UNIFORM DECAY OF WEAK SOLUTIONS TO A VON KÁRMÁN PLATE WITH NONLINEAR BOUNDARY DISSIPATION

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Uniform Decay of Weak Solutions to a von Kármán Plate With Nonlinear Boundary Dissipation

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Abstract

Asymptotic behavior of solutions to a von Kármán model with $\gamma \equiv 0$, i.e., without accounting for rotational forces, is considered. It is shown that in the presence of boundary damping all weak solutions decay to zero in the energy norm.

1 Introduction

1.1 Statement of the Problem

Let Ω be an open bounded domain in R^2 with a sufficiently smooth (e.g., C^∞) boundary, Γ. In Ω, we consider the following von Kármán system in the variables $w(t,x)$ and $\chi(w(t,x))$ with nonlinear feedback controls, f and g:

$$w_{tt} + \Delta^2 w + b(x)w_t = [w, \chi(w)] \quad \text{in } Q_\infty = (0,\infty)\times\Omega \tag{1.1.a}$$

$$\left.\begin{aligned} w(0,\cdot) &= w_0 \\ w_t(0,\cdot) &= w_1 \end{aligned}\right\} \quad \text{in } \Omega \tag{1.1.b}$$

$$\Delta w + (1-\mu)B_1 w = -f(\frac{\partial}{\partial\nu} w_t) \quad \text{on } \Sigma_\infty = (0,\infty)\times\Gamma \tag{1.1.c}$$

$$\frac{\partial}{\partial\nu}\Delta w + (1-\mu)B_2 w - w = g(w_t) \quad \text{on } \Sigma_\infty = (0,\infty)\times\Gamma, \tag{1.1.d}$$

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where $b(x) \in L^\infty(\Omega)$ satisfies $b(x) > 0$ *a.e.* in Ω, $0 < \mu < \frac{1}{2}$ is Poisson's ratio, the operators B_1 and B_2 are given by

$$\begin{aligned} B_1 w &= 2n_1 n_2 w_{xy} - n_1^2 w_{yy} - n_2^2 w_{xx} \\ B_2 w &= \tfrac{\partial}{\partial \tau}[(n_1^2 - n_2^2) w_{xy} + n_1 n_2 (w_{yy} - w_{xx})], \end{aligned} \tag{1.2}$$

where τ is the tangential direction. The controls, f and g are continuous, monotone functions and are subject to the following constraints:

$$\left.\begin{array}{ll} f(s)s > 0 & for\ s \neq 0 \\ g(s)s > 0 & for\ s \neq 0 \\ m|s| \leq |f(s)| \leq M|s| & for\ |s| > 1 \\ m|s| \leq |g(s)| \leq M|s| & for\ |s| > 1. \end{array}\right\} (H)$$

Note that no assumptions are made on the growth of f and g at the origin.

In (1.1), the Airy's stress function, $\chi(w)$, satisfies the system of equations

$$\left.\begin{array}{rll} \Delta^2 \chi & = -[w, w] & in\ \Omega \\ \chi & = \frac{\partial}{\partial \nu}\chi = 0 & on\ \Gamma, \end{array}\right\} \tag{1.3}$$

where

$$[\phi, \psi] = \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 \psi}{\partial x^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 \psi}{\partial x \partial y}. \tag{1.4}$$

Define the bilinear form

$$a(w, v) = \int_\Omega (\Delta w \Delta v + (1-\mu)(2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}))d\Omega. \tag{1.5}$$

We define the energy functional by

$$E_w(t) = \frac{1}{2}\int_\Omega \{|w_t(t)|^2 + |\Delta\chi(w(t))|^2\}d\Omega + \frac{1}{2}a(w(t), w(t)) + \frac{1}{2}\int_\Gamma w^2(t)d\Gamma \equiv E_{w,1}(t) + E_{w,2}(t), \tag{1.6}$$

where $E_{w,2}(t)$ is defined by

$$E_{w,2}(t) \equiv \frac{1}{2}\int_{\Omega} |\Delta\chi(w(t))|^2 d\Omega. \tag{1.7}$$

It is known (see [6], [11]) that $a(w,w) + \int_{\Gamma} w^2 d\Gamma$ is equivalent to the $H^2(\Omega)$ norm and

$$E_{w,2}(t) \leq C\|w(t)\|^4_{H^2(\Omega)}. \tag{1.8}$$

In view of this, the associated space of finite energy is $\mathcal{H} \equiv H^2(\Omega) \times L_2(\Omega)$.

Our goal is to show that the boundary controls, f and g, cause the energy of our system, (1.6), to decay uniformly with respect to the initial energy as time increases.

In recent years, the problem of boundary stabilization has attracted considerable attention (see [2], [5], [6], [7], [11], [13], [14], [9] and references therein). Although literature exists on a variety of models, we shall focus our attention on the results most pertinent to model (1.1).

In the context of control theory and, in particular, stabilization theory, the von Kármán model was introduced for the first time in [6], where exponential decay rates for the energy function, (1.6), associated with the solutions to (1.1) with $b \equiv 0$ and *linear* feedbacks were established. Uniform decay rates for von Kármán models accounting for rotational forces are given in [2], [5]. This latter problem, due to the higher regularity of the velocity ($w_t \in H^1(\Omega)$) is different from (1.1), both in terms of the results and the techniques used. Thus, the closest to our problem are the exponential decay results of [6] which hold under the following hypotheses:

(i.) Geometric conditions of "star-shaped" type are imposed on Ω,

(ii.) the solution to (1.1) for which decay estimates apply are assumed to be regular, i.e.,

$w \in C(0,T;H^4(\Omega))$, $w_t \in C(0,T;H^2(\Omega))$,

(iii.) boundary feedbacks are *linear* and are of the following form:

$$\begin{aligned} \Delta w + (1-\mu)B_1 w &= -\alpha(h\cdot\nu)\tfrac{\partial}{\partial\nu}w \\ \tfrac{\partial}{\partial\nu}\Delta w + (1-\mu)B_2 w &= \lambda(h\cdot\nu)w_t + \beta(h\cdot\nu)w - \alpha\tfrac{\partial}{\partial\tau}((h\cdot\nu)\tfrac{\partial}{\partial\tau}w), \end{aligned} \tag{1.9}$$

where $h \equiv x - x_0$ for some $x_0 \in R^2$ and the constants $\alpha, \lambda, \beta > 0$ are such that $0 < \alpha < \alpha_0$, $0 < \beta < \beta_0(\alpha)$

for some $\alpha_0 > 0$, $\beta_0(\alpha) > 0$.

Geometric conditions imposed on Ω were instrumental in [6] to control the sign of certain boundary integrals which could not be bounded by the energy. The regularity of solutions in point *(ii.)* was needed in [6] to justify partial differential equation calculations which otherwise are only formal. While the issue of existence of regular solutions to (1.1) has been recently resolved in [4] (in the case of *homogeneous* boundary conditions see [16], [15], [3]); this still does not settle the problem of how to obtain decay rates valid for all weak solutions (whose existence in the case of linear boundary damping was established in [6]). One of the obstacles was related to a potential lack of a uniqueness result for (1.1) with initial data in the basic energy space $H^2(\Omega) \times L_2(\Omega)$ (see [6], p. 113). Indeed, since weak solutions may not depend continuously (in $H^2(\Omega) \times L_2(\Omega)$) on the initial data in $H^2(\Omega) \times L_2(\Omega)$, the usual density argument used on decay rates obtained for regular solutions becomes inapplicable. Similar difficulty has been noted (see [14]) when exponential decay rates are proven for regular solutions only. (It should be noted that the situation is different when one considers the von Kármán model accounting for rotational forces, where the uniqueness result has been known.)

The main goal of this paper is to prove that the energy of the system (1.1), with *nonlinear* boundary feedback, decays uniformly for all *weak* solutions of finite energy (i.e., in $H^2(\Omega) \times L_2(\Omega)$). Moreover, we shall show this decay result holds *without any geometric conditions* imposed on Ω. Thus, our results dispense entirely with all three hypotheses listed above in points *(i.)-(iii.)* and assumed in previous work on this problem. It should, however, be noted that the boundary conditions in (1.1) require *velocity* feedback in the first boundary condition (1.1.c). This is in contrast to (1.9) where only *position* feedback is used in this boundary condition. The presence of velocity feedback in (1.1) is needed at the level of the microlocal analysis result (see Proposition 4.2) which, in turn, allows us to dispense with the geometric conditions. The proof of our results relies on a combination of the techniques of [5] (where uniform decay rates were obtained for the model accounting for rotational inertia) with a recent regularity result (see Lemma 2.1) on the smoothness of the Airy's stress function. In fact, by using this result, one can prove that weak solutions to (1.1) are *unique*.

1.2 Statement of Main Results

We shall start with a well-posedness result for (1.1) which is proven in section 2. Denote $\Sigma_T \equiv \Gamma \times (0,T)$, $Q_T \equiv \Omega \times (0,T)$.

Theorem 1.1 *For any $w_0 \in H^2(\Omega)$, $w_1 \in L_2(\Omega)$, and $T > 0$, there exists a unique weak solution to (1.1), $w \in C(0,T; H^2(\Omega)) \cap C^1(0,T; L_2(\Omega))$, such that $\frac{\partial}{\partial \nu} w_t$, $w_t \in L_2(\Sigma_T)$ and*

$$||w_t||_{L_2(\Sigma_T)} + ||\frac{\partial}{\partial \nu} w_t||_{L_2(\Sigma_T)} \leq C(||w_0||_{H^2(\Omega)}, ||w_1||_{L_2(\Omega)}), \tag{1.10}$$

where $C(u,v)$ denotes a function which is bounded for bounded values of the arguments.

Remark 1.1: Notice that the regularity property in (1.10) does not follow from a priori interior regularity of w (i.e., $w_t \in L_2(\Omega)$). It is an independent regularity result.

Remark 1.2: The uniqueness statement in Theorem 1.1 is new even in the case of zero boundary conditions (see [7] and [11]).

