# Structure of positive solution to $(-\Delta + V)u = 0$ in $R^n$

Minoru Murata

Department of Mathematics

Tokyo Metropolitan University

Fukazawa, Setagaya

Tokyo, 158 Japan

# Table of Contents

RUNNING HEAD:

Positive solutions of Schrödinger Equations

PLEASE SEND ALL MAIL TO:

Minoru Murata

Department of Mathematics

Tokyo Metropolitan University

Fukazawa, Setagaya

Tokyo, 158 Japan

## §1. Introduction

Positive solution of second order elliptic equations have been fascinating many mathematicians (see [1~5,7,9,11~13,16,19~21]). In this paper we investigate the structure of all positive solutions of a stationary Schrödinger equation

$$(1.1)\qquad (-\Delta + V)u = 0 \quad \text{on} \quad R^n ,$$

where $V$ is a real-valued function belonging to $L_{p,loc}(R^n)$ with $p > n/2$ for $n \geq 2$ and $p = 1$ for $n = 1$. We are interested in establishing an explicit representation formula for positive solutions, determing explicity their asymptotics at infinity, and solving the Dirichlet problem with boundary condition at infinity. We are also interested in the stability and unstability of structures of positive solutions: how the structures are influenced by the change of potentials.

Our interest in these problems comes from [13,14,15], where asymptotic behavior as $t \to \infty$ of a Schrödinger semigroup with very short range potential and unique existence theorems for ellictic equations like $(-\Delta)^m$ on $R^n$ were investigated. In extending the results there to a Schrödinger operator with more general potential, we think it is necessary to obtain an explicit representation formula for all positive solutions of it. As for close connection between the positive solutions of a Schrödinger equation and the asymptotic behavior as $t \to \infty$ of the corresponding Schrödinger semigroup, see [13,18,19] and references therein.

The investigation of integral representations of positive solutions goes back to Martin's fascinating work [11], where he gave a method to represent any positive harmonic functions in an arbitrary domain in $R^3$ by an integral on the minimal Martin boundary.

Subsequently, his method has been extended to second order elliptic operators with zero potential (see, [21] and references in [7]). However, explicit construction of representation formulas is still isolated. Here we mention a few papers in this direction. Karpelevic [9] gave explicit representation formulas for positive solutions of $(-\Delta+\lambda)u = 0$ on a symmetric space, where $\Delta$ is the Laplace-Beltrami operator and $\lambda$ is a constant; in the special case where $\Delta$ is the usual Lapacian on $R^n$, Caffarelli and Littman [5] showed in a more elementary way that any positive solution of $(-\Delta+1)u = 0$ in $R^n$ is represented uniquely by

$$(1.2) \qquad u(x) = \int_{S^{n-1}} e^{x\omega} d\mu(\omega) \ ,$$

where $d\mu$ is a nonnegative measure on the unit sphere $S^{n-1}$, which is the minimal Martin boundary in this case. An analogous representation formula for a second order elliptic operator with periodic coefficients on $R^n$ was given by Agmon [1]. In [7], the minimal Martin boundary for the operator $\Delta/2 + Bx\cdot\nabla$ on $R^2$, B a $2\times 2$ nonsingular constant matrix, was constructed explicity. In [4], the structure of positive harmonic functions on a complete manifold of negative curvature was investigated precisely.

The aim of this paper is to throw light on how a potential determines the structure of positive solutions of (1.1). The aim is accomplished completely, in Section 5 and Appendix 1, for "principally" radially symmetric potentials on $R^n$, $n \geq 2$, and potentials on $R^1$. As for potentials which are essentially different from radially symmetric ones, however, the stage of accomplishment is rudimentary. We take an initial step in this direction is Section 7, and give some potentials for which the structure of positive solutions is quite different from that for principally radially symmetric potentials.

The plan of this paper is as follows. In Section 2 we introduce basic notions and establish fundamental theorems concerning the structure of positive solutions:

We classify potentials into three classes: (i) $V$ is subcritical if $-\Delta+V$ has a positive Green's function; (ii) $V$ is critical if $-\Delta+V$ is nonnegative and does not have a positive Green's function; and (iii) $V$ is supercritical if $-\Delta+V$ is not nonnegative. (It is well-known, see [2,3,12], that (1.1) has a positive solution iff $-\Delta+V \geq 0$.) We characterize a subcritical (or supercritical) potential by stability under small local perturbations, a critical one by unstability (cf. Theorem 2.4, and see also Theorems 2.5~2.7). We show that if $V$ is critical, then (1.1) has only one positive solution modulo constant multiple with specific decay property at infinity; and that if $V$ is subcritical, then any positive solution of (1.1) can be represented by an integral on the minimal Martin boundary (see Theorems 2.2

and 2.3). We introduce the notion of P-equivalence concerning the similarity of structures of positive solutions (see Definition 2.9) and give criteria for P-equivalence (see Theorems 2.10 and 2.11).

For some reasons (see Section 7, for example), we develop the general theory in Section 2 for Schrödinger equations in arbitrary domains of $R^n$, although we are interested in positive solutions on $R^n$.

Throughout the present paper except for Section 2 and Appendix 1 we assume that $n \geq 2$. In Appendix 1 we deal with one dimensional Schrödinger equations and give complete answers to problems raised in this paper. In Sections 3~7 we treat the higher dimensional case.

Sections 3 and 4 are devoted to a precise description of the structure of positive solutions for radially symmetric potentials on $R^n$, $n \geq 2$. These sections are also preliminaries for Sections 5 and 7. In Section 5 we deal with potentials which are not so much different from radially symmetric ones. We call such potentials principally radially symmetric potentials, an example of which is a very short range potentials, i.e., a potential $V$ such that $(1+|x|)^{\rho-n/p} V(x)$ belong to $L_p(R^n)$ for some $p > n/2$ and $\rho > 2$. We establish (see Thoerems 5.3 and 5.4) that if a principally radially symmetric potential $V$ is subcritical, then either the following (I) or (II) holds:

(I) The equation (1.1) has a unique positive solution modulo constant multiple with specific asymptotics as $x$ goes to infinity.

(II) Any positive solution $u$ of (1.1) is represented uniquely by (1.2) with $e^{x\omega}$ replaced by a positive solution with specific asymptotic property.

We give a criterion for (I) or (II) to hold (see Theorems 3.2 and 3.3); for example, very short range potentials belong to the class (I) (see Theorem 5.6). We also solve the Dirichlet problem at infinity (see Theorem 5.5). A significant feature of the problem is that the boundary value at infinity of a solution $u$ needs to be prescribed by the limit of quotient

$$u(x)/g(x) ,$$

where $g$ is a specified positive solution.

Section 6 is a preparation for Section 7 and concerned with positive solutions in a half space. In Section 7 we discuss a potential on $R^n$ which is not radially symmetric in a drastic way, and obtain rather striking phenomena (see Theorem 7.1 and Example 7.11):

The minimal Martin boundary for the potential is a finite union of disjoint closed intervals.

Finally, in Appendix 2, we prove Theorems 4.9 and 4.10 which play a crucial role in establishing Theorem 7.1.

## §2. Classification of potentials, integral representation, and stability

In this section we give some fundamental results for positive solutions in a domain of $R^n$ of a Schrödinger equation with potential satisfying only a local condition.

Let $D$ be a connected open set in $R^n$ and $V$ satisfy the condition

(2.1) $V$ is a real-valued function in $L_{p,loc}(D)$, where $p > n/2$ for $n \geq 2$ and $p = 1$ for $n = 1$.

Consider the Schrödinger equation

$$(-\Delta+V)u = 0 \quad \text{in} \quad D . \tag{2.2}$$

We are interested in the structure of all positive solutions of (2.2). A positive solution of (2.2) means a continuous positive function in $D$ satisfying (2.2) in the distribution sense. It is known (cf. [20]) that any weak solution is Hölder continuous, and that if $u$ is a nonnegative solution in $D$, then $u$ is strictly positive in $D$ or identically zero.

Following Martin [11], we call $u$ a minimal solution when $u$ is a positive solution such that for any positive solution $v$ with $v(x) \leq u(x)$ in $D$ there exists a constant $c$ such that $v(x) = cu(x)$ in $D$. Theorems 2.2 and 2.3 below state that any positive solution can be represented uniquely as "a linear combination" of minimal solutions.

In order to state the theorems we introduce a classification of potentials.

Definition 2.1. (i) We say that $V$ is subcritical if there exists, for any $y$ in $D$, a positive solution $H(x,y)$ satisfying

$$(2.3) \qquad (-\Delta+V(x))H(x,y) = \delta(x-y) \quad \text{in } D,$$

where $\delta$ is Dirac's function.

(ii) We say that $V$ is critical if there exists no positive solution $H(x,y)$ satisfying (2.3) for some $y$ in $D$ and $-\Delta+V \geq 0$, i.e.,

$$(2.4) \qquad \int |\nabla\phi(x)|^2 + V(x)|\phi(x)|^2 dx \geq 0 \quad \text{for any } \phi \in C_0^\infty(D).$$

(iii) We say that $V$ is supercritical if (2.4) does not hold.

Here we note that if $V$ is subcritical, then $-\Delta+V \geq 0$; because the condition (2.4) is a necessary and sufficient condition for the existence of a positive solution (cf. [2],[3], [12],[19]). The terms supercritical, critical, and subcritical were introduced by Simon [18] in connection with large time asymptotics of Schrödinger semigroups. His and our definitions are equivalent in some cases (see [13]).

For illustrating the calssification we give simple examples.

Example (i) Let $D = R^n$ and $V = 0$. Then $V$ is subcritical if $n \geq 3$, and critical if $n = 1$ or $2$ (see, for example, Theorem 3.1 in §3).

(ii) Let $D$ be a bounded domain with smooth boundary, and let $V$ be smooth on $\bar{D}$. Denote by $m$ the smallest eigenvalue of the operator $-\Delta+V$ in $D$ with zero Dirichlet condition. Then $V$ is subcritical iff $m > 0$, critical iff $m = 0$, and supercritical iff $m < 0$.

Now, choose a point $x^0$ in $D$ and a sequence of bounded domains $D_j$, $j = 1,2,\ldots,$ with smooth boundaries such that

$$(2.5) \qquad x^0 \in D_1 \ , \ \bar{D}_j \subset D_{j+1} \ , \text{ and } \bigcup_{j=1}^{\infty} D_j = D \ ,$$

where $\bar{D}_j$ is the closure of $D_j$. Suppose that (2.4) holds. Then, since $-\Delta+V$ is coercive in any bounded domain $D_j$, $j = 1,2,\ldots,$ there exists a unique positive Green's function $G_j(x,y)$ in $D_j$ such that for any $y$ in $D_j$

$$(2.6) \qquad (-\Delta + V(x))G_j(x,y) = \delta(x-y) \text{ in } D_j, \ G_j(x,y)\big|_{\partial D_j} = 0$$

(cf. [19]). If we extend $G_j$ to $D$ by putting $G_j(x,y) = 0$ for $x$ in $D\backslash D_j$, $G_j$ is a nondecreasing nonnegative function. By the Harnack inequality, either $G_j(x,y)$ converges at any $x$ in $D\backslash\{y\}$ or diverges at any $x$ in $D\backslash\{y\}$. If $V$ is subcritical, the minimal Green's function $G(x,y)$ can be defined by

$$(2.7) \qquad G(x,y) = \lim_{j\to\infty} G_j(x,y) \ ,$$

since $(H-G_j)\big|_{\partial D_j} > 0$ and $(-\Delta+V)(H-G_j) = 0$ in $D_j$, and so, by the generalized maximum principle (cf. [17]), $G_j(x,y) \leq H(x,y)$. It is clear that $G$ satisfies (2.3) with $H$ replaced by $G$, is positive in $D$, and does not depend on the choice of $\{D_j\}$. If $V$ is critical, the minimal Green's function does not exist.

We say that a solution $u$ is normalized if $u(x^0) = 1$.

We are now ready to state the first theorem.

Theorem 2.2. If $V$ is critical, then there exists a unique normalized positive solution $u$ satisfying

$$(2.8) \qquad u(x) = \int G(x,y)W(y)u(y)dy ,$$

where $W$ is a nonnegative function in $C_0^\infty(D)$ not identically zero and $G(x,y)$ is the minimal Green's function for $V + W$.

This theorem will be proved by making use of a characterization of a critical potential (Theorem 2.4) after Theorems 2.3 ~2.6. It follows from Theorem 2.5 that $V+W$ is subcritical. We will see later that the uniqueness also holds for a class of subcritical potentials.

Our second theorem asserts, roughly speaking, that if $V$ is subcritical, then Martin's results can be extended to $-\Delta+V$ in $D$.

Suppose that $V$ is subcritical, and let $G$ be the minimal Green's function. Put

$$(2.9) \qquad P(x,y) = \begin{cases} G(x,y)/G(x^0,y) , & x^0 \neq y , \\ 0 , & x^0 = y \text{ and } x \neq y , \\ 1 , & x^0 = y = x . \end{cases}$$

Consider the set of all fundamental sequences $\{y_j\}$, i.e., $\{y_j\}$

has no point of accumulation in $D$ and the series $\{P(x,y_j)\}$ of functions of $x$ converges to a positive solutions in $D$. Two fundamental sequences $\{y_j\}$ and $\{y_j'\}$ are called equivalent if $\lim_{j\to\infty} P(x,y_j) = \lim_{j\to\infty} P(x,y_j')$. We denote by $\Sigma$ the set of all equivalence classes of fundamental sequences. For $\omega$ in $\Sigma$, we put

$$(2.10) \qquad P(x,\omega) = \lim_{j\to\infty} P(x,y_j) \ ,$$

where $\{y_j\}$ is a representative of $\omega$. Select a fixed closed ball $B$ with center $x^0$ such that $B \subset D_1$, and define $d(z,z')$ for $z$ and $z'$ in $D+\Sigma$ by

$$(2.11) \qquad d(z,z') = \int_B \frac{|P(x,z) - P(x,z')|}{1+|P(x,z) - P(x,z')|} dx \ .$$

Then the following theorem can be shown along the line given by Martin [11].

Theorem 2.3. (i) $d$ is a metric on $D+\Sigma$, which is compact with respect to the topology induced by the metric $d$, and $\Sigma$ is the boundary of $D+\Sigma$.

(ii) $P(x,z)$ is continuous on $D\times(D+\Sigma)\setminus\{(x,x);x \in D\}$.

(iii) Any minimal solution is equal to $P(x,\omega)$ for some $\omega$ in $\Sigma$.

(iv) The set $\sigma = \{\omega \in \Sigma;\ P(x,\omega)$ is a minimal solution$\}$ is a countable union of closed subsets of $\Sigma$.

(v) For any normalized positive solution $u$ there exists a unique nonnegative Borel measure $\mu$ on $\Sigma$ such that

$\mu(\Sigma\backslash\sigma) = 0$, $\mu(\sigma) = 1$, and

$$(2.12) \qquad u(x) = \int_\sigma P(x,\omega)d\mu(\omega) \quad .$$

We call $\sigma$ the minimal (Martin) boundary for $V$ in $D$, whose identification and explicit construction of $P(x,\omega)$ is one of our main purposes.

Our third theorem is a characterization of a subcritical/critical potential.

Theorem 2.4. Let $W$ be a nonnegative function with compact support in $D$ which is positive on a set of positive measure and satisfies (2.1). Then one has:

(i) $V$ is critical iff $-\Delta+V \geq 0$ and $-\Delta+V - \varepsilon W$ is not nonnegative for all $\varepsilon > 0$.

(ii) $V$ is subcritical iff $-\Delta+V \geq 0$ and $-\Delta+V - \varepsilon W \geq 0$ for any sufficiently small $\varepsilon > 0$.

Proof. It suffices to show that if $-\Delta+V - \varepsilon W \geq 0$ for some $\varepsilon > 0$, then $V$ is subcritical, and that if $V$ is subcritical, then $-\Delta+V - \varepsilon W \geq 0$ for any sufficiently small $\varepsilon > 0$.

Suppose that $V - \varepsilon W \geq 0$ for some $\varepsilon > 0$. By the assumption, we can choose a bounded Borel set $A$ of positive measure such that $\bar{A} \subset D$ and

$$a = \inf\{W(x) \; ; \; x \in A\} > 0 \; .$$

Put $w(x) = \varepsilon a \chi_A(x)$, where $\chi_A$ is the characteristic function of $A$. Denote by $G_j$ the Green's function for the operator

$-\Delta+V - \varepsilon W + w$ in $D_j$ with Dirichlet zero boundary condition. For $j$ with $\bar{A} \subset D_j$, define a function $h_j$ on $D_j$ by

$$(2.13) \qquad h_j(x) = \int G_j(x,y)w(y)dy \; .$$

We have that $h_j$ belongs to $H_0^1(D_j)$, where $H_0^1(D_j)$ is the completion of $C_0^\infty(D_j)$ in terms of the norm of Sobolev space of order 1. By (2.4)

$$\int_{D_j} [(-\Delta+V-\varepsilon W)h_j(x)]h_j(x)dx \geq 0.$$

Noting that $(-\Delta+V-\varepsilon W+w)h_j = w$, we have that

$$\int wh_j^2 dx \leq \int wh_j dx \leq \left(\int wh_j^2 dx\right)^{1/2}\left(\int w \, dx\right)^{1/2} \; .$$

Thus

$$(2.14) \qquad \int wh_j^2 \, dx \leq \int w \, dx \; ,$$

which implies

$$(2.15) \qquad (\varepsilon a)^2 \int_A dx \left(\int_A G_j(x,y)dy\right)^2 \leq \int_A dx \; .$$

This shows that $G_j$ converges, which means that $V-\varepsilon W+w$ is subcritical. Denoting by $H_j$ the Green's function for the operator $-\Delta+V$ in $D_j$ with zero Dirichlet boundary condition, we have

$$(2.16) \qquad (-\Delta+V-\varepsilon W+w)(G_j - H_j) = (\varepsilon W - w)H_j \; .$$

Since $\varepsilon W - w \geq 0$ and $H_j \geq 0$, we thus obtain that $H_j \leq G_j$. This together with the convergence of $G_j$ shows that $V$ is subcritical.

Conversely, suppose that $V$ is subcritical. We first claim that there exists $\varepsilon > 0$ such that for any $t$ with $|t| < \varepsilon$, any domain $E$ with smooth compact boundary in $D$, and any $g$ in $C_0^\infty(E)$, the equation

$$(2.17) \qquad (-\Delta + V + tW)u = g \quad \text{in} \quad E \ , \ u \in H_0^1(E) \ ,$$

has a solution. Denote by $G_E$ the integral operator with kernel $G_E(x,y)$, the Green's function for $-\Delta+V$ in $E$ with zero Dirichlet boundary condition. Then (2.17) is equivalent to

$$(2.18.1) \qquad u = -t\, G_E W^{1/2}(W^{1/2}u) + G_E g \ ,$$

$$(2.18.2) \qquad (1 + t\, W^{1/2} G_E W^{1/2})(W^{1/2}u) = W^{1/2} G_E g \ ,$$

where $W^{1/2}$ is the multiplication operator by $(W(x))^{1/2}$. Note that $W^{1/2}\, G_E W^{1/2}$ is a bounded operator in $L_2(E)$ with norm smaller than

$$\| W^{1/2}\; G W^{1/2} \|_{L_2(D) \to L_2(D)} \ ,$$

where $G$ is the integral operator with kernel $G(x,y)$, the minimal Green's function for $V$. Thus there exists a positive number $\varepsilon > 0$ such that the equation (2.18.2) admits a solution for $t$ with $|t| < \varepsilon$. Now, suppose that there exist $t$ and

$\phi$ in $C_0^\infty(D)$ such that $|t| < \varepsilon$ and

$$\int |\nabla\phi(x)|^2 + (V + tW)|\phi(x)|^2 dx < 0 .$$

Since $-\Delta + V + tW > 0$ on a sufficiently small domain, this implies that, for some domain $E$, the operator $-\Delta + V + tW$ on $E$ with zero Dirichlet boundary condition has the eigenvalue $0$. This is a contradiction to the first claim. The proof is now complete. Q.E.D.

Incidentally, the proof of Theorem 2.4 yields

Theorem 2.5. Let $W$ be a nonnegative function which is positive on a set of positive measure and satisfies (2.1) with $V$ replaced by $W$. Then we have:

(i) Suppose that $V$ is critical. Then $V + W$ is subcritical, and $V - W$ is supercritical.

(ii) If $V$ is subcritical, then $V + W$ is subcritical.

A direct consequence of Theorem 2.5 is worth mentioning.

Theorem 2.6. Let $D'$ and $D''$ be domains such that $D' \subset D \subset D''$, $D\backslash D'$ and $D''\backslash D$ have an interior point. Let $V$ be extended to $D''$ with the same property. Then we have:

(i) Suppose that $V$ is critical in $D$. Then $V$ is subcritical in $D'$, and supercritical in $D''$.

(ii) If $V$ is subcritical in $D$, then $V$ is subcritical in $D'$.

Proof. (ii) is clear. Let us show (i). Choose a function $W$ in $C_0^\infty(D\backslash D')$ such that $W \geq 0$ and $W \neq 0$. Let $H$ be the minimal Green's function for $-\Delta + V + W$ in $D$. Then $H$ satisfies (2.3) with $D$ replaced by $D'$. Thus $V$ is subcritical in $D'$. This together with (ii) shows that $V$ is supercritical in $D''$. Q.E.D.

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Choose a closed ball $B$ with center $x^0$ and $B \subset D_1$, and put $w_j(x) = \chi_B(x)/j$. Denote by $\lambda_j^k$ the principal eigenvalue of the operator $-\Delta + V - w_j$ on $D_k$ with zero Dirichelet condition. Then the mini-max principle says that $\{\lambda_j^k\}_k$ is a decreasing sequence. Furthermore, by Theorem 2.4, $\lambda_j^k < 0$ for sufficiently large $k$. Since $-\Delta + V$ is strictly positive in $D_1$, $\lambda_j^1 > 0$ for sufficiently large $j$. Thus, for such $j$, we can choose $k(j)$ such that $\lambda_j^{k(j)} \geq 0$ and $\lambda_j^{k(j)+1} < 0$. Clearly, $k(j) \to \infty$ as $j \to \infty$. Hence there exists a domain $E_j$ and an eigenfunction $u_j$ such that $D_{k(j)} \subset E_j \subset D_{k(j)+1}$, $u_j \in H_0^1(E_j)$, $u_j > 0$ in $E_j$, $u_j(x^0) = 1$, and

$$(2.19) \qquad (-\Delta + V - w_j)u_j = 0 \qquad \text{in } E_j.$$

By the Harnack inequality, the function $u(x) = \lim_{j\to\infty} u_j(x)$ is a positive solution of $(-\Delta+V)u = 0$ in $D$. Denoting by $H_j$ the Green's function for $-\Delta+V+W$ in $E_j$ with zero Dirichlet boundary condition, we have

$$u_j(x) = \int H_j(x,y)(w_j(y) + W(y))u_j(y)dy .$$

Letting $j\to\infty$ , we thus obtain (2.8).

