|}, & \text{if } \alpha > 0 \\ 0, & \text{if } \alpha = 0 . \end{cases} \tag{27}$$

The first inequality is (14) in the proof of Theorem 2 and the second inequality is established by noticing that $D_2H_F(t,a)$ is the solution of the equation

$$\dot{w} = F'\big(\xi(t;a,F)\big)w + F''\big(\xi(t;a,F)\big)H_F(t,a)H_F(t,a)$$

with initial condition $w(0) = 0$. In fact, from this equation and inequality (26) we obtain the differential inequality $|\dot{w}| \leqslant \alpha|w| + \beta e^{2\alpha|t|}$. Solving this differential inequality with initial condition $w(0) = 0$, leads to (27). In the proof of Theorem 2, it was shown that h_g is bounded by some $D > 0$ satisfying $D \leqslant \delta$. Consequently, using inequalities (26), (27) in formula (24), we get

$$\|v_{i+1}\| = \|T_g(v_i)\| \leqslant r\,\frac{\beta}{\alpha}\,e^{2\alpha r}(e^{\alpha r} + \|v_i\|)\Big(D \sup_{\phi\in V}|g'(\phi)| + \sup_{\phi\in B_0}|g(\phi)|\Big)$$
$$+ r e^{\alpha r}(e^{\alpha r} + \|v_i\|)^2 \sup_{\phi\in V}|g'(\phi)| \quad ,$$

where the first term on the right-hand side vanishes if $\alpha = 0$. It is clear that, given $\sigma < 1/2$, if $\sup_{\phi\in B_0}|g(\phi)|$ and $\sup_{\phi\in V}|g'(\phi)|$ are sufficiently small, then $\|v_{i+1}\| \leqslant (1 + \|v_i\| + \|v_i\|^2)\sigma/2$. Since $\|v_1\| = 0$, it follows that

$$\|v_i\| < \sigma/2 < 1/2 \quad , \qquad \text{for } i = 1,2,\ldots \quad . \tag{28}$$

On the other hand,

$$\|v_{i+1} - v_i\| = \|T_g(v_i) - T_g(v_{i-1})\| \leqslant \Big[r\,\frac{\beta}{\alpha}\,e^{2\alpha r}\Big(D \sup_{\phi\in V}|g'(\phi)| + \sup_{\phi\in B_0}|g(\phi)|\Big)$$
$$+ r e^{\alpha r}(2e^{\alpha r} + \|v_i\| + \|v_{i-1}\|) \sup_{\phi\in V}|g'(\phi)|\Big]\|v_i - v_{i-1}\| \quad ,$$

where the first term inside square brackets vanishes if $\alpha = 0$. Consequently, for sufficiently small $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$, we have $\|v_{i+1} - v_i\| < (1/2)\|v_i - v_{i-1}\|$, for all $i = 1, 2, \ldots$. This shows that $\{v_i\}$ is a Cauchy sequence and, consequently, it converges to some v, as $i \to \infty$. Clearly, v is a fixed point of T_g, i.e., $v = T_g(v)$.

Now, we can prove that v is, indeed, the derivative $\partial h_g/\partial a$. Fix $a \in \mathbb{R}^n$ and let γ be a function defined for small values of $\epsilon > 0$ by

$$\gamma(\epsilon) = \sup_{\substack{\theta \in [-r, 0] \\ |b| \leq \epsilon}} \frac{|h_g(\theta, a+b) - h_g(\theta, a) - v(\theta, a)b|}{|b|} \tag{29}$$

In order to prove that $\partial h_g/\partial a$ exists and is equal to v, we need to prove that $\gamma(\epsilon) \to 0$ as $\epsilon \to 0$. Denoting

$$y(s, b) = z(s, a+b) - z(s, a) - H_F(s, a)b - v(s, a)b \ , \tag{30}$$

it follows from formulas (23) and (24), using a first order Taylor expansion, that

$$\begin{aligned} h_g(\theta, a+b) - h_g(\theta, a) - v(\theta, a)b = \int_0^\theta \Big\{ & D_2 H_F\big(\theta-s,\ z(s,a)\big)\, y(s,b)\, g\big(z(s+\cdot\,, a)\big) \\ & + H_F\big(\theta-s,\ z(s,a)\big)\, g'\big(z(s+\cdot\,, a)\big) \Big[H_F\big(\cdot\,,\ z(s,a)\big)\, y(s,b) + h_g\big(\cdot\,,\ z(s,a+b)\big) \\ & - h_g\big(\cdot\,,\ z(s,a)\big) - v\big(\cdot\,,\ z(s,a)\big)\big(H_F(s,a)b + v(s,a)b\big) + \mathcal{O}\big(|z(s,a+b) - z(s,a)|\big)\Big] \\ & + \mathcal{O}\Big(|z(s,a+b) - z(s,a)| + \big|h_g\big(\cdot\,,\ z(s,a+b)\big) - h_g\big(\cdot\,,\ z(s,a)\big)\big|\Big)\Big\} ds \ . \end{aligned} \tag{31}$$

From formulas (22) and (30), taking into account the definition of H_F and formula (29), we get for $s \in [-r, 0]$

$$\begin{aligned} |y(s,b)| &\leq \big|\xi(s; a+b, F) - \xi(s; a, F) - H_F(s,a)b\big| + \big|h_g(s, a+b) \\ &\qquad\qquad - h_g(s,a) - v(s,a)b\big| \qquad (32) \\ &\leq \mathcal{O}(|b|) + \gamma(|b|)\,|b| \ , \end{aligned}$$

as $|b| \to 0$. On the other hand, recalling that it was shown in the proof of Theorem 2 that h_g is lipschitzian in the second argument with Lipschitz constant $\Delta > 0$, an application of inequality (13) to formula (22) gives

$$|z(s,a+b) - z(s,a)| \leqslant (e^{\alpha r} + \Delta)|b| \quad , \qquad \text{for } s \in [-r, 0] \tag{33}$$

and, then

$$\mathcal{O}\Big(|z(s,a+b) - z(s,a)| + \big|h_g\big(\cdot, z(s,a+b)\big) - h_g\big(\cdot, z(s,a)\big)\big|\Big) = \mathcal{O}(|b|) \ , \tag{34}$$

as $|b| \to 0$. It also follows from formulas (29) and (30) and inequalities (27), (32), (33), that