To state our stability result, we will need the following notation. Let the function $h(x)$ be defined by:

$$h(x) \equiv h_1(x) + h_2(x), \tag{1.11}$$

where $h_i(x)$ are concave, strictly increasing functions with $h_i(0) = 0$ such that

$$\begin{aligned} h_1(sf(s)) &\geq s^2 + f^2(s) \quad |s| \leq 1 \\ h_2(sg(s)) &\geq s^2 + g^2(s) \quad |s| \leq 1. \end{aligned} \tag{1.12}$$

(Such functions can be easily constructed. See [8].) Then $h(x)$ enjoys the same properties, i.e., it is concave, strictly increasing, and $h(0) = 0$. Define

$$\tilde{h}(x) \equiv h(\frac{x}{mes\ \Sigma_T}). \tag{1.13}$$

Since $\tilde{h}$ is monotone increasing, for every $c \geq 0$, $cI + \tilde{h}$ is invertible. Setting

$$p(x) \equiv (cI + \tilde{h})^{-1}(Kx), \tag{1.14}$$

where K is a positive constant, we see that p is a positive, continuous, strictly increasing function with $p(0) = 0$.

We are now in a position to state our stability result.

Theorem 1.2 *Assume hypothesis (H) holds. Let w be the solution to system (1.1). Then for some $T_0 > 0$,*

$$E_w(t) \leq \mathcal{S}(\frac{t}{T_0} - 1) \; for \; t > T_0, \tag{1.15}$$

where $\mathcal{S}(t) \to 0$ as $t \to \infty$ and is the solution (contraction semigroup) of the differential equation

$$\begin{cases} \frac{d}{dt}\mathcal{S}(t) + q(\mathcal{S}(t)) = 0 \\ \mathcal{S}(0) = E_w(0), \end{cases} \tag{1.16}$$

and $q(x)$ is given by

$$q(x) \equiv x - (I + p)^{-1}(x) \; for \; x > 0. \tag{1.17}$$

In this case, the constant K will generally depend on $E_w(0)$ and the constant $c = \frac{1}{mes \Sigma_T}(m^{-1} + M)$.

Remark 1.3: It is interesting to note the role played by the light internal damping represented by the term $b(x)w_t$. This damping alone (without boundary dissipation) would not suffice to achieve *uniform* decay of energy. ("Strong" internal damping which would cause the energy to decay *uniformly* would have to be coercive, i.e., it should satisfy $b(x) \geq b_0 > 0$.) On the other hand, the presence of this mild damping term seems to be essential in proving the effectiveness of the boundary dissipation. Thus, the combination of both the light interior damping and the boundary dissipation provides the desired energy decay. We note that the presence of light interior damping is physically motivated, since most vibrating materials possess some degree of interior damping.

Remark 1.4: The result of Theorem 1.2 remains valid if the term $b(x)w_t$ is replaced by a more general "light" damping of the form Dw_t, where the operator $D \in \mathcal{L}(L_2(\Omega))$ is self-adjoint, nonnegative with the property $Dz = 0 \Longrightarrow z = 0$.

Remark 1.5: If the domain Ω is "star-shaped", then the result of Theorem 1.2 holds with $b \equiv 0$ in (1.1). Indeed, the presence of the term bw_t is needed only at the level of the uniqueness argument in section 4.2. On the other hand, in the case of "star-shaped" domains, this uniqueness property holds true without any interior damping. This last assertion follows by adapting multiplier arguments in section 5.2 of [6].

2 Proof of Theorem 1.1

We note that the existence, uniqueness and regularity of solutions to (1.1) under stronger differentiability hypotheses imposed on the nonlinear functions f and g was proven in [4] by using a contraction mapping argument. Below, we shall give a proof based on a nonlinear Galerkin method which requires only hypothesis (H). One should note, however, that in both cases a critical role is played by the regularity result of Lemma 2.1. The first step in the proof is standard and based on the application of a nonlinear Galerkin method (see [11]). Let $V_h \subset H^2(\Omega)$ be a finite dimensional subspace of $H^2(\Omega)$ with the following approximation property. For every $u \in H^2(\Omega)$, there exists $u_h \in V_h$ such that $||u - u_h||_{H^2(\Omega)} \to 0$ as $h \to 0$. We consider the following semidiscrete equations in the variable $w^h(t)$.

$$\begin{aligned} \int_\Omega w^h_{tt}\phi^h d\Omega + \int_\Omega \Delta w^h \Delta\phi^h d\Omega \quad &+ \int_\Omega bw^h_t\phi^h d\Omega + \int_\Gamma f(\tfrac{\partial}{\partial\nu}w^h_t)\tfrac{\partial}{\partial\nu}\phi^h d\Gamma + \int_\Gamma g(w^h_t)\phi^h d\Gamma \\ &= \int_\Omega [w^h, \chi(w^h)]\phi^h d\Omega \qquad \forall \phi^h \in V_h, \end{aligned} \tag{2.1}$$

with $w^h(0) = w_{0h}$, $w^h_t(0) = w_{1h}$, where $||w_{0h} - w_0||_{H^2(\Omega)} \to 0$, $||w_{1h} - w_1||_{L_2(\Omega)} \to 0$.

Setting $\phi^h = w^h_t$ in (2.1) yields a familiar a priori bound:

$$E_{w^h}(t) + \int_0^t \int_\Gamma f(\tfrac{\partial}{\partial\nu}w^h_t)\tfrac{\partial}{\partial\nu}w^h_t d\Gamma dt + \int_0^t \int_\Gamma g(w^h_t)w^h_t d\Gamma dt \le E_{w^h}(0). \tag{2.2}$$

By combining (2.2) with (1.8) and hypothesis (H), we infer the a priori bound

$$||w_t^h(t)||^2_{L_2(\Omega)} + ||w^h(t)||^2_{H^2(\Omega)} \le C(||w_0||_{H^2(\Omega)}, ||w_1||_{L_2(\Omega)}). \tag{2.3}$$

By standard ordinary differential equation techniques, we obtain the existence and uniqueness of the solution $(w^h, w_t^h) \in C(0,T; H^2(\Omega) \times L_2(\Omega))$.

Let w^{h_1} and w^{h_2} be two solutions of (2.1). Writing down the equation for the difference $\tilde{w}^h \equiv w^{h_1} - w^{h_2}$ and using as a test function $\phi^h = \tilde{w}_t^h$, we obtain

$$\begin{aligned} ||\tilde{w}^h(t)||^2_{H^2(\Omega)} + ||\tilde{w}_t^h(t)||^2_{L_2(\Omega)} \quad & + \int_0^t \int_\Gamma [f(\tfrac{\partial}{\partial\nu} w_t^{h_1}) - f(\tfrac{\partial}{\partial\nu} w_t^{h_2})] \tfrac{\partial}{\partial\nu} \tilde{w}_t^h d\Gamma dt + \int_0^t \int_\Gamma [g(w_t^{h_1}) - g(w_t^{h_2})] \tilde{w}_t^h d\Gamma dt \\ & \le [||w^{h_1}(0) - w^{h_2}(0)||^2_{H^2(\Omega)} + ||w_t^{h_1}(0) - w_t^{h_2}(0)||^2_{L_2(\Omega)} \\ & \quad + \int_0^t \int_\Omega \{[\tilde{w}^h, \chi(w^{1h})]\tilde{w}_t^h + [w^{h_2}, \chi(w^{h_1}) - \chi(w^{h_2})]\tilde{w}_t^h\} d\Omega dt. \end{aligned} \tag{2.4}$$

The following result is critical.

Lemma 2.1 *(See Theorem 5.1 in [4].) Let* $w \in H^2(\Omega)$*. Then*

$$||\chi(w)||_{W^2_\infty(\Omega)} \le C||w||^2_{H^2(\Omega)}. \tag{2.5}$$

Remark 2.1: We note that a standard result (see [11], [6]) says that if $w \in H^2(\Omega)$, then $\chi(w) \in H^{3-\epsilon}(\Omega)$. This regularity does not imply that $\chi(w) \in W^2_\infty(\Omega)$.

From Lemma 2.1 and (2.3), we obtain

$$||\chi(w^{1h}(t))||_{W^2_\infty(\Omega)} \le C(||w_0||_{H^2(\Omega)}, ||w_1||_{L_2(\Omega)}). \tag{2.6}$$

From Theorem 5.1 in [4] and (2.3), it follows also that

$$\begin{aligned} ||\chi(w^{h_1}(t)) - \chi(w^{h_2}(t))||_{W^2_\infty(\Omega)} & \le C(||w^{h_1}(t)||_{H^2(\Omega)}, ||w^{h_2}(t)||_{H^2(\Omega)})||w^{h_1}(t) - w^{h_2}(t)||_{H^2(\Omega)} \\ & \le C(||w_0||_{H^2(\Omega)}, ||w_1||_{L_2(\Omega)})||\tilde{w}^h(t)||_{H^2(\Omega)}. \end{aligned} \tag{2.7}$$

Using (2.6) and (2.7) in (2.4) followed by Gronwall's inequality yields, for $t \le T$,

$$\|\tilde{w}^h(t)\|^2_{H^2(\Omega)} + \|\tilde{w}^h_t\|^2_{L_2(\Omega)} \quad + \int_0^t \int_\Gamma [f(\tfrac{\partial}{\partial\nu} w_t^{h_1}) - f(\tfrac{\partial}{\partial\nu} w_t^{h_2})] \tfrac{\partial}{\partial\nu}\tilde{w}^h_t d\Gamma dt + \int_0^t \int_\Gamma [g(w_t^{h_1}) - g(w_t^{h_2})]\tilde{w}^h_t d\Gamma dt$$
$$\le C_T(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L_2(\Omega)})[\|w_0^{h_1} - w_0^{h_2}\|^2_{H^2(\Omega)} + \|w_1^{h_1} - w_1^{h_2}\|^2_{L_2(\Omega)}]. \tag{2.8}$$

Since f and g are monotone increasing, we infer from (2.8) that there exists $w^* \in C(0,T;H^2(\Omega)) \cap C^1(0,T;L_2(\Omega))$ such that

$$\begin{aligned} w^h &\to w^* \quad \textit{in } C(0,T;H^2(\Omega)) \\ w^h_t &\to w^*_t \quad \textit{in } C(0,T;L_2(\Omega)). \end{aligned} \tag{2.9}$$

Also, from (2.2), hypothesis (H) and (2.9), one can show that on a subsequence (denoted by the same symbol for simplicity)

$$\begin{aligned} \tfrac{\partial}{\partial\nu} w^h_t|_\Gamma &\to \tfrac{\partial}{\partial\nu} w^*_t|_\Gamma \quad \textit{weakly in } L_2(\Sigma_T) \\ w^h_t|_\Gamma &\to w^*_t|_\Gamma \qquad \textit{weakly in } L_2(\Sigma_T). \end{aligned} \tag{2.10}$$