Now, let $v$ be any normalized positive solution of $(-\Delta+V)v = 0$ in $D$. Then we have (cf. [12]) that

$$\int_{E_j} u_j(-\Delta+V)u_j dx = \int_{E_j} |\nabla(u_j/v)|^2 v^2 \, dx.$$

Thus, for any fixed $D_i$ , we have

$$(2.20) \qquad 0 = \lim_{j\to\infty}\int_D w_j u_j^2 \, dx \geq \lim_{j\to\infty}\int_{D_i} |\nabla(u_j/v)|^2 v^2 \, dx$$
$$= \int_{D_i} |\nabla(u/v)|^2 v^2 \, dx.$$

Hence $v = u$. Q.E.D.

Incidentally, Theorem 2.4 and the proof of Theorem 2.2 yield another criterion of criticality. In what follows $C(D)$ denotes the set of all continuous functions on $D$.

Theorem 2.7. $V$ is critical iff there exist a positive solution $u$ and a sequence $\{\phi_j\}_j$ in $H_0^1(D) \cap C(D)$ such that $\phi_j$ converges to $u$ as $j\to\infty$ uniformly on any compact subset of $D$ and

$$(2.21) \qquad \int |\nabla\phi_j|^2 + V|\phi_j|^2 \, dx \to 0 \quad \text{as} \quad j\to\infty .$$

Proof. Assume that there exist $u$ and $\phi_j$ satisfying the properties in the theorem. Suppose that $V$ is subcritical. Then Theorem 2.4 implies that there exists a function $w$ in $C_0^\infty(D)$ such that $w(x^0) > 0$ and $-\Delta + V - w \geq 0$. Since

$$\int w|\phi_j|^2 dx \leq \int |\nabla\phi_j|^2 + V|\phi_j|^2 dx ,$$

we have that $\int w\, u^2 dx \leq 0$. This is a contradiction. The converse assertion is clear from the proof of Theorem 2.2. Q.E.D.

Remark. Let $-\Delta + V \geq 0$ and $H$ be a nonnegative self-adjoint realization in $L_2(D)$ of the operator $-\Delta + V$ with domain $C_0^\infty(D)$. Then Theorem 2.7 implies that if $V$ is critical, then $\inf \mathrm{spec}(H) = 0$, and that if $0$ is an eigenvalue of $H$, then $V$ is critical. Note that $\inf \mathrm{spec}(H) = 0$ does not imply that $V$ is critical. Such potentials as $\inf \mathrm{spec}(H) = 0$ are more interesting than the others from the point of asymptotic behavior as $t\to\infty$ of $e^{-tH}$; in studying it the positive solutions of the equation

$$(H - \inf \mathrm{spec}(H))u = 0$$

play a crucial role (see [13] and [15]).

Now, suppose that $V$ is subcritical and $W$ is the non-negative function as in Theorem 2.5. Put

$$\mu = \inf\{\lambda \in R^1 ; V + \lambda W \text{ is subcritical}\} . \tag{2.22}$$

Since $V + \lambda W$ is supercritical for negative $\lambda$ with $|\lambda| >> 1$, Theorem 2.5 implies that $\mu$ is a nonpositive number such that

$V + \lambda W$ is subcritical for $\lambda > \mu$, supercritical for $\lambda < \mu$, and critical or subcritical for $\lambda = \mu$. The following example shows that $V + \mu W$ can be subcritical.

Example. Let $D = R^n$, $n \geq 3$, $V = 0$, and $W(x) = (1+x_1)^{\beta}\chi(x)$ for some $\beta > -2$, where $\chi$ is the characteristic function of the set $\{x = (x_1, x') \in R^n ; x_1 > 1, |x'| < 1\}$. Then $\mu = 0$ and $V + \mu W$ is subcritical (cf. [2],[3],[8]).

But, if $W$ has compact support, then $V + \mu W$ is critical. This is guranteed by the following theorem, which also gives a method to construct critical potentials from a subcritical potential.

Theorem 2.8. Let $V$ be subcritical and $W$ a real-valued function with compact support in $D$ which is not zero on a Borel set of positive measure and belongs to $L_p(D)$, where $p > n/2$ for $n \geq 2$ and $p = 1$ for $n = 1$. Put

$$(2.23)\qquad \begin{aligned} S_+ &= \{\lambda \in R^1 ; V + \lambda W \text{ is subcritical}\}, \\ S_0 &= \{\lambda \in R^1 ; V + \lambda W \text{ is critical}\}, \\ S_- &= \{\lambda \in R^1 ; V + \lambda W \text{ is supercritical}\}. \end{aligned}$$

Then $S_\pm$ are nonempty open sets, and $S_0$ is a nonempty finite set. If we assume, in addition, that $W \geq 0$, then

$$(2.24)\qquad S_0 = \{-1/M\},$$

where $M$ is the greatest eigenvalue of the compact self-adjoint operator $T$ in $L_2(D)$ defined by

$$(2.25) \qquad T = W^{1/2}\, GW^{1/2} .$$

Here $W^{1/2}$ is the multiplication operator by $W(x)^{1/2}$ and $G$ is the integral operator with kernel $G(x,y)$, the minimal Green's function for $V$.

Proof. We first treat the case where $W \geq 0$. Since $S_-$ is clearly a nonempty open set, Theorems 2.4 and 2.5 show that $S_+$ is a nonempty open set and $S_0 = \{\mu\}$, where $\mu$ is the number defined by (2.22). We see from the proof of Theorem 2.4 that $\lambda$ is in $S_+$ if $\lambda > -1/M$. Thus, in order to show (2.24) we have only to prove that $\lambda$ is in $S_-$ if $\lambda < -1/M$. Denote by $T_j$ the integral operator in $L_2(D_j)$ defined by (2.25) with $G$ replaced by $G_j$. Since $T_j \uparrow T$ in the operator norm and $1 + \lambda T$ has negative eignevalue, there exist $j$, $\varepsilon > 0$, and $v_j$ in $L_2(D_j)$ such that

$$(1 + \lambda T_j)v_j = -\varepsilon v_j .$$

Put $u_j = -\lambda G_j W^{1/2} v_j$. We have that $W^{1/2} u_j = (1+\varepsilon) v_j$ . Thus

$$(-\Delta + V + \lambda W) u_j = \varepsilon\lambda(1+\varepsilon)^{-1} u_j \ , \ u_j \in H_0^1(D_j) \ .$$

Hence (2.4) does not hold for some $\phi$ in $C_0^\infty(D_j)$, which implies that $V + \lambda W$ is supercritical. The proof of (2.24) is now complete.

We next treat the general case where both $W_\pm(x) = \max(\pm W(x),0)$ do not vanish identically. Since

$$\pm \int W\phi^2 dx > 0$$

for some $C_0^\infty$-function approximating the characteristic function of $\mathrm{Supp}(\pm W)$, we obtain that $\lambda$ belongs to $S_-$ if $|\lambda| >> 1$. Clearly, $S_\pm$ are nonempty open sets. It remains to prove discreteness of $S_0$. For $t \geq 0$, we denote by $f(t)$ the greatest eigenvalue of the operator $W_-^{1/2} H_t W_-^{1/2}$, where $H_t$ is the integral operator with kernel $H_t(x,y)$, the minimal Green's function for $-\Delta + V + tW_+$. It is known (see [10, p.386]) that $f(t)$ is real analytic on $[0,\infty)$. On the other hand, (2.24) implies that $\lambda$ belongs to $S_0$ iff $\lambda = 1/f(\lambda)$. Hence $S_0 \cap [0,\infty)$ is discrete. Similarly, it can be shown that $S_0 \cap (-\infty,0]$ is also discrete. Q.E.D.

So far we have concentrated on difference of structures of positive solutions. Now we investigate the similarity of structures.

Definition 2.9. Let $V_1$ and $V_2$ satisfy (2.1) and (2.4). $V_1$ and $V_2$ are said to be P-equivalent if the following statement (i) or (ii) holds:

(i) Both $V_1$ and $V_2$ are critical and the normalized positive solutions $u_1$ and $u_2$ for $V_1$ and $V_2$, respectively, are equivalent, i.e., $cu_1 \leq u_2 \leq Cu_1$ for some positive constants $c$ and $C$.

(ii) Both $V_1$ and $V_2$ are subcritical, the minimal boundaries $\sigma_1$ and $\sigma_2$ for $V_1$ and $V_2$, respectively, are homeomorphic, and there exist positive constants $c$ and $C$ such that for any $\omega$ in $\sigma_1$

(2.25) $$cP_1(x,\omega) \le P_2(x,I\omega) \le CP_1(x,\omega) \quad \text{in } D,$$

where $I$ is the homeomorphism from $\sigma_1$ to $\sigma_2$, and $P_1(x,\omega)$ and $P_2(x,I\omega)$ are the minimal solutions for $V_1$ and $V_2$, respectively.

Theorem 2.10. Let $V$ and $V+W$ satisfy (2.1), and $-\Delta + V \ge 0$. Then one has:

(i) If $V$ and $V+W$ are P-equivalent, then $W$ is of the form

(2.26) $$W = \{\Delta f + 2(\nabla u/u)\cdot\nabla f\}/f$$

for a positive solution $u$ of $(-\Delta+V)u = 0$ and a continuous function $f$ in $D$ such that $0 < \inf f \le \sup f < \infty$ and $\Delta f \in L_{p,loc}(D)$.

(ii) If $W$ has the form (2.26), then both $V$ and $V+W$ are critical or subcritical.

(iii) If $W$ has the form (2.26) and $V$ is critical, then $V$ and $V+W$ and P-equivalent.

Proof. (i) Suppose that $V$ and $V+W$ are P-equivalent. Then there are positive solutions $u$ and $v$ for $V$ and $V+W$, respectively, such that $cu \le v \le Cu$ for some $c$ and $C$. Put $f = v/u$. Then we get (2.26), since $V = \Delta u/u$ and $V+W = \Delta(fu)/fu$.

(ii) Suppose that $V$ is critical. Let $\{\phi_j\}_j$ be the sequence in $H_0^1(D) \cap C(D)$ having the properties described in Theorem 2.7. Put $v = fu$ and $\psi_j = f\phi_j$. Clearly, $v$ is a positive solution of the equation $(-\Delta+V+W)v = 0$ in $D$, and $\psi_j$ belongs to $H_0^1(D) \cap C(D)$ and converges to $v$ as $j \to \infty$ uniformly on any compact subset of $D$. We have

$$\int |\nabla\psi_j|^2 + (V+W)|\psi_j|^2 \, dx = \int |\nabla(f\phi_j/fu)|^2 (fu)^2 \, dx$$

$$\leq C^2 \int |\nabla(\phi_j/u)|^2 u^2 \, dx = C^2 \int |\nabla\phi_j|^2 + V|\phi_j|^2 \, dx .$$

Thus $\psi_j$ satisfies the condition (2.21) with $V$ and $\phi_j$ replaced by $V+W$ and $\psi_j$, respectively. Hence Theorem 2.7 shows that $V+W$ is critical. Conversely, suppose that $V+W$ is critical. Then $v = fu$ must be a constant multiple of the normalized positive solution $w$ of the equation $(-\Delta+V+W)w = 0$. Hence the same argument as above shows that $V$ is critical. This proves (ii).

(iii) directly follows from (ii). Q.E.D.

Remark. Even if $V$ and $V+W$ are P-equivalent, $W$ can behave very wildly at infinity. For example, let $D = R^2$ and $W(x) = \Delta(\sin g(x))/(2 + \sin g(x))$ for a real-valued $C^\infty$-function on $R^2$. Then $W$ and $0$ are P-equivalent.

Theorem 2.11. Suppose that $V$ and $V+W$ satisfy (2.1) and (2.4). If $W$ has compact support and is of the form (2.26), then $V$ and $V+W$ are P-equivalent.

Proof. It suffices to show that if both $V$ and $V+W$ are subcritical, then $V$ and $V+W$ are P-equivalent. Put $W_\pm(x) = \max(\pm W(x), 0)$. Since $V$, $V+W_+$, and $(V + W_+)-W_-$ are subcritical and $W_\pm$ have compact support, we may and shall assume that $W \geq 0$ and $W \not\equiv 0$.

We denote by $G_0$, $\Sigma_0$, $\sigma_0$, $P_0$ the minimal Green's function, Martin boundary, minimal Martin boundary, minimal solution for $V$, respectively; the corresponding ones for $V+W$ are denoted by $G$, $\Sigma$, $\sigma$, $P$. We have

$$(2.27)\qquad \begin{aligned} G(x,y) &= G_0(x,y) - \int G_0(x,z)W(z)G(z,y)dz \\ &= G_0(x,y) - \int G(x,z)W(z)G_0(z,y)dz. \end{aligned}$$

For $\omega$ in $\Sigma_0$, choose a fundamental sequence $\{y_j\}_j$ which is a representative of $\omega$. Then we have

$$(2.28)\qquad \lim_{j\to\infty} G(x,y_j)/G_0(x^0,y_j) = P_0(x,\omega) - \int G(x,z)W(z)P_0(z,\omega)dz.$$

We claim that the right hand side of (2.28) is not identically zero. Suppose that it is so. Then $v(x) = W(x)^{1/2}\, P_0(x,\omega)$ is a nontrivial solution of $Tv = v$, where $T$ is the operator (2.25). This together with Theorem 2.8 contradicts subcriticality of $V$. This proves the claim. Thus, by Harnack's inequality, the right hand side of (2.28) is a positive solution of the equation

$(-\Delta + V + W)u = 0$. Hence $\{y_j\}_j$ is also a fundamental sequence for $V+W$ which satisfies

$$(2.29)\quad P(x,I\omega) \equiv \lim_{j\to\infty} \frac{G(x,y_j)}{G(x^0,y_j)} = \frac{P_0(x,\omega)-\int G(x,z)W(z)P_0(z,\omega)dz}{P_0(x^0,\omega)-\int G(X^0,z)W(z)P_0(z,\omega)dz},$$

where $I$ is a mapping from $\Sigma_0$ to $\Sigma$ such that $I\omega$ is an equivalence class of $\{y_j\}$. We first claim that $I$ is an injection, that is, $P(x,I\omega) \not\equiv P(x,I\omega')$ if $\omega \neq \omega'$. Suppose that $P(x,I\omega) \equiv P(x,I\omega')$. Then we have by (2.29) that for some positive constants $c(\omega)$ and $c(\omega')$

$$c(\omega)\{P_0(x,\omega) - \int G(x,z)W(z)P_0(z,\omega)dz\}$$

$$= c(\omega')\{P_0(x,\omega') - \int G(x,z)W(z)P_0(z,\omega')dz\}.$$

Thus $v(x) = W(x)^{1/2}[c(1)P_0(x,\omega) - c(1')P_0(x,\omega')]$ is not identically zero and satisfies the equation $Tv = v$, which is a contradiction. This proves the claim.

The same argument as above shows that if $\{y_j\}$ is a fundamental sequence for $V+W$, then it is a fundamental sequence for $V$. This implies that $I$ is onto. On the other hand, we see by (2.29) and (2.11) that $I$ is continuous. Hence $I$ is a homeomorphism from $\Sigma_0$ to $\Sigma$. Let us show that $I$ is a bijection from $\sigma_0$ to $\sigma$. Denote by $\mathbb{P}_0$ and $\mathbb{P}$ the set of all positive solutions for $V$ and $V+W$, respectively. We define an operator $T$ from $\mathbb{P}_0$ to $\mathbb{P}$ by

$$(2.30) \qquad Tu(x) = u(x) - \int G(x,z)W(z)u(z)dz,$$

and an operator $S$ from $\mathbb{P}$ to $\mathbb{P}_0$ by

$$(2.31) \qquad Sv(x) = v(x) + \int G_0(x,z)W(z)v(z)dz .$$

We have by the resolvent equation that

$$STu(x) = u(x) - \int G(x,z)W(z)u(z)dz + \int G_0(x,z)W(z)u(z)dz$$

$$+ \iint G_0(x,z)W(z)G(z,z')W(z')u(z')dzdz'$$

$$= u(x).$$

Similarly, $TS$ is the identiry map on $\mathbb{P}$ . Hence $S$ is a bijection from $\mathbb{P}$ to $\mathbb{P}_0$ . Clearly, $S$ is order preserving: If $v_1 \leq v_2$, then $Sv_1 \leq Sv_2$. Suppose that $u_1 \leq u_2$. Since either $u_1 = u_2$ or $u_2 - u_1 > 0$, we have that $T(u_2-u_1) \geq 0$, which implies that $T$ is order preserving. Now, suppose that $v$ is a minimal solution in $\mathbb{P}$ , and that $u \leq Sv$. Then we have that $Tu \leq T(Sv) = v$, which implies that $Tu = cv$ for some $c$. Thus $u = cSv$, which proves the minimality of $Sv$. Similarly, if $u$ is a minimal solution in $\mathbb{P}_0$ , then $Tu$ is a minimal solution in $\mathbb{P}$ . Since $T$ is considered to be an extention of the map $I$, this proves that $I$ is a bijection from $\sigma_0$ to $\sigma$. It remains to show that there exists a constant

$C$ such that

$$(2.32) \qquad C^{-1} P(x,\omega) \le P_0(x, I^{-1}\omega) \le CP(x,\omega)$$

for all $x$ in $D$ and $\omega$ in $\sigma$, where $I^{-1}$ is the inverse map of $I$. Choose a domain $B$ such that $\mathrm{Supp}\, W \subset B$ and $\bar{B}$ is a compact set in $D$. Since $P(x^0,\omega) = 1$, the Harnack inequality implies that

$$(2.33) \qquad C_1^{-1} \le P(x,\omega) \le C_1, \qquad x \in B,\ \omega \in \Sigma,$$

for some $C_1 > 0$. Denoting by $H(x,z)$ the standard Green's function for $-\Delta$, we have

$$(2.34) \qquad G_0(x,z) \le C_2(|H(x,z)| + 1), \qquad x,z \in B,$$

for some $C_2$. Combining (2.33) and (2.34), we obtain that

$$(2.35) \qquad 1 \le SP(x,\omega)/P(x,\omega) \le C_3, \qquad x \in B\ , \quad \omega \in \Sigma,$$

for some $C_3$ , where $S$ is the operator defined by (2.31). On the other hand, for $z$ in $\mathrm{Supp}\, W$

$$(-\Delta + V + W)G_0(x,z) = 0 \quad \text{in} \quad D\backslash\bar{B}\ .$$

Since (2.33) and (2.34) yield $\sup\{G_0(x,z)/P(x,\omega)\ ;\ x \in \partial B\ ,$

$z \in \text{Supp } W$ , $\omega \in \Sigma\} < \infty$, we thus obtain that

$$(2.36) \qquad G_0(x,z)/P(x,\omega) \leq C_4, \qquad x \in D\backslash B \ , \quad \omega \in \Sigma,$$

for some $C_4$. This implies that $1 \leq SP(x,\omega)/P(x,\omega) \leq C_5$ for all $x$ in $D\backslash B$ and $\omega$ in $\Sigma$. This together with (2.35) shows that $1 \leq SP(x,\omega)/P(x,\omega) \leq C_6$ for all $x$ in $D$ and $\omega$ in $\Sigma$. Since

$$P_0(x,I^{-1}\omega) = SP(x,\omega)/SP(x^0,\omega) \ ,$$

we thus get (2.32) with $C = C_6$. Q.E.D.

We close this section with a conjecture concerning Dirichlet problem at infinity.

Conjecture 2.12. Suppose that $V$ is subcritical. Then there exists a nonnegative measure $\nu$ on $\sigma$ such that

$$(2.37) \qquad \lim_{x\to\omega'} \int_\sigma P(x,\omega)\phi(\omega)d\nu(\omega)/\int_\sigma P(x,\omega)d\nu(\omega) = \phi(\omega')$$

for any continuous function $\phi$ on $\sigma$.

This conjecture is clearly valid if $D$ is a bounded domain with smooth boundary and $V$ belongs to $L_p(D)$. We will show later that the conjecture is also valid in the case where $D = R^n$ and $V$ is a pricipally radially symmetric potential, or $D$ is an interval in $R^1$ (see Theorems 3.4, 5.5, and A9).

## §3. Radially symmetric potentials.

This section is devoted to the study of the structure of positive solutions on $R^n$ of a Schrödinger equation with radially symmetric potential. In studying it we use the separation of variables technique.

Throughout the section we assume that

(3.1) $V$ is a real-valued radially symmetric function in $L_{p,loc}(R^n)$ for some $p > n/2$, where $n \geq 2$.

In what follows a positive solution means a positive continuous function satisfying the Schrödinger equation

$$(3.2) \qquad (-\Delta + V)u = 0 \quad \text{on} \quad R^n .$$

We say that $u$ is normalized when $u(o) = 1$.

It is easily seen that for any $j = 0,1,2,..$ there exists a unique solution $g_j$ of the initial value problem

$$(3.3) \quad -g_j''(r) - \frac{n-1}{r} g_j'(r) + \{\frac{j(j+n-2)}{r^2} + V(r)\}g_j(r) = 0 \quad \text{on} \quad (0,\infty),$$

$$(3.4) \qquad g_j(r) = r^j + o(r^{j+1}) \quad \text{as} \quad r \to 0 .$$

The following theorems are main results of this section.

Theorem 3.1. (i) There holds the condition (2.4) with $D$ replaced by $R^n$, i.e., $-\Delta + V \geq 0$, if and only if $g_0(r) > 0$

for any $r > 0$.

(ii) $V$ is subcritical iff $g_0 > 0$ and

$$(3.5) \qquad \int_1^\infty r^{1-n}(g_0(r))^{-2}dr < \infty .$$

(iii) $V$ is critical iff $g_0 > 0$ and the integral of the left hand side of (3.5) diverges; in this case, $u(x) = g_0(|x|)$ is a unique normalized positive solution.

Theorem 3.2. Suppose that $V$ is subcritical. Then there exists a unique normalized positive solution iff

$$(3.6) \qquad \int_1^\infty t^{n-3} g_0(t)^2 [\int_t^\infty s^{1-n} g_0(s)^{-2} ds] dt = \infty .$$

In this case, $\sum = \sigma = \{\infty\}$ and $P(x,\infty) = g_0(|x|)$. Here $\sum$, $\sigma$, and $P$ are those appearing in Theorem 2.3.

Theorem 3.3. Suppose that $V$ is subcritical and (3.6) does not hold. Then $\sum = \sigma = S^{n-1}$, where $S^{n-1} = \{\omega \in R^n;$ $|\omega| = 1\}$ ; for any $j$ ,

$$(3.7) \qquad a_j = \lim_{r\to\infty} g_0(r)/g_j(r)$$

exists and is a positive number; and $P(x,\omega)$ , $\omega \in S^{n-1}$ , is determined as follows:

(i) For $n \geq 3$ ,

$$(3.8)\qquad P(x,\omega) = \sum_{j=0}^{\infty} \frac{2j+n-2}{n-2} a_j g_j(|x|) P_j(x\cdot\omega/|x|) ,$$

where $P_j(t)$ are Gegenbauer's polynomials defined by

$$(3.9)\qquad \sum_{j=0}^{\infty} P_j(t) z^j = (1-2tz+z^2)^{-(n-2)/2}, \quad |t| \le 1 \text{ and } |z| < 1.$$

(ii) For $n = 2$ ,

$$(3.10)\qquad P(x,\omega) = g_0(|x|) + \sum_{j=1}^{\infty} 2j a_j g_j(|x|) P_j(x\cdot\omega/|x|) ,$$

where $P_j(t)$ are polynomials defined by

$$(3.11)\qquad \sum_{j=1}^{\infty} P_j(t) z^j = -\log(1-2tz+z^2) , \quad |t| \le 1 \text{ and } |z| < 1.$$

Theorem 3.4. Suppose that $V$ is subcritical. Then one has:

(i) If (3.6) holds, then the Dirichlet problem at infinity

$$(3.12)\qquad (-\Delta + V)u = 0 \text{ on } R^n, \quad \lim_{|x|\to\infty} u(x)/g_0(|x|) = c ,$$

where $c$ is a constant, admits a unique solution $u(x) = cP(x,\infty)$.