$$\begin{aligned}
&\Big|h_g\big(\cdot, z(s,a+b)\big) - h_g\big(\cdot, z(s,a)\big) - v\big(\cdot, z(s,a)\big)\big(H_F(s,a)b + v(s,a)b\big)\Big| \\
&= \Big|h_g\big(\cdot, z(s,a+b)\big) - h_g\big(\cdot, z(s,a)\big) - v\big(\cdot, z(s,a)\big)\big[z(s,a+b) - z(s,a)\big] \\
&\qquad + v\big(\cdot, z(s,a)\big)\ y(s,b)\Big| \\
&\leqslant \gamma\big((e^{\alpha r} + \Delta)|b|\big)(e^{\alpha r} + \Delta)|b| + \gamma(|b|)|b| + \mathcal{O}(|b|) \quad ,
\end{aligned} \tag{35}$$

as $|b| \to 0$. Using inequalities (32) - (35) and (26),(27) in formula (31), and recalling the definition of γ in (29), we get

$$\gamma(\epsilon) \leqslant C_1\gamma(\epsilon) + C_2\gamma\big((e^{\alpha r} + \Delta)\epsilon\big) + \mathcal{O}(1) \quad ,$$

as $\epsilon \to 0$, where $C_1, C_2 \geqslant 0$ are constants which can be made as small as desired by making $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$ sufficiently small. In particular, we can take $C_1 < 1$, $A \overset{\text{def}}{=} C_2/(1 - C_1) < 1$ and denote $B = (e^{\alpha r} + \Delta)$, to get

$$\gamma(\epsilon) \leqslant A\,\gamma(B\epsilon) + \mathcal{O}(1) \quad , \tag{36}$$

as $\epsilon \to 0$. Using this inequality repeatedly, we get

$$\gamma(B^{-i}\epsilon) \leqslant A^i\gamma(\epsilon) + (A^{i-1} + A^{i-2} + \ldots + 1)\mathcal{O}(1) \leqslant A^i\gamma(\epsilon) + (1-A)^{-1}\mathcal{O}(1) \quad ,$$

as $\epsilon \to 0$. From inequality (28) we have $\|v\| < 1/2$ and from the fact that h_g is lipschitzian in the second variable with Lipschitz constant Δ, the definition of γ in (29) implies that $\gamma(\epsilon) \leqslant \Delta + 1/2$. Since $\gamma(\epsilon)$ is bounded and $0 \leqslant A < 1$, if we denote by $\{\epsilon_i\}$ any sequence of positive numbers converging to zero as $i \to \infty$, and let $i \to \infty$, we obtain $\gamma(B^{-i}\epsilon_i) \to 0$ as $i \to \infty$. As the function $\gamma(\epsilon)$ is nondecreasing for $\epsilon > 0$, it follows that $\gamma(\epsilon) \to 0$ as $\epsilon \to 0$. This proves the theorem for $k = 1$.

In order to prove the theorem for $k = 2$, we notice that each element of the sequence $\{v_i(\theta, a)\}$ is differentiable in a, because, as we have just proved, h_g is differentiable in a. Therefore, the derivative relative to a of $v_{i+1}(\theta, a) = T_g(v_i)(\theta, a)$, as given by formula (24), exists provided $v_i(\theta, a)$ is itself differentiable relative to a. Since we start with $v_1 = 0$, all the v_i are differentiable relative to a, and

$$\frac{\partial v_{i+1}}{\partial a}(\theta, a) = \int_0^\theta \Bigg\{ D_2 H_F\big(\theta - s, z(s,a)\big) \frac{\partial v_i}{\partial a}(s,a)\, g\big(z(s+\cdot\,, a)\big)$$

$$+ H_F\big(\theta - s, z(s,r)\big)\, g'\big(z(s+\cdot\,, a)\big) \Bigg\{ \frac{\partial v_i}{\partial a}\big(\cdot\,, z(s,a)\big)\big[H_F(s,a) + v_i(s,a)\big]^2$$

$$+ \big[H_F\big(\cdot\,, z(s,a)\big) + v_i\big(\cdot\,, z(s,a)\big)\big] \frac{\partial v_i}{\partial a}(s,a)\Bigg\}\Bigg\}\, ds$$

$$+ (\text{terms not involving derivatives of } v\text{'s}) \,. \tag{37}$$

The terms that do not involve derivatives of v's are formed of g and its first and second derivatives, H_F and its first and second derivatives relative to the second argument, and the functions v_i. Consequently, we have to verify the existence of $\partial^2 H_F(t,a)/\partial a^2$ for $t \in [-r, 0]$. It is not difficult to show that this derivative does indeed exist and satisfies the nonhomogeneous linear variational equation obtained by formally differentiating the variational equation $\dot{w} = F'\big(\xi(t; a, F)\big)w$ twice relative to a, i.e., $\partial^2 H_F(t,a)/\partial a^2$ satisfies the equation

$$\dot{w} = F'\big(\xi(t;a,F)\big)w + F''\big(\xi(t;a,F)\big)$$

$$+ \left[\frac{\partial H_F}{\partial a}(t,a)H_F(t,a) + 2H_F(t,a)\frac{\partial H_F}{\partial a}(t,a)\right] + F'''\big(\xi(t;a,F)\big)\left[H_F(t,a)\right]^3$$

with initial condition $w(0) = 0$. It is then possible to verify that $\partial H_F(t,a)/\partial a^2$ is defined and bounded over the interval $[-r, 0]$. Using inequalities (26) - (28) in a similar way as done above for the sequence $\{v_i\}$, we get in the usual uniform operator norm

$$\left\| \frac{\partial v_{i+1}}{\partial a} - \frac{\partial v_i}{\partial a} \right\| \leqslant C_3 \left\| \frac{\partial v_i}{\partial a} - \frac{\partial v_{i-1}}{\partial a} \right\| + C_4 \left\| v_i - v_{i-1} \right\| \quad , \tag{38}$$

where $C_3, C_4 > 0$ and C_3 can be made as small as desired by making $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g(\phi)|$ sufficiently small. Since we already know that $\|v_i - v_{i-1}\| < (1/2)^i$, it follows that $\{\partial v_i/\partial a\}$ is a Cauchy sequence. Arguing as before for the sequence $\{v_i\}$, we can show that $\partial v/\partial a = \partial^2 h_g/\partial a$ exists and is equal to the limit of $\{\partial v_i/\partial a\}$. This proves the theorem for $k = 2$.