Convergence in (2.9) together with the continuity of the Airy's stress function $\chi(w)$ in $H^{-2}(\Omega)$ with respect to week topology in $H^2(\Omega)$ allow us to deduce (in a standard manner) that w^* satisfies

$$\begin{aligned} \lim_{h\to 0} \int_\Omega (w^h_{tt}\phi^h + \Delta w^h \Delta\phi^h + b w^h_t \phi^h) d\Omega &= \int_\Omega (w^*_{tt}\phi^* + \Delta w^* \Delta \phi^* + b w^*_t \phi^*) d\Omega, \\ \lim_{h\to 0} \int_\Omega [w^h, \chi(w^h)]\phi^h d\Omega &= \int_\Omega [w^*, \chi(w^*)]\phi d\Omega, \end{aligned} \qquad \textit{in } \mathcal{D}'(0,T) \tag{2.11}$$

for all $\phi \in H^2_0(\Omega)$ and $\phi^h \to \phi$ in $H^2_0(\Omega)$. The main point is to pass to the limit on nonlinear boundary conditions. To accomplish this, we recall (2.8) which, together with monotonicity of f and g, in particular, yields

$$\begin{aligned} \lim_{h_1,h_2\to 0} \int_0^t \int_\Gamma [f(\tfrac{\partial}{\partial\nu} w_t^{h_1}) - f(\tfrac{\partial}{\partial\nu} w_t^{h_2})] \tfrac{\partial}{\partial\nu} \tilde{w}^h_t d\Gamma dt &\to 0 \\ \lim_{h_1,h_2\to 0} \int_0^t \int_\Gamma [g(w_t^{h_1}) - g(w_t^{h_2})] \tilde{w}^h_t d\Gamma dt &\to 0. \end{aligned} \tag{2.12}$$

From (2.10) combined with hypothesis (H), we infer

$$\begin{aligned} f(\tfrac{\partial}{\partial\nu} w^h_t) &\to f_0 \quad \textit{weakly in } L_2(\Sigma_T) \\ g(w^h_t) &\to g_0 \quad \textit{weakly in } L_2(\Sigma_T). \end{aligned} \tag{2.13}$$

Now, by using the monotonicity of f, (2.10), and (2.13) together with Lemma 2.1 in [1], we deduce that

$$\begin{aligned} f_0 &= f(\tfrac{\partial}{\partial\nu}w_t^*), \\ g_0 &= g(w_t^*), \end{aligned} \tag{2.14}$$

which allows us to pass to the (weak) limit on the boundary conditions and after taking into account (2.1) and (2.11) yields

$$\int_0^T\int_\Omega(w_{tt}^*\phi+\Delta w^*\Delta\phi+bw_t^*\phi)d\Omega dt+\int_0^T\int_\Gamma f(\tfrac{\partial}{\partial\nu}w_t^*)\tfrac{\partial}{\partial\nu}\phi d\Gamma dt+\int_0^T g(w_t^*)\phi d\Gamma dt=\int_0^T\int_\Omega[w^*,\chi(w^*)]\phi d\Omega dt,$$
$$\forall\phi\in\mathcal{D}'(0,T;H^2(\Omega)). \tag{2.15}$$

This proves that w^* is a weak solution of (1.1). Since, as it is shown in Lemma 3.2, the estimate identical to this in (2.8) is satisfied with $\tilde{w}=w^1-w^2$ (where w^1, w^2 are two (potentially) different solutions corresponding to the same initial data w_0, w_1), the uniqueness of w^* follows. Regularity in (1.10) follows from (2.10). □

3 Preliminary Energy Estimates

Our goal is to prove energy decay rates for problem (1.1). In order to do this, one needs to perform certain partial differential equation calculations on the problem. These calculations require regularity of the solutions higher than is available from Theorem 1.1. Since our nonlinear problem may not have a sufficiently regular solution (even if the initial data are smooth), we resort to an approximation argument which was used earlier in the context of wave equations in [8]. In fact, the idea here is to approximate solutions to the nonlinear problem (1.1) by solutions to different (linear) problems. Since this linear problem admits regular solutions for smooth initial data, the partial differential equation calculations can be performed on this problem. Final passage to the limit on the approximation problem allows us to obtain needed energy identities for the original nonlinear problem.

To follow our program, we start by defining the following approximations. To do this, we note the following corollary of Theorem 1.1.

Corollary 3.1 *Let w be any solution of (1.1) with regularity property (1.10). Then*

$$f(\frac{\partial}{\partial \nu} w_t) \in L_2(\Sigma_T) \tag{3.1}$$

and

$$g(w_t) \in L_2(\Sigma_T). \tag{3.2}$$

Proof of Corollary 3.1: Follows from hypothesis (H) together with (1.10). □

Let w be the solution of the original problem (1.1). By using the regularity properties in (1.10), Lemma 2.1 (which gives $[w, \chi(w)] \in L_2(\Omega)$ for $w \in H^2(\Omega)$), (3.1) and (3.2), along with dense imbeddings of appropriate (see below) Sobolev spaces, we are in a position to define the following sequences of functions:

$$f_n \in H^{1,1}(Q_T); \quad ||f_n - [w, \chi(w)]||_{L_2(Q_T)} \longrightarrow 0 \tag{3.3}$$

$$f_{1n} \in H^{1,1}(\Sigma_T); \quad ||f_{1n} - f(\tfrac{\partial}{\partial \nu} w_t)||_{L_2(\Sigma_T)} \longrightarrow 0 \tag{3.4}$$

$$f_{2n} \in H^{1,1}(\Sigma_T); \quad ||f_{2n} - g(w_t)||_{L_2(\Sigma_T)} \longrightarrow 0 \tag{3.5}$$

$$\alpha_n \in H^{1,1}(\Sigma_T); \quad ||\alpha_n - \tfrac{\partial}{\partial \nu} w_t||_{L_2(\Sigma_T)} \longrightarrow 0 \tag{3.6}$$

$$\beta_n \in H^{1,1}(\Sigma_T); \quad ||\beta_n - w_t||_{L_2(\Sigma_T)} \longrightarrow 0, \tag{3.7}$$

where, we recall, $Q_T \equiv \Omega \times (0,T)$ and $\Sigma_T \equiv \Gamma \times (0,T)$. We consider the following approximating problem:

$$\begin{cases} w_{n,tt} + \Delta^2 w_n + b w_{n,t} = f_n \\ \quad w_n(0) = w_{n,0}; \quad w_{n,t}(0) = w_{n,1} \\ \Delta w_n + (1-\mu) B_1 w_n + \frac{\partial}{\partial \nu} w_{n,t}|_\Gamma = -f_{1n} + \alpha_n \\ \frac{\partial}{\partial \nu} \Delta w_n + (1-\mu) B_2 w_n - w_n - w_{n,t}|_\Gamma = f_{2n} - \beta_n, \end{cases} \tag{3.8}$$

where

$$||w_{n,0} - w_0||_{H^2(\Omega)} \rightarrow 0; \quad ||w_{n,1} - w_1||_{L_2(\Omega)} \rightarrow 0, \tag{3.9}$$

and $(w_{n,0}, w_{n,1}) \in \mathcal{D}$, where $\mathcal{D}$, as dense set of $\mathcal{H}$, consists of $w_{n,0} \in H^4(\Omega)$, $w_{n,1} \in H^2(\Omega)$, where $w_{n,0}, w_{n,1}$

satisfy the appropriate compatibility conditions on the boundary. By standard linear semigroup methods, one easily shows that the linear problem, (3.8), admits a classical solution,

$$w_n \in C(0,T;H^4(\Omega)) \cap C^1(0,T;H^2(\Omega)). \tag{3.10}$$

The following proposition plays an important role in our development.

Proposition 3.1 *Let w_n (respectively, w) be a solution of (3.8) (respectively, (1.1)). Then as $n \to \infty$, the following convergence holds.*

$$w_n \to w \text{ in } C(0,T;H^2(\Omega)) \cap C^1(0,T;L_2(\Omega)) \tag{3.11}$$

$$\left.\begin{array}{l} \frac{\partial}{\partial\nu} w_{n,t}|_\Gamma \to \frac{\partial}{\partial\nu} w_t \\ w_{n,t}|_\Gamma \to w_t \end{array}\right\} \quad \text{in } L_2(\Sigma_T). \tag{3.12}$$

Proof: Consider the equation satisfied by the difference $w_n - w_m$. Taking the inner product of this equation with $w_{n,t} - w_{m,t}$ and integrating the result from 0 to T yields

$$\begin{aligned} E_{w_n - w_m,1}(T) \quad &+ \int_0^T \int_\Gamma [\hat{w}_t^2 + |\tfrac{\partial}{\partial\nu}\hat{w}_t|^2] d\Gamma dt + \int_0^T \int_\Omega b\hat{w}_t^2 d\Omega dt \\ &= \int_0^T \int_\Omega (f_n - f_m)\hat{w}_t d\Omega dt + \int_0^T \int_\Gamma (f_{1n} + \alpha_n - f_{1m} - \alpha_m)\tfrac{\partial}{\partial\nu}\hat{w}_t d\Gamma dt \\ &\quad + \int_0^T \int_\Gamma (f_{2n} + \beta_n - f_{2m} - \beta_m)\hat{w}_t d\Gamma dt + E_{w_n - w_m,1}(0), \end{aligned} \tag{3.13}$$

where $\hat{w} \equiv w_n - w_m$. Hence, for all $t \le T$,

$$\begin{aligned} C_0[||\hat{w}(t)||^2_{H^2(\Omega)} \quad &+ ||\hat{w}_t(t)||^2_{L_2(\Omega)}] + ||\tfrac{\partial}{\partial\nu}\hat{w}_t||^2_{L_2(\Sigma_T)} + ||\hat{w}_t||^2_{L_2(\Sigma_T)} \\ &\le \tfrac{1}{2}||f_n - f_m||^2_{L_2(Q_T)} + \tfrac{1}{2}||\hat{w}_t(t)||^2_{L_2(Q_T)} \\ &\quad + \tfrac{1}{2}||f_{1n} - \alpha_n - f_{1m} + \alpha_m||^2_{L_2(\Sigma_T)} + \tfrac{1}{2}||\tfrac{\partial}{\partial\nu}\hat{w}_t||^2_{L_2(\Sigma_T)} \\ &\quad + \tfrac{1}{2}||f_{2n} - \beta_n - f_{2m} + \beta_m||^2_{L_2(\Sigma_T)} + \tfrac{1}{2}||\hat{w}_t||^2_{L_2(\Sigma_T)} + E_{w_n - w_m,1}(0), \end{aligned} \tag{3.14}$$