(ii) Assume that (3.6) does not hold. Then $x$ converges to $\omega \in \sigma$ in the topology induced by the metric (2.11) iff $|x| \to \infty$ and $x/|x| \to \omega$, and the Dirichlet problem at infinity

$$(3.13)\qquad (-\Delta + V)u = 0 \text{ on } R^n, \quad \lim_{x\to\omega} u(x)/g_0(|x|) = \phi(\omega) ,$$

where $\phi$ is a continuous function on $S^{n-1}$, admits a unique solution $u$ given by

$$(3.14) \qquad u(x) = \int_{|\omega|=1} P(x,\omega)\phi(\omega)\,d\omega/A_n .$$

Here $d\omega$ is the standard surface element on $S^{n-1}$ and $A_n$ is the area of $S^{n-1}$.

Remark. Suppose that $g_0 > 0$. Let $W$ be a radially symmetric function of the form

$$W = \{f'' + [(n-1)/r + 2g_0'/g_0]f'\}/f$$

for some radially symmetric function $f$ with $0 < \inf f \leq \sup f < \infty$ and $\Delta f \in L_{p,loc}(R^n)$. Then it is easily seen from Theorems 3.1 and 3.2 that if (3.6) holds, then $V$ and $V+W$ are P-equivalent (see Definition 2.9). As for the case where (3.6) does not hold, however, we do not know whether $V$ and $V+W$ are P-equivalent (cf. Theorems 2.10 and 2.11).

The proof of Theorems 3.1 to 3.4 is decomposed into a few steps.

Lemma 3.5. $-\Delta + V \geq 0$ iff $g_0 > 0$.

Proof. Recall that $-\Delta + V \geq 0$ iff there exists a positive solution of (3.2), and note that $u(x) = g_0(|x|)$ is a unique radially symmetric normalized solution of (3.2). Thus, it is clear that if $g_0 > 0$, then $-\Delta + V \geq 0$. Next, suppose that there exists a positive solution $u$ of (3.2). Put

$$g(x) = \int_{|\omega|=1} u(|x|\omega)\,d\omega/u(0)A_n.$$

Then $g_0(x) = g(x) > 0$. Q.E.D.

From now on we assume that $g_0 > 0$.

For $j \geq 1$, put $h_j(r) = g_j(r)/g_0(r)$. Then we have

Lemma 3.6. $h_j$ is a strictly increasing differentiable function with $h_j(r) > 0$ for any $r > 0$. Furthermore,

$$h_j(r) = j(j+n-2)\int_0^r g(r,t)h_j(t)\,dt\ , \tag{3.15}$$

where

$$g(r,t) = \int_t^r t^{-2}(t/s)^{n-1}(g_0(t)/g_0(s))^2 ds. \tag{3.16}$$

Proof. we have

$$h_j'' + (2g_0'/g_0 + (n-1)/r)h_j' - j(j+n-2)r^{-2}h_j = 0\ ,$$

$$h_j(r) = r^j + o(r^{j+1}) \quad \text{as} \quad r \to 0.$$

Form this we obtain (3.15) and (3.16), which imply that $h_j(r) > 0$ and $h_j'(r) > 0$ for $r > 0$. Q.E.D.

Lemma 3.7. (i) If $\lim_{r\to\infty} h_1(r) = \infty$, then $\lim_{r\to\infty} h_j(r) = \infty$ for any $j$. (ii) If $\lim_{r\to\infty} h_1(r) < \infty$, then $\lim_{r\to\infty} h_j(r) < \infty$ for any $j$. (iii) (3.6) holds iff $\lim_{r\to\infty} h_1(r) = \infty$.

Proof. (i) Suppose that $\lim_{r\to\infty} h_1(r) = \infty$. We see in the same way as in the proof of Lemma 3.6 that $g_j(r)/g_1(r)$ , $j = 2,3,\ldots$, are strictly increasing and positive for $r > 0$. Thus $h_j(r) = [g_j(r)/g_1(r)][g_1(r)/g_0(r)]$ goes to infinity as $r \to \infty$.

(ii) Suppose that $b = \lim_{r\to\infty} h_1(r) > \infty$. Put

$$g(\infty,t) = \lim_{r\to\infty} g(r,t) = t^{n-3} g_0(t)^2 \int_t^\infty s^{1-n} g_0(s)^{-2} ds.$$

Then (3.15) and the monotone convergence theorem show that

$$h_1(1)\int_1^\infty g(\infty,t)dt \leq \int_1^\infty g(\infty,t)h_1(t)dt \leq b/(n-1).$$

Choose $R$ so large that

$$j(j+n-2)\int_R^\infty g(\infty,t)dt < 1/2.$$

Then we obtain that for any $r > R$

$$h_j(r)/2 \leq j(j+n-2)\int_0^R g(\infty,t)h_j(t)dt. \tag{3.17}$$

But the right hand side of (3.17) is finite, since $h_j(t)/h_1(t)$ is bounded on $[0,R]$. This completes the proof of (ii).

(iii) follows easily from the proof of (ii). Q.E.D.

For $R > 0$ and $j = 0,1,\ldots,$ we define a function $f_{jR}$ by

$$(3.18) \qquad f_{jR}(r) = \max(2j+n-2,1)g_j(r)\int_r^R t^{1-n}g_j(t)^{-2}dt.$$

Clearly, $f_{jR}$ is a solution of the differential equation (3.3) such that $f_{jR}(R) = 0$, $f_{jR}(r) > 0$ for $r$ with $0 < r < R$, and

$$(3.19) \qquad (g_j'f_{jR} - f_{jR}'g_j)(r) = \max(2j+n-2,1)r^{1-n} .$$

Define $G_R(x,y)$ by

$$(3.20) \quad G_R(x,y) = c_n \sum_{j=0}^{\infty} g_j(|x|)f_{jR}(|y|)P_j(x\cdot y/|x||y|) \qquad \text{for } |x| \leq |y| \leq R,$$

$$= c_n \sum_{j=0}^{\infty} f_{jR}(|x|)g_j(|y|)P_j(x\cdot y/|x||y|) \qquad \text{for } |y| \leq |x| \leq R,$$

where $c_n$ and $P_j$ are as follows: (i) For $n \geq 3$, $c_n = \Gamma(n/2)/2(n-2)\pi^{n/2}$ with $\Gamma$ being the gamma function and $P_j$ are the polynomials defined by (3.9); and (ii) for $n = 2$, $c_n = 1/2\pi$, $P_0 = 1$, and $P_j(j \geq 1)$ are the polynomials defined by (3.11).

The following theorem is a key to the proof of Theorems 3.1 to 3.3.

Theorem 3.8. $G_R(x,y)$ is the positive Green's function for $-\Delta + V$ on $B_R = \{x;\ |x| < R\}$ with zero Dirichlet condition.

For the proof of Theorem 3.8 we prepare two elementary lemmas.

Lemma 3.9. The following asymptotic formulas as $j \to \infty$ hold uniformly for $r$ with $0 < r \leq R$.

$$g_j(r)/r^j = 1 + O(j^{-1}). \tag{3.21}$$

$$g_j'(r)/jr^{j-1} = 1 + O(j^{-1}). \tag{3.22}$$

$$f_{jR}(r)/r^{-j-n+2} = 1 + O(j^{-1}). \tag{3.23}$$

$$f_{jR}'(r)/(-j-n+2)r^{-j-n+1} = 1 + O(j^{-1}). \tag{3.24}$$

Proof. Put $w_j(r) = g_j(r)r^{-j}$. Then we have that for $2j + n - 2 > 0$

$$w_j(r) = 1 + \int_0^r \frac{1-(t/r)^{2j+n-2}}{2j+n-2} tV(t)w_j(t)dt. \tag{3.25}$$

The asymptotics (3.21) is easily derived from (3.25). (3.23) to (3.24) are shown similarly. Q.E.D.

Lemma 3.10. $\max_{-1\leq t\leq 1} |P_j(t)| = O(j^{n-3})$ as $j \to \infty$.

Proof. With $e^{i\tau} = t + i(1-t^2)^{1/2}$, we have that $1 - 2tz$

$+ z^2 = (z-e^{i\tau})(z-e^{-i\tau})$. For $n \geq 3$, Cauchy's formula yields

$$(3.26) \quad P_j(t) = \frac{1}{2\pi i}\int_{|z|=1-1/j} [(z-e^{i\tau})(z-e^{-i\tau})]^{-(n-2)/2} z^{-j-1}dz.$$

The lemma for $n \geq 3$ is derived from (3.26). The lemma for $n = 2$ is shown similarly. Q.E.D.

Now, let us prove Theorem 3.8.

Proof of Theorem 3.8. We have only to show that for any $\phi$ in $C_0^\infty(B_R)$

$$(3.27) \qquad \int G_R(x,y)(-\Delta+V(x))\phi(x)dx = \phi(y).$$

Since $G_R(x,0) = c_n f_{0R}(|x|)$, (3.27) for $y = 0$ can be easily shown. Let us treat the case where $|y| > 0$. For $j = 0,1,\ldots,$ put

$$(3.28) \qquad \phi_j(r;y/|y|) = \int_{|\omega|=1} \phi(r\omega)P_j(\omega\cdot y/|y|)d\omega,$$

and

$$\Phi_j(r) = -\frac{d}{dr}\left(r^{n-1}\frac{d}{dr}\phi_j(r;y/|y|)\right)$$
$$+ [j(j+n-2)r^{-2} + V(r)]r^{n-1}\phi_j(r;y/|y|).$$

Note that

$$-\Delta_x P_j(x\cdot y/|x||y|) = -|x|^{-2}\Lambda_\omega P_j(\omega\cdot y/|y|)$$
$$= j(j+n-2)|x|^{-2}P_j(\omega\cdot y/|y|),$$

where $\omega = x/|x|$ and $\Lambda$ is the Laplace-Beltrami operator on $S^{n-1}$. This together with Lemma 3.10 implies that $\Phi_j(r) = O(j^{-\nu})$ as $j \to \infty$ for any $\nu > 0$. Thus we have by Lemma 3.9 that

$$\int G_R(x,y)(-\Delta+V(x))\phi(x)dx$$
$$= \sum_{j=0}^{\infty} c_n[\int_0^c g_j(r)f_{jR}(c)\Phi_j(r)dr + \int_c^R g_j(c)f_{jR}(r)\Phi_j(r)dr],$$

where $c = |y|$. Integration by parts together with (3.19) then show that the left hand side of (3.27) is equal to

$$\sum_{j=0}^{\infty} \max(2j+n-2,1)c_n\phi_j(|y|;y/|y|). \tag{3.29}$$

Now, let $n \geq 3$. A fundamental solution for $-\Delta$ is given by $c_n|x-y|^{2-n}$, and

$$c_n|x-y|^{2-n} = c_n \sum_{j=0}^{\infty} |x|^j|y|^{-j-n+2} P_j(x\cdot y/|x||y|), \quad |x| < |y|$$
$$= c_n \sum_{j=0}^{\infty} |x|^{-j-n+2}|y|^j P_j(x\cdot y/|x||y|), \quad |y| < |x|.$$

Thus the same computation as above shows that

$$\int c_n|x-y|^{2-n}(-\Delta)\phi(x)dx$$

is equal to (3.29), which implies (3.27). As for $n = 2$, consider $-(\log|x-y|)/2\pi$. Then the same argument shows (3.27).

Q.E.D.

Remark. It is seen from the proof of Theorem 3.8 that for any $\phi$ in $C^\infty(S^{n-1})$

$$(3.30) \qquad \phi(\omega) = c_n \sum_{j=0}^{\infty} \max(2j+n-2,1) \int \phi(\omega')P_j(\omega\cdot\omega')d\omega'.$$

We are now ready to prove Theorems 3.1 to 3.3.

Proof of Theorem 3.1. (i) has been proved already. We have that $G_R(0,y) = c_n f_{0R}(|y|)$ for any fixed $y \neq 0$. The Harnack inequality implies that $V$ is subcritical iff

$$\lim_{R\to\infty} G_R(0,y) < \infty .$$

But, by (3.18), this is equivalent to (3.5), which shows (ii). (iii) is shown similarly. Q.E.D.

Theorems 3.1 and 3.8 and Lemma 3.6 yield

Theorem 3.11. Suppose that $V$ is subcritical. Put

$$(3.31) \qquad f_j(r) = \max(2j+n-2,1)g_j(r)\int_r^\infty t^{1-n}g_j(t)^{-2}dt.$$

Then the minimal Green's function $G(x,y)$ is given by

$$(3.32) \quad G(x,y) = c_n \sum_{j=0}^{\infty} g_j(|x|)f_j(|y|)P_j(x\cdot y/|x||y|) \text{ for } |x| \leq |y|,$$
$$= c_n \sum_{j=0}^{\infty} g_j(|y|)f_j(|x|)P_j(x\cdot y/|x||y|) \text{ for } |y| \leq |x|.$$

Proof of Theorems 3.2 and 3.3. We have that for any $x$

with $|x| < r$

$$(3.33)\qquad \frac{G(x,r\omega)}{G(0,r\omega)} = \sum_{j=0}^{\infty} \frac{f_j(r)}{f_0(r)} g_j(|x|)P_j(x\cdot\omega/|x|).$$

We have that with $b_j = \max(2j+n-2,1)/\max(n-2,1)$

$$f_j(r)/f_0(r) = b_j[g_0(r)/g_j(r)]^{-1}$$
$$\times[\int_r^{\infty} t^{1-n}g_0(t)^{-2}\{g_0(t)/g_j(t)\}^2dt]/[\int_r^{\infty} t^{1-n}g_0(t)^{-2}dt].$$

Thus, by Lemma 3.6, $f_j(r)/f_0(r) \leq b_jg_0(r)/g_j(r)$. Lemma 3.7 now shows that (3.6) is equivalent to

$$(3.34)\qquad \lim_{r\to\infty} f_j(r)/f_0(r) = 0 \quad \text{for any} \quad j \geq 1.$$

Hence (3.6) holds iff

$$(3.35)\qquad \lim_{|y_j|\to\infty} G(x,y_j)/G(0,y_j) = g_0(|x|).$$

This together with Theorem 2.3 proves Theorem 3.2. Similarly, if (3.6) does not hold, then $a_j > 0$ and

$$(3.36)\qquad \lim_{r\to\infty} f_j(r)/f_0(r) = a_jb_j .$$

This yields

$$(3.37)\qquad \lim_{\substack{|y_j|\to\infty \\ y_j/|y_j|\to\omega}} \frac{G(x,y_j)}{G(0,y_j)} = \sum_{j=0}^{\infty} a_jb_jg_j(|x|)P_j(x\cdot\omega/|x|) = P(x,\omega).$$

Since any sequence $\{y_j\}_j$ having no accumulation point in $R^n$ includes a subsequence $\{y_{j(k)}\}_k$ such that $|y_{j(k)}| \to \infty$ and $y_{j(k)}/|y_{j(k)}| \to \omega$ for some $\omega$ in $S^{n-1}$ as $k \to \infty$, Theorem 2.3 and (3.37) prove Theorem 3.3 except for the equality $\Sigma = \sigma$. It remains to show that $P(x,\omega)$ is minimal for any $\omega$ in $S^{n-1}$. Note that if $u(x)$ is a positive solution of (3.2), $u(Tx)$ is also a positive solution for any orthogonal transformation $T$. By Theorem 2.3, there exists $\omega^0$ in $S^{n-1}$ such that $P(x,\omega^0)$ is a minimal solution. Suppose that $u(x) \leq P(x,\omega)$. Then, choosing an orthogonal transformation $T$ such that $P(Tx,\omega) = P(x,\omega^0)$, we have that

$$u(Tx) \leq P(x,\omega^0).$$

Thus $u(Tx) = cP(x,\omega^0)$ for some constant $c$, which implies $u(x) = cP(x,\omega)$. This proves that $P(x,\omega)$ is minimal. Q.E.D.

It remain to prove Theorem 3.4. To this end, we need to investigate properties of $G(x,y)$ and $P(x,\omega)$. In the rest of this section we assume that $V$ is subcritical.

Theorem 3.12. Considering the minimal Green's function $G$ as a function of $|x|$, $|y|$, and $t = x \cdot y/|x||y|$, one has

$$\frac{\partial G}{\partial t}(|x|,|y|,t) > 0 \tag{3.38}$$

for $x, y$, and $t$ with $x \neq y$, $|x||y| \neq 0$, and $-1 < t \leq 1$.

Proof. We have

$$\frac{\partial G}{\partial t}(|x|,|y|,t) = \sum_{j=0}^{\infty} c_n f_j(|x|) g_j(|y|) P_j'(t) \tag{3.39}$$

for $|x| \geq |y|$. By (3.9) and (3.11),

$$\sum_{j=0}^{\infty} P_j'(t) z^j = a_n z(1-2tz+z^2)^{-n/2}, \tag{3.40}$$

where $a_n = n-2$ for $n \geq 3$ and $a_2 = 2$. This implies that $P_j'(1) > 0$ for $j \geq 1$, which together with (3.39) yields (3.38) with $t = 1$.

We next claim that for any fixed $R > 0$,

$$\frac{\partial G_R}{\partial t}(|x|,|y|,t) \geq 0 \tag{3.41}$$

for $x$ and $y$ with $x \neq y$, $|x||y| \neq 0$, and $|x|,|y| \leq R$. To this end we may and shall asssume that $V$ is smooth on $[0,R]$. Now assume that $n = 2$. Putting

$$H_R(x,y) = G_R(x,y) + (2\pi)^{-1}\log |x-y|,$$

we have that $(-\Delta+V)H_R(x,y) = 0$ in $B_R = \{x;\ |x| < R\}$ and $H_R(x,y) = (2\pi)^{-1}\log |x-y|$ for $x$ with $|x| = R$. Thus $H_R$ is smooth on $\bar{B}_R$. On the other hand,

$$(\partial/\partial t)(-\log |x-y|)\big|_{|x-y|=\varepsilon} = |x||y|\varepsilon^{-2}$$

for sufficiently small $\varepsilon > 0$. Therefore there exists a positive constant $\delta$ such that (3.41) holds for $x$ and $y$ with $0 < |x-y| \leq \delta$. Now we introduce the polar coordinates $(r,\theta)$ by

$$r = |x| \quad \text{and} \quad \cos\theta = t.$$

Considering $G_R$ as a function of $r$ and $\theta$, we have

$$(3.42) \qquad [-\partial^2/\partial r - r^{-1}\partial/\partial r - r^{-2}\partial^2/\partial\theta^2 + V(r)]G_R = 0$$

in $\Omega = \{(r,\theta);\ r < R,\ 0 < \theta < \pi\}\backslash\{x;\ |x-y| \leq \varepsilon\}$. Differentiating both hands of (3.42) we have

$$(3.43) \qquad (-\Delta+V)(\partial G_R/\partial\theta) = 0 \quad \text{in} \quad \Omega.$$

Since $\partial G_R/\partial\theta = -\sin\theta(\partial G_R/\partial t)$ and $G_R = 0$ for $|x| = R$, we see that $\partial G_R/\partial\theta \leq 0$ on the boundary of $\Omega$. This implies that $\partial G_R/\partial\theta \leq 0$ in $\Omega$, for $-\Delta + V > 0$ in $B_R$. The proof of the claim for $n = 2$ is now complete. Ir remains to prove the claim for $n \geq 3$. The same argument as above shows that (3.41) holds for $x$ and $y$ with $0 < |x-y| \leq \delta$, and that

$$(3.44) \qquad [-\Delta + V + (n-2)(r\sin\theta)^{-2}](\partial G_R/\partial\theta) = 0$$

in $B_R\backslash\{x;\ |x-y| \leq \delta$ or $\theta = 0$ or $\theta = \pi\}$. Let $\chi_\varepsilon$ be the characteristic function of the set

$$\{x \in R^n\ ;\ |x-sy| \leq \varepsilon \ \text{ for some } \ s \ \text{ in } \ R^1\}.$$

Put $F_\varepsilon = -\chi_\varepsilon(n-2)(r \sin\theta)^{-2}(\partial G_R/\partial\theta)$. Then we have that

$$[-\Delta + V + (n-2)(1-\chi_\varepsilon)(r \sin\theta)^{-2}](\partial G_R/\partial\theta) = F_\varepsilon$$

in $\Omega = B_R \backslash \{x;\ |x-y| \le \delta\}$, and that

$$\int_\Omega |F_\varepsilon| dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Denote by $H(x,y)$ the Green's function for the operator $-\Delta + V + (n-2)(1-\chi_\varepsilon)(r \sin\theta)^{-2}$ in $\Omega$ with zero Dirichlet condition. Since $\partial G_R/\partial\theta \le 0$ on the boundary of $\Omega$, we then have that

$$(\partial G_R/\partial\theta)(x,y) \le \int_\Omega H(x,z)F_\varepsilon(z,y)dz.$$

Noting that $H(x,z) \le I(x,z)$ with $I$ being the Green's function for $-\Delta + V$ in $\Omega$ with zero Dirichlet condition, we obtain that for $x$ with $x \ne sy$ for any $s$,

$$(\partial G_R/\partial\theta)(x,y) \le \lim_{\varepsilon\to 0} \int_\Omega H(x,z)F_\varepsilon(z,y)dz = 0.$$

This proves the claim for $n \ge 3$.

It follows from (3.41) that $\partial G/\partial t \ge 0$. But $\partial G/\partial t = -(\partial G/\partial\theta)(\sin\theta)^{-1}$ for $\sin\theta \ne 0$, and $\partial G/\partial\theta \not\equiv 0$ satisfies the equation (3.43) or (3.44). Hence (3.38) holds for $t$ with $-1 < t < 1$. This proves the theorem. Q.E.D.

Theorem 3.13. Let $W$ be a function satisfying (3.1) and $V \leq W$. Denote by $G^V$ and $G^W$ the minimal Green's functions for $V$ and $W$, respectively. Then

$$G^V \geq G^W \quad \text{and} \quad \partial G^V/\partial t \leq \partial G^W/\partial t. \tag{3.45}$$

Proof. The first half of (3.45) is clear. The second half will be proved if we show that $H_R \equiv \partial G_R^V/\partial\theta - \partial G_R^W/\partial\theta \geq 0$. The same argument as in the proof of Theorem 3.12 shows that

$$(-\Delta+V+(n-2)(r\sin\theta)^{-2})H_R = -(W-V)\partial G_R^W/\partial\theta \geq 0$$

in $B_R$ and $H_R|_{\partial B_R} = 0$. This implies that $H_R \geq 0$. Q.E.D.