The proof of the theorem for $k > 2$ follows by induction. Assume the theorem holds for a certain $k \geqslant 2$ and the hypothesis is satisfied for $k + 1$, and let $w = \partial^k h_g/\partial a^k$. Differentiating equation (23) k times, we get

$$w(\theta, a) = \int_0^\theta \Big\{ D_2 H_F\big(\theta - s, z(s,a)\big) w(s,a)\, g\big(z(s+\cdot, a)\big)$$

$$+ H_F\big(\theta - s, z(s,a)\big) g'\big(z(s+\cdot, a)\big) \Big\{ w\big(\cdot, z(s,a)\big)\left[H_F(s,a) + v(s,a)\right]^k$$

$$+ \left[H_F\big(\cdot, z(s,a)\big) + v\big(\cdot, z(s,a)\big)\right] w(s,a) \Big\}\Big\}\, ds$$

$$+ \text{(terms not involving } w\text{)} \quad . \tag{39}$$

The terms not involving w contain derivatives of g and the derivatives of $h_g(\theta, a)$ and $H_F(\theta, a)$ relative to a, up to order k. We must therefore verify that the derivatives of $H_F(\theta, a)$ relative to a exist and are bounded, up to order k, for $\theta \in [-r, 0]$, $a \in \mathbb{R}^n$. It is not hard to show that this is the case and that these derivatives satisfy nonhomogeneous linear variational equations with initial condition zero at the orgin. These equations are obtained by formally differentiating the linear variational equation $\dot{w} = F'(\xi(t; a, F))\, w$ realtive to a, as many times as the order of the derivative of H_F in question. Now, we can let $\mathfrak{I}$ be the mapping transforming w into the function $\mathfrak{I}(w)$ of (θ, a) according to the right-hand side of (39), and define recursively the sequence $\{w_i\}$ by $w_1 = 0$ and $w_{i+1} = \mathfrak{I}(w_i)$. It follows from inequalities (26) - (28), as before for the sequence $\{v_i\}$, that $\{w_i\}$ is a Cauchy sequence, in the usual uniform norm, provided $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$ are sufficiently small. Consequently, $\{w_i\}$ converges to a fixed point w of $\mathfrak{I}$, and $w = \partial^k h_g / \partial a^k$. Clearly, the functions $w_i(\theta, a)$ are differentiable in a, and, consequently, $(\partial w_{i+1}/\partial a)(\theta, a)$ is equal to the right-hand side of (39) with w replaced by $\partial w_i / \partial a$.

Using again inequalities (26) - (28), we obtain an inequality like (38) with v_i replaced by w_i. Arguing as before in the sequel of inequality (38), we prove that $\partial w / \partial a = \partial^{k+1} h_g / \partial a^{k+1}$ exists and is equal to the limit of $\{\partial w_i / \partial a\}$. This proves that the theorem holds for $k+1$ if it holds for some $k \geqslant 2$, thus completing the induction step. Q.E.D.

5. Stability

This section discusses the stability properties of the invariant manifold B_g of Theorems 2 and 4. The following result is stated independently of the theorem on stability because it is also used in the next section.

Lemma 5

Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 function with bounded first and second derivatives and let $B_0 = \{\xi(\cdot\,; a, F) \in C : a \in \mathbb{R}^n\}$. If $g : C \to \mathbb{R}^n$ is a continuously differentiable function with first derivative bounded in a neighborhood of B_0, $V = \{\phi \in C : \inf_{\psi \in B_0} |\phi - \psi| \leqslant \delta\}$ for some $\delta > 0$, then, for sufficiently small $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V'} |g'(\phi)|$, the orbit of any solution $x(t)$ of the equation

$$\dot{x}(t) = F[x(t)] + g(x_t) \quad , \tag{40}$$

which satisfies $|x_t - \xi(\cdot\,; x(t), F+K_g)| < \delta$ for all $t \in [0, r]$, stays in V for all $t \geqslant r$ and approaches, exponentially fast, the manifold B_g where the flow of (40) is given by the ODE $\dot{x} = (F+K_g)(x)$. More precisely, there exists $\sigma > 0$ such that

$$|x_t - \xi(\cdot\,; x(t), F+K_g)| < \delta \quad \text{and} \quad \inf_{\psi \in B_g} |x_t - \psi| \leqslant \delta e^{-\sigma t} \ , \quad t \geqslant r \ .$$

Proof

Let $\dot{x} = (F+K_g)(x)$ be the ODE which gives the flow of (40) on B_g. According to formulas (9) - (10) of Theorem 2, we see that $x(t)$ satisfies the equation

$$\dot{x}(t) = [F+K_g](x(t)) + g(x_t) - g[\xi(\cdot\,; x(t), F+K_g)]$$

and we have

$$\xi(\theta; a, F+K_g) = \xi(\theta; a, F) + h_g(\theta, a) \ , \qquad \text{for } \theta \in [-r, 0],\ a \in \mathbb{R}^n \ .$$

The proofs of Theorems 2 and 4 guarantee that the function h_g is bounded and continuously differentiable with bounded derivative relative to the second argument, and the bounds $\left|h_g(\theta, a)\right| \leqslant D$ and $\left|\partial h_g(\theta, a)/\partial a\right| \leqslant \Delta$ can be taken arbitrarily small, by requiring $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$ to be sufficiently small. In particular, we can make $D > 0$ so small that B_g lies inside an arbitrarily small neighborhood of B_0. Denoting

$$z(t, \theta) = x(t+\theta) - \xi\big(\theta; x(t), F+K_g\big) , \quad \text{for } t \geqslant 0, \ \theta \in [-r, 0] ,$$

the nonlinear variation of constants formula of Lemma 1 gives

$$z(t, \theta) = \int_t^{t+\theta} H_{F+K_g}\big(t+\theta - s, x(s)\big)\Big[g(x_s) - g\Big(\xi\big(\cdot\, ; x(s), F+K_g\big)\Big)\Big] ds , \tag{41}$$

$$\text{for } t+\theta \geqslant 0, \ \theta \in [-r, 0],$$

where $H_{F+K_g}(t, a) = \partial\xi(t; a, F+K_g)/\partial a$. Clearly,

$$H_{F+K_g}(\theta, a) = H_F(\theta, a) + \frac{\partial h_g}{\partial a}(\theta, a) \quad , \quad \theta \in [-r, 0], \ a \in \mathbb{R}^n ,$$

and, from the inequality (14) of the proof of Theorem 2, we get

$$\left|H_{F+K_g}(\theta, a)\right| \leqslant e^{-\alpha\theta} + \Delta \quad , \qquad \theta \in [-r, 0] \quad ,$$

where $\alpha = \sup\left\{|F'(a)| : a \in \mathbb{R}^n\right\}$. Let $\epsilon > 0$ be such that $\left|x_s - \xi\big(\cdot\, ; x(s), F+K_g\big)\right| \leqslant \delta - \epsilon$ for all $s \in [0, r]$ and define

$$T = \sup\left\{t \in \mathbb{R}^+ : \left|x_s - \xi\big(\cdot\, ; x(s), F+K_g\big)\right| \leqslant \delta - \epsilon , \quad \text{for all } 0 \leqslant s \leqslant t\right\} .$$