and by Gronwall's inequality

$$\begin{aligned}
\|\hat{w}(t)\|^2_{H^2(\Omega)} & +\|\hat{w}_t(t)\|^2_{L_2(\Omega)} + \|\tfrac{\partial}{\partial\nu}\hat{w}_t\|^2_{L_2(\Sigma_T)} + \|\hat{w}_t\|^2_{L_2(\Sigma_T)} \\
& \leq C_T[\|f_n - f_m\|^2_{L_2(0,T;L_2(\Omega))} + \|f_{1n} - f_{1m}\|^2_{L_2(\Sigma_T)} + \|\alpha_n - \alpha_m\|^2_{L_2(\Sigma_T)} \\
& \quad +\|f_{2n} - f_{2m}\|^2_{L_2(\Sigma_T)} + \|\beta_n - \beta_m\|^2_{L_2(\Sigma_T)} + E_{w_n - w_m,1}(0)] \\
& \longrightarrow 0 \ as\ n \rightarrow \infty,
\end{aligned} \tag{3.15}$$

where the limit follows by using (3.3)-(3.7) and (3.9). Thus, by Corollary 3.1,

$$\begin{aligned}
w_n \rightarrow w^* \ in\ C(0,T;H^2(\Omega)) \cap C^1(0,T;L_2(\Omega)) \\
\tfrac{\partial}{\partial\nu} w_{n,t}|_\Gamma \rightarrow \tfrac{\partial}{\partial\nu} w^*_t|_\Gamma \ in\ L_2(\Sigma_T) \\
w_{n,t} \rightarrow w^*_t \ in\ L_2(\Sigma_T).
\end{aligned} \tag{3.16}$$

This allows us to pass with the limit on the linear equation, (3.8). We obtain

$$\begin{cases}
w^*_{tt} + \Delta^2 w^* + b w^*_t = [w, \chi(w)] \\
w^*(0) = w_0; \quad w^*_t(0) = w_1 \\
\Delta w^* + (1-\mu) B_1 w^* + \tfrac{\partial}{\partial\nu} w^*_t|_\Gamma = -f(\tfrac{\partial}{\partial\nu} w_t) + \tfrac{\partial}{\partial\nu} w_t \\
\tfrac{\partial}{\partial\nu}\Delta w^* + (1-\mu) B_2 w^* - w^* - w^*_t|_\Gamma = g(w_t) - w_t.
\end{cases} \tag{3.17}$$

Since w satisfies (3.17) and the solution to (3.17) is *unique,* we infer that $w \equiv w^*$ and

$$\begin{aligned}
w_n \rightarrow w \ in\ C(0,T;H^2(\Omega)) \cap C^1(0,T;L_2(\Omega)) \\
\tfrac{\partial}{\partial\nu} w_{n,t}|_\Gamma \rightarrow \tfrac{\partial}{\partial\nu} w_t|_\Gamma \ in\ L_2(\Sigma_T) \\
w_{n,t} \rightarrow w_t \ in\ L_2(\Sigma_T),
\end{aligned} \tag{3.18}$$

as desired. □

Now we are in a position to prove the fundamental energy relation for problem (1.1).

Lemma 3.1 *(Energy Identity) Let w be the solution to (1.1). Then the following energy identity holds.*

$$E_w(T) - E_w(0) + \int_{\Sigma_T} [g(w_t)w_t + f(\frac{\partial}{\partial\nu}w_t)\frac{\partial}{\partial\nu}w_t]d\Gamma dt + \int_{Q_T} b(x)w_t^2 d\Omega dt = 0. \tag{3.19}$$

Proof: We first prove this energy identity for the solution, w_n, of the approximating problem, (3.8). Indeed, by applying a standard energy argument to (3.8), we obtain

$$\begin{aligned} E_{w_n,1}(T) - E_{w_n,1}(0) \quad & + \int_{\Sigma_T} |\tfrac{\partial}{\partial\nu} w_{n,t}|^2 d\Gamma dt + \int_{\Sigma_T} (w_{n,t})^2 d\Gamma dt + \int_{Q_T} b(x) w_{n,t}^2 d\Omega dt \\ & = \int_{Q_T} f_n w_{n,t} d\Omega dt - \int_{\Sigma_T} (f_{1n} - \alpha_n) \tfrac{\partial}{\partial\nu} w_{n,t} d\Gamma dt - \int_{\Sigma_T} (f_{2n} - \beta_n) w_{n,t} d\Gamma dt. \end{aligned} \tag{3.20}$$

Using convergence properties (3.3)-(3.7) and the result of Proposition 3.1 allow us to pass with the limit as $n \to \infty$ and we obtain

$$\begin{aligned} E_{w,1}(T) - E_{w,1}(0) \quad & + \int_{\Sigma_T} |\tfrac{\partial}{\partial\nu} w_t|^2 d\Gamma dt + \int_{\Sigma_T} w_t^2 d\Gamma dt + \int_{Q_T} b(x) w_t^2 d\Omega dt \\ & = \int_0^T \int_\Omega [w, \chi(w)] w_t d\Omega dt + \int_{\Sigma_T} [-f(\tfrac{\partial}{\partial\nu} w_t) + \tfrac{\partial}{\partial\nu} w_t] \tfrac{\partial}{\partial\nu} w_t d\Gamma dt \\ & \quad - \int_{\Sigma_T} g(w_t) w_t d\Gamma dt + \int_{\Sigma_T} w_t^2 d\Gamma dt. \end{aligned} \tag{3.21}$$

After canceling boundary terms, we have

$$\begin{aligned} E_{w,1}(T) - E_{w,1}(0) \quad & + \int_{Q_T} b(x) w_t^2 d\Omega dt \\ & = \int_0^T \int_\Omega [w, \chi(w)] w_t d\Omega dt - \int_{\Sigma_T} f(\tfrac{\partial}{\partial\nu} w_t) \tfrac{\partial}{\partial\nu} w_t d\Gamma dt - \int_{\Sigma_T} g(w_t) w_t d\Gamma dt. \end{aligned} \tag{3.22}$$

Integrating by parts, using equation (1.3) and symmetricity of the trilinear form ([6], Lemma 5.2.1) yields

$$\begin{aligned} \int_0^T \int_\Omega [w, \chi(w)] w_t d\Omega dt \quad & = \int_\Omega [w, \chi(w)] w d\Omega |_0^T - \int_0^T \int_\Omega [w_t, \chi(w)] w d\Omega dt \\ & = - \int_\Omega |\Delta \chi(w)|^2 d\Omega |_0^T - \int_0^T \int_\Omega [w_t, \chi(w)] w d\Omega dt \\ & \Longrightarrow \int_0^T \int_\Omega [w, \chi(w)] w_t d\Omega dt = E_{w,2}(0) - E_{w,2}(T). \end{aligned} \tag{3.23}$$

Our desired result follows now directly from (3.22) and (3.23). □

The following inequality result is used to prove uniqueness of solutions to (1.1).

Lemma 3.2 *Let w and $\hat{w}$ be any two solutions corresponding to (1.1) which satisfy regularity requirements* *of Theorem 1.1, including (1.10). Let* $\tilde{w} \equiv w - \hat{w}$. *Then*

$$\begin{aligned} ||\tilde{w}(t)||^2_{H^2(\Omega)} &+ ||\tilde{w}_t(t)||^2_{L_2(\Omega)} + \int_0^t \int_\Gamma [f(\tfrac{\partial}{\partial\nu} w_t) - f(\tfrac{\partial}{\partial\nu}\hat{w}_t)]\tfrac{\partial}{\partial\nu}\tilde{w}_t d\Gamma dt + \int_0^t \int_\Gamma [g(w_t) - g(\hat{w}_t)]\tilde{w}_t d\Gamma dt \\ &\le C_T(||w_0||_{H^2(\Omega)}, ||\hat{w}_0||_{H^2(\Omega)}, ||w_1||_{L_2(\Omega)}, ||\hat{w}_1||_{L_2(\Omega)})[||w_0 - \hat{w}_0||_{H^2(\Omega)} + ||w_1 - \hat{w}_1||_{L_2(\Omega)}] \end{aligned} \tag{3.24}$$

Proof: Let w_n (respectively, $\hat{w}_n$) be a solution to (3.8) with $(f_n, f_{1n}, f_{2n}, \alpha_n, \beta_n)$ satisfying (3.3)-(3.7) (respectively, $(f_n, f_{1n}, f_{2n}, \alpha_n, \beta_n)$ satisfying (3.3)-(3.7) with w replaced by $\hat{w}$).

Denote $\tilde{w}_n \equiv w_n - \hat{w}_n$. Then

$$\begin{cases} \tilde{w}_{n,tt} + \Delta^2 \tilde{w}_n + b\tilde{w}_{n,t} = f_n - \hat{f}_n \\ \tilde{w}_n(0) = w_{n,0}; \quad \tilde{w}_{n,t}(0) = w_{n,1} \\ \Delta\tilde{w}_n + (1-\mu)B_1\tilde{w}_n + \frac{\partial}{\partial\nu}\tilde{w}_{n,t}|_\Gamma = -f_{1n} + \hat{f}_{1n} + \alpha_n - \hat{\alpha}_n \\ \frac{\partial}{\partial\nu}\Delta\tilde{w}_n + (1-\mu)B_2\tilde{w}_n - \tilde{w}_n - \tilde{w}_{n,t}|_\Gamma = f_{2n} - \hat{f}_{2n} - \beta_n + \hat{\beta}_n. \end{cases} \tag{3.25}$$

Standard energy estimates applied to this problem yield

$$\begin{aligned} E_{\tilde{w}_n,1}(t) - E_{\tilde{w}_n,1}(0) &+ \int_0^t \int_\Omega b\tilde{w}^2_{n,t} d\Omega dt + \int_0^t \int_\Gamma (|\tfrac{\partial}{\partial\nu}\tilde{w}_{n,t}|^2 + \tilde{w}^2_{n,t}) d\Gamma dt \\ &= \int_0^t \int_\Omega (f_n - \hat{f}_n)\tilde{w}_{n,t} d\Omega dt + \int_0^t \int_\Gamma \{(\alpha_n - \hat{\alpha}_n)\tfrac{\partial}{\partial\nu}\tilde{w}_{n,t} + (\beta_n - \hat{\beta}_n)\tilde{w}_{n,t} \\ &\quad -(f_{1n} - \hat{f}_{1n})\tfrac{\partial}{\partial\nu}\tilde{w}_{n,t} - (f_{2n} - \hat{f}_{2n})\tilde{w}_{n,t}\} d\Gamma dt. \end{aligned} \tag{3.26}$$