Theorem 3.14. Suppose that (3.6) does not hold. Considering $P(x,\omega)$ as a function of $|x|$ and $t = x\cdot\omega/|x|$, one has

$$\frac{\partial P}{\partial t}(|x|,t) > 0 \tag{3.46}$$

for $|x| > 0$ and $-1 < t \leq 1$.

Proof. Theorem 3.12 and its proof imply that $\partial P/\partial t \geq 0$, $\partial P/\partial t(|x|,1) > 0$, and

$$[-\Delta + V + (n-2)(r\sin\theta)^{-2}](\partial P/\partial\theta) = 0 \tag{3.47}$$

in $R^n$, from which Theorem 3.14 follows. Q.E.D.

The following theorem is a direct consequence of the formula (3.32).

Theorem 3.15. For $|x| \leq |y|$,

$$(3.48) \qquad \int_{S^{n-1}} G(x,|y|\omega)\,d\omega/A_n = \int_{S^{n-1}} G(|x|\omega',y)\,d\omega'/A_n$$

$$= c_n g_0(|x|) f_0 |y|),$$

where $A_n$ is the area of $S^{n-1}$.

Concerning the asymptotic behavior as $|x| \to \infty$ of $P(x,\omega)$ we have

Theorem 3.16. Suppose that (3.6) does not hold. Then one has: (i)

$$(3.49) \qquad \int_{S^{n-1}} P(r\omega,\omega')\,d\omega/A_n = \int_{S^{n-1}} P(r\omega,\omega')\,d\omega'/A_n = g_0(r);$$

(ii) $P(r\omega,\omega)/A_n g_0(r)$ diverges monotonically to infinity as $r \to \infty$; (iii) $P(r\omega,\omega')/A_n g_0(r)$ converges to zero as $r \to \infty$ uniformly for $\omega$ and $\omega'$ with $\omega\cdot\omega' \leq 1-\varepsilon$, where $\varepsilon > 0$; and (iv) for any continuous function $\phi$ on $S^{n-1}$,

$$(3.50) \qquad \lim_{\substack{|x|\to\infty \\ x/|x|\to\omega}} \int_{S^{n-1}} \frac{P(x,\omega')}{A_n g_0 |x|} \phi(\omega')\,d\omega' = \phi(\omega).$$

Proof. (3.49) is a direct consequence of (3.8) to (3.11).

We put $Q_x(\omega') = P(x,\omega')/A_n g_0(|x|)$. For any $\phi$ in $C^\infty(S^{n-1})$, the left hand side of (3.50) is equal to

$$\lim_{r\to\infty} \int \sum_{j=0}^{\infty} \frac{\max(n-2+2j,1)}{\max(n-2,1)A_n} \frac{g_j(r)}{g_0(r)} a_j P_j(\omega\cdot\omega')\phi(\omega')d\omega'$$

$$= c_n \sum_{j=0}^{\infty} \max(n-2+2j,1)\int P_j(\omega\cdot\omega')\phi(\omega')d\omega' = \phi(\omega).$$

Here the last equality is nothing but the equality (3.30) given in the Remark after the proof of Theorem 3.8. On the other hand, (3.49) implies that any sequence $\{Q_{x_j}\}_j$ with $|x_j| \to \infty$ and $x_j/|x_j| \to \omega$ has a subsequence converging weakly to a measure on $S^{n-1}$. Thus we get (iv). For any sufficiently samll $\varepsilon > 0$, Put

$$\Omega = \{\omega' \in S^{n-1} ; 1 - \varepsilon < \omega'\cdot\omega < 1 - \varepsilon/2\}.$$

Then Theorem 3.14 implies

$$\text{(3.52)} \qquad \int_\Omega Q_{r\omega}(\omega')d\omega' \geq Q_{r\omega}(\theta)\int_\Omega d\omega',$$

where $\theta\cdot\omega = 1 - \varepsilon$. By (3.50), the left hand side of (3.52) converges to zero as $r \to \infty$. Thus, $Q_{r\omega}(\theta) \to 0$ as $r \to \infty$. This together with Theorem 3.14 shows (iii). Since $P_j(1) > 0$ for any $j$, (ii) follows from Lemma 3.6, Theorem 3.14, and (3.49). Q.E.D.

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. Suppose that (3.6) does not hold. The first assertion of (ii) follows easily from (3.8) to (3.11) and (3.32). Since Theorem 3.16 implies that the function $u$ given by (3.14) is a solution of the problem (3.13), the second assertion of (ii) is proved if we show that $u$ is the unique solution of (3.13). Let $v$ be a solution of (3.13), and put $w(x) = \{v(x) - u(x)\}/g_0(|x|)$. Then $w$ satisfies the equation

$$\{-\Delta + 2(\nabla g_0/g_0)\cdot\nabla\}w = 0 \quad \text{on} \quad R^n,$$

and $w(x) \to 0$ as $|x| \to \infty$. Thus the maximum principle shows that $w = 0$, i.e., $v = u$. This completes the proof of (ii). (i) can be shown similarly and easily. Q.E.D.

§4. Examples.

In this section we give a few examples of radially symmetric potentials on $R^n$, $n \geq 2$, to which results in Sections 2 and 3 are applied. The potentials we discuss on are: (i) Highly oscillatory potentials for which some pathological phenomana happen; (ii) potentials with compact support; and (iii) potentials which are equal to $ar^\beta$ for $r \geq 1$.

Throughout the section a positive solution means a positive solution $u$ normalized by $u(0) = 1$.

4.1. Highly oscillatory potentials. Minimal Green's function might be expected to be bounded. But there is a potential for which the minimal Green's function grows very rapidly at infinity.

Example 4.1. Let $g_0$ be a positive $C^\infty$-function on $[0,\infty)$ such that $g_0(r) \geq r^{(3-n)/2}$ for $r \geq 1$, $g_0(r) = e^r$ for $r$ in

$$\bigcup_{k=1}^{\infty} \{r > 0;\ |r - 4k| < 1/2\},$$

and $g_0(r) = r^{(3-n)/2}$ for $r$ in

$$[1,\infty) \setminus \bigcup_{k=1}^{\infty} \{r > 0;\ |r - 4k| < 1\}.$$

Let $V(x) = \Delta g_0(|x|)/g_0(|x|)$ on $R^n$. Then Theorem 3.1 shows that $V$ is a subcritical potential. Let $G(x,y)$ be the minimal Green's function. By Theorem 3.11,

$$G(x,0) = d_n g_0(|x|) \int_{|x|}^{\infty} t^{1-n} g_0(t)^{-2} dt$$

for some constant $d_n$. For $r$ with $|r - 4k| < 1/2$, we have

$$\int_r^\infty t^{1-n} g_0(t)^{-2} dt \geq \int_{4k+1}^{4k+3} t^{-2} dt \geq 1/6r^2.$$

Thus, $G(x,0) \geq d_n e^{|x|}/6|x|^2$ for $x$ with $||x|-4k| < 1/2$. By the Harnack inequality, for any compact set $K$ there exists a constant $C$ such that

$$G(x,y) \geq C\, e^{|x|}|x|^{-2} \tag{4.1}$$

for any $y$ in $K$ and any $x$ such that $||x|-4k| < 1/2$ for some $k$ and $|x| > \sup\{|y| + 1;\ y \in K\}$.

Theorem 2.7 seems to suggest that a positive solution corresponding to a critical potential has decay property similar to that of an $L_2$-eigenfunction. But Theorem 3.1 says that a positive solution $u$ of (3.2) with critical potential decays at infinity only in the sense that

$$\int_{|x|>1} \left( u(x)|x|^{n-1} \right)^{-2} dx = \infty. \tag{4.2}$$

Indeed, the following example shows that the positive solution can grow very rapidly at infinity.

Example 4.2. Let $V$ be the same as in Example 4.1. Choose, by Theorem 2.8, a nonnegative function $W$ in $C_0^\infty(R^n)$ such that $V - W$ is critical. Theorem 2.2 asserts that there exists a unique positive solution $u$ of the equation $(-\Delta+V-W)u = 0$ on $R^n$. Then (4.1) and (2.8) show that there exists a constant $C$ such that

$$u(x) \geq Ce^{|x|}|x|^{-2} \tag{4.3}$$

for any $x$ with $|x| > \sup\{|y| + 1;\ y \in \mathrm{Supp}\ W\}$ and $||x|-4k| < 1/2$ for some $k = 1,2,\ldots$ .

4.2. Potentials with compact support. For $a$ and $b$ with $-\infty < a < \infty$ and $0 < b < 1$, put

$$(4.4)\qquad V = \begin{cases} ar^{-2} & , \ b \leq r \leq 1 \\ 0 & , \ \text{otherwise.} \end{cases}$$

Then we have the following theorems.

Theorem 4.3. Let $n = 2$. Then there hold the following statements (i)~(iii).

(i) If $a < 0$, then $V$ is supercritical.

(ii) If $a = 0$, then $V$ is critical and the unique positive solution is equal to 1.

(iii) If $a > 0$, then $V$ is subcritical. Furthermore, $\sum = \{\infty\}$ and the unique positive solution $P(x,\infty)$ satisfies, for $|x| \geq 1$,

$$(4.5)\qquad P(x,\infty) = C \log |x| + D,$$

where $C = A(b^{-A}-b^{A})/2$ , $A = a^{1/2}$ , and $D = (b^{-A}+b^{A})/2$.

Theorem 4.4. Let $n \geq 3$. Then there hold the following statements.

(i) If $a < -(n-2)^2/4$ and $b < c$, then $V$ is supercritical. Here

$$(4.6)\qquad c = \exp[-2(\pi-2\theta)(-4a - (n-2)^2)^{-1/2}],$$

$$\theta = \tan^{-1}[(-4a-(n-2)^2)^{1/2}/(n-2)], \quad 0 < \theta < \pi/2.$$

(ii) If $a < -(n-2)^2/4$ and $b = c$, then $V$ is critical and the unique positive solution $u(x)$ satisfies, for $|x| \geq 1$,

$$u(x) = d|x|^{2-n}, \tag{4.7}$$

where $d$ is a positive constant.

(iii) If $a < -(n-2)^2/4$ and $b > c$, or $a \geq -(n-2)^2/4$, then $V$ is subcritical, $\sum = \{\infty\}$, and the unique positive solution $P(x,\infty)$ satisfies, for $|x| \geq 1$,

$$P(x,\infty) = C + D|x|^{2-n}, \tag{4.8}$$

where $C > 0$ and $D$ is a real constant.

Theorem 4.5. If $V$ is subcritical, then the minimal Green's function $G(x,y)$ has the following properties:

(i) For $n \geq 3$,

$$G(x,y) = c_n|x-y|^{2-n} + g(x,y), \tag{4.9}$$

where $g$ is a function on $R^{2n}$ such that

$$|g(x,y)| \leq M(|x||y|)^{2-n} \quad \text{on} \quad \{(x,y);\ |x|,|y| > 2\}$$

for some constant $M$ and $|x-y|^{n-3}g(x,y)$ is bounded on any compact subset of $R^{2n}$.

(ii) For $n = 2$,

$$G(x,y) = (2\pi)^{-1}(-\log|x-y| + \log|x| + \log|y| + g(x,y)), \tag{4.10}$$

where $g$ is a function on $R^4$ such that

$$|g(x,y)| \le M \quad \text{on} \quad \{(x,y);\ |x|,|y| > 1\}$$

for some constant $M$, and the function $\log |x| + \log |y| + g(x,y)$ is bounded on any compact subset of $R^4$.

Proof of Theorems 4.3~4.5. Assume that $n = 2$. The statements (i) and (ii) follows from Theorem 3.1. Let us consider the case where $a > 0$. Solve the initial value problem (3.3) and (3.4) with $j = 0$, keeping in mind that the solution is continuously differentiable at $r = b$ and $r = 1$. Then we have that

$$(4.11) \qquad g_0(r) = C \log r + D \quad \text{for} \quad r \ge 1.$$

Noting that $(g_0'f_0 - f_0'g_0)(r) = r^{-1}$, we obtain by (3.31) that

$$(4.12) \qquad f_0(r) = 1/C \quad \text{for} \quad r \ge 1.$$

As for $j \ge 1$, similar computations yield

$$(4.13) \qquad g_j(r) = C_j r^j + D_j r^{-j} \quad \text{for} \quad r \ge 1,$$

$$(4.14) \qquad f_j(r) = C_j^{-1} r^{-j} \quad \text{for} \quad r \ge 1.$$

where $C_j = (2+A_j/j + j/A_j)b^{j-A_j}/4 + (2-A_j/j - j/A_j)b^{j+A_j}/4$, $A_j = (j^2+a)^{1/2}$, and $D_j = (j/A_j - A_j/j)(b^{j-A_j} - b^{j+A_j})/4$. Theorem 4.3(iii) follows from Theorem 3.2 and (4.11). Theorem 3.11 and (4.11)~(4.14) show that (4.10) holds on $\{(x,y);\ |x|,|y| > 1\}$ with

$$(4.15) \qquad g(x,y) = \sum_{j=0}^{\infty} (D_j/C_j)(|x||y|)^{-j} P_j(x\cdot y/|x||y|),$$

where $C_0 = C$ and $D_0 = D$. The function $g$ is clearly bounded on $\{(x,y);\ |x|,|y| > 1\}$. The other properties of $G$ can be easily shown. This proves Theorem 4.5(ii).

Next, assume that $n \geq 3$. Concerning Theorems 4.4 and 4.5(i) we only indicate how to determine the constant $c$ given by (4.6), since the other statements can be shown in the same way as above. Theorem 2.4 and (2.8) imply that if $V$ is critical, then $g_0(r) = dr^{2-n}$ on $[1,\infty)$ for some constant $d > 0$. Thus we can find $c$ and $d$ as numbers such that the boundary value problem

$$(4.16)\qquad \begin{gathered} g_0''(r) + (n-1)r^{-1}g_0'(r) - ar^{-2}g_0(r) = 0,\ c \leq r \leq 1, \\ g_0(c) = 1,\ g_0'(c) = 0,\ g_0(1) = d,\ g_0'(1) = (2-n)d \end{gathered}$$

has a positive solution. Q.E.D.

4.3. $V = O(r^{-2})$. For a real number $a$, put

$$(4.17)\qquad V = \begin{cases} ar^{-2} & , \ r \geq 1 \\ 0 & , \ r < 1. \end{cases}$$

Then we have

Theorem 4.6. Let $b = -(n-2)^2/4$. Then one has:

(i) If $a < b$, then $V$ is supercritical.

(ii) If $a = b$ and $n = 2$, then $V$ is critical and the positive solution $u$ is equal to $1$.

(iii) If $a > b$ and $n = 2$, or $a \geq b$ and $n \geq 3$, then $V$

is subcritical. Furthermore, $\sum = \{\infty\}$ and the unuque positive solution $P(x,\infty)$ has the form, for $|x| \geq 1$,

(4.18) $P(x,\infty) = [(A+n-2)|x|^{A} + A|x|^{-A-n+2}]/(2A+n-2)$ when $a > b$,

(4.19) $P(x,\infty) = |x|^{A}(-A \log |x| + 1)$ when $a = b$ and $n \geq 3$,

where $A = (a-b)^{1/2} - (n-2)/2$.

We omit the proof, since it is similar to the proof of Theorems 4.3 and 4.4.

The following estimates for the minimal Green's function is of independent interest and useful when we consider perturbations of $V$.

Theorem 4.7. If $V$ is subcritical, then the minimal Green's function $G$ satisfies the following estimate:

(i) If $n \geq 3$ and $a > -(n-2)^2/4$, then

(4.20) $G(x,y) \leq M\,|x-y|^{2-n}$ , if $|x|,|y| \leq 1$ or $|y|/2 \leq |x| \leq 2|y|$,

$\leq M \min(\langle x\rangle^{A}\langle y\rangle^{-A'},\ \langle x\rangle^{-A'}\langle y\rangle^{A})$ , otherwise,

for some $M > 0$, where $A = [a + (n-2)^2/4]^{1/2} - (n-2)/2$, $A' = A + n-2$, and $\langle x\rangle = (1+|x|^2)^{1/2}$.

(ii) If $n \geq 3$ and $a = -(n-2)^2/4$, then

(4.21) $G(x,y) \leq M|x-y|^{2-n} + M(\langle x\rangle\langle y\rangle)^{(2-n)/2}\min(\log 2\langle x\rangle, \log 2\langle x\rangle)$.

(iii) If $n = 2$ and $a > 0$, then

(4.22) $G(x,y) \leq M(-\log \langle y\rangle|x-y|^{-1}+M)$, if $|x|,|y| \leq 1$ or $|y|/2 \leq |x| \leq 2|y|$,

$\leq M \min(\langle x\rangle^{-A}\langle y\rangle^{A},\ \langle x\rangle^{A}\langle y\rangle^{-A})$ , otherwise.

Proof. (i) Assume that $a > -(n-2)^2/4$ and $n \geq 3$. Solving (3.3) and (3.4), we have that $g_j(r) = r^j$ for $r \leq 1$ and

(4.23) $$g_j(r) = C_j r^{A_j} + D_j r^{-B_j} \quad \text{for} \quad r \geq 1,$$

where $A_j = [a + (2j+n-2)^2/4]^{1/2} - (n-2)/2$, $B_j = A_j + n-2$, $C_j = (B_j+j)/(A_j+B_j)$, and $D_j = (A_j-j)/(A_j+B_j)$. By (3.31),

(4.24) $$f_j(r) = E_j r^{-B_j} \quad \text{for} \quad r \geq 1,$$

where $E_j = (2j+n-2)/(B_j+j)$. Theorem 3.11 implies that $G(x,y) = |x|^{-A'}|y|^A g(x,y)$ for $x$ and $y$ with $|x| \geq 2|y| \geq 4$, where

$$g(x,y) = c_n \sum_{j=0}^{\infty} E_j |x|^{-(B_j-B_0)} (C_j |y|^{A_j-A_0} + D_j |y|^{-B_j-A_0}) P_j(x\cdot y/|x||y|).$$

Noting that $B_j - B_0 = O(j)$ and $A_j - A_0 = O(j)$ as $j \to \infty$, we obtain the estimate (4.20) for $(x,y)$ with $|x| \geq 2|y| \geq 4$. The estimate (4.20) for $(x,y)$ with $|y| \geq 2|x| \geq 4$ can be obtained by $G(x,y) = G(y,x)$, and (4.20) for $|x|$ or $|y| \leq 2$ is clear. It remains to prove (4.20) for $(x,y)$ with $1 \leq |y|/2 \leq |x| \leq 2|y|$, which clearly holds if $a \geq 0$ because $G(x,y) \leq c_n|x-y|^{2-n}$ in this case. Now assume that $a < 0$, and consider the operator $-\Delta + V_t$, $t > 0$, with

(4.25) $$V_t = ar^{-2} \quad \text{for} \quad r \geq t, \quad \text{and} \quad V_t = 0 \quad \text{for} \quad r < t.$$

Then $V_t$ is subcritical for $t$ with $0 < t \leq 1$. Denote the corresponding minimal Green's function by $G_t$. By changing independent variables we have

(4.26) $$G(x,y) = |y|^{2-n} G_{1/|y|}(x/|y|, y/|y|).$$

We obtain that for $z$ and $\omega$ with $|z| = 2$ and $|\omega| = 1$

$$G_t(z,\omega) = c_n \sum_{j=0}^{\infty} E_j 2^{-B_j}(C_j + D_j t^{A_j+B_j}) P_j(z\cdot\omega/2).$$

Thus, $\sup\{G_t(z,\omega);\ 0 < t \le 1/2,\ |z| = 2,\ |\omega| = 1\} < \infty$. Similarly, $\sup\{G_t(z,\omega);\ |z| = 1/2,\ |\omega| = 1,\ 0 < t \le 1/2\} < \infty$. Now consider the Green's function $H(z,\omega)$ for the operator $-\Delta + a|z|^{-2}$ on $\{x;\ 1/2 < |z| < 2\}$ with zero Dirichlet condition. Then it is well known (cf. [20]]) that

$$H(z,\omega) \le M|z-\omega|^{2-n}$$

for some $M > 0$. Putting $F_t(z,\omega) = G_t(z,\omega) - H(z,\omega)$, we have

$$(-\Delta+a|z|^{-2})F_t(z,\omega) = 0 \quad \text{on} \quad \{z;\ 1/2 < |z| < 2\},$$

$$\sup\{F_t(z,\omega);\ |z| = 1/2 \ \text{ or } \ 2,\ |\omega| = 1,\ 0 < t \le 1/2\} < \infty.$$

This implies that $F_t(z,\omega) \le M$ for some $M$ independent of $t, z$ and $\omega$, which shows that

(4.27) $$G_t(z,\omega) \le M(|z-\omega|^{2-n} + 1)$$

for $1/2 \le |z| \le 2$, $|\omega| = 1$, and $0 < t \le 1/2$. The estimate (4.20) for $1 \le |y|/2 \le |x| \le 2|y|$ now follows from (4.26) and (4.27).

(ii) Assume that $n \ge 3$ and $a = -(n-2)^2/4$. With the same notation as in (i) we have

$$G_t(z,\omega)/c_n = -\log t + 2/(n-2) + \sum_{j=1}^{\infty} E_j 2^{-B_j}(C_j+D_j t^{A_j+B_j})P_j(z\cdot\omega/|z|)$$

for $|z| = 2$, $|\omega| = 1$, and $0 < t \leq 1/2$. A similar formula holds for $|z| = 1/2$. The same argument as in (i) now shows that

$$(4.28) \qquad G_t(z,\omega) \leq M(|z-\omega|^{2-n} - \log t)$$

for $1/2 \leq |z| \leq 2$, $|\omega| = 1$, and $0 < t \leq 1/2$. The estimate (4.21) for $1 \leq |y|/2 \leq |x| \leq 2|y|$ follows from (4.26) and (4.28). The estimate for the other case can be obtained easily.

(iii) can be shown similarly. Q.E.D.

4.4. $V = O(r^\beta)$ with $\beta > -2$. Let $V$ be a function on $R^n$ of the form

$$(4.29) \qquad V(x) = a|x|^\beta$$

for real numbers $a$ and $\beta > -2$. Then we have

Theorem 4.8. (i) If $a < 0$, then $V$ is supercritical.

(ii) If $a = 0$, then $V$ is critical when $n = 2$, and subcritical when $n \geq 3$.