Making $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$ so small that we can take $D < \delta$, the mean value theorem can be applied to formula (41) yielding

$$|z(t,\theta)| \leqslant \int_{t+\theta}^{t} (e^{-\alpha(t+\theta-s)} + \Delta) \sup_{\psi \in V} |g'(\psi)| \max_{\tau \in [-r,0]} |z(s,\tau)| ds \tag{42}$$

$$\leqslant \left(\frac{e^{\alpha r}-1}{\alpha} + \Delta r\right) \sup_{\psi \in V} |g'(\psi)| (\delta - \epsilon) \quad , \quad \text{for } r \leqslant t \leqslant T, \ \theta \in [-r,0] \ .$$

by taking $\sup_{\psi \in V} |g'(\psi)|$ sufficiently small, we can make $|z(t,\theta)| < (\delta - \epsilon)/2$ for all $r \leqslant t \leqslant T$, $\theta \in [-r,0]$ and, consequently, $T = +\infty$. This implies that $|x_t - \xi(\cdot; x(t), F+K_g)| < \delta$ for all $t \geqslant r$. Using inequality (42) in the variation of constants formula (41), as before, we get

$$|z(t,\theta)| < \delta \left[\left(\frac{e^{\alpha r}-1}{\alpha} + \Delta r\right) \sup_{\psi \in V} |g'(\psi)|\right]^2 \quad , \quad \text{for } t \geqslant 2r, \ \theta \in [-r,0] \ .$$

Proceeding in this way through further intervals of length r, we obtain

$$|z(t,\theta)| < \delta \left[\left(\frac{e^{\alpha r}-1}{\alpha} + \Delta r\right) \sup_{\psi \in V} |g'(\psi)|\right]^m \quad , \quad \text{for } t \geqslant mr, \ \theta \in [-r,0] \ .$$

Consequently, for $\sup_{\psi \in V} |g'(\psi)|$ sufficiently small, we have

$$|z(t,\theta)| < \delta \beta^{t/r} \quad , \quad \text{for } t \geqslant r \ ,$$

with $0 < \beta < 1$. Since $B_g = \{\xi(\cdot; a, F+K_g) \in C : a \in \mathbb{R}^n\}$, we get the exponential estimate appearing in the statement of the theorem by setting $\sigma = -(\ell n \beta)/r$.

Q. E. D.

Theorem 6

Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2-function with bounded first and second derivatives and let $B_0 = \{\xi(\cdot; a, F) \in C : a \in \mathbb{R}^n\}$. If $g : C \to \mathbb{R}^n$ is a continuously differentiable function with first derivative bounded in a neighborhood of B_0, $V = \{\phi \in C : \inf_{\psi \in B_0} |\phi - \psi| \leqslant \delta\}$ for some $\delta > 0$, then, for sufficiently small

$\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$, the manifold B_g is exponentially asymptotically stable under the equation

$$\dot{x}(t) = F[x(t)] + g(x_t) \ .$$

More precisely, given any neighborhood V' of B_g, there exist $L, \sigma > 0$ and a neighborhood U of B_g such that

$$x_t(\phi, g) \in V' \quad \text{and} \quad \inf_{\psi \in B_g} |x_y(\phi, g) - \psi| \leqslant L e^{-\sigma t}$$

for all $\phi \in U$ and $t \geqslant 0$.

<u>Proof</u>

Without loss of generality, we assume that V' is a neighborhood of B_g of the form $V' = \left\{\phi \in C : \inf_{\psi \in B_0} |\phi - \psi| \leqslant \delta'\right\}$ for some $\alpha\delta' \leqslant \delta$, and we take U to be an ϵ-neighborhood of B_g, i.e., $U = \left\{\phi \in C : \inf_{\psi \in B_g} |\phi - \psi| \leqslant \epsilon\right\}$. If $\phi \in U$, then *for* each $\nu > 0$ there exists $a \in \mathbb{R}^n$ such that $|\phi - \xi(\cdot\,; a, F+K_g)| \leqslant \epsilon + \nu$. Let $\alpha = \sup\left\{|F'(a)| : a \in \mathbb{R}^n\right\}$ and $\Delta = \sup\left\{|\partial h_g(\theta, a)/\partial a| : \theta \in [-r, 0], a \in \mathbb{R}^n\right\}$. Then

$$\begin{aligned} \left|\xi(\cdot\,; a, F+K_g) - \xi\left(\cdot\,; \phi(0), F+K_g\right)\right| &\leqslant (e^{\alpha r} + \Delta)\,|\phi(0) - a| \\ &\leqslant (e^{\alpha r} + \Delta)(\epsilon + \nu) \end{aligned}$$

and, consequently,

$$\begin{aligned} \left|\phi - \xi\left(\cdot\,; \phi(0), F+K_g\right)\right| &\leqslant \left|\phi - \xi(\cdot\,; a, F+K_g)\right| + \left|\xi(\cdot\,; a, F+K_g) - \xi\left(\cdot\,; \phi(0), F+K_g\right)\right| \\ &\leqslant (\epsilon + \nu)(1 + e^{\alpha r} + \Delta) \ . \end{aligned}$$

Since $\nu > 0$ can be taken arbitrarily small, we have

$$\left|\phi - \xi\left(\cdot\,; \phi(0), F+K_g\right)\right| \leqslant \epsilon(1 + e^{\alpha r} + \Delta) \ .$$

The variation of constants formula of Lemma 1 gives

$$x(t;\phi,g) = \xi\big(t;\phi(0),F\big) + \int_0^t H_F\big(t-s, x(s;\phi,g)\big)g\big[x_s(\phi,g)\big]\,ds \ .$$

If we define $T = \sup\big\{t\in[0,r] : x_s(\phi,g)\in V' \text{ for all } s\in[0,t]\big\}$ and use the inequality (14) of Theorem 2, we get

$$\big|x(t;\phi,g) - \xi\big(t;\phi(0),F\big)\big| \leqslant \frac{(e^{\alpha r}-1)}{\alpha}\Big[\sup_{\psi\in B_0}|g(\psi)| + \delta' \sup_{\psi\in V'}|g'(\psi)|\Big], \quad \text{for } t\in[0,T] \ .$$