Using the result of Proposition 3.1, passing with the limit as $n \to \infty$ and noticing that by (3.6), (3.7), (3.12) and (1.10),

$$\begin{aligned} \int_0^t \int_\Gamma (\alpha_n - \hat{\alpha}_n)\tfrac{\partial}{\partial\nu}\tilde{w}_{n,t} d\Gamma dt &\longrightarrow \int_0^t \int_\Gamma |\tfrac{\partial}{\partial\nu}\tilde{w}_t|^2 d\Gamma dt \\ \int_0^t \int_\Gamma (\beta_n - \hat{\beta}_n)\tilde{w}_{n,t} d\Gamma dt &\longrightarrow \int_0^t \int_\Gamma \tilde{w}_t^2 d\Gamma dt, \end{aligned} \tag{3.27}$$

we obtain

$$\begin{aligned} E_{\tilde{w},1}(t) - E_{\tilde{w},1}(0) &+ \int_0^t \int_\Omega b\tilde{w}_t^2 d\Omega dt \\ &+ \int_0^t \int_\Gamma [f(\tfrac{\partial}{\partial\nu} w_t) - f(\tfrac{\partial}{\partial\nu}\hat{w}_t)]\tfrac{\partial}{\partial\nu}\tilde{w}_t d\Gamma dt + \int_0^t \int_\Gamma [g(w_t) - g(\hat{w}_t)]\tilde{w}_t d\Gamma dt \\ &= \int_0^t \int_\Omega ([\chi(w), w] - [\chi(\hat{w}), \hat{w}])\tilde{w}_t d\Omega dt. \end{aligned} \tag{3.28}$$

But

$$[\chi(w), w] - [\chi(\hat{w}), \hat{w}] = [\chi(w) - \chi(\hat{w}), w] + [\chi(\hat{w}), \tilde{w}], \tag{3.29}$$

and by (2.5) in Lemma 2.1 and (2.7) combined with a priori bounds resulting from Lemma 3.1, we obtain

$$||[\chi(w(t)), w(t)] - [\chi(\hat{w}(t)), \hat{w}]||_{L_2(\Omega)} \leq C(||w_0||_{H^2(\Omega)}, ||\hat{w}_0||_{H^2(\Omega)}, ||w_1||_{L_2(\Omega)}, ||\hat{w}_1||_{L_2(\Omega)})||\tilde{w}(t)||_{H^2(\Omega)}. \tag{3.30}$$

From (3.28) and (3.30),

$$\begin{aligned} ||\tilde{w}(t)||^2_{H^2(\Omega)} + ||\tilde{w}_t(t)||^2_{L_2(\Omega)} \quad & + \int_0^t \int_\Gamma [f(\tfrac{\partial}{\partial\nu} w_t) - f(\tfrac{\partial}{\partial\nu}\hat{w}_t)] \tfrac{\partial}{\partial\nu}\tilde{w}_t d\Gamma dt + \int_0^t \int_\Gamma [g(w_t) - g(\hat{w}_t)]\tilde{w}_t d\Gamma dt \\ & \leq C[||\tilde{w}(0)||^2_{H^2(\Omega)} + ||\tilde{w}_t(0)||^2_{L_2(\Omega)}] \\ & \quad + C(||w_0||_{H^2(\Omega)}, ||\hat{w}_0||_{H^2(\Omega)}, ||w_1||_{L_2(\Omega)}, ||\hat{w}_1||_{L_2(\Omega)}) \int_0^t ||\tilde{w}(t)||^2_{H^2(\Omega)} dt. \end{aligned} \tag{3.31}$$

Application of Gronwall's inequality implies the result of Lemma 3.2

4 A Priori Estimates

To prove Theorem 1.2, the following inequality is fundamental.

Lemma 4.1 *Let w be the solution to (1.1), $0 < \alpha < T/2$ and $\epsilon > 0$ be arbitrary. Then there exist constants, C and $C_{T,\alpha,\epsilon}(E_w(0))$ such that the following inequality holds:*

$$\begin{aligned} \int_\alpha^{T-\alpha} E_w(t)dt - CE_w(0) \quad & \leq C_{T,\alpha,\epsilon}(E_w(0))\{\int_{\Sigma_T} |w_t|^2 d\Gamma dt + \int_{\Sigma_T} |\tfrac{\partial}{\partial\nu} w_t|^2 d\Gamma dt \\ & \quad + ||f(\tfrac{\partial}{\partial\nu} w_t)||^2_{L_2(\Sigma_T)} + ||g(w_t)||^2_{L_2(\Sigma_T)} + \int_{Q_T} b(x) w_t^2 d\Omega dt\}, \end{aligned} \tag{4.1}$$

where $C_{T,\alpha,\epsilon}(E_w(0))$ is an increasing function of $E_w(0)$.

4.1 Multiplier Methods

To prove Lemma 4.1, we begin by using a multiplier method on the approximation problem (3.8) to prove the following preliminary estimate.

Proposition 4.1 *Let $(w_{n,0}, w_{n,1}) \in \mathcal{D}$, $0 < \alpha < T/2$ and $0 < \epsilon < 1/2$. Then the energy of system (3.8) as given by (1.6) satisfies the following estimate:*

$$\begin{aligned}\int_\alpha^{T-\alpha} E_{w_n,1}(t)dt \quad & -\tfrac{1}{2}\int_\alpha^{T-\alpha}\int_\Omega f_n\vec{h}\cdot\nabla w_n d\Omega dt - C_1 E_{w_n,1}(0) \\ & \le C_T\{||\alpha_n - \tfrac{\partial}{\partial\nu}w_{n,t}||^2_{L_2(\Sigma_T)} + ||\beta_n - w_{n,t}||^2_{L_2(\Sigma_T)} + ||w_{n,t}||^2_{L_2(\Sigma_T)} + ||\tfrac{\partial}{\partial\nu}w_{n,t}||^2_{L_2(\Sigma_T)} \\ & \qquad ||f_n||^2_{L_2(0,T;H^{-\epsilon}(\Omega))} + ||f_{1n}||^2_{L_2(\Sigma_T)} + ||f_{2n}||^2_{L_2(\Sigma_T)} + \int_{Q_T} b(x)w^2_{n,t}d\Omega dt + l.o.(w_n)\},\end{aligned} \tag{4.2}$$

where

$$l.o.(w_n) \equiv ||w_n||^2_{L_2(0,T;H^{2-\epsilon}(\Omega))} + ||w_{n,t}||^2_{L_2(0,T;H^{-\epsilon}(\Omega))}, \tag{4.3}$$

$\vec{h} \equiv x - x_0$ for some $x_0 \in R^2$. Here, the constant C_1 does not depend on T.

Proof of Proposition 4.1: *Step 1:* Multiplying equation (3.8) by $\vec{h}\cdot\nabla w_n$, integrating by parts and adapting the arguments of [6] yields

$$\begin{aligned}\int_0^T E_{w_n,1}(t)dt \quad & -\int_0^T\int_\Omega f_n\vec{h}\cdot\nabla w_n d\Omega dt \\ & \le C[(w_{n,t}, \vec{h}\cdot\nabla w_n)_{L_2(\Omega)}|_0^T \\ & \quad +|\int_{\Sigma_T}[\tfrac{\partial}{\partial\nu}(\vec{h}\cdot\nabla w_n)f_{1n} + (\vec{h}\cdot\nabla w_n)f_{2n}]d\Gamma dt| \\ & \quad +|\int_{\Sigma_T}[\tfrac{\partial}{\partial\nu}(\vec{h}\cdot\nabla w_n)(\alpha_n - \tfrac{\partial}{\partial\nu}w_{n,t}) - (\vec{h}\cdot\nabla w_n)(\beta_n - w_{n,t})]d\Gamma dt| \\ & \quad +|\int_{\Sigma_T}(\vec{h}\cdot\nabla w_n)w_n d\Gamma dt| + \int_{\Sigma_T} w^2_{n,t}d\Gamma dt \\ & \quad +|\tfrac{1}{2}\int_{\Sigma_T}\vec{h}\cdot\nu[w^2_{n,xx} + w^2_{n,yy} + 2\mu w_{n,xx}w_{n,yy} + 2(1-\mu)w^2_{n,xy}]d\Gamma dt| \\ & \quad +|\int_{Q_T} b(x)w_{n,t}\vec{h}\cdot\nabla w_n d\Omega dt|.\end{aligned} \tag{4.4}$$

Notice that the regularity of the solution given by (3.10) allows us to justify the calculations in [6].

Step 2: Bounding Linear Terms. To bound the last line of (4.4), we write

$$|\int_{Q_T} b(x)w_{n,t}\vec{h}\cdot\nabla w_n d\Omega dt| \le C_1\int_{Q_T} b(x)w^2_{n,t}d\Omega dt + C_2 l.o.(w_n). \tag{4.5}$$

All terms which need to be evaluated at 0 and T can be bounded by

$$C_1 E_{w_n,1}(T) + C_2 E_{w_n,1}(0). \tag{4.6}$$

Finally, by using duality to split the terms involving α_n and β_n, noting that all boundary terms involving second derivatives of the solution can be bounded by second order traces of the solution w_n and $l.o.(w_n)$, and taking into account the above estimates, we obtain the following estimate,

$$\begin{aligned}\int_0^T E_{w_n,1}(t)dt - \int_0^T \int_\Omega f_n \vec{h} \cdot \nabla w_n d\Omega dt \quad &\leq C\{E_{w_n,1}(T) + E_{w_n,1}(0) \\ &+ \int_{\Sigma_T} (f_{1n}^2 + f_{2n}^2) d\Gamma dt \\ &+ ||\alpha_n - \tfrac{\partial}{\partial \nu} w_{n,t}||^2_{L_2(\Sigma_T)} + ||\beta_n - w_{n,t}||^2_{L_2(\Sigma_T)} \\ &+ \int_{\Sigma_T} (|\tfrac{\partial^2}{\partial \nu^2} w_n|^2 + |\tfrac{\partial^2}{\partial \tau \partial \nu} w_n|^2 + |\tfrac{\partial^2}{\partial \tau^2} w_n|^2 d\Gamma dt \\ &+ \int_{Q_T} b w_{n,t}^2 d\Omega dt + \int_{\Sigma_T} w_{n,t}^2 d\Gamma dt + l.o.(w_n)\}.\end{aligned} \tag{4.7}$$

Our next step is to estimate second order traces of the function w_n on the boundary. To accomplish this, it is critical to use the following "regularity" result obtained by microlocal analysis methods.