(iii) If $a > 0$, then $V$ is subcritical, $\sigma = S^{n-1}$, and the minimal solution $P(x,\omega)$, $\omega \in S^{n-1}$, is given by

$$(4.30) \qquad P(x,\omega) = \sum_{j=0}^{\infty} \frac{\max(n-2+2j,1)}{\max(n-2,1)} \Gamma\left(\frac{n-2}{\beta+2}+1\right)\left(\frac{s}{2}\right)^{(2-n)/(\beta+2)} \times I_{(n-2+2j)/(\beta+2)}(s)P_j(x\cdot\omega/|x|),$$

where $s = a^{1/2}(\beta/2+1)^{-1}|x|^{\beta/2+1}$, $\Gamma$ is the gamma function, and

$I_\nu(s)$ is the modified Bassel function of the first kind defined by

$$I_\nu(s) = \sum_{m=0}^{\infty} (s/2)^{\nu+2m}/m!\Gamma(\nu+m+1).$$

In particular, when $\beta = 0$,

$$(4.31) \qquad P(x,\omega) = \exp(a^{1/2}x\cdot\omega).$$

Proof. (i) is well-known (see, for example, [2],[3], and [8 , Theorem 7.1, p. 362]). (ii) is proved already. Let us prove (iii). First, assume that $\beta = 0$. Then the minimal Green's function $G(x,y)$ has the asymptotic formula

$$G(x,y) = C|x-y|^{-(n-1)/2} \exp[-a^{1/2}|x-y|](1+o(1))$$

as $|y| \to \infty$, where $c$ is a positive constant. Thus

$$P(x,\omega) = \lim_{r\to\infty} G(x,r\omega)/G(0,r\omega) = \exp(a^{1/2}x\cdot\omega).$$

Next we treat the general case. Consider the equation

$$(4.32) \qquad g_j'' + (n-1)r^{-1}g_j' - [j(j+n-2)r^{-2} + ar^{\beta}]g_j = 0.$$

By the change of variable : $s = a^{1/2}(\beta/2+1)^{-1}r^{\beta/2+1}$, we have that with $h_j(s) = g_j(r)$

$$(4.33) \qquad h_j''(s) + \frac{\beta+2n-2}{\beta+2}\frac{1}{s}h_j'(s) - [1 + \frac{4j(j+n-2)}{(\beta+2)^2}\frac{1}{s^2}]h_j(s) = 0.$$

Putting $i_j(s) = s^{(n-2)/(\beta+2)}h_j(s)$, we have

$$(4.34) \qquad i_j''(s) + s^{-1}i_j'(s) - \{1 + [(n-2+2j)/(\beta+2)s]^2\}i_j(s) = 0.$$

Thus,

$$(4.35)\qquad g_j(r) = C_j s^{(2-n)/(\beta+2)} I_{(n-2+2j)/(\beta+2)}(s)$$

for some $C_j > 0$. Making use of the initial condition $g_j(r)r^{-j} \to 1$ as $r \to 0$, we obtain that

$$(4.36)\qquad C_j = 2^{(n-2)/(\beta+2)} \left(\frac{a^{1/2}}{\beta+2}\right)^{-2j/(\beta+2)} \Gamma\left(\frac{n-2+2j}{\beta+2}+1\right).$$

Thus

$$(4.37)\qquad \lim_{r\to\infty} g_0(r)/g_j(r) = C_0/C_j .$$

Theorem 3.3, (4.35) and (4.37) now show (iii). Q.E.D.

In the rest of this subsection we assume that $a > 0$, and give results concerening asymptotic behaviors as $x \to \infty$ of $P(x,\omega)$ and $G(x,y)$. Theorems 4.9 and 4.10 below will be used in Sections 5 and 7 and proved in Appendix 2. As for properties of $P$ and $G$ enjoyed by any radially symmetric potential not satisfying (3.6), see Theorem 3.12 to 3.16.

Theorem 4.9. Let $\cos\theta = x\cdot\omega/|x|$ and $0 \leq \theta < \min(\pi, \pi/(\beta/2+1))$. Then

$$(4.38)\qquad P(x,\omega) = c(n,a,\beta)\left[\frac{\sin\theta}{\sin(\beta/2+1)\theta}\right]^{\delta_n/2} |x|^{(n-2)\beta/4}$$

$$\times \exp\left[\frac{a^{1/2}}{\beta/2+1}|x|^{\beta/2+1}\cos\left(\frac{\beta}{2}+1\right)\theta\right](1+O(|x|^{-\beta/2-1}))$$

as $|x| \to \infty$, where $c(n,a,\beta)$ is a positive constant depending on $n,a$ and $\beta$, and $\delta_n = 0$ for $n$ even and $\delta_n = 1$ for $n$ odd. Furthermore, $\partial P/\partial\theta$ has the asymptotics as $|x| \to \infty$ that is obtained by differentiating (4.38) formally with respect to $\theta$.

Theorem 4.10. Put

$$d(x,y) = [X^2 + Y^2 - 2XY\cos(\beta/2+1)\theta]^{1/2}, \tag{4.39}$$

where $X = a^{1/2}(\beta/2+1)^{-1}|x|^{\beta/2+1}$, $Y = a^{1/2}(\beta/2+1)^{-1}|y|^{\beta/2+1}$, and $\cos\theta = x\cdot y/|x||y|$ with $0 \le \theta \le \pi$. Then the minimal Green's function $G(x,y)$ has the following properties:

(i) For $x$ and $y$ with $d(x,y) \ge 1$, $0 \le \theta < \min(\pi, \pi/(\beta/2+1))$, and $\max(|x|,|y|) \ge 1$,

$$G(x,y) \le M(\langle x\rangle\langle y\rangle)^{(n-2)\beta/4} d(x,y)^{-(n-1)/2} e^{-d(x,y)}, \tag{4.40}$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$ and $M$ is a positive constant independent of $x$ and $y$. Furthermore,

$$G(x,y) = \alpha(|x||y|)^{(n-2)\beta/4}\, d^{-(n-1)/2}\, e^{-d}(1+O(d^{-1})) \tag{4.41}$$

as $d = d(x,y) \to \infty$ with $|x|,|y| \ge 1$, where $\alpha$ is some positive constant.

(ii) For $x$ and $y$ with $d(x,y) \le 1$ and $\min(|x|,|y|) \ge 1$,

$$(4.42) \qquad G(x,y) \le M|x-y|^{2-n} , \qquad n \ge 3 ,$$

$$\le M(\log |x-y|^{-1} - \beta \log |x||y|), \quad n = 2 .$$

Furthermore,

$$(4.43) \quad G(x,y) = \gamma(|x||y|)^{(n-2)\beta/4} d^{2-n}(1+O(d)), \qquad n \ge 3 ,$$

$$= \gamma \log d^{-1} + O(d) , \qquad n = 2 ,$$

as $d = d(x,y) \to 0$, where $\gamma$ is some positive constant.

## §5. Principally radially symmetric potentials.

In this section we consider potentials $V$ of the form

$$(5.1) \qquad V = V_0 + V_1 + V_2 ,$$

where $V_0$, $V_1$ and $V_2$ are functions satisfying (2.1) with $n \geq 2$ such that $V_0$ is radially symmetric and subcritical, $V_1$ is a "small" perturbation of $V_0$ (for the definition of small perturbation, see below), and $V_2$ has compact support. $V_1$ and $V_2$ are, in general, non-radially symmetric. Throughout this section we assume that $V$ has the form (5.1).

We start with a definition of small perturbation.

Definition 5.1. We say that $V_1$ is a small perturbation of $V_0$ if $V_0$ and $V_1$ satisfy the following conditions:

(i) For any $j = 1,2,\ldots,$ put

$$(5.2) \qquad G^j(x,y;|V_1|) = \int G^0(x,z)|V_1(z)|G^{j-1}(z,y;|V_1|)dz,$$

where $G^0(x,y)$ is the minimal Green's function for $V_0$. Then

$$(5.3) \qquad \sum_{j=1}^{\infty} G^j(x,y;|V_1|) < \infty$$

for any $(x,y)$ with $x \neq y$; (ii)

$$(5.4) \lim_{y\to\omega} \sum_{j=0}^{\infty} (-1)^j G^j(x,y;V_1)/G^0(0,y) \equiv \tilde{P}(x,\omega)$$

exists for any $(x,\omega)$ in $R^n \times \sigma$, where $G^0(x,y;V_1) = G^0(x,y)$,

$\sigma = S^{n-1}$ or $\{\infty\}$, and $y \to \omega$ means that $|y| \to \infty$ if $\omega = \infty$, and $|y| \to \infty$ and $y/|y| \to \omega$ if $\omega \in S^{n-1}$; and (iii)

$$(5.5) \qquad \tilde{P}(x,\omega) - P^0(x,\omega) = o(g_0(|x|)) \quad \text{as} \quad |x| \to \infty ,$$

where $g_0(r)$ is the solution of (3.3) and (3.4) with $j=0$ and $V=V_0$ and $P^0(x,\omega)$ is the minimal solution for $V_0$.

Lemma 5.2. The condition (5.3) holds iff $V_0 - |V_1|$ is subcritical; in this case,

$$(5.6) \qquad G(x,y;V_1) = \sum_{j=0}^{\infty} (-1)^j \, G^j(x,y;V_1)$$

is the minimal Green's function for $V_0 + V_1$.

Proof. Suppose that $V_0-|V_1|$ is subcritical. Denoting by $G_R^0$ and $G_R$ Green's functions for $-\Delta+V_0$ and $-\Delta+V_0-|V_1|$ in $B_R = \{x; |x| < R\}$ with zero Dirichlet boundary condition, respectively, we have

$$(5.7) \; G_R(x,y) = G_R^0(x,y) + \int_{B_R} G_R^0(x,z)|V_1(z)|G_R(z,y)dz.$$

Letting $R \to \infty$, we have

$$(5.8) \qquad G(x,y) = G^0(x,y) + \int_{R^n} G^0(x,z)|V_1(z)|G(z,y)dz ,$$

where $G(x,y)$ is the minimal Green's function for $V_0-|V_1|$.

Since $G^0 \leq G$, (5.8) implies

$$(5.9) \qquad \sum_{j=0}^{\infty} G^j(x,y;|V_1|) \leq G(x,y).$$

This shows (5.3). Conversely, suppose that (5.3) holds. Then we see that Green's function $G_R$ for $-\Delta+V_0-|V_1|$ in $B_R$ exists and is given by

$$(5.10) \qquad G_R(x,y) = \sum_{j=0}^{\infty} G_R^j(x,y;|V_1|) ,$$

where $G_R^j$ are defined by (5.2) with $G^0$ and $|V_1|$ replaced by $G_R^0$ and $|V_1|\chi_{B_R}$. Since $G_R^j \to G^j$, we then obtain that the left hand side of (5.9) is the minimal Green's function for $V_0-|V_1|$. This proves that $V_0-|V_1|$ is subcritical. The second assertion of the lemma is now clear. Q.E.D.

In what follows a positive solution means a positive continuous function $u$ on $R^n$ satisfying the equation

$$(-\Delta + V)u = 0 \quad \text{on} \quad R^n$$

with $u(0) = 1$, where $V$ is of the form (5.1).

The following theorem is one of main results in this section, and concerned with the potential $V_0$ for which $\sum = \{\infty\}$.

Theorem 5.3. Suppose that $V_0$ satisfies (3.6). Then one has:

(i) If $V$ is subcritical, then $\sum = \sigma = \{\infty\}$ and the unique positive solution $P(x,\infty)$ has the asymptotics

$$(5.11) \qquad P(x,\infty) = cg_0(|x|)(1 + o(1)) \quad \text{as} \quad |x| \to \infty$$

for some positive constant $c$.

(ii) If $V$ is critical, then the unique positive solution $u(x)$ has the asymptotic

$$(5.12) \qquad u(x) = cf_0(|x|)(1 + o(1)) \quad \text{as} \quad |x| \to \infty$$

for some positive constant $c$, where $f_0$ is given by (3.31) with $j=0$.

Proof. (i) We denote by $G(x,y)$ the minimal Green's function for $V$. Noting that $V_2$ has compact support we have that

$$(5.13) \qquad G(x,y) = G(x,y;V_1) - \int G(x,z)V_2(z)G(z,y;V_1)dz,$$

$$(5.14) \qquad G(x,y) = G(x,y;V_1) - \int G(x,z;V_1)V_2(z)G(z,y)dz.$$

We have by (5.4), (5.6), and (5.13) that

$$(5.15) \qquad \lim_{y\to\infty} \frac{G(x,y)}{G^0(0,y)} = \tilde{P}(x,\infty) - \int G(x,z)V_2(z)\tilde{P}(z,\infty)dz.$$

This together with the Harnack inequality implies that

$$C \equiv \lim_{y\to\infty} G(0,y)/G^0(0,y) > 0.$$

Thus

(5.16) $$P(x,\infty) = \lim_{y\to\infty} G(x,y)/G(0,y)$$

$$= C^{-1}[\tilde{P}(x,\infty) - \int G(x,z)V_2(z)\tilde{P}(z,\infty)dz].$$

Since (5.4), (5.6) and (5.14) imply that

(5.17) $$G(x,y) = O(G^0(x,0)) \quad \text{as} \quad |x| \to \infty$$

uniformly for $y$ in any compact set, a consequence of Theorems 3.2 and 3.11

(5.18) $$G^0(x,0) = o(P^0(x,\infty)) \quad \text{as} \quad |x| \to \infty$$

shows that $G(x,y) = o(P^0(x,\infty))$ as $|x| \to \infty$. Hence (5.16) and (5.5) yield (5.11).

(ii) Put $U(x) = \max(V_2(x),0)$ and $W(x) = \max(-V_2(x),0)$. Then we see that $V_0 + V_1 + U$ is subcritical, $V_2 = U - W$, and $W \not\equiv 0$. Denoting the minimal Green's function for $V_0 + V_1 + U$ by $H(x,y)$, we have by (2.8) that

(5.19) $$u(x) = \int H(x,y)W(y)u(y)dy.$$

In the proof of (i) we have proved (see (5.15)) that

$$Q(y) \equiv \lim_{x\to\infty} H(x,y)/G^0(x,0) > 0.$$

Hence

$$(5.20) \qquad u(x) = G^0(x,0)[\int QWu\,dy + o(1)] \quad \text{as} \quad |x| \to \infty .$$

Since $Q > 0$ , $u > 0$ , and $W \not\equiv 0$ , this proves (5.12). Q.E.D.

The following theorem is concerned with the potential $V_0$ for which $\sigma = S^{n-1}$. As for properties of the minimal solution $P^0(x,\omega)$ for $V_0$, see Theorem 3.16.

Theorem 5.4. Suppose that $V_0$ does not satisfy (3.6). Then one has:

(i) If $V$ is subcritical, then $\sum = \sigma = S^{n-1}$ and the minimal solutions $P(x,\omega)$, $\omega \in S^{n-1}$, satisfy

$$(5.21) \qquad P(x,\omega) = C(\omega)P^0(x,\omega) + o(g_0|x|) \quad \text{as} \quad |x| \to \infty$$

for some positive continuous function $C(\omega)$ on $S^{n-1}$.

(ii) If $V$ is critical, then the unique positive solution $u$ has the asymptotics (5.12).

Proof. (ii) can be shown in the same way as in the proof of Theorem 5.3(ii). Now assume that $V$ is subcritical. Denote by $G(x,y)$ the minimal Green's function for $V$. Then we have by (5.13), (5.4), and (5.6) that

$$(5.22) \qquad \lim_{y\to\omega} \frac{G(x,y)}{G^0(0,y)} = \tilde{P}(x,\omega) - \int G(x,z)V_2(z)\tilde{P}(z,\omega)dz.$$

Since $G(x,z) = O(f_0(|x|)) = o(g_0(|x|))$ as $|x| \to \infty$ uniformly for $z$ in a compact set, we have by (5.5) and Theorem 3.16 that the right hand side of (5.22) is a positive solution of $(-\Delta+V)u = 0$ on $R^n$. Thus the function

$$c(\omega) = \tilde{P}(0,\omega) - \int G(0,z)V_2(z)\tilde{P}(z,\omega)dz \tag{5.23}$$

is a positive continuous function on $S^{n-1}$. Hence

$$P(x,\omega) = \lim_{y\to\omega}\frac{G(x,y)}{G(0,y)} = c(\omega)^{-1}\{P^0(x,\omega) + o(g_0(|x|))\} \tag{5.24}$$

as $|x| \to \infty$. This proves the second half of the conclusion in (i). It remain to prove that $P(x,\omega)$ is minimal for any $\omega$ in $S^{n-1}$. Suppose that a positive solution $u$ satisfies the inequality

$$u(x) \leq P(x,\omega)$$

for any $x$ in $R^n$. By Theorem 2.3, there exists a measure $d\mu$ on $S^{n-1}$ such that

$$u(x) = \int P(x,\omega')d\mu(\omega')$$

Let $\phi$ be any function in $C_0^\infty(S^{n-1}\setminus\{\omega\})$ such that $0 \leq \phi \leq 1$. Then we have

$$\int \phi(\omega')P(x,\omega')d\mu(\omega') \leq P(x,\omega). \tag{5.24}$$

Choose a $C^\infty$-function $\psi$ on $S^{n-1}$ such that $0 \leq \psi \leq 1$, $\psi(x) = 1$ on Supp $\phi$, and $\psi(\omega) = 0$. We then have that

$$(5.25)\quad \iint \psi(\omega'')\phi(\omega')c(\omega'')P(r\omega'',\omega')d\mu(\omega')d\omega''/A_n g_0(r)$$

$$\leq \int \psi(\omega'')c(\omega'')P(r\omega'',\omega)d\omega''/A_n g_0(r).$$

Letting $r \to \infty$ we get

$$(5.26)\qquad \int \phi(\omega')d\mu(\omega') \leq 0$$

This implies that $d\mu(\omega') = m\delta(\omega'-\omega)$ for some $m > 0$, which shows that $u(x) = mP(x,\omega)$. This concludes that $P(x,\omega)$ is minimal. Q.E.D.

The following theorem asserts that the conjecture 2.12 is true for principally radially symmetric potentials, and follows directly from Theorems 5.3, 5.4, and 3.4.

Theorems 5.5. Suppose that $V$ is subcritical. Then the conclusion of Theorem 3.4 holds with (3.14) replaced by

$$(5.27)\qquad u(x) = \int_{|\omega|=1} P(x,\omega)\phi(\omega)[A_n C(\omega)]^{-1}d\omega.$$

In the rest of this section we give some applications of Theorems 5.3 and 5.5. To this end we introduce weighted $L_p$-spaces. For $s$ and $p$ with $-\infty < s < \infty$ and $1 \leq p \leq \infty$, we put

$$(5.28)\qquad L_p^s = \{f \in L_{p,loc}(R^n);\ \langle x\rangle^s f(x) \in L_p(R^n)\}.$$

In what follows we always impose on $p$ the condition: $p > n/2$.

Theorem 5.6. Suppose that $V$ is a real-valued function belonging to $L_p^{\rho-n/p}$ for some $\rho > 2$ and $p$. Then one has:

(i) If $V$ is subcritical, then $\sigma = \{\infty\}$ and the unique positive solution $P(x,\infty)$ has the asymptotics

$$(5.29)\qquad \begin{aligned} P(x,\infty) &= c + o(1) &&\text{as } |x| \to \infty,\ n \geq 3, \\ &= c \log |x| + O(1) &&\text{as } |x| \to \infty,\ n = 2, \end{aligned}$$

for some $c > 0$.

(ii) If $V$ is critical, then the unique positive solution $u$ has the following asymptotics as $|x| \to \infty$:

$$(5.30)\qquad u(x) = a|x|^{2-n}(1 + o(1))$$

for some constant $a > 0$.

Remark. This theorem was shown in [16] by a different method.

Proof. Let $n \geq 3$. Put $V_0 = 0$. Choosing a sufficiently large number $N$, put $V_1(x) = V(x)$ for $|x| \geq N$ and $V_1(x) = 0$ for $|x| < N$. We claim that $V_1$ is a small perturbation of $0$. We have

$$\begin{aligned} &\int |x-z|^{2-n}|V_1(z)||z-y|^{2-n}dz \\ &\leq \|V_1\| [\int (|x-z||z-y|)^{(2-n)p'}\langle z\rangle^{(-\rho+n/p)p'}dz]^{1/p'} \\ &\leq C\|V_1\|(\langle x\rangle^{(2-\rho)p'} + \langle y\rangle^{(2-\rho)p'})^{1/p'}|x-y|^{2-n}, \end{aligned}$$

where $\|V_1\|$ is the norm of $V_1$ in $L_p^{\rho-n/p}$ and $p'$ is the conjugate exponent of $p$. Thus the mathematical induction shows that for any $j = 0,1,\ldots$

$$(5.31) \qquad G^j(x,y;|V_1|) \leq (M\|V_1\|)^j |x-y|^{2-n},$$

where $M$ is a constant independent of $x,y$ and $j$. Since we may assume that $p < \infty$ by choosing another $\rho$ if necessary, (5.31) implies (5.3) if we choose $N$ so large that $M\|V_1\| < 1/2$. Now let us show (5.5), where $P^0(x,\infty) = 1$ in our case. With $x$ fixed we have that

$$\int_{|z|>|y|/2} (|x-z|\,|z-y|/|y|)^{2-n}|V_1(z)|dz = O(|y|)^{-\varepsilon})$$

as $|y| \to \infty$, where $\varepsilon = \rho - 2$, and that for $|z| \leq |y|/2$

$$(|x-z|\,|z-y|/|y|)^{2-n}|V_1(z)| \leq (|x-z|/2)^{2-n}|V_1(z)|.$$

Thus Lebesgue's dominated convergence theorem and (5.31) show that the limits

$$(5.32) \qquad \lim_{y\to\infty} G^j(x,y;V_1)/G^0(0,y) \equiv Q^j(x)$$

exist and satisfy

$$(5.33) \qquad Q^j(x) = \int c_n|x-z|^{2-n}V_1(z)Q^{j-1}(z)dz\ , \quad j \geq 1\ ,$$

and $Q^0(x) = 1$. We then have

$$Q^j(x) \le 2^{-j}\langle x\rangle^{-\varepsilon}, \quad j = 1,2,\ldots .$$

Hence

$$\lim_{y\to\infty} \frac{G(x,y;V_1)}{G^0(x,y)} = \sum_{j=0}^{\infty} Q^j(x) = 1 + O(|x|^{-\varepsilon}) \quad \text{as} \quad |x| \to \infty .$$

This proves (5.4) and (5.5). Now we can apply Theorem 5.3 to get Theorem 5.6 for $n \ge 3$.

When $n = 2$, choose a function $V_0$ of the form (4.4) with $a > 0$ and $0 < b < 1$, i.e., $V_0(x) = a|x|^{-2}$ for $b \le |x| \le 1$ and $V_0(x) = 0$ otherwise. Then we can show the theorem for $n = 2$ in the same way as above by making use of Theorems 4.3, 4.5, and 5.3 Q.E.D.

Similarly, we have the following theorem by Theorems 4.6, 4.7, and 5.3.

Theorem 5.7. Let $V_0$ be a function of the form (4.17) with $a > 0$ if $n = 2$ and $a \ge -(n-2)^2/4$ if $n \ge 3$: $V_0(x) = a|x|^{-2}$ for $|x| \ge 1$ and $V_0(x) = 0$ for $|x| < 1$. Suppose that $V - V_0$ is a real-valued function belonging to $L_p^{\rho-n/p}$ for some $\rho > 2$ and $p$. Then one has:

(i) If $V$ is subcritical, then $\sigma = \{\infty\}$ and the unique positive solution $P(x,\infty)$ has the following asymptotics as $|x| \to \infty$:

(5.34) $\quad P(x,\infty) = c|x|^A(1 + o(1))\,, \qquad a > -(n-2)^2/4\,,$

(5.35) $\quad P(x,\infty) = c|x|^A(\log|x|)(1 + o(1))\,, \quad a = -(n-2)^2/4$ and $n \ge 3\,,$

for some $c > 0$, where $A = [a + (n-2)^2/4]^{1/2} - (n-2)/2$.