By taking $\sup_{\psi\in B_0}|g(\psi)|$ and $\sup_{\psi\in V'}|g'(\psi)|$ sufficiently small, we can make

$$\big|x(t;\phi,g) - \xi\big(t;\phi(0),F\big)\big| \leqslant \frac{\delta'}{2}e^{-\alpha r} \quad , \quad \text{for all } t\in[0,T] \ . \tag{43}$$

We notice that

$$\big|x_t(\phi,g) - \xi\big(t+\cdot\,;\phi(0),F\big)\big|$$

$$= \max\Big\{\big|x(s;\phi,g) - \xi\big(s;\phi(0),g\big)\big|, \ \big|\phi(\tau) - \xi\big(\tau;\phi(0),F\big)\big| : s\in[0,t], \ \tau\in[t-r,0]\Big\}.$$

By choosing $\epsilon > 0$ sufficiently small and restricting $\sup_{\psi\in B_0}|g(\psi)|$ and $\sup_{\psi\in V}|g'(\psi)|$ to be smaller, if necessary, we can make $\big|\phi - \xi\big(\cdot\,;\phi(0),F\big)\big| \leqslant \delta'/2$. It follows that $\big|x_t(\phi,g) - \xi\big(t+\cdot\,;\phi(0),F\big)\big| \leqslant \delta'/2$ for all $t\in[0,T]$. This implies that $T = r$ and, consequently,

$$\big|x_t(\phi,g) - \xi\big(\cdot\,;x(t;\phi,g),F+K_g\big)\big| \leqslant \big|x_t(\phi,g) - \xi\big(t+\cdot\,;\phi(0),F\big)\big|$$

$$+\big|\xi\big(t+\cdot\,;\phi(0),F\big) - \xi\big(\cdot\,;x(t;\phi,g),F\big)\big| + \big|\xi\big(\cdot\,;x(t;\phi,g),F\big) - \xi\big(\cdot\,;x(t;\phi,g),F+K_g\big)\big|$$

$$\leqslant \frac{\delta'}{2}e^{-\alpha r} + e^{\alpha r}\big|x(t;\phi,g) - \xi\big(t;\phi(0),F\big)\big| + D \quad , \quad \text{for all } t\in[0,r] \ ,$$

where $D = \sup\{|h_g(\theta, a)| : \theta \in [-r, 0],\ a \in \mathbb{R}^n\}$. It is known from Theorem 2 that $D > 0$ can be made arbitrarily small by restricting $\sup_{\psi \in B_0} |g(\psi)|$ and $\sup_{\psi \in V} |g'(\psi)|$ to be sufficiently small. Applying this and inequality (43), we get

$$\left|x_t(\phi, g) - \xi\left(\cdot\, ; x(t; \phi, g),\ F + K_g\right)\right| < \delta' \quad , \qquad \text{for all } t \in [0, r] .$$

Now, we apply Lemma 5 to get the result. Q.E.D.

6. Manifolds of Initial Data for Global Solutions

The set of initial data for global solutions has an important role in the study of geometric properties of FDEs. It is of interest to know when this set is a differentiable manifold and what is its order of smoothness. The previous results could be used for that purpose, provided the manifold B_g, whose existence, smoothness and stability were established in the preceding theorems, could be shown to be equal to the whole set A_g of initial data for global solutions of the perturbed equation. Although A_g is not in general equal to B_g (even if $A_0 = B_0$), as shown in an example given after the next result, the following local statement is true.

Theorem 7

Assume that $F: \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 function with bounded first and second derivatives and let $B_0 = \{\xi(\cdot\,; a, F) : a \in \mathbb{R}^n\}$. If $g: C \to \mathbb{R}^n$ is a continuously differentiable function with first derivative bounded in a neighborhood of B_0, $V = \{\phi \in C : \inf_{\psi \in B_0} |\phi - \psi| \leqslant \delta\}$ for some $\delta > 0$, then there exists a neighborhood U of B_g such that, for sufficiently small $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$, the orbits of all global solutions of the equation

$$\dot{x}(t) = F[x(t)] + g(x_t)$$

which lie in U for all $t \in \mathbb{R}$ are, in fact, contained in B_g, or, in other words, if A_g^U is the set of initial data for global solutions whose orbits stay in U, then $A_g^U = B_g$.

Proof

By Theorem 6, there exist $\sigma, L > 0$ and a neighborhood U of B_g such that

$$x_t(\phi, g) \in V \quad \text{and} \quad \inf_{\psi \in B_g} |x_t(\phi, g) - \psi| \leqslant L e^{-\sigma t}$$

for all $\phi \in U$ and $t \geqslant 0$.

Let us assume that $\phi \in A_g^U$ and $\phi \notin B_g$, and denote $\nu = \inf_{\psi \in B_g} |\phi - \psi| > 0$. Take $T > 0$ so large that $Le^{-\sigma T} < \nu$. Since A_g^U is the set of initial data for global solutions whose orbits stay in U for all $t \in \mathbb{R}$, there exists a point $\zeta \in A_g^U \subset U$ such that $x_T(\zeta, g) = \phi$. Thus, $\inf_{\psi \in B_g} |\phi - \psi| < \nu$, contradicting the previous assumption.

This proves that $A_g^U \subset B_g$. Since $B_g \subset U$, Theorem 2 implies that $B_g \subset A_g^U$, and it follows that $B_g = A_g^U$. Q.E.D.

The following example shows that, in general, A_g is not equal to B_g.

Example 2

Consider the following scalar linear equation

$$\dot{x}(t) = -\epsilon\left[\dot{x}(t) - x(t-r)\right] \tag{44_ϵ}$$

on $C = C\left([-r, 0]; \mathbb{R}^n\right)$, with $\epsilon \geqslant 0$ small. This is an equation of the type discussed in the previous theorems with $F(x) = 0$ and $g(\phi) = g_\epsilon(\phi) = -\epsilon\left[\phi(0) - \phi(-r)\right]$. Consequently, Theorems 2 and 4 guarantee the existence of a C^∞-manifold, B_{g_ϵ} where the flow of (44_ϵ) is given by a C^∞ ODE, provided ϵ is sufficiently small, and that B_{g_ϵ} approaches, as $\epsilon \to 0$, the set $A_0 = B_0$ of initial data for global solutions of the equation (44_0), which consists of the constant functions defined on $[-r, 0]$. Actually, it is obvious, in this case, that $B_{g_\epsilon} = A_0$.

One of the characteristic values of (44_ϵ) sits at the origin of the complex plane as ϵ changes, but there exists one characteristic value with real part decreasing to $-\infty$ as $\epsilon \to 0$. This last characteristic value gives characteristic solutions which are global solutions of (44_ϵ) and have initial conditions which do not approach, as $\epsilon \to 0$, the set $A_0 = B_0$ of initial data for global solutions of (44_ϵ). For this example, we have $B_{g_\epsilon} = A_0$ and $B_{g_\epsilon} \subsetneqq A_{g_\epsilon}$.