Proposition 4.2 *([10], **Theorem 1.1**) Let $p(t,x)$ be a solution to the following linear problem (in the sense of distributions)*

$$\left.\begin{aligned} p_{tt} + \Delta^2 p = F \quad & \text{in } Q_T \\ p(0,\cdot) = p_0;\ p_t(0,\cdot) = p_1 \quad & \text{in } \Omega \\ \Delta p + (1-\mu) B_1 p = g_1 \quad & \text{on } \Sigma_T \\ \tfrac{\partial}{\partial \nu} \Delta p + (1-\mu) B_2 p - p = g_2 \quad & \text{on } \Sigma_T. \end{aligned}\right\} \tag{4.8}$$

For every $T > \alpha > 0$ and $\frac{1}{2} > \epsilon > 0$, the following estimate holds:

$$\begin{aligned}\int_\alpha^{T-\alpha} \int_\Gamma (|\tfrac{\partial^2 p}{\partial \tau^2}|^2 \quad &+ |\tfrac{\partial^2 p}{\partial \nu^2}|^2 + |\tfrac{\partial^2 p}{\partial \nu \partial \tau}|^2) d\Gamma dt \\ &\leq C_{T,\alpha}\{||F||^2_{L_2([0,T];H^{-\epsilon}(\Omega))} + ||g_1||^2_{L_2(\Sigma_T)} + ||g_2||^2_{L_2([0,T];H^{-1}(\Gamma))} \\ &+ ||p_t||^2_{L_2(\Sigma_T)} + ||\tfrac{\partial}{\partial \nu} p_t||^2_{L_2(0,T;H^{-1}(\Gamma)} + ||p||^2_{L_2([0,T];H^{3/2+\epsilon}(\Omega))}\}.\end{aligned} \tag{4.9}$$

Using the result of Proposition 4.2, we shall prove

Proposition 4.3 *Let w_n be the solution to (3.8). Then for any $\alpha > 0$ and $\epsilon > 0$, w_n satisfies the following inequality:*

$$\begin{aligned} \int_\alpha^{T-\alpha} \int_\Gamma (|\frac{\partial^2 w_n}{\partial \tau^2}|^2 & +|\frac{\partial^2 w_n}{\partial \nu^2}|^2 + |\frac{\partial^2 w_n}{\partial \nu \partial \tau}|^2) d\Gamma dt \\ & \leq C_{T,\alpha,\epsilon}\{||f_n||_{L_2(0,T;H^{-\epsilon}(\Omega))} + ||\alpha_n - \frac{\partial}{\partial \nu} w_{n,t}||^2_{L_2(\Sigma_T)} + ||f_{1n}||^2_{L_2(\Sigma_T)} \\ & \quad +||\beta_n - w_{n,t}||^2_{L_2(0,T;H^{-1}(\Gamma))} + ||f_{2n}||^2_{L_2(0,T;H^{-1}(\Gamma))} \\ & \quad \int_0^T \int_\Omega b w_{n,t}^2 d\Omega dt + ||w_{n,t}||^2_{L_2(\Sigma_T)} + ||\frac{\partial}{\partial \nu} w_{n,t}||^2_{L_2(\Sigma_T)} + l.o.(w_n). \end{aligned} \tag{4.10}$$

Proof: We apply the result of Proposition 4.2 to system (3.8) with

$$\begin{aligned} F &\equiv -b(x) w_{n,t} + f_n \\ g_1 &\equiv \alpha_n - \frac{\partial}{\partial \nu} w_{n,t} - f_{1n} \\ g_2 &\equiv -\beta_n + w_{n,t} + f_{2n}, \end{aligned} \tag{4.11}$$

and note that $||\cdot||_{L_2(0,T;H^{-1}(\Gamma))} \leq C||\cdot||_{L_2(\Sigma_T)}$. □

Completion of Proof of Proposition 4.1: Now the result of Proposition 4.1 follows by combining (4.7) applied on $[\alpha, T-\alpha]$ with Proposition 4.3 and using the fact that the energy is nonincreasing. □

Recalling Proposition 3.1, we take the limit of (4.2) as $n \to \infty$ to obtain a similar inequality for the solution to (1.1), which we state in the following proposition.

Proposition 4.4 *Let $(w_0, w_1) \in \mathcal{H}$, $0 < \alpha < T/2$ and $0 < \epsilon < 1/2$. Then the energy of system (1.1) as given by (1.6) satisfies the following estimate:*

$$\begin{aligned} \int_\alpha^{T-\alpha} E_w(t) dt & \quad -C_1 E_w(0) \\ & \leq C_T\{\int_{\Sigma_T} (|w_t|^2 + |\frac{\partial}{\partial \nu} w_t|^2) d\Gamma dt + E_w^2(0) \int_{\Sigma_T} |\Delta \chi(w)| d\Gamma dt \\ & \quad + \int_{\Sigma_T} |f(\frac{\partial}{\partial \nu} w_t)|^2 d\Gamma dt + \int_{\Sigma_T} |g(w_t)|^2 d\Gamma dt \\ & \quad +C(E_w^2(0)) \int_0^T ||\chi(w)||_{H^{3-\epsilon}(\Omega)} dt \\ & \quad + \int_{Q_T} b(x) w_t^2 d\Omega dt + l.o.(w)\}. \end{aligned} \tag{4.12}$$

Proof: *Step 1: Approximation Results.* Taking the limit as $n \to \infty$ in (4.2), by virtue of (3.3)-(3.7) and Proposition 3.1, we find

$$\begin{aligned}\int_\alpha^{T-\alpha} E_{w,1}(t)dt - \int_\alpha^{T-\alpha} \int_\Omega [w,\chi(w)]\vec{h}\cdot\nabla w d\Omega dt \quad & -C_1 E_{w,1}(0) \\ & \leq C_T\{\|w_t\|^2_{L_2(\Sigma_T)} + \|\tfrac{\partial}{\partial\nu} w_t\|^2_{L_2(\Sigma_T)} \\ & \quad + \int_{\Sigma_T} |f(\tfrac{\partial}{\partial\nu} w_t)|^2 d\Gamma dt + \int_{\Sigma_T} |g(w_t|^2 d\Gamma dt \\ & \quad + \int_0^T \|[w,\chi(w)]\|^2_{H^{-\epsilon}(\Omega)} dt \\ & \quad + \int_{Q_T} b(x) w_t^2 d\Omega dt + l.o.(w)\}.\end{aligned} \tag{4.13}$$

Step 2: Bounding Nonlinear Terms. From the regularity of the solution, w, given in Theorem 1.1 and from elliptic regularity (see [12]), we infer

$$\chi \in H^{3-\epsilon}(\Omega) \cap H_0^2(\Omega) \qquad \forall \epsilon > 0. \tag{4.14}$$

Hence, $\Delta^2\chi \in H^{-1-\epsilon}(\Omega)$, which, combined with $\chi \in H_0^{1+\epsilon}(\Omega)$, allows us to define

$$(\Delta^2\chi, \chi)_{L_2(\Omega)} \tag{4.15}$$

as a duality pairing between $H_0^{1+\epsilon}(\Omega)$ and $H^{-1-\epsilon}(\Omega)$. Application of Green's formula together with regularity in (4.14) gives

$$(\Delta^2\chi, \chi)_{L_2(\Omega)} = \|\Delta\chi\|^2_{L_2(\Omega)}. \tag{4.16}$$

Similarly, from (4.14), the regularity $w \in H^2(\Omega)$, and properties of the von Kármán nonlinearity, $[\ ,\ ]$, we also obtain that

$$[w, \chi(w)] \in H^{-\epsilon}(\Omega). \tag{4.17}$$

Hence,

$$\int_\Omega [w, \chi(w)]\vec{h} \cdot \nabla w d\Omega \tag{4.18}$$

can be defined as a duality pairing between $H^{\epsilon}(\Omega)$ and $H^{-\epsilon}(\Omega)$, $\epsilon < \frac{1}{2}$. This, in turn, allows us to justify the computation in [6] (pg. 114), based on the application of the Divergence Theorem, which then leads to

$$([w,\chi(w)],\vec{h}\cdot\nabla w)_{L_2(\Omega)} = -\frac{1}{2}||\Delta\chi||^2_{L_2(\Omega)} - \frac{1}{2}\int_\Gamma \vec{h}\cdot\nu(\Delta\chi)^2 d\Gamma. \tag{4.19}$$

From [2], we find

$$\int_{\Sigma_T} |\Delta\chi(w)|^2 d\Gamma dt \le \epsilon C E_w^2(0)\int_0^T E_w(t)dt + \frac{1}{4\epsilon}\int_{\Sigma_T}|\Delta\chi(w)|d\Gamma dt, \tag{4.20}$$

and

$$\int_0^T ||[w,\chi(w)]||^2_{H^{-\epsilon}(\Omega)}dt \le C[l.o.(w) + E_w^2(0)\int_0^T ||\chi(w)||_{H^{3-\epsilon_1}(\Omega)}dt]. \tag{4.21}$$

Combining (4.19), (4.20) and (4.21) with (4.13), and choosing ϵ to be sufficiently small, we arrive at our desired result. □

4.2 Compactness-Uniqueness Argument

From trace theory (see [2], (2.3.1)), we have that for $0 < \epsilon < \frac{1}{2}$,

$$\int_{\Sigma_T}|\Delta\chi(w)|d\Gamma dt \le C\int_0^T ||\chi(w)||_{H^{3-\epsilon}(\Omega)}dt. \tag{4.22}$$

Thus, to remove terms involving $\chi(w)$ and lower-order terms, we need the following result.

Proposition 4.5 *Let w be the solution to (1.1). Then w satisfies the following inequality:*

$$\begin{aligned}\int_0^T ||\chi(w)||_{H^{3-\epsilon}(\Omega)}dt + l.o.(w) \quad &\le C(E_w(0))\{||w_t||^2_{L_2(\Sigma_T)} + ||\tfrac{\partial}{\partial\nu}w_t||^2_{L_2(\Sigma_T)} + \int_{Q_T} b(x)w_t^2 d\Omega dt \\ &+||f(\tfrac{\partial}{\partial\nu}w_t)||^2_{L_2(\Sigma_T)} + ||g(w_t)||^2_{L_2(\Sigma_T)}\}\end{aligned} \tag{4.23}$$

where $C(E_w(0))$ is an increasing function of $E_w(0)$.