(ii) If $V$ is critical, then the unique positive solution $u$ has the following asymptotics as $|x| \to \infty$:

(5.36) $$u(x) = c|x|^{-A'}(1 + o(1))$$

for some $c > 0$, where $A' = \{[4a + (n-2)^2]^{1/2} + n-2\}/2$.

We close this section with the following thoerem.

Theorem 5.8. Let $V_0(x) = a|x|^\beta$ for some $a > 0$ and $\beta > -2$. Suppose that $V - V_0$ is a real-valued function belonging to $L_p^{\rho-n/p}$ for some $p \geq n$ and $\rho > 1 - \beta/2$. Then one has:

(i) If $V$ is subcritical, then $\sigma = S^{n-1}$ and the minimal solution $P(x,\omega)$, $\omega \in S^{n-1}$, has the following asymptotics as $|x| \to \infty$:

(5.37) $$P(x,\omega) = F(\omega)\left[\frac{\sin\theta}{\sin(\beta/2 + 1)\theta}\right]^{\delta_n/2}|x|^{(n-2)\beta/4}$$
$$\times \exp\left[\frac{a^{1/2}}{\beta/2+1}|x|^{\beta/2+1}\cos\left(\frac{\beta}{2}+1\right)\theta\right](1 + o(1)),$$

where $\cos\theta = x\cdot\omega/|x|$, $0 \leq \theta < \min(\pi,\pi/(\beta/2+1))$, $\delta_n = 0$ for $n$ even $\delta_n = 1$ for $n$ odd, $F(\omega)$ is a positive continuous function on $S^{n-1}$. Furthermore, when $\beta > 0$, for any sufficiently small $\varepsilon > 0$ there exists a constant $C$ such that

(5.38) $$P(x,\omega) \leq C|x|^{(n-2)\beta/4}\exp\left[-\frac{a^{1/2}}{\beta/2+1}|x|^{\beta/2+1}\cos\varepsilon\right]$$

for any $\theta \geq (\pi-\varepsilon)/(\beta/2+1)$.

(ii) If $V$ is critical, then the unique positive solution $u$ has the asymptotics

$$(5.39) \qquad u(x) = c|x|^{-(n-1)/2-\beta/4}\exp[-\frac{a^{1/2}}{\beta/2+1}|x|^{\beta/2+1}](1+o(1))$$

as $|x| \to \infty$, where $c$ is a positive constant.

Proof. We shall show the theorem only for $n \geq 3$, since the case $n = 2$ can be treated similarly. Let $n \geq 3$. Choosing a sufficiently large $N$, put $V_1(x) = (V - V_0)(x)$ for $|x| \geq N$ and $V_1(x) = 0$ for $|x| < N$. We claim that $V_1$ is a small perturbation of $V_0$ if $N$ is sufficiently large. With

$$(5.40) \qquad \tilde{x} = \frac{a^{1/2}}{\beta/2+1}|x|^{\beta/2+1}\, e^{i(\beta/2+1)\theta},$$

$$\tilde{y} = \frac{a^{1/2}}{\beta/2+1}|y|^{\beta/2+1}\, e^{i(\beta/2+1)\theta},$$

(4.39) becomes

$$(5.41) \qquad d(x,y) = |\tilde{x}-\tilde{y}|.$$

This implies that $d(x,y) \leq d(x,z) + d(z,y)$ for any $x,y$ and $z$. We denote by $G^0(x,y)$ the minimal Green's function for $V_0$. By Theorems 4.10 and 3.12, $G^0$ has the estimates:

(i) For $x$ and $y$ with $d(x,y) \leq 1$ or $X,Y \leq 1$

(5.42) $$G^0(x,y) \leq M|x-y|^{2-n} .$$

Here $X = a^{1/2}(\beta/2+1)^{-1}|x|^{\beta/2+1}$ and $Y = a^{1/2}(\beta/2+1)^{-1}|y|^{\beta/2+1}$.

(ii) For $x$ and $y$ with $d(x,y) > 1$, $0 \leq \theta < \min(\pi, \pi((\beta/2+1))$, and $X$ or $Y \geq 1$,

(5.43) $$G^0(x,y) \leq M(\langle X\rangle\langle Y\rangle)^{\gamma}\, d(x,y)^{-(n-1)/2} e^{-d(x,y)} ,$$

where $\langle X\rangle = (X^2+1)^{1/2}$ and $\gamma = (n/2-1)\beta/(\beta+2)$ .

(iii) Let $\varepsilon$ be a positive number such that $\pi\beta/(\beta+2) < \varepsilon < \pi$ . Then, for $x$ and $y$ with $d(x,y) > 1$, $(\pi-\varepsilon)/(\beta/2+1) \leq \theta \leq \pi$, and $X$ or $Y \geq 1$,

(5.44) $$G^0(x,y) \leq M(\langle X\rangle\langle Y\rangle)^{\gamma} d_{\varepsilon}(x,y)^{-(n-1)/2} e^{-d_{\varepsilon}(x,y)}$$

where $d_{\varepsilon}(x,y) = (X^2 + Y^2 + 2XY \cos \varepsilon)^{1/2}$ .

Now we claim that

$$G^1(x,y;|V_1|) = \int G^0(x,z)|V_1(z)|G^0(z,y)dz$$

has the same estimates as $G^0(x,y)$. We have by Hölder's inequality that

(5.45) $$G^1(x,y) \leq \|V_1\| [\int (G^0(x,z)G^0(z,y)\langle z\rangle^{-\rho+n/p})^{p'} dz]^{1/p'},$$

where $\|V_1\|$ is the norm of $V_1$ in $L_p^{\rho-n/p}$. We may and shall assume that $n \le p < \infty$ and $\rho > 1-\beta/2$. We first treat the case where $d(x,y) \le 1$ or $X,Y \le 1$. We put

$$(5.46)\quad I = \int_{D_1 \cup D_2 \cup D_3} (G^0(x,z)G^0(z,y)\langle z\rangle^{-\rho+n/p})^{p'}dz = I_1 + I_2 + I_3,$$

where $D_1 = \{z \in R^n: d(x,z) \le d(x,y)/2\}$, $D_2 = \{z;\ d(x,y)/2 < d(x,z) \le 2\}$, and $D_3 = \{z;\ d(x,z) > 2\}$. Since $d(z,y) \ge d(x,y) - d(z,x) \ge d(x,y)/2$ if $d(x,z) \le d(x,y)/2$, we have by (5.42) that

$$(5.47)\qquad I_1 \le (M|x-y|^{2-n})^{p'}\int_{D_1} (G^0(x,z)\langle z\rangle^{-\rho+n/p})^{p'}dz.$$

we have that for $x$ and $z$ with $d(x,z) \le 1/2$

$$(5.48)\qquad d(x,z) \le M\langle x\rangle^{\beta/2}|x-z|.$$

Thus

$$(5.49)\quad \int_{D_1} (G^0(x,z)\langle z\rangle^{-\rho+n/p})^{p'}dz \le C\int_E (\langle z\rangle^{-\rho+n/p}|x-z|^{-n+2})^{p'}dz,$$

where $C$ is a constant independent of $x$ and $E = \{z;\ |x-z| \le C\langle x\rangle^{-\beta/2}d(x,y)\}$. Noting that $(-n+2)p' + n > 0$, we have that the above integral is estimated by

$$C'\langle x\rangle^{p'\delta/2}d(x,y)^{p'(-n+2)+n}$$

with

(5.50) $$\delta = 2(-\rho+n/p) - \beta(-n+2) - n\beta/p'.$$

Since $-\rho < -1 + \beta/2$, we get $\delta < (n/p-1)(\beta+2) \leq 0$. Thus

(5.51) $$I_1^{1/p'} \leq C_1|x-y|^{2-n}.$$

Similarly,

(5.52) $$I_2^{1/p'} \leq C_2|x-y|^{2-n}.$$

When $d(x,y) \leq 1$ and $d(x,z) > 2$, we have by (5.43) that

$$G^0(x,z)G^0(z,y) \leq C(\langle X\rangle\langle Y\rangle)^{\gamma}\langle Z\rangle^{2\gamma}D(x,z)^{-n+1}e^{-2D(x,z)} ,$$

where $D(x,z) = d(x,z)$ for $\theta$ with $\cos\theta = x\cdot z/|x||z|$ and $0 \leq \theta < \min(\pi,(\pi-\varepsilon)/(\beta/2+1))$, and $D(x,z) = d_{\varepsilon}(x,z)$ for $\theta$ with $(\pi-\varepsilon)/(\beta/2+1) \leq \theta \leq \pi$. Here $\varepsilon$ is a sufficiently small positive number. Thus

(5.53) $$I_3 \leq C(\langle X\rangle\langle Y\rangle)^{\gamma p'}(J_1+J_2) ,$$

$$J_k = \int_{E_k} [\langle Z\rangle^{2\gamma+2(-\rho+n/p)/(\beta+2)}D(x,z)^{-n+1}e^{-2D(x,z)}]^{p'}dz ,$$

where $E_1 = \{z;\ d(x,z) > 2,\ |X-Z| \leq 1/2\}$ and $E_2 = \{z;\ d(x,z) > 2,\ |X-Z| > 1/2\}$. Since $d(x,z) \leq CX\theta + 1/2$ for $z$ in $E_1$, we have by changing the variable $|z|$ to $Z$ after using the polar coordinates for $z$ that

$$J_1 \leq M\langle X\rangle^{\{2\gamma+2(-\rho+n/p)/(\beta+2)\}p'+2n/(\beta+2)-1}$$

$$\times \left[\int_{\Phi}^{2/CX} e^{-2p'D(x,z)}\theta^{n-2}d\theta + e^{-2p'X}\right],$$

where $\Phi = \min(\pi,(\pi-\varepsilon)/(\beta/2+1))$. Noting that

$$D(x,z)^2 = (X-Z)^2 + (X^2+Z^2)((\beta/2+1)^2 2^{-1}\theta^2 + O(\theta^4)) \quad \text{as} \quad \theta \to 0,$$

we get

$$(5.54) \qquad J_1 \leq M\langle X\rangle^{\delta p'/(\beta+2)}$$

with $\delta = 2\gamma(\beta+2) + 2(-\rho+n/p) + 2n/p' - (\beta+2)/p' + (-n+1)(\beta+2)/p'$, which is equal to (5.50). We have

$$(5.55) \quad J_2 \leq Me^{-2p'X} + M\langle X\rangle^{\{2\gamma+2(-\rho+n/p)/(\beta+2)\}p'}$$

$$\times \int_{\substack{|X-Z|>1/2 \\ Z>1}} D(x,z)^{(-n+1)p'}e^{-2p'D(x,z)}dz .$$

Since

$$D(x,z) = |X-Z| + \{(\beta/2+1)^2 XZ/2|X-Z|\}\{\theta^2 + O(\theta^4)\} \quad \text{as} \quad \theta \to 0,$$

the stationary phase method shows that

$$\int_{|\omega|=1} D(x,|z|\omega)^{(-n+1)p'} e^{-2p'D(x,|z|\omega)} d\omega$$

$$\leq M|X-Z|^{(-n+1)p'} e^{-2p|X-Z|} (XZ/|X-Z|)^{-(n-1)/2} .$$

Thus

$$(5.56) \qquad J_2 \leq Me^{-2p'x} + M\langle X\rangle^{\delta p'/(\beta+2)}$$

$$\times \int_{|X-Z|>1/2} |X-Z|^{(-n+1)(p'+1/2)} e^{-2p'|X-Z|} dz$$

$$\leq M'\langle X\rangle^{\delta p'/2} ,$$

where $\delta$ is the negative number given by (5.50). Combining (5.53), (5.54), and (5.56) we get

$$(5.57) \qquad I_3^{1/p'} \leq C'(\langle X\rangle\langle Y\rangle)^{\gamma}\langle x\rangle^{\delta/2}$$

$$\leq C'(\langle X\rangle\langle Y\rangle)^{\gamma} d(x,y)^{-n+2} \leq C_3|x-y|^{2-n} .$$

Hence we obtain by (5.45), (5.46), (5.51), (5.52), and (5.57) that

$$(5.58) \qquad G^1(x,y) \leq M\|V_1\| \, |x-y|^{2-n}$$

for $x$ and $y$ with $d(x,y) \leq 1$ or $X,Y \leq 1$. We next estimate the integral (5.46) in the case (ii) where $d(x,y) > 1$, $0 \leq \theta <$

$\min(\pi, \pi/(\beta/2+1))$, and $X$ or $Y \geq 1$. The same calculations as in the estimate of $I_1$ show that

$$(5.59) \qquad I \leq MG^0(x,y)^{p'}$$

$$+ \int_{\substack{d(x,z)\geq 1/2 \\ d(y,z)\geq 1/2}} (G^0(x,z)G^0(z,y)\langle z\rangle^{-\rho+n/p})^{p'} dz \ .$$

Denoting the above integral by $J$ we have

$$J \leq M(\langle X\rangle\langle Y\rangle)^{\gamma p'}$$

$$\times \int_{\substack{D(x,z)\geq 1/2 \\ D(y,z)\geq 1/2}} [\langle Z\rangle^{2\gamma+2(-\rho+n/p)/(\beta+2)}$$

$$(D(x,z)D(y,z))^{(-n+1)/2} e^{-D(x,z)-D(y,z)}]^{p'} dz,$$

where $D(x,z)$ is the same as in (5.53). Since the minimum

$$d(x,y)$$

of $D(x,z) + D(y,z)$ is atained at $z = x$ or $y$, the stationary phase method shows that

$$(5.60) \qquad J^{1/p'} \leq M'(\langle X\rangle\langle Y\rangle)^{\gamma}\langle x\rangle^{\delta/2} d(x,y)^{(-n+1)/2} e^{-d(x,y)} \ .$$

Combining (5.45), (5.59), and (5.60), we get

$$(5.61) \qquad G^1(x,y) < M\|V_1\|(\langle X\rangle\langle Y\rangle)^\gamma d(x,y)^{(-n+1)/2} e^{-d(x,y)}$$

for $x$ and $y$ with $d(x,y) > 1$, $0 \leq \theta < \min(\pi, \pi/(\beta/2+1))$, and $X$ or $Y \geq 1$. Similarly,

$$(5.62) \qquad G^1(x,y) \leq M\|V_1\|(\langle X\rangle\langle Y\rangle)^\gamma d_\varepsilon(x,y)^{(-n+1)/2} e^{-d_\varepsilon(x,y)}$$

for $x$ and $y$ with $d(x,y) > 1$, $(\pi-\varepsilon)/(\beta/2+1) \leq \theta \leq \pi$, and $X$ or $Y \geq 1$. This completes the proof of the claim that $G^1(x,y)$ has the same estimate as $G^0(x,y)$; moreover, the factor $\|V_1\|$ does appear in the estimate of $G_1$. Hence we obtain by by mathematical induction that

$$(5.63) \qquad G^j(x,y) \leq (M\|V_1\|)^j F(x,y), \qquad j = 0,1,\dots,$$

where $F(x,y)$ is the function appearing in the right hand sides of (5.42)~(5.44). This shows (5.3) if we choose $N$ so large that $M\|V_1\| < 1/2$ (note that we may assume that $p < \infty$). (5.4) and (5.5) can be shown similarly along the line given in the proof of Theorem 5.6. Thus $V_1$ is a small purturbation of $V_0$. Hence Theorem 5.4 yields the theorem except for (5.37) and (5.38). But they can be shown along the line given in the proof of Theorem 5.4, since (5.37) and (5.38) hold with $P$ replaced by $\tilde{P}$. Q.E.D.

## §6. Positive solutions in a half space

In this section we give some results concerning positive solutions in a half space of a Schrödinger equation with radially symmetric potential satisfying (3.1). The results here will be used in Section 7. Throughout the section we assume the condition (3.1) and use the same notation as in Section 3. We write $R_+^n = \{x \in R^n;\ x_n > 0\}$.

Theorem 6.1. (i) $-\Delta+V \geq 0$ in $R_+^n$ if and only if $g_1(r) > 0$ for any $r > 0$.

(ii) $V$ is subcritical in $R_+^n$ if and only if $g_1(r) > 0$ for $r > 0$ and

$$\int_1^\infty r^{1-n} g_1(r)^{-2} dr < \infty. \tag{6.1}$$

(iii) $V$ is critical in $R_+^n$ iff $g_1(r) > 0$ for $r > 0$ and the left hand side of (6.1) diverges. Furthermore, any positive solution is a constant multiple of

$$u(x) = g_1(|x|)x_n/|x|. \tag{6.2}$$

Proof. (i) If $g_1(r) > 0$ for $r > 0$, then $u(x) = g_1(|x|)x_n/|x|$ is a positive solution of $(-\Delta+V(|x|))u(x) = 0$ in $R_+^n$. This implies $-\Delta+V \geq 0$ in $R_+^n$. Conversely, suppose that $-\Delta+V \geq 0$ in $R_+^n$. Then, for any half ball $B_R^+ = \{x;\ |x| < R,\ x_n > 0\}$, the operator $-\Delta+V$ is strictly positive; for if the smalest eigenvalue of $-\Delta+V$ in $B_t^+$ were zero for some

$t > 0$, then there would exist $u_t$ and $u_{2t}$ such that for $j = 1,2$, $(-\Delta+V)u_{jt} = 0$ in $B_{jt}^+$, $u_{jt}(x) > 0$ in $B_{jt}^+$, and $u_{jt}(x) = 0$ on $\partial B_{jt}^+$, which is a contradiction. Now, $u(x) = g_1(|x|)x_n/|x|$ is a solution in $B_R^+$ for any $R > 0$, and $\pm u(x) \geq 0$ on $\partial B_R^+$ if $\pm g_1(R) \geq 0$. On the other hand, since $g_1(r) = r + o(r)$ as $r \to 0$, $u(x) > 0$ for sufficiently small $x$ in $B_R^+$. Thus the generalized maximum principle shows that $u(x) \geq 0$ in $B_R^+$ for any $R > 0$, which implies $g_1(R) > 0$.

(ii) & (iii) Let $-\Delta+V \geq 0$ in $R_+^n$. Denote by $G_{R,+}$ the Green's function for $-\Delta+V$ in $R_+^n$ with zero Dirichlet boundary condition. Then we have by Theorem 3.8 and the reflection principle that

$$(6.3) \qquad G_{R,+}(x,y)$$

$$= c_n \sum_{j=1}^{\infty} g_j(|x|)f_{jR}(|y|)\{P_j(x\cdot y/|x||y|)-P_j(x\cdot\tilde{y}/|x||y|)\}$$

for $|x| \leq |y| < R$, where $\tilde{y} = (y_1,\ldots,y_{n-1}, -y_n)$. Thus the same argument as in the proof of Theorem 3.1 shows (ii) and the first half of (iii). The second half of (iii) follows from Theorem 2.2. Q.E.D.

From the proof of Theorem 6.1 we have

Theorem 6.2. If $V$ is subcritical, then the minimal Green's function $G_+(x,y)$ is given by

$$(6.4)\quad G_+(x,y) = c_n \sum_{j=1}^{\infty} g_j(|x|) f_j(|y|)\{P_j(x\cdot y/|x||y|) - P_j(x\cdot\tilde{y}/|x||y|)\}$$

for $|x| \leq |y|$, and $G_+(y,x) = G_+(x,y)$.

From now on we consider the case where $V$ is subcritical and normalize positive solutions $u$ by $u(x^0) = 1$ with $x^0 = (0,\ldots,0,1)$.

The following two theorems can be shown in a way similar to that in the proof of Theorems 3.2 and 3.3.

Theorem 6.3. Suppose that $V$ is subcritical and

$$(6.5)\qquad \int_1^{\infty} t^{n-3}\, g_1(t)^2 \Big[\int_t^{\infty} s^{1-n}\, g_1(s)^{-2} ds\Big] dt = \infty .$$

Then $\sigma = \partial R_+^n \cup \{\infty\}$ and the minimal solutions $P_+$ are given by

$$(6.6)\qquad P_+(x,\infty) = g_1(|x|)x_n/g_1(1)|x| ,$$

$$(6.7)\quad P_+(x,\omega) = K(x,\omega)/K(x^0,\omega),$$

$$K(x,\omega) = -(2/|\omega|)\partial G/\partial\theta(|x|,|\omega|,\theta), \quad \omega \in \partial R_+^n ,$$

where $G$ is the function given by (3.32), $0 \leq \theta \leq \pi$, and $\cos\theta = x\cdot(\omega,0)/|x||\omega|$ .

Theorem 6.4. Suppose that $V$ is subcritical and (6.5) does not hold. Then $\sigma = \partial R_+^n \cup \bar{S}_+^{n-1}$, $\bar{S}_+^{n-1} = \{\omega \in S^{n-1};\ \omega_n \geq 0\}$, the minimal solutions $P_+(x,\omega)$ for $\omega$ in $\partial R_+^n$ are given by

(6.7), and $P_+(x,\omega)$ for $\omega$ in $\bar{S}_+^{n-1}$ are

$$(6.8) \qquad P_+(x,\omega) = \{Q(x,\omega) - Q(x,\tilde{\omega})\}/\{Q(x^0,\omega) - Q(x^0,\tilde{\omega})\},$$

$$Q(x,\omega) = \sum_{j=1}^{\infty} \frac{2j+n-2}{n}\left[\lim_{r\to\infty} \frac{g_1(r)}{g_j(r)}\right] g_j(|x|)P_j\left(\frac{x\cdot\omega}{|x|}\right) .$$

Remark 6.5. (i) If $g_0 > 0$ and (3.6) does not hold, then $Q(x,\omega)$ in (6.8) can be replaced by $P(x,\omega)$ in Theorem 3.3.

(ii) If a positive solution $u$ in $R_+^n$ is continuous up to the boundary, then Theorems 2.3, 6.3, and 6.4 imply that

$$(6.9) \qquad \int_{R^{n-1}} u(\omega,0)\; K(x^0,\omega)d\omega < \infty .$$

## §7. Wildly non-radially symmetric potentials.

The purpose of this section is to give examples of non-radially symmetric potentials on $R^2$ for which the structures of positive solutions are quite different from those for radially symmetric potentials.