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It is clear from Theorem 7 that we have $A_g = B_g$ provided the orbits of all solutions of the perturbed equation enter a sufficiently small neighborhood of B_g after some fixed finite time. A sufficient condition for this to happen is given in the following result.

Theorem 8

Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 function with bounded first and second derivatives, let $B_0 = \{\xi(\cdot\,; a, F) \in C : a \in \mathbb{R}^n\}$ and $\alpha = \sup\{|F'(a)| : a \in \mathbb{R}^n\}$. If $g : C \to \mathbb{R}^n$ is a bounded continuously differentiable function with first derivative bounded in a neighborhood of B_0, $V = \{\phi \in C : \inf_{\psi \in B_0} |\phi - \psi| \leqslant \delta\}$ for some δ satisfying $0 < \delta \leqslant +\infty$, and such that $\sup_{\phi \in C} |g(\phi)| < \delta\alpha/2(e^{\alpha r} - 1)$, then, for sufficiently small $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$, the C^1-manifold B_g of theorems 2 and 4 is equal to the set A_g of initial data for global solutions of the equation

$$\dot{x}(t) = F[x(t)] + g(x_t) \quad .$$

Proof

Proceeding as in the proof of Lemma 5, and denoting $x(t) = x(t; \phi, g)$,

$$z(t, \theta) = x(t+\theta) - \xi(\theta; x(t), F + K_g) \quad , \quad \text{for } t \geqslant 0, \ \theta \in [-r, 0] \ ,$$

we get, for $t + \theta \geqslant 0$ and $\theta \in [-r, 0]$,

$$z(t, \theta) = \int_t^{t+\theta} H_{F+K_g}(t+\theta - s, x(s)) \left[g(x_s) - g\left(\xi(\cdot\,; x(s), F + K_g)\right)\right] ds \ ,$$

where

$$\left|H_{F+K_g}(\theta, a)\right| \leqslant e^{-\alpha\theta} + \Delta \quad , \qquad \theta \in [-r, 0] \quad ,$$

and $\Delta > 0$ can be made arbitrarily small by making $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$

sufficiently small. Using this inequality in the preceding equation, we obtain

$$|z(t,\theta)| \leqslant \int_{t+\theta}^{t} (e^{-\alpha(t+\theta-s)} + \Delta)\, 2 \sup_{\psi\in C} |g(\psi)| \leqslant 2\left(\frac{e^{\alpha r}-1}{\alpha} + \Delta r\right) \sup_{\psi\in C} |g(\psi)| \quad ,$$

for $t \geqslant r$, $\theta \in [-r,0]$. By hypothesis, $\sup_{\psi\in C} |g(\psi)| < \delta\alpha/2(e^{\alpha r}-1)$, so that, for $\sup_{\psi\in B_0} |g(\psi)|$ and $\sup_{\psi\in V} |g'(\psi)|$ both sufficiently small, we have $|z(t,\theta)| < \delta$ for all $t \geqslant r$, $\theta \in [-r,0]$. This implies that $|x_t - \xi(\cdot\,; x(t), F+K_g)| < \delta$ for all $t \geqslant r$. Consequently, Lemma 5 can be applied to guarantee that there exists a $\sigma > 0$, independent of $\phi \in C$, such that

$$\inf_{\psi\in B_g} |x_t(\phi,g) - \psi| \leqslant \delta e^{-\sigma t} \quad , \qquad \text{for all } t \geqslant 2r,\ \phi\in C \ .$$

Now, we proceed in a way similar to the proof of Theorem 7. Let us assume that $\phi \in A_g$ and $\phi \notin B_g$, and denote $\nu = \inf_{\psi\in B_g} |\phi - \psi| > 0$. Take $T > 0$ so large that $\delta e^{-\sigma t} < \nu$. Since A_g is the set of initial data for global solutions, there exists a point $\zeta \in A_g$ such that $x_{T+r}(\zeta, g) = \phi$. Consequently, $\sup_{\psi\in B_g} |\phi - \psi| > \nu$, contradicting the assumption. This proves that $A_g \subset B_g$, the reverse inclusion results from Theorem 2. Q.E.D.

As a corollary, we obtain the following result about the smoothness and stability of the set of orbits of global solutions.

<u>Theorem 9</u>

Let us consider the equation

$$\dot{x}(t) = F[x(t)] + g(x_t) \ , \tag{45}$$

where $F: \mathbb{R}^n \to \mathbb{R}^n$ is a C^{k+1} $(k \geqslant 1)$ function with bounded first and second derivatives, and let $g: C \to \mathbb{R}^n$ be a bounded C^k function with first derivative bounded in a neighborhood $V = \left\{\phi \in C : \inf_{\psi\in B_0} |\phi - \psi| \leqslant \delta\right\}$ for some $\delta > 0$. Suppose that $\sup_{\phi \in C} |g(\phi)| < \delta\alpha/2(e^{\alpha r} - 1)$, where $\alpha = \sup_{a \in \mathbb{R}^n} |F'(a)|$. If both $\sup_{\phi \in B_0} |g(\phi)|$ and $\sup_{\phi \in V} |g'(\phi)|$ are sufficiently small, then the set A_g of initial data for global solutions of the equation (45) is a C^k-manifold diffeomorphic to $\mathbb{R}^n$, which is exponentially asymptotically stable under the flow of (45) and approaches A_0 in the Hausdorff metric, as $\sup_{\phi \in B_0} |g(\phi)|$ converges to zero.

7. Perturbations of Linear Retarded Equations

It was illustrated at the end of Section 2 how the results for perturbations of ODEs, presented in this paper, can be applied for perturbations of certain equations containing retardations (see Corollary 3 and Example 1). When the unperturbed equations are linear, the conditions for applicability of the techniques used in this paper can be somewhat relaxed. This is done in the present section.

If f is a continuous linear function from $C = C([-r,0];\mathbb{R}^n)$ into $\mathbb{R}^n$, then there exists an $n \times n$ matrix-valued function of bounded variation η defined on $[-r,0]$ such that

$$f(\phi) = \int_{-r}^{0} [d\eta(\theta)]\phi(\theta) \qquad \text{for all } \phi \in C .$$

We can separate the contribution of the atom at zero of the measure in the previous integral as

$$f(\phi) = E_f\phi(0) + \int_{-r}^{0} [d\eta_c(\theta)]\phi(\theta) \qquad \text{for all } \phi \in C ,$$

where E_f is an $n \times n$ matrix and η_c is continuous at zero.