Proof of Proposition 4.5: By contradiction, assume (4.23) does not hold. Then there exists a sequence of functions, $\{w_n(t)\} \in \mathcal{H}$ such that each $w_n(t)$ satisfies the system

$$\begin{aligned} w_{n,tt} + \Delta^2 w_n + b w_{n,t} &= [w_n, \chi(w_n)] && \text{in } Q_T \\ w_n(0,\cdot) = w_{n,0};\ w_{n,t}(0,\cdot) &= w_{n,1} && \text{in } \Omega \\ \Delta w_n + (1-\mu)B_1 w_n &= -f(\tfrac{\partial}{\partial\nu} w_{n,t}) && \text{on } \Sigma_T \\ \tfrac{\partial}{\partial\nu}\Delta w_n + (1-\mu)B_2 w_n - w_n &= g(w_{n,t}) && \text{on } \Sigma_T \end{aligned} \tag{4.24}$$

and such that

$$\lim_{n\to\infty} \frac{\int_0^T \|\chi(w_n)\|_{H^{3-\epsilon}(\Omega)} dt + l.o.(w_n)}{\mathcal{P}(w_n)} = \infty, \tag{4.25}$$

where

$$\mathcal{P}(w_n) \equiv \|w_{n,t}\|^2_{L_2(\Sigma_T)} + \|\tfrac{\partial}{\partial\nu} w_{n,t}\|^2_{L_2(\Sigma_T)} + \int_{Q_T} b(x) w_{n,t}^2 d\Omega dt + \|f(\frac{\partial}{\partial\nu} w_{n,t})\|^2_{L_2(\Sigma_T)} + \|g(w_{n,t})\|^2_{L_2(\Sigma_T)}, \tag{4.26}$$

and where the initial energy of (4.24) is uniformly bounded in n. Hence, by Lemma 3.1 and the same arguments as in Proposition 2.5 of [2], we know the following convergence properties hold:

$$\left.\begin{aligned} w_n &\xrightarrow{w} w \ in\ L_2([0,T]; H^2(\Omega)) \\ w_{n,t} &\xrightarrow{w} w_t \ in\ L_2([0,T]; L_2(\Omega)) \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} & l.o.(w_n) \longrightarrow l.o.(w) \\ & \int_0^T \|\chi(w_n)\|_{H^{3-\epsilon}(\Omega)} dt \longrightarrow \int_0^T \|\chi(w)\|_{H^{3-\epsilon}(\Omega)} dt. \end{aligned}\right. \tag{4.27}$$

Case 1: Assume $\int_0^T \|\chi(w)\|_{H^{3-\epsilon}(\Omega)} dt + l.o.(w) \neq 0$. Then

$$\mathcal{P}(w_n) \longrightarrow 0, \tag{4.28}$$

which in turn implies that

$$\begin{aligned} w_{n,t} &\longrightarrow 0 \ in\ L_2(Q_T), \\ f(\tfrac{\partial}{\partial\nu} w_{n,t}) &\longrightarrow 0 \ in\ L_2(\Sigma_T), \\ g(w_{n,t}) &\longrightarrow 0 \ in\ L_2(\Sigma_T). \end{aligned} \tag{4.29}$$

Thus, by passing with the limit as $n \longrightarrow \infty$ on (4.24), we obtain the limit system

$$\begin{aligned} &\Delta^2 w = [w, \chi(w)] && \text{in } Q_T \\ &w(0,\cdot) = w_0;\ w_t(0,\cdot) = w_1 && \text{in } \Omega \\ &\Delta w + (1-\mu)B_1 w = 0 && \text{on } \Sigma_T \\ &\tfrac{\partial}{\partial\nu}\Delta w + (1-\mu)B_2 w - w = 0 && \text{on } \Sigma_T. \end{aligned} \tag{4.30}$$

By the uniqueness result in [2], the solution w to (4.30) is identically zero. Hence

$$\int_0^T \|\chi(w_n)\|_{H^{3-\epsilon}(\Omega)} dt + l.o.(w_n) \longrightarrow 0, \tag{4.31}$$

which contradicts our assumption. Thus Proposition 4.5 holds for Case 1.

Case 2: Assume $\int_0^T \|\chi(w)\|_{H^{3-\epsilon}(\Omega)} dt + l.o.(w) = 0$. Define

$$c_n \equiv \{\int_0^T \|\chi(w_n)\|_{H^{3-\epsilon}(\Omega)} dt + l.o.(w_n)\}^{1/2}, \tag{4.32}$$

and

$$v_n \equiv \frac{w_n}{c_n} \ \textit{where}\ c_n \longrightarrow 0. \tag{4.33}$$

Then v_n satisfies the system

$$\begin{aligned} &v_{n,tt} + \Delta^2 v_n + b v_{n,t} = [v_n, \chi(w_n)] && \text{in } Q_T \\ &v_n(0,\cdot) = v_{n,0};\ v_{n,t}(0,\cdot) = v_{n,1} && \text{in } \Omega \\ &\Delta v_n + (1-\mu)B_1 v_n = -\tfrac{1}{c_n} f(\tfrac{\partial}{\partial\nu} w_{n,t}) && \text{on } \Sigma_T \\ &\tfrac{\partial}{\partial\nu}\Delta v_n + (1-\mu)B_2 v_n - v_n = \tfrac{1}{c_n} g(w_{n,t}) && \text{on } \Sigma_T \end{aligned} \tag{4.34}$$

By the quadratic dependence of χ on w_n, v_n satisfies

$$\int_0^T \|\chi(v_n)\|_{H^{3-\epsilon}(\Omega)} dt + l.o.(v_n) \equiv 1. \tag{4.35}$$

By dividing both sides of (4.25) by c_n^2 and using (4.26) and (4.27), we obtain

$$||v_{n,t}||^2_{L_2(\Sigma_T)} + ||\frac{\partial}{\partial\nu} v_{n,t}||^2_{L_2(\Sigma_T)} + \int_{Q_T} b(x) v_{n,t}^2 d\Omega dt + \frac{1}{c_n^2}||f(\frac{\partial}{\partial\nu} w_{n,t})||^2_{L_2(\Sigma_T)} + \frac{1}{c_n^2}||g(w_{n,t})||^2_{L_2(\Sigma_T)} \longrightarrow 0. \quad (4.36)$$

Using the energy identity, (3.19), we can remove $E_{w_n}(0)$ from the left-hand side of (4.12) as follows:

$$\begin{aligned} \int_\alpha^{T-\alpha} E_{w_n}(t)dt - C_1 E_{w_n}(T) \quad & \leq C_T\{\int_{\Sigma_T}(|w_{n,t}|^2 + |\tfrac{\partial}{\partial\nu} w_{n,t}|^2) d\Gamma dt \\ & + \int_{\Sigma_T} |f(\tfrac{\partial}{\partial\nu} w_{n,t})|^2 d\Gamma dt + \int_{\Sigma_T} |g(w_{n,t})|^2 d\Gamma dt \\ & + C(E^2_{w_n}(0)) \int_0^T ||\chi(w_n)||_{H^{3-\epsilon}(\Omega)} dt + \int_{Q_T} b(x) w^2_{n,t} d\Omega dt + l.o.(w_n)\} \\ & + C_1 \int_{\Sigma_T} f(\tfrac{\partial}{\partial\nu} w_{n,t}) \tfrac{\partial}{\partial\nu} w_{n,t} d\Gamma dt + C_1 \int_{\Sigma_T} g(w_{n,t}) w_{n,t} d\Gamma dt. \end{aligned} \quad (4.37)$$

Using Hölder's inequality, we can split the terms on the last line of the above inequality and absorb them into the remaining terms on the right-hand side. We also know, since the energy of our system is nonincreasing, that

$$(T - 2\alpha - C_1) E_{w_n}(T) \leq \int_\alpha^{T-\alpha} E_{w_n}(t)dt - C_1 E_{w_n}(T). \quad (4.38)$$

Therefore, by choosing T to be appropriately large, we find

$$\begin{aligned} E_{w_n}(T) \quad & \leq C_T\{\int_{\Sigma_T}(|w_{n,t}|^2 + |\tfrac{\partial}{\partial\nu} w_{n,t}|^2) d\Gamma dt \\ & + \int_{\Sigma_T} |f(\tfrac{\partial}{\partial\nu} w_{n,t})|^2 d\Gamma dt + \int_{\Sigma_T} |g(w_{n,t})|^2 d\Gamma dt \\ & + C(E^2_{w_n}(0)) \int_0^T ||\chi(w_n)||_{H^{3-\epsilon}(\Omega)} dt + \int_{Q_T} b(x) w^2_{n,t} d\Omega dt + l.o.(w_n)\}. \end{aligned} \quad (4.39)$$

From (4.39), after division by c_n^2 and recalling (4.35) and (4.36), we have

$$||v_n||^2_{C(0,T;H^2(\Omega))} + ||v_{n,t}||^2_{C(0,T;L_2(\Omega))} \leq C(E_{w_n}(0)). \quad (4.40)$$

Since $E_{w_n}(0)$ are uniformly bounded for all n, $C(E_{w_n}(0))$ is uniformly bounded for all n as well and

$$\begin{aligned} v_n & \xrightarrow{w^*} v \ in \ L_\infty(0,T;H^2(\Omega)) \ \ weakly \ * \\ v_{n,t} & \xrightarrow{w^*} \ in \ L_\infty(0,T;H^2(\Omega)) \ \ weakly \ *. \end{aligned} \quad (4.41)$$

The convergence of (4.36) implies

$$\begin{aligned} \tfrac{1}{c_n} g(w_{n,t}) &\longrightarrow 0 \; in \; L_2(\Sigma_T) \\ \tfrac{1}{c_n} f(\tfrac{\partial}{\partial\nu} w_{n,t}) &\longrightarrow 0 \; in \; L_2(\Sigma_T) \\ v_{n,t} &\longrightarrow 0 \; in \; L_2(Q_T). \end{aligned} \tag{4.42}$$