We use polar coordinates $(r,\phi)$ for a point in $R^2$. Let $V$ be a real valued function on $R^2$ such that

$$(7.1) \quad \begin{aligned} V(r,\phi) &= r^{\beta_{2j}} \quad \text{for } \phi \text{ with } \Theta_{2j-1} \le \phi \le \Theta_{2j}, \quad j=1,\dots,k, \\ &= 0, \qquad\qquad \text{otherwise,} \end{aligned}$$

where $k$ is a natural number, $\beta_{2j} > -2$ for $j=1,\dots,k$, $0 = \Theta_0 < \Theta_1 < \dots < \Theta_{2k} = 2\pi$. Put $\theta_i = \Theta_i - \Theta_{i-1}$, $i=1,\dots,2k$. Then we have

Theorem 7.1. $V$ is subcritical and the corresponding minimal boundary $\sigma$ is homeomorphic to

$$(7.2) \qquad \bigcup_{j=1}^{k} \{\Theta_{2j-2} + \theta_{2j-1}/2\} \cup \sigma_{2j} ,$$

$$\sigma_{2j} = \{\psi \in [0,2\pi); |\psi - \Theta_{2j-1} - \theta_{2j}/2| \le \theta_{2j}/2 - \pi/(\beta_{2j} + 2)\}.$$

Furthermore, the minimal solutions $P(r,\phi;\psi)$, $\psi \in \sigma$, have the following growth properties as $r \to \infty$:

(i) Let $\psi = \Theta_{2j-2} + \theta_{2j-1}/2$. Then

$$(7.3)\quad P(r,\phi;\psi) = C(\psi)\chi_{2j-1}(\phi)r^{\pi/\theta_{2j-1}}\sin\{\pi(\phi-\Theta_{2j-2})/\theta_{2j-1}\}$$
$$+\ p(r,\phi;\psi)$$

as $r \to \infty$, where $C(\psi)$ is a positive constant $\chi_i(\phi)$ is a characteristic function of the set $[\Theta_{i-1}\ ,\ \Theta_i]$ and $p(r,\phi;\psi)$ is a function having the following asymptotic property: For $\phi$ with $\Theta_{2i-2} < \phi < \Theta_{2i-1}$ ,

$$(7.4)\qquad p(r,\phi;\psi) = o(r^{\pi/\theta_{2i-1}}) \quad \text{as} \quad r \to \infty\ ;$$

and for $\phi$ with $\Theta_{2i-1} < \phi < \Theta_{2i}$ ,

$$(7.5)\quad p(r,\phi;\psi) = O(\exp[-q_{2i}(\phi)r^{\beta_{2i}/2+1}]) \quad \text{as} \quad r \to \infty$$

with $q_{2i}(\phi)$ being a positive continuous function in $(\Theta_{2i-1}\ ,\ \Theta_{2i})$.

(ii) Let $\psi$ be in $\sigma_{2j}$. Then

$$(7.6)\quad P(r,\phi;\psi) = C(\psi)\chi_{2j}(\phi)\exp[\frac{r^{\beta_{2j}/2+1}}{\beta_{2j}/2+1}\cos(\beta_{2j}/2+1)(\phi-\psi)](1+o(1))$$
$$+\ p(r,\phi;\psi)$$

as $r \to \infty$, where $p(r,\phi,\psi)$ is a function having the asymptotics (7.4) and (7.5).

Remark. We should note that the discontinuity of $V$ is not a reason for the disconnectedness of $\sigma$. In fact, the perturbation technique employed in Section 5 and the proof of Theorem 7.1 below show that for some smooth potential the same conclusion as in Theorem 7.1 holds for a continuous potential.

The following theorem, which can be shown easily from Theorem 7.1, is closely connected with the conjecture 2.12.

Theorem 7.2. Let $\nu$ be the measure on $\sigma$ defined by

$$(7.7) \qquad \nu(\{\Theta_{2j-2} + \theta_{2j-1}/2\}) = \theta_{2j-1}\,,$$

$$\nu(\beta) = |B| \qquad \text{for} \quad B \subset \bigcup_{j=1}^{k} \sigma_{2j}\,,$$

where $|B|$ is the usual Lebesque measure. Then one has that for any continuous function $f$ on $\sigma$

$$(7.8) \quad \lim_{r\to\infty} \frac{\int_\sigma P(r,\phi;\psi) f(\psi) d\nu(\psi)}{\int_\sigma P(r,\phi;\psi) d\nu(\psi)} = f(\phi), \qquad \phi \in \sigma.$$

Furthermore, the left hand side $F(\phi)$ of (7.8) satisfies: (i) $F(\phi) = f(\Theta_{2j-2} + \theta_{2j-1}/2)$ for $\phi$ in $(\Theta_{2j-2}, \Theta_{2j-1})$; (ii) $F(\phi) = f(\Theta_{2j-1} + \pi/(\beta_{2j} + 2))$ for $\phi$ in $(\Theta_{2j-1}\,, \Theta_{2j-1} + \pi/(\beta_{2j+2}))$; and (iii) $F(\phi) = f(\Theta_{2j} - \pi/(\beta_{2j} + 2))$ for $\phi$ in $(\Theta_{2j} - \pi/(\beta_{2j} + 2), \Theta_{2j})$.

Remark. The problem of identifying the topology of $R^2 \cup \sigma$ induced by the metric (2.11) and obtaining a unique existence theorem, like Theorems 3.4 and 5.5, for the Dirichlet problem at infinity is left open.

The proof of Theorem 7.1 is somewhat long, and is decomposed into a few lemmas.

Lemma 7.3. Let $D = \{(r,\phi);\ 0 < \phi < \theta,\ r > 0\}$ and $W$ be a real-valued radially symmetric function in $L_{p,loc}(R^2)$ for some $p > 1$. Put

$$(7.9) \qquad W_+(r) = \alpha^{-2}\, r^{2/\alpha-2}\, W(r^{1/2})$$

with $\alpha = \pi/\theta$. Then one has:

(i) $W$ in $D$ is subcritical or critical or supercritical iff $W_+$ in $R_+^2$ is so.

(ii) Assume that $W$ is subcritical in $D$. Then $\sigma = \partial D \cup \{\infty\}$ or $\sigma = \partial D \cup [0,\theta]$ and the minimal solutions $P_D(x;\omega)$, $\omega \in \sigma$, normalized by $P_D(x^0;\omega) = 1$ with $x^0 = (1,\ \theta/2)$ are given by:

(1) For $\omega$ in $\partial D$,

$$(7.10) \qquad P_0(x;\omega) = K_D(x;\omega)/K_D(x^0;\omega) \;,$$

$$K_D(r,\phi;t,\psi) = 2\varepsilon(\psi)\,t^{-1}(\partial G/\partial\zeta)(r^\alpha,t^\alpha,\zeta)\Big|_{\zeta=\alpha(\phi-\psi)} \;,$$

where $\varepsilon(\psi) = -1$ for $\psi = 0$, $\varepsilon(\psi) = 1$ for $\psi = \theta$, and $G$ is the function in (6.7) with $V$ replaced by $W_+$, which has the form (3.32).

(2) For $\omega = \infty$ ,

$$P_D(r,\phi;\infty) = g_1(r^\alpha)\sin \alpha\phi/g_1(1) \quad (7.11)$$

where $g_1$ is the solution of (3.3) and (3.4) with $j=1$, $n=2$, and $V = W_+$ .

(3) For $\omega = \psi$ with $0 \le \psi \le \theta$ ,

$$P_D(r,\phi;\psi) = P_+(r^\alpha,\alpha\phi;\alpha\psi) \quad (7.12)$$

where $P_+$ is the function given by (6.8).

Proof. By changing the variable $(r,\phi)$ to $(r^\alpha,\alpha\phi)$ we have:

(a) Any positive solution $u$ of $(-\Delta+W)u = 0$ in $D$ is represented uniquely by $u(r,\phi) = u_+(r^\alpha,\alpha\phi)$, where $u_+$ is a solution of $(-\Delta+W_+)u_+ = 0$ in $R^2_+$ .

(b) The Green's function $G_{R,D}$ in $D \cap B_R$ for $W$ is represented by

$$G_{R,D}(r,\phi;t,\psi) = G_{R,+}(r^\alpha,\alpha\phi;t^\alpha,\alpha\psi) \quad (7.13)$$

where $G_{R,+}$ is the Green's function in $R^2_+ \cap B_R$ for $W_+$ .

The assertion (i) follows from (a) and (b). The assertion (ii) follows from Theorems 6.3, 6.4, and the equality

$$G_D(r,\phi;t,\psi) = G_+(r^\alpha,\alpha\phi;t^\alpha,\alpha\psi) \quad (7.14)$$

where $G_D$ is the minimal Green's function for $W$ in $D$ and $G_+$ is the minimal Green's function for $W_+$ in $R_+^2$ given by (6.4).

Q.E.D.

Now let $u$ be a nonnegative solution of $(-\Delta+V)u = 0$ in $R^2$. Then Lemma 7.3 and Theorem 2.3 show that $u$ is represented uniquely as follows:

(i) For $\phi$ in $[\Theta_{2j-1}, \Theta_{2j}]$, $j=1,\ldots,k$,

$$(7.15)\quad u(r,\phi) = \sum_{i=2j-1}^{2j} (-1)^i \int_0^\infty u(t,\Theta_i)\, \frac{2}{t}\, \frac{\partial G_{2j}}{\partial \zeta}(r^{\alpha_{2j}}, t^{\alpha_{2j}}, \zeta)\Big|_{\zeta=\alpha_{2j}(\phi-\Theta_i)} dt + \int_0^{2\pi} Q_{2j}(r^{\alpha_{2j}}, \alpha_{2j}\phi; \alpha_{2j}\psi)\, d\mu_{2j}(\psi) \;,$$

where $\alpha_{2j} = \pi/\theta_{2j}$ , $G_{2j}$ is the minimal Green's function on $R^2$ for the potential

$$V_{2j} = \alpha_{2j}^{-2} |x|^{(\beta_{2j}+2)/\alpha_{2j} - 2} \;,$$

$\mu_{2j}$ is a nonnegative measure such that Supp $\mu_{2j} \subset [\Theta_{2j-1}, \Theta_{2j}]$ and $\mu_{2j}[\Theta_{2j-1}, \Theta_{2j}] < \infty$, and

$$(7.16)\quad Q_{2j}(s,\theta;\theta') = \frac{P_{2j}(s,\theta-\gamma_{2j};\theta'-\gamma_{2j}) - P_{2j}(s,\theta-\gamma_{2j};\gamma_{2j}-\theta')}{P_{2j}(1,\pi/2;\theta'-\gamma_{2j}) - P_{2j}(1,\pi/2;\gamma_{2j}-\theta')}$$

with $\gamma_{2j} = \alpha_{2j}\Theta_{2j-1}$ and $P_{2j}$ being the minimal solution in $R^2$ for $V_{2j}$ .

(ii) For $\phi$ in $[\Theta_{2j-2}, \Theta_{2j-1}]$,

$$(7.17)\quad u(r,\phi) = \sum_{i=2j-2}^{2j-1} (-1)^i \int_0^\infty \frac{u(t,\Theta_i)\alpha(rt)^\alpha \sin\alpha(\phi-\Theta_i)\,dt}{\pi t[r^{2\alpha}+t^{2\alpha}-2(rt)^\alpha \cos\alpha(\phi-\Theta_i)]}$$

$$+ c_{2j-1}\, r^\alpha \sin\alpha(\phi-\Theta_{2j-2}) ,$$

where $\alpha = \alpha_{2j-1} = \pi/\theta_{2j-1}$ and $c_{2j-1}$ is a positive constant.

Define a nonnegative measure $d\mu$ on $[0,2\pi]$ by

$$(7.18)\qquad d\mu(\psi) = \sum_{j=1}^{k} \{d\mu_{2j}(\psi) + c_{2j-1}\delta(\psi - \Theta_{2j-2} - \theta_{2j-1}/2)\},$$

where $\delta$ is the Dirac measure, and put $Tu = d\mu$. Then $T$ is mapping from the set of all nonnegative solutions to the set of all nonnegative measures $d\mu$ on $[0,2\pi]$ such that $\mu([0,2\pi]) < \infty$ and

$$(7.19)\qquad \text{Supp } d\mu \subset \bigcup_{j=1}^{2k} \{\Theta_{2j-2} + \theta_{2j-1}/2\} \cup [\Theta_{2j-1}, \Theta_{2j}].$$

Lemma 7.4. If $\text{Supp } Tu \cap [\Theta_{2j-3}, \Theta_{2j}]$ is empty, then

$$(7.20)\quad R^{\beta_{2j-2}/2}\int_R^{2R} u(t,\Theta_{2j-2})\,dt + R^{\beta_{2j}/2}\int_R^{2R} u(t,\Theta_{2j-1})\,dt \le CR^{-\alpha_{2j-1}}$$

for any $R > 1$, where $C$ is a constant independent of $R$ .

Proof. Choose $C^\infty$-functions $g$ and $h$ on $R^1$ such that $g \ge 0$, $g(t) = 1$ for $1 \le t \le 2$, $g(t) = 0$ for $t \le 1/2$ or $t \ge 4$, $h \ge 0$, $h(\phi) = 1$ for $\Theta_{2j-2} - \theta_{2j-3}/8 \le \phi \le \Theta_{2j-1} + \theta_{2j}/8$, and $h(\phi) = 0$ for $\phi \le \Theta_{2j-2} - \theta_{2j-3}/4$ or $\phi \ge \Theta_{2j-1} + \theta_{2j}/4$. Put

$$\chi_R(r,\phi) = g(r/R)h(\phi).$$

Then we have that

$$(7.21)\qquad \int_0^\infty\int_0^{2\pi} Vu\chi_R r d\phi dr = \int_0^\infty\int_0^{2\pi} u\Delta\chi_R r d\phi dr$$

$$\le CR^{-1}\int_{R/2}^{4R}\int_{\Theta_{2j-2}-\theta_{2j-3}/4}^{\Theta_{2j-1}+\theta_{2j}/4} u(r,\phi)d\phi dr ,$$

where $C$ is a positive constant independent of $R$. By (7.17),

$$(7.22)\int_{\Theta_{2j-2}}^{\Theta_{2j-1}} u(r,\phi)d\phi = \int_0^\infty \frac{\alpha}{\pi t}\log\frac{|r^\alpha+t^\alpha|}{|r^\alpha-t^\alpha|}\{u(t,\Theta_{2j-2}) + u(t,\Theta_{2j-1})\}dt,$$

where $\alpha = \alpha_{2j-1}$. We write $\alpha' = \alpha_{2j-2}$, $\alpha'' = \alpha_{2j}$, and

$J_\delta = [\Theta_{2j-2} - \delta\theta_{2j-3}, \Theta_{2j-2}] \cup [\Theta_{2j-1}, \Theta_{2j-1} + \delta\theta_{2j}]$ for $0 < \delta \le 1$.

We then have by (7.15) that

$$(7.23)\qquad \int_{J_\delta} u(r,\phi)d\phi$$

$$= \int_0^\infty \frac{2\alpha'}{t}\{G_{2j-2}(r^{\alpha'},t^{\alpha'},0) - G_{2j-2}(r^{\alpha'},t^{\alpha'},\delta\pi)\}u(t,\Theta_{2j-2})dt$$

$$+ \int_0^\infty \frac{2\alpha'}{t}\{G_{2j-2}(r^{\alpha'},t^{\alpha'},\delta\pi) - G_{2j-2}(r^{\alpha'},t^{\alpha'},\pi)\}u(t,\Theta_{2j-3})dt$$

$$+\int_0^\infty \frac{2\alpha''}{t}\{G_{2j}(r^{\alpha''},t^{\alpha''},0) - G_{2j}(r^{\alpha''},t^{\alpha''},\delta\pi)\}u(t,\Theta_{2j-1})dt$$

$$+\int_0^\infty \frac{2\alpha''}{t}\{G_{2j}(r^{\alpha''},t^{\alpha''},\delta\pi) - G_{2j}(r^{\alpha''},t^{\alpha''},\pi)\}u(t,\Theta_{2j})dt.$$

Note that (7.17) implies that

$$(7.24)\qquad \int_1^\infty t^{-\alpha_i-1}\{u(t,\Theta_{i-1}) + u(t,\Theta_i)\}dt < \infty$$

for $i=1,3,\ldots,2k-1$. On the other hand, Theorem 4.10 implies that

$$(7.25)\qquad G_i(r^{\alpha_i},t^{\alpha_i},\delta\pi) \leq C\exp[-\varepsilon(r^{\beta_i/2+1} + t^{\beta_i/2+1})]$$

for $r >> 1$, where $i=2,4,\ldots,2k$, $C$ and $\varepsilon$ are positive constants depending only on $\delta$. Similarly,

$$(7.26)\qquad G_i(r^{\alpha_i},t^{\alpha_i},0) \leq C\exp[-\varepsilon(r^{\beta_i/2+1} + t^{\beta_i/2+1})]$$

for $r,t$ with $|r-t| > r/4$ and $r >> 1$. Combining (7.21) to (7.26) we obtain that

$$(7.27)\quad \int_{R/4}^{8R} dt\, u(t,\Theta_{2j-2})\int_0^\infty (2\alpha'/t)(r^{\beta_{2j-2}+1} - CR^{-1})$$

$$\times\, G_{2j-2}(r^{\alpha'},t^{\alpha'},0)\chi(r/R)dr$$

$$+ \int_{R/4}^{8R} dt\, u(t,\Theta_{2j-1}) \int_0^\infty (2\alpha''/t)(r^{\beta_{2j}+1} - CR^{-1})$$

$$\times\, G_{2j}(r^{\alpha''}, t^{\alpha''}, 0)\chi(r/R)\,dr$$

$$\leq C \exp(-\varepsilon R^\gamma) + CR^{-1}\int_0^\infty dt\{u(t,\Theta_{2j-2}) + u(t,\Theta_{2j-1})\}$$

$$\times \int_{R/2}^{4R} \frac{\alpha}{\pi t} \log \frac{|r^\alpha + t^\alpha|}{|r^\alpha - t^\alpha|}\, dr$$

for some positive constants $\varepsilon$, $\gamma$, and $C$. We have

$$(7.28) \qquad \int_{R/2}^{4R} \frac{\alpha}{\pi t R} \log \frac{|r^\alpha + t^\alpha|}{|r^\alpha - t^\alpha|}\, dr \leq \begin{cases} Ct^{\alpha-1}R^{-\alpha}, & t \leq R, \\ \\ Ct^{-\alpha-1}R^{\alpha}, & t \geq R. \end{cases}$$

Since

$$\int_{t^{\beta/2}|r-t|\leq 1} r^\beta \log(t^{\beta/2}|r-t|)^{-1}\, dr \geq ct^{\beta/2}, \qquad t >> 1\,,$$

we have by Theorem 4.10 that for $t$ with $R \leq t \leq 2R$ and $i = 2j-2$ , $2j$ ,

$$(7.29) \qquad \int_0^\infty \frac{2\alpha_i}{t}(r^{\beta_i+1} - CR^{-1})G_i(r^{\alpha_i}, t^{\alpha_i}, 0)\chi(r/R)\,dr \geq ct^{\beta_i/2}\,,$$

where $c$ is a positive constant independent of $R$. (7.27) to (7.29) yield

$$(7.30)\qquad \int_R^{2R} t^{\beta_{2j-2}/2}\, u(t,\Theta_{2j-2}) + t^{\beta_{2j}/2}\, u(t,\Theta_{2j-1})dt$$

$$\leq CR^{-\alpha} + C\int_1^R R^{-\alpha}t^{\alpha-1}U(t)dt + C\int_R^\infty R^{\alpha}t^{-\alpha-1}U(t)dt\ ,$$

where $U(t) = u(t,\Theta_{2j-2}) + u(t,\Theta_{2j-1})$. With $\beta = \min(\beta_{2j-2},\beta_{2j})$ we have by (7.30) and (7.24) that

$$\int_R^{2R} U(t)dt \leq C_1 R^{\alpha-\beta/2}\ ,$$

which implies that

$$\int_1^\infty t^{-\alpha-1+\varepsilon}\, U(t)dt < \infty\ ,$$

where $0 < \varepsilon < \beta/2 + 1$. This together with (7.30) yields

$$\int_R^{2R} U(t)dt \leq CR^{-\alpha-\beta/2} + C_2 \max(R^{\alpha-\beta/2-\varepsilon},\ R^{-\alpha-\beta/2})$$

$$+ C_2 R^{\,\alpha-\beta/2-\varepsilon}\ ,$$

which implies

$$\int_1^\infty t^{-\alpha-1+2\varepsilon}\, U(t)dt < \infty$$

if $-\alpha-1+2\varepsilon < \alpha + \beta/2$. After rpeating this process k-times, $k > 2\alpha/\varepsilon + 1$, we get

$$\int_R^{2R} U(t)dt \leq C_k R^{-\alpha-\beta/2}\ .$$

Hence the right hand side of (7.30) is estimated from above by $C_k R^{-\alpha}$, which proves the lemma. Q.E.D.

Let $G(x,y)$ be the minimal Green's function for $V$ in $R^2$. Since $u(x) = G(x,0)$ is also represented by (7.15) and (7.17) with $d\mu = 0$ and $(-\Delta+V)G(x,0) = \delta(x)$, we can apply Lemma 7.4 to $G(x,0)$ to get

Lemma 7.5. The estimate (7.20) holds for any $j=1,\ldots k$ with $u(x) = G(x,0)$.

Lemma 7.6. Let $u$ be a nonnegative solution of $(-\Delta+V)u = 0$ in $R^2$. If $Tu = 0$, then $u = 0$.

Proof. Choose a $C^\infty$-function $f$ on $R^1$ such that $f(t) = 1$ for $t \leq 1$, $f(t) = 0$ for $t \geq 2$, and $f \geq 0$. Put $f_R(r) = f(r/R)$. Then we have

$$(7.31) \qquad \int Vuf_R dx \leq CR^{-2} \int_{R\leq|x|\leq 2R} udx.$$

But Lemma 7.4 and its proof show that the right hand side of (7.31) is estimated from above by $CR^{-\gamma}$, $\gamma = \min(\alpha_1,\ldots,\alpha_{2k})$. Letting $R \to \infty$ we thus obtain that $u = 0$. Q.E.D.

This following lemma is a key to the proof of Theorem 7.1.