Theorem 10

Let $f: C \to \mathbb{R}^n$ be a continuous linear function, and let $B_0 \subset C$ be a manifold where the flow of

$$\dot{x}(t) = f(x_t)$$

is given by a linear ODE in $\mathbb{R}^n$, $\dot{x} = Ax$, where A is an $n \times n$ real matrix. If $\alpha = \sup\{|Ax| : x \in \mathbb{R}^n, |x| = 1\}$ and $\beta = \sup\{|f(\phi) - E_f\phi(0)| : \phi \in C, |\phi| = 1\}$ satisfy $re^{\alpha r}\beta < 1$, and if $g: C \to \mathbb{R}^n$ is a uniformly lipschitzian function on a neighborhood of B_0, $V = \{\phi \in C : \inf_{\psi \in B_0} |\phi - \psi| \leqslant \delta\}$ for some δ satisfying $0 < \delta \leqslant +\infty$, with

$\sup_{\phi \in B_0} |g(\phi)|$ and the Lipschitz constant of g on V both sufficiently small, then there exists a C^0-manifold B_g where the flow of the perturbed equation

$$\dot{x}(t) = f(x_t) + g(x_t)$$

is given by a C^0 ODE in $\mathbb{R}^n$, $\dot{x} = Ax + K_g(x)$, such that the function $h_g(\theta, a) = \xi(\theta; a, AI+K_g) - e^{A\theta} a$ is continuous, bounded and globally lipschitzian in the second argument. There exists $\Delta_0 > 0$ such that B_g is the unique subset of V with these properties for which the Lipschitz constant of h_g relative to the second argument is smaller than Δ_0. Furthermore, B_g and the associated ODE depend continuously on g, in the sense of Theorem 2.

If, in addition to the above hypothesis, g is continuously differentiable on V, then B_g is an exponentially asymptotically stable C^1-manifold where the flow of the perturbed equation is given by a C^1 ODE in $\mathbb{R}^n$.

Proof

As in Corollary 3, we write the perturbed equation in the form

$$\dot{x}(t) = Ax(t) + \left\{ f(x_t) - Ax(t) + g(x_t) \right\} .$$

According to Theorem 2, in order to prove the theorem for $k = 0$, it is sufficient to show that the mapping

$$G_g(h)(\theta, a) = \int_0^\theta e^{A(\theta - s)} \Big\{ (f+g) \left[e^{A\cdot} \left(e^{As} a + h(s,a)\right) + h\left(\cdot, e^{As} a + h(s,a)\right)\right] - A\left(e^{As} a + h(s,a)\right)\Big\} \, ds$$

is a contraction mapping on the set $S(D, \Delta)$ defined in the proof of Theorem 2.

For all $\phi \in B_0$, we have $f(\phi) = A\phi(0)$. Thus,

$$f\left[e^{A\cdot}\left(e^{As}a + h(s,a)\right) + h\left(\cdot, e^{As}a + h(s,a)\right)\right] = A\left(e^{As}a + h(s,a)\right)$$
$$+ f\left[h\left(\cdot, e^{As}a + h(s,a)\right)\right] ,$$

and, consequently,

$$G_g(h)(\theta,a) = \int_0^\theta e^{A(\theta-s)}\left\{f\left[h\left(\cdot, e^{As}a + h(s,a)\right)\right]\right.$$
$$\left. + g\left[e^{A\cdot}\left(e^{As}a + h(s,a)\right) + h\left(\cdot, e^{As}a + h(s,a)\right)\right]\right\} ds .$$

If $\alpha \neq 0$, we can derive the following estimates (refer to the proof of Theorem 2 for the terms related to g) valid for all $\theta \in [-r, 0]$, $a, \bar{a} \in \mathbb{R}^n$, $h, \bar{h} \in S(D, \Delta)$ with $0 < D \leqslant \delta$, and where $L(g)$ denotes the Lipschitz constant of g on V:

$$\left|G_g(h)(\theta,a)\right| \leqslant \frac{e^{\alpha r}-1}{\alpha}\left[\sup_{\phi \in B_0} |g(\phi)| + D\,L(g)\right] + \frac{e^{\alpha r}-1}{\alpha} D\beta$$

$$\left|G_g(h)(\theta,a) - G_g(h)(\theta,\bar{a})\right| \leqslant \left\{\frac{1}{\alpha}\left(\alpha r\, e^{\alpha r} + (e^{\alpha r}-1)\Delta\right)(e^{\alpha r} + \Delta)\, L(g)\right.$$
$$\left. + \frac{\Delta}{\alpha}\left(\alpha r\, e^{\alpha r} + (e^{\alpha r}-1)\,\Delta\right)\beta\right\} |a - \bar{a}|$$

$$\left\|G_g(h) - G_g(\bar{h})\right\| < \left\{\frac{e^{\alpha r}-1}{\alpha}\left[e^{\alpha r} + 1 + \Delta\right] L(g) + \frac{(e^{\alpha r}-1)}{\alpha}(1+\Delta)\beta\right\} \|h - \bar{h}\|$$

and, with $\alpha = 0$, we have

$$\left|G_g(h)(\theta,a)\right| \leqslant r\left[\sup_{\phi \in B_0} |g(\phi)| + D\,L(g)\right] + rD\beta$$

$$\left|G_g(h)(\theta,a) - G_g(h)(\theta,\bar{a})\right| \leqslant \left\{r(1+\Delta)^2 L(g) + r\Delta(1+\Delta)\beta\right\} |a - \bar{a}|$$

$$\|G_g(h) - G_g(\bar{h})\| \leqslant \{r(2+\Delta)L(g) + r(1+\Delta)\beta\} \|h - \bar{h}\| .$$

In order to get G_g to be a contraction on $S(D,\Delta)$ for $\sup_{\phi \in B_0} |g(\phi)|$, $L(g)$ and $\Delta > 0$ all sufficiently small, it is sufficient to have $r e^{\alpha r}\beta < 1$. This is the condition contained in the hypothesis of the theorem. Proceeding as in the proof of Theorem 2, we get the first part of the stated result.