Thus, we obtain the limit system

$$\begin{aligned} &\Delta^2 v = [v, \chi(w)] && \text{in } Q_T \\ &v(0,\cdot) = v_0; \; v_t(0,\cdot) = v_1 && \text{in } \Omega \\ &\Delta v + (1-\mu)B_1 v = 0 && \text{on } \Sigma_T \\ &\tfrac{\partial}{\partial\nu}\Delta v + (1-\mu)B_2 v - v = 0 && \text{on } \Sigma_T. \end{aligned} \tag{4.43}$$

Again, by the result of [2], $v \equiv 0$, implying

$$\int_0^T ||\chi(v_n)||_{H^{3-\epsilon}(\Omega)} dt + l.o.(v_n) \longrightarrow 0, \tag{4.44}$$

which is a contradiction. Thus, Proposition 4.5 holds in Case 2 and our proof is complete. □

Conclusion of Proof of Lemma 4.1: Substituting the result of Proposition 4.5 into (4.12), we obtain our desired result,

$$\begin{aligned} \textstyle\int_\alpha^{T-\alpha} E_w(t)dt - CE_w(0) \;\; &\leq C_{T,\alpha,\epsilon}(E_w(0))\{\textstyle\int_{\Sigma_T} |w_t|^2 d\Gamma dt + \int_{\Sigma_T} |\tfrac{\partial}{\partial\nu} w_t|^2 d\Gamma dt \\ &+ ||f(\tfrac{\partial}{\partial\nu} w_t)||^2_{L_2(\Sigma_T)} + ||g(w_t)||^2_{L_2(\Sigma_T)} + \textstyle\int_{Q_T} b(x) w_t^2 d\Omega dt\}. \square \end{aligned} \tag{4.45}$$

5 Final Estimates

Let the functions $h(x)$, $h_i(x)$, $i = 1, 2$, and $\tilde{h}(x)$ be defined as in (1.11), (1.12), and (1.13), respectively.

$$h(x) \equiv h_1(x) + h_2(x), \tag{5.1}$$

where $h_i(x)$ are concave, strictly increasing functions with $h_i(0) = 0$. Then $h(x)$ enjoys the same properties. Moreover, we assume that

$$\begin{aligned} h_1(sf(s)) &\geq s^2 + f^2(s) \quad |s| \leq 1 \\ h_2(sg(s)) &\geq s^2 + g^2(s) \quad |s| \leq 1. \end{aligned} \tag{5.2}$$

By the hypotheses imposed on functions $h_i(x)$, we obtain

$$\int_{\Sigma_T} |f(\frac{\partial}{\partial\nu}w_t)|^2 d\Gamma dt = \int_{\Sigma_{A_1}} |f(\frac{\partial}{\partial\nu}w_t)|^2 d\Gamma dt + \int_{\Sigma_{B_1}} |f(\frac{\partial}{\partial\nu}w_t)|^2 d\Gamma dt, \tag{5.3}$$

where $\Sigma_{A_1} \equiv \{(t,x) \in \Sigma_T : |\frac{\partial}{\partial\nu}w_t| \leq 1\}$ and $\Sigma_{B_1} \equiv \Sigma_T \backslash \Sigma_{A_1}$. Hence, using hypothesis (H) on Σ_{B_1}, we find

$$\begin{aligned} \int_{\Sigma_T} |\tfrac{\partial}{\partial\nu}w_t|^2 d\Gamma dt \quad &+ \int_{\Sigma_T} |f(\tfrac{\partial}{\partial\nu}w_t)|^2 d\Gamma dt \\ &\leq \int_{\Sigma_{A_1}} [|\tfrac{\partial}{\partial\nu}w_t|^2 + |f(\tfrac{\partial}{\partial\nu}w_t)|^2] d\Gamma dt + (M + \tfrac{1}{m}) \int_{\Sigma_{B_1}} f(\tfrac{\partial}{\partial\nu}w_t)\tfrac{\partial}{\partial\nu}w_t d\Gamma dt \\ &\leq \int_{\Sigma_{A_1}} h_1(\tfrac{\partial}{\partial\nu}w_t f(\tfrac{\partial}{\partial\nu}w_t)) d\Gamma dt + (M + \tfrac{1}{m}) \int_{\Sigma_T} f(\tfrac{\partial}{\partial\nu}w_t)\tfrac{\partial}{\partial\nu}w_t d\Gamma dt. \end{aligned} \tag{5.4}$$

Similarly, the same argument applied to g yields

$$\int_{\Sigma_T} |g(w_t)|^2 d\Gamma dt + \int_{\Sigma_T} |w_t|^2 d\Gamma dt \leq \int_{\Sigma_T} h_2(w_t g(w_t)) d\Gamma dt. \tag{5.5}$$

Define

$$\tilde{h}_i(x) \equiv h_i(\frac{x}{mes\ \Sigma_T}). \tag{5.6}$$

Then, by Jensen's inequality,

$$\begin{aligned} \int_{\Sigma_T} \{|w_t|^2 \quad &+ |\tfrac{\partial}{\partial\nu}w_t|^2 + |f(\tfrac{\partial}{\partial\nu}w_t)|^2 + |g(w_t)|^2\} d\Gamma dt \\ &\leq C_1 \int_{\Sigma_T} \{g(w_t)w_t + f(\tfrac{\partial}{\partial\nu}w_t)\tfrac{\partial}{\partial\nu}w_t\} d\Gamma dt \\ &\quad + C_2 \left[\tilde{h}_2(\int_{\Sigma_T} g(w_t)w_t d\Gamma dt) + \tilde{h}_1(\int_{\Sigma_T} f(\tfrac{\partial}{\partial\nu}w_t)\tfrac{\partial}{\partial\nu}w_t d\Gamma dt)\right] \\ &\leq C_1 \int_{\Sigma_T} \{g(w_t)w_t + f(\tfrac{\partial}{\partial\nu}w_t)\tfrac{\partial}{\partial\nu}w_t\} d\Gamma dt \\ &\quad + C_2 \textstyle\sum_{i=1}^2 \tilde{h}_i(\int_{\Sigma_T} \{g(w_t)w_t + f(\tfrac{\partial}{\partial\nu}w_t)\tfrac{\partial}{\partial\nu}w_t\} d\Gamma dt + \int_{Q_T} b(x) w_t^2 d\Omega dt), \end{aligned} \tag{5.7}$$

where the last inequality follows from the monotonicity of the functions $\tilde{h}_i$.

Denoting $\mathcal{F} \equiv \int_{\Sigma_T}\{g(w_t)w_t+f(\frac{\partial}{\partial\nu}w_t)\frac{\partial}{\partial\nu}w_t\}d\Gamma dt+\int_{Q_T} b(x)w_t^2 d\Omega dt$, we obtain from Lemma 4.1 and (5.7), using the monotonicity of $\tilde{h}$ once more to include $\int_{Q_T} b(x)w_t^2 d\Omega dt$,

$$\int_\alpha^{T-\alpha} E_w(t)dt - C_1 E_w(0) \leq C_{T,\alpha,\epsilon}(E_w(0))[\mathcal{F}+\tilde{h}(\mathcal{F})]. \tag{5.8}$$

Since

$$\int_0^\alpha E_w(t)dt + \int_{T-\alpha}^T E_w(t)dt \leq 2\alpha E_w(0), \tag{5.9}$$

we find

$$\int_0^T E_w(t)dt - C_{1,\alpha}E_w(0) \leq C_{T,\alpha,\epsilon}(E_w(0))[\mathcal{F}+\tilde{h}\mathcal{F}], \tag{5.10}$$

and by Lemma 3.1,

$$\begin{aligned} &\textstyle\int_0^T E_w(t)dt \leq C_{T,\alpha,\epsilon}(E_w(0))[\mathcal{F}+\tilde{h}\mathcal{F}] + C_{1,\alpha}E_w(0) \\ &\Longrightarrow (T-C_{1,\alpha})E_w(T) \leq C_{T,\alpha,\epsilon}(E_w(0))[\mathcal{F}+\tilde{h}(\mathcal{F})] \\ &\Longrightarrow E_w(T) \leq C_T(E_w(0))[\mathcal{F}+\tilde{h}(\mathcal{F})]. \end{aligned} \tag{5.11}$$

Hence, recalling (3.1),

$$(I+\tilde{h})^{-1}(\frac{E_w(T)}{C_T(E_w(0))}) \leq \mathcal{F} = E_w(0) - E_w(T). \tag{5.12}$$

Setting

$$p(s) \equiv (I+\tilde{h})^{-1}(\frac{s}{C_T(E_w(0))}), \tag{5.13}$$

we have proven the following proposition.

Proposition 5.1 *Let w be the solution to (1.1) and $E_w(t)$ be the corresponding energy at time t. For every $T>0$, there exists a monotone increasing function, p, such that*

$$p(E_w(T)) + E_w(T) \leq E_w(0). \tag{5.14}$$

To arrive at the conclusion of Theorem 1.2, we need to apply the result of Lemma 3.3 in [8].

Lemma 5.1 ***([8], Lemma 3.3)*** *Let p be a positive, increasing function such that $p(0)=0$. Since p is* *increasing, we can define a function q such that $q(x) = x - (I+p)^{-1}(x)$. Notice that q is also an increasing function. Consider a sequence s_n of positive numbers which satisfy:*

$$s_{m+1} + p(s_{m+1}) \leq s_m. \tag{5.15}$$

Then $s_m \leq \mathcal{S}(m)$, where $\mathcal{S}(t)$ is a solution of a differential equation

$$\begin{cases} \frac{d}{dt}\mathcal{S}(t) + q(\mathcal{S}(t)) = 0 \\ \mathcal{S}(0) = s_0. \end{cases} \tag{5.16}$$

Moreover, if $p(x) > 0$ for $x > 0$, then $\lim_{t\to\infty} \mathcal{S}(t) = 0$.

Applying the result of Proposition 5.1, we obtain

$$E_w(m(T+1)) + p(E_w(m(T+1))) \leq E_w(mT), \tag{5.17}$$

for $m = 0, 1, \ldots$ Thus, applying Lemma 5.1 with

$$s_m \equiv E_w(mT), \tag{5.18}$$

yields

$$E_w(mT) \leq \mathcal{S}(m), \;\; m = 0, 1, \ldots \tag{5.19}$$

Setting $t = mT + \tau$, $0 \leq \tau < T$, and recalling the evolution property gives

$$E_w(t) \leq E_w(mT) \leq \mathcal{S}(m) \leq \mathcal{S}(\frac{t-\tau}{T}) \leq \mathcal{S}(\frac{t}{T} - 1) \; for \; t > T, \tag{5.20}$$

which completes the proof of Theorem 1.2. □

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