In order to prove the second part of the statement, one proceeds as in the proof of Theorem 4. Let h_g denote the fixed point of the mapping G_g in the set $S(D,\Delta)$, with $0 < D \leqslant \delta$ and $\Delta > 0$ chosen so that G_g is a contraction mapping on this set for $\sup_{\phi \in B_0} |g(\phi)|$ and $L(g)$ sufficiently small. Using the notation

$$z(\theta, a) = e^{A\theta} a + h_g(\theta, a) \quad , \qquad \theta \in [-r, 0],$$

we can write

$$h_g(\theta, a) = \int_0^\theta e^{A(\theta - s)} \left\{ f\left[h_g(\cdot, z(s,a))\right] + g\left[e^{A\cdot} z(s,a) + h_g(\cdot, z(s,a))\right] \right\} ds .$$

Let us assume that g is continuously differentiable. If the derivative $\partial h_g/\partial a$ exists, it must be a fixed point of the mapping

$$\left[T_g(v)b\right](\theta, a) = \int_0^\theta e^{A(\theta - s)} g'\left[z(\cdot, z(s,a))\right] \left[e^{A\cdot} + v(\cdot, z(s,a))\right] \left[e^{As} b + v(s,a)b\right] ds$$

$$+ \int_0^\theta e^{A(\theta - s)} f'\left[h_g(\cdot, z(s,a))\right] v(\cdot, z(s,a)) \left[e^{As} b + v(s,a)b\right] ds .$$

It is now easy to check that if $re^{\alpha r}\beta < 1$ and $L(g)$ is sufficiently small, then the sequence defined inductively by

$$v_i = 0 \qquad \text{and} \qquad v_{i+1} = T_g(v_i)$$

satisfies

$$\|v_i\| < 1/2 \qquad \text{and} \qquad \|v_{i+1} - v_i\| < \lambda \|v_i - v_{i-1}\| \quad ,$$

for some $\lambda < 1$. Consequently, $\{v_i\}$ is a Cauchy sequence converging to some v, as $i \to \infty$. Now, we need to prove that v is the derivative $\partial h_g / \partial a$. For this, we consider the function γ defined as in formula (29) of the proof of Theorem 4. In the same way as in the proof of that theorem, we obtain

$$\gamma(\epsilon) \leqslant c_1 \gamma(\epsilon) + c_2 \gamma\left((e^{\alpha r} + \Delta)\epsilon\right) + \mathcal{O}(1) \quad ,$$

as $\epsilon \to 0$, where $c_1, c_2 \geqslant 0$ are constants which can be made arbitrarily small by making $\sup_{\phi \in V} |g'(\phi)|$ sufficiently small. Proceeding as in the proof of Theorem 4, we can see that $\gamma(\epsilon) \to 0$ as $\epsilon \to 0$, proving that $h_g(\theta, a)$ is continuously differentiable in a. This implies that B_g is a C^1-manifold where the flow of the perturbed equation is given by a C^1 ODE in $\mathbb{R}^n$. The stability of B_g results from Theorem 6. Q.E.D.

Example 3

Let us consider the equation

$$\dot{x}(t) = x(t) - x(t-r)$$

with $r > 0$. The manifold $B_0 \subset C([-r, 0]; \mathbb{R}^n)$ consisting of the constant functions defined in $[-r, 0]$, is a manifold where the flow is given by the ODE $\dot{x} = Ax \equiv 0$. The values of α and β in the previous theorem are, for this example, $\alpha = 0$ and $\beta = 1$. Consequently, if $r < 1$ the theorem guarantees that for all globally lipschitzian functions $g : C \to \mathbb{R}^n$ with $\sup_{\phi \in B_0} |g(\phi)|$ and the Lipschitz constant both sufficiently small, there exists a manifold B_g with the properties in the first part of the statement of the theorem, where the flow of the perturbed equation

$$\dot{x}(t) = x(t) - x(t-r) + g(x_t)$$

is given by a C^0 ODE in $\mathbb{R}^n$.

If, in addition, g is continuously differentiable, the second part of the theorem guarantees that B_g is a C^1-manifold where the flow of the perturbed equation is given by a C^1 ODE in $\mathbb{R}^n$.

One may notice that $r = 1$ is the smallest value of $r = 0$ for which B_0 is not a hyperbolic invariant set. In fact, the linearized equation around B_0 has zero as a double characteristic value when $r = 1$ and as a single characteristic value when $r \neq 1$.

Example 4

Let us consider the FDE in $\mathbb{R}^n$

$$\dot{x}(t) = (1-e^{-r})^{-1} \int_{-r}^{0} x(t+\theta)d\theta \ .$$

For this equation, $B_0 = \left\{\phi \in C : \phi(\theta) = e^{\theta}a, \text{ with } a \in \mathbb{R}^n\right\}$ is an invariant manifold where the flow is given by the ODE $\dot{x} = x$. The constants α and β in Theorem 7 are, for this example, $\alpha = 1$ and $\beta = r/(1-e^{-r})$. Thus, if $r^2 e^{2r} < (e^r - 1)$, Theorem 7 guarantees that for all globally lipschitzian functions $g: C \to \mathbb{R}^n$ with $\sup_{\phi \in B_0} |g(\phi)|$ and Lipschitz constant both sufficiently small, there exists a manifold B_g, with the properties of the first part of the theorem, where the flow of the perturbed equation

$$\dot{x}(t) = (1-e^{-r})^{-1} \int_{-r}^{0} x(t+\theta)d\theta + g(x_t)$$

is given by a C^0 ODE in $\mathbb{R}^n$. If, in addition, g is continuously differentiable, then B_g is a C^1-manifold where the flow of the perturbed equation is given by a C^1 ODE in $\mathbb{R}^n$. It is easy to check that the inequality $r^2 e^{2r} < (e^r - 1)$ holds for r in a nonempty interval of positive numbers.

Remarks

1. The essence of the persistence and smoothness of invariant manifolds under small perturbations of dynamical systems, lies in the hyperbolicity of the unperturbed manifold. Consequently, the questions in this paper can be directly addressed by general results on perturbations of hyperbolic manifolds. However, the hyperbolicity condition requires a comparison of the rates of the flow on the manifold with the rates in normal directions to the manifold. In particular applications, these conditions may be difficult to verify. When this happens, the approach presented in this paper, though applying to a restricted class of problems, should be useful. In addition to this remark, one should note that the proofs are significantly easier using the techniques introduced here than the usual techniques for perturbations of general hyperbolic manifolds.

2. The ideas used in the present paper can be extended to the case where the FDE is defined in a compact manifold $M \subset \mathbb{R}^n$, instead of the whole space $\mathbb{R}^n$. One then considers invariant manifolds where the flow is given by an ODE in M. Results establishing the equality of the manifold B_g with the set of initial data for global solutions, as in Theorems 7 and 8, can be somewhat simplified, in this context, since the boundedness of the perturbation g follows from the compactness of the manifold M.

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