Lemma 7.7. $Tu = d\mu$ iff $u = \lim_{R\to\infty} u_R$, where $u_R$ is the solution of the Dirichlet problem

(7.32) $\quad (-\Delta + V)u = 0$ in $B_R$ , $u_R|_{\partial B_R} = Q|_{\partial B_R}$ ;

(7.33) $$Q(r,\phi) = c_{2j-1} r^{\alpha_{2j-1}} \sin \alpha_{2j-1}(\phi - \Theta_{2j-2}),\ \Theta_{2j-2} \le \phi \le \Theta_{2j-1},$$

$$= \int_0^{2\pi} Q_{2j}(r^{\alpha_{2j}}, \alpha_{2j}\phi; \alpha_{2j}\psi)\, d\mu_{2j}(\psi),\ \Theta_{2j-1} \le \phi \le \Theta_{2j} .$$

Proof. Suppose that $u = \lim_{R\to\infty} u_R$ . Set $u_R = Q + v_R$ . Then we have that $v_R(R,\phi) = 0$ and

(7.34) $$(-\Delta + V)v_R(r,\phi) = \sum_{i=0}^{2k-1} R_i(r) \otimes \delta(\phi - \Theta_i) ,$$

$$R_{2j-1}(r) = \frac{1}{r}\frac{\partial}{\partial\phi}\int_0^{2\pi} Q_{2j}(r^{\alpha_{2j}}, \alpha_{2j}\phi; \alpha_{2j}\psi)\, d\mu_{2j}(\psi)\Big|_{\phi=\Theta_{2j-1}}$$

$$+ \frac{1}{r} c_{2j-1}\alpha_{2j-1} r^{\alpha_{2j-1}} ,$$

$$R_{2j}(r) = -\frac{1}{r}\frac{\partial}{\partial\phi}\int_0^{2\pi} Q_{2j}(r^{\alpha_{2j}}, \alpha_{2j}\phi; \alpha_{2j}\psi)\, d\mu_{2j}(\psi)\Big|_{\phi=\Theta_{2j}}$$

$$+ \frac{1}{r} c_{2j+1}\alpha_{2j+1} r^{\alpha_{2j+1}} ,$$

where $\delta(\cdot)$ is Dirac's function. Note that $R_i(r) \ge 0$ . We then have

(7.35) $$v_R(r,\phi) = \sum_{i=0}^{2k-1} \int_0^R G_R(r,\phi; t, \Theta_i) R_i(t)\, dt .$$

Thus, $u = Q + v$ with

$$(7.36)\qquad v(r,\phi) = \lim_{R\to\infty} v_R(r,\phi) = \sum_{i=0}^{2k-1} \int_0^\infty G(r,\phi;t,\Theta_i)R_i(t)dt.$$

On the other hand, since $v_R(R,\phi) = 0$, we have another expression of $v_R$ : For $\phi$ in $[\Theta_{j-1}, \Theta_j]$,

$$(7.37)\ v_R(r,\phi)$$

$$= \sum_{i=j-1}^{j} (-)^i \int_0^\infty v_R(t,\Theta_i)\ \frac{2}{t}\ \frac{\partial G_{j,R}}{\partial\zeta}(r^{\alpha_j},t^{\alpha_j},\zeta)\Big|_{\zeta=\alpha_j(\phi-\Theta_i)}dt,$$

where $G_{2p,R}$ and $G_{2p-1,R}$ are the minimal Green's functions in $B_R$ for the potentials

$$\alpha_{2p}^{-2}|x|^{(\beta_{2p}+2)/\alpha_{2p}-2} \quad \text{and} \quad 0\ ,$$

respectively. Since $v_R$ and $\partial G_{j,R}/\partial\zeta$ are increasing as $R \to \infty$, the limit $v$ of $v_R$ is represented by (7.15) and (7.7) with $d\mu_{2j} = 0$ and $c_{2j-1} = 0$. This proves that if $u = \lim_{R\to\infty} u_R$, then $Tu = d\mu$.

Conversely, suppose that $Tu = d\mu$. Let $u_R$ be the solution of (7.32). Since $u(R,\phi) \geq u_R(R,\phi)$, we have that $u_R \leq u$ in $B_R$. Noting that $u_R$ is increasing as $R \to \infty$, we see that $u_R$ has a limit $w$ such that $u-w$ is a nonnegative solution with $T(u-w) = 0$. Lemma 7.5 now implies that $u = w = \lim_{R\to\infty} u_R$ .

Q.E.D.

Corollary 7.8. $T$ is an injection.

Lemma 7.9. If

$$\operatorname{Supp} d\mu \subset \bigcup_{j=1}^{k} \{\Theta_{2j-2} + \theta_{2j-1}/2\} \cup \sigma_{2j} ,$$

then there exists a nonnegative solusion $u$ such that $Tu = d\mu$.

Proof. Let $v_R$ be the function (7.35). It is easily seen from the proof of Lemma 7.6 that we have only to show $v_R(0,0)$ is bounded for $R > 1$. By Theorem 4.9 and (7.34),

$$R_{2j-1}(r) \le C(\langle r\rangle^{\beta_{2j}/2} + \langle r\rangle^{\alpha_{2j-1}}), \tag{7.38}$$

$$R_{2j}(r) \le C(\langle r\rangle^{\beta_{2j}/2} + \langle r\rangle^{\alpha_{2j+1}}).$$

This together with Lemma 7.5 and (7.35) implies the boundedness of $v_R(0,0)$. Q.E.D.

Lemma 7.10. $\operatorname{Supp} \; Tu \subset \bigcup_{j=1}^{k} \{\Theta_{2j-2} + \theta_{2j-1}/2\} \cup \sigma_{2j}$ .

Proof. Suppose that $\operatorname{Supp} Tu \ni \psi$ for some $\psi$ in $[\Theta_{2j-1}, \Theta_{2j}]\setminus\sigma_{2j}$ . Then we have by Theorem 4.9 that

$$\frac{\partial}{\partial\phi}\int_0^{2\pi} Q_{2j}(r^{\alpha_{2j}},\alpha_{2j}\phi;\alpha_{2j}\psi)\,d\mu_{2j}(\psi)\Big|_{\phi=\Theta_i} ,$$

$i=2j-1$ or $2j$, grows exponentially as $r \to \infty$. Since $u(r,\Theta_{2j-1})$ and $u(r,\Theta_{2j})$ grow at most polynomially as $r \to \infty$, we have by (7.15) that $(\partial u/\partial\phi)(r,\Theta_{2j-1})$ or $(\partial u/\partial\phi)(r,\Theta_{2j})$ is not a tempered distribution with respect to $r$. But (7.17) implies that both $(\partial u/\partial\phi)(r,\Theta_{2j-1})$ and $(\partial u/\partial\phi)(r,\Theta_{2j})$ are tempered distributions. This is a contradiction. Q.E.D.

Completion of the proof of Theorem 7.1. Lemma 7.9 implies that for $\psi$ in $\sigma_{2j}$, we can define $P(r,\phi;\psi)$ by

(7.39) $$P(r,\phi;\psi) = q(r,\phi;\psi)/q(0,0;\psi),\ q = \lim_{R\to\infty} u_R ,$$

where $u_R$ is the solution of (7.32) with

$$\begin{aligned} Q(r,\phi) &= Q_{2j}(r^{\alpha_{2j}},\alpha_{2j}\phi;\alpha_{2j}\psi),\quad \Theta_{2j-1} \le \phi \le \Theta_{2j} , \\ &= 0 , \qquad\qquad \text{otherwise.} \end{aligned}$$

For $\psi = \Theta_{2j-2} + \Theta_{2j-1}/2$ , we define $P(r,\phi;\psi)$ by (7.39) with $u_R$ being the solution of (7.32) with

$$\begin{aligned} Q(r,\phi) &= r^{\alpha_{2j-1}} \sin \alpha_{2j-1}(\phi-\Theta_{2j-2}),\quad \Theta_{2j-2} \le \phi \le \Theta_{2j} , \\ &= 0 , \qquad\qquad \text{otherwise.} \end{aligned}$$

Then Lemma 7.7 implies the growth properties of $P(r,\phi;\psi)$, from which it follows that $P(r,\phi;\psi)$ is minimal. On the other hand, Lemmas 7.7 and 7.10 assert that $\{P(r,\phi;\psi);\psi \in \sigma\}$ is the set of all minimal solutions. Furthermore, it is clear that the topology of $\sigma$ induced by the metric (2.11) is equal to the usual one.

Q.E.D.

We close this section with an example showing that even if $V$ and $W$ satisfy $cV \le W \le CV$ for some positive constants $c$

and C, the minimal boundaries for $V$ and $W$ can be different.

Example 7.11. Let $V(x) = a^2$ in $\bar{R}^2_+$ and $V(x) = b^2$ in $R^2_-$, where $0 < a < b$. Then $V$ is subcritical and $\sigma = [0,\pi] \cup [\pi+\theta, 2\pi-\theta]$, where $\theta = \cos^{-1} a/b$. Let us prove this assertion. Denoting by $G(x,y)$ the minimal Green's function for $V$, we have

$$(7.40) \qquad G(x,0) = \int e^{ix_1\xi} g(\xi,x_2)d\xi ,$$

$$g(\xi,x_2) = [(a^2+\xi^2)^{1/2} + (b^2+\xi^2)^{1/2}]\exp[\mp(a^2+\xi^2)^{1/2}x_2]/(a^2-b^2),$$

$$\pm x_2 \geq 0.$$

By changing the path of integration in (7.40) in the complex plane, we have, for some $c > 0$,

$$(7.41) \qquad K(x_1) \equiv G(x_1,0,0,0) = c|x_1|^{-3/2}e^{-a|x_1|}(1 + o(1))$$

as $|x_1| \to \infty$. On the other hand, the same argument as in the proof of Theorem 7.1 shows that any nonnegative solution $u$ of $(-\Delta+V)u = 0$ in $R^2$ must satisfy

$$(7.42) \qquad \int_{-\infty}^{\infty} u(t,0)\langle t\rangle^{-3/2}e^{-a|t|}dt < \infty.$$

Let us define a measure $Tu$ in the same way as in the proof of Theorem 7.1. We claim that if $Tu = 0$, then $u = 0$. Since

$$\lim_{x_2 \uparrow 0} \partial u/\partial x_2(x) = \int (b^2+\xi^2)^{1/2} \hat{f}(\xi) e^{ix_1\xi} d\xi/2\pi$$

$$= \lim_{x_2 \downarrow 0} \partial u/\partial x_2(x) = \int -(a^2+\xi^2)^{1/2} \hat{f}(\xi) e^{ix_1\xi} d\xi/2\pi$$

with $\hat{f}$ being the Fourier transform of $u(x_1,0)$, we obtain that

$$\int K(t-s)u(s,0)dx = 0.$$

This together with positivity of $K$ proves the claim. We can now show the assertion in the same way as in the proof of Theorem 7.1.

Appendix 1. The one-dimensional case.

In this appendix we study positive solutions of a Schrödinger equation

$$\text{(A.1)} \qquad (-d^2/dx^2 + V(x))u(x) = 0 \quad \text{in} \quad (a,b),$$

where $-\infty \leq a < b \leq \infty$ and $V$ is a real-valued function in $L_{1,loc}(a,b)$.

Choose $\{a_j\}_{j=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$, and $c$ such that $a < a_{j+1} < a_j < c < b_j < b_{j+1} < b$ for any $j$, $\lim_{j\to\infty} a_j = a$ and $\lim_{j\to\infty} b_j = b$. Let $h_j(x)$ be a solution of the initial value problem

$$\text{(A.2)} \quad -h_j''(x) + V(x)h_j(x) = 0 \quad \text{in} \quad (a,b),\ h_j(a_j) = 0,\ h_j'(a_j) = 1.$$

Then we have

Theorem A.1. $-d^2/dx^2 + V \geq 0$ in $(a,b)$ iff for any $j$ $h_j(x) > 0$ for $x$ in $(a_j, b)$.

Proof. Suppose that $-\Delta+V \geq 0$. Let $d_j = \inf\{t; h_j(x) > 0$ for $x$ in $(a_j, t)\}$. Clearly, $a_j < d_j$. If $d_j < b$ for some $j$, then zero is an eigenvalue of $-\Delta+V$ in $(a_j, d_j)$ with zero Dirichlet boundary condition, which is a contradiction. Conversely, suppose that $h_j(x) > 0$ for $x$ in $(a_j, b)$, $j=1,2,\ldots$ . Put

$$\text{(A.3)} \qquad h_+(x) = \lim_{j\to\infty} h_j(x)/h_j(c).$$

Then $h_+$ is a positive solution, which implies $-\Delta+V \geq 0$.

Q.E.D.

Lemma A.2. The positive solution $h_+$ defined by (A.3) satisfies

$$\int_a^c h_+(t)^{-2}dt = \infty \ . \tag{A.4}$$

Proof. Suppose that (A.4) does not hold. Then

$$g(x) = h_+(x)\int_a^x h_+(t)^{-2}dt[h_+(c)\int_a^c h_+(t)^{-2}dt]^{-1}$$

is another positive solution such that $\lim_{x\to a} g(x)/h_+(x) = 0$. But, since $g(a_j) > 0$, $h_j(a_j)/h_j(c) = 0$, and $g(c) = 1$, we have that $h_j(x)/h_j(c) \leq g(x)$ for $x$ with $a_j < x < c$. Thus, $h_+(x) \leq g(x)$ for any $x$ in $(a,c)$, which is a contradiction. Q.E.D.

This lemma implies that $h_+(x)$ and

$$H_+(x) = h_+(x) \int_x^c h_+(t)^{-2}dt$$

are linearly independent solutions which are positive near $x = a$ and

$$\lim_{x\to a} h_+(x)/H_+(x) = 0.$$

This property indeed characterizes $h_+$ . That is, we have

Lemma A.3. If there exist two linearly independent solutions $f$ and $g$ such that $f$ and $g$ are positive near $x = a$ and $\lim_{x\to a} g(x)/f(x) = 0$, then $h_+(x) = Cg(x)$ for some $C > 0$.

Proof. We have that $h_+ = C_1 g + C_2 f$ and $H_+ = C_3 g + C_4 f$ for some constants $C_j$. This together with the property of the functions implies that $C_1 > 0$, $C_2 = 0$, and $C_4 > 0$. Q.E.D.

The same argument as above shows that if $-\Delta+V \geq 0$, then there exists a positive solution $h_-$ such that $h_-(c) = 1$ and

$$\int_c^b h_-(t)^{-2} dt = \infty \; . \tag{A.5}$$

Lemma A.3 yields the following relation between $h_+$ and $h_-$ :

Lemma A.4. (i) $h_+ = h_-$ if and only if

$$\int_c^b h_+(t)^{-2} dt = \infty \quad \text{or} \quad \int_a^c h_-(t)^{-2} dt = \infty \; . \tag{A.6}$$

(ii) If (A.6) does not hold, then

$$h_-(x) = h_+(x)\int_x^b h_+(t)^{-2} dt \left[\int_c^b h_+(t)^{-2} dt\right]^{-1} , \tag{A.7$_-$}$$

$$h_+(x) = h_-(x)\int_a^x h_-(x)^{-2} dt \left[\int_a^c h_-(t)^{-2} dt\right]^{-1} . \tag{A.7$_+$}$$

Theorem A.5. Suppose that $-\Delta+V \geq 0$. Then $V$ is critical iff (A.6) holds or $h_+ = h_-$ .

Proof. Let $G_j$ be the Green's function for $-\Delta+V$ in $(a_j, b_j)$ with zero Dirichlet boundary condition. Then we obtain that

$$\text{(A.8)} \qquad G_j(x,y) = h_+(x)h_+(y)\int_{a_j}^{x} h_+(t)^{-2}dt \int_{y}^{b_j} h_+(t)^{-2}dt$$
$$\times \left[ \int_{a_j}^{b_j} h_+(t)^{-2}dt \right]^{-1}$$

for $a_j \leq x \leq y \leq b_j$, and $G_j(y,x) = G_j(x,y)$. This together with (A.4) implies that $\lim_{j\to\infty} G_j(x,y) = \infty$ for any $x$ and $y$ iff (A.6) holds. Q.E.D.

Incidentally, we have obtained

Theorem A.6. If $V$ is subcritical, then the minimal Green's function $G(x,y)$ is given by

$$\text{(A.9)} \qquad G(x,y) = Ch_+(x)h_-(y), \qquad a < x \leq y < b,$$
$$= Ch_+(y)h_-(x), \qquad a < y \leq x < b,$$

where $C = \int_c^b h_+(t)^{-2}dt$.

This theorem and Theorem 2.3 yield

Theorem A.7. If $V$ is subcritical, then $\sigma = \{a,b\}$, $P(x,a) = h_-(x)$, and $P(x,b) = h_+(x)$.

Since $V = h_+''/h_+$, Theorems A.5 and A.7 and Lemma A.4 imply

Theorem A.8. Suppose that $-\Delta+V \geq 0$ and let $h = h_+$ with $h_+$ being the function (A.3). Let $W$ be a real-valued function in $L_{1,loc}(a,b)$. Then $V+W$ and $V$ are P-equivalent iff

(A.10) $$W = \{f'' + 2(h'/h)f'\}/f$$

for a positive continuous function $f$ such that $f$ and $f^{-1}$ are in $L_\infty(a,b)$, and $f''$ is in $L_{1,loc}(a,b)$.

The following theorem asserts that the Conjecture 2.12 is true in the one dimensional case, and can be shown easily from Lemma A.4. (ii), (A.4), and (A.5).

Theorem A.9. Suppose that $V$ is subcritical. Let $g(x) = h_-(x) + h_+(x)$. Then the Dirichlet problem at infinity

(A.11) $$(-d^2/dx^2 + V)u = 0 \quad \text{in} \quad (a,b),$$

$$\lim_{x\to a} u(x)/g(x) = A, \quad \lim_{x\to b} u(x)/g(x) = B,$$

admits a unique solution $u$ given by $u(x) = Ah_-(x) + Bh_+(x)$.

In the rest of this appendix we give some examples illustrating the results.

Example A.10. Suppose that $a > -\infty$ and $V$ belong to $L_1(a,c)$. Let $h$ be a solution of the initial value problem : $-h'' + Vh = 0$ in $(a,b)$, $h(a) = 0$, and $h'(a) = 1$. Then it is clear that (i) $-\Delta+V \geq 0$ iff $h > 0$ in $(a,b)$; and (ii) if $-\Delta+V \geq 0$, then $h_+(x) = h(x)/h(c)$.

Example A.11. Suppose that $(a,b) = R^1$ and $V(x) = V(|x|)$. Let $c = 0$ and $g_0$ be an even function such that $-g_0'' + Vg_0'' = 0$ in $R^1$ and $g_0(0) = 1$. Then we see that (i) $-\Delta+V \geq 0$ iff $g_0 > 0$; and (ii) if $g_0 > 0$, (A.12) $$h_{\pm}(x) = g_0(x) \lim_{R\to\infty} \left[\int_{\mp R}^{x} g_0(t)^{-2}dt \Big/ \int_{\mp R}^{0} g_0(t)^{-2}dt\right].$$

Example A.12. Suppose that $(a,b) = R^1$ and $V$ is a real-valued function such that

(A.13) $$\int_{-\infty}^{\infty} (|t|+1)|V(t)|dt < \infty.$$

Then elementary calculations yield a solution $h$ such that

(A.14) $$\lim_{x\to-\infty} h(x) = 1.$$

We easily obtain that (i) $-\Delta+V \geq 0$ iff $h > 0$; (ii) If $h > 0$, then $h_+(x) = h(x)/h(c)$; (iii) $V$ is critical iff $h(x)$ approaches a positive constant as $x \to \infty$; (iv) $V$ is subcritical iff $h(x)/x$ approaches a positive constant as $x \to \infty$; and (v) if $h > 0$ and

$$W = \{e^{2x}\sin e^x - e^x[1 + 2h'/h] \cos e^x\}/(2 - \sin e^x),$$

then $V+W$ and $V$ are P-equivalent.

Appendix 2. Proof of Theorems 4.9 and 4.10.

Proof of Theorem 4.9. Put

$$(A.15)\qquad Q(s,t) = \sum_{j=0}^{\infty} \frac{\max(2j+n-2,1)}{\max(n-2,1)} I_{(n-2+2j)/(\beta+2)}(s)P_j(t),$$

where $s = [a^{1/2}/(\beta/2+1)]|x|^{\beta/2+1}$ and $t = \cos\theta$. By (4.30),

$$(A.16)\qquad P(x,\omega) = \Gamma(\frac{n-2}{\beta+2}+1)(\frac{s}{2})^{(2-n)/(\beta+2)}Q(s,t).$$

We first investigate $Q$ in the case where $n$ is odd and $0 < \theta < \min(\pi,\pi/(\beta/2+1))$. By deforming the contour in the formula 5 in [6, p. 15] we have

$$J_\nu(z) = \frac{1}{2\pi i}\{\int_{i\infty}^{iR} + \int_C + \int_{iR}^{i\infty}\}u^{-\nu-1}\exp[\frac{z}{2}(u - \frac{1}{u})]du$$

for $\nu > 0$ and $\operatorname{Im} z \geq 0$, where $R > 1$, $C$ denotes the circle $|u| = R$, and $\arg u = -3\pi/2$ on the first path of integration while $\arg u = \pi/2$ on the third path. Since $I_\nu(s) = i^{-\nu}J_\nu(is)$, we have by changing the variable $u$ to $v = iu$ that

$$(A.17)\qquad I_\nu(s) = \frac{1}{2\pi i}\{\int_{-\infty}^{-R} + \int_C + \int_{-R}^{-\infty}\}v^{-\nu-1}\exp[\frac{s}{2}(v + \frac{1}{v})]dv$$

for $\nu \geq 0$ and $s > 0$, where $\arg v = -\pi$ on the first path of integration while $\arg v = \pi$ on the third path. On the other hand, differentiation of (3.9) with respect to $z$ yields

$$\sum_{j=0}^{\infty} [2j/(n-2)]P_j(t)z^{j-1} = 2(t-z)(1-2tz+z^2)^{-n/2}.$$

Thus

(A.18) $$\sum_{j=0}^{\infty} (1+2j/(n-2))P_j(t)z^{j+n/2-1} = (z-2t+1/z)^{-n/2}(1/z-z),$$

where the branch of $(z-2t+1/z)^{-n/2}$ is chosen so that $(z-2t+1/z)^{-n/2} > 0$ for $0 < z < 1$. Combining (A.15), (A.17) and (A.18), we have that

$$Q = \frac{1}{2\pi i}\{\int_{-\infty}^{-R} + \int_C + \int_{-R}^{-\infty}\}(z-2t+\frac{1}{z})^{-n/2}(\frac{1}{z} - z)\frac{1}{v}\exp[\frac{s}{2}(v + \frac{1}{v})]dv$$

where $z = v^{-1/(\beta/2+1)}$. Changing the variable $v$ to $z = v^{-1/(\beta/2+1)}$ we then obtain that

(A.19) $$Q = \frac{\beta/2+1}{2\pi i}\int_{\gamma_1+\gamma_2+\gamma_3}(z-2t+\frac{1}{z})^{-n/2}(z^{-2}-1)\exp[\frac{s}{2}(z^{\beta/2+1}+z^{-\beta/2-1})]dz,$$

where $\gamma_1$, $\gamma_2$ and $\gamma_3$ are the contours defined by

(A.20) $$\begin{aligned}\gamma_1 &= \{\gamma e^{-i\pi/(\beta/2+1)}\,;\ \gamma : 0 \to b\},\\ \gamma_2 &= \{be^{i\phi}\,;\ \phi : -\pi/(\beta/2+1) \to \pi/(\beta/2+1)\},\\ \gamma_3 &= \{\gamma e^{i\pi/(\beta/2+1)}\,;\ \gamma : b \to 0\},\end{aligned}$$

where $0 < b < 1$. Letting $b$ approach $1$ after integrating by parts we get from (A.19)

$$(A.21)\qquad Q = [(\tfrac{n}{2}-1)(\tfrac{n}{2}-2)\cdots\tfrac{1}{2}]^{-1}[(\tfrac{\beta}{2}+1)\tfrac{s}{2}]^{(n-1)/2}(\tfrac{\beta}{2}+1)R,$$

$$R = \frac{1}{2\pi i}\int_{\Gamma_1+\Gamma_2+\Gamma_3}(z-2t+z^{-1})^{-1/2}(z^{-2}-1)\exp[\tfrac{s}{2}(z^{\beta/2+1}+z^{-\beta/2-1})]dz$$

$$\equiv R_1 + R_2 + R_3\ ,$$

where $\Gamma_j(i=1,2,3,)$ are equal to $\gamma_j$ with $b$ replaced by 1. It is easily seen that

$$(A.22)\qquad R_1 + R_3 = O(e^{-s}) \quad \text{as} \quad s \to \infty\ .$$

By symmetry of the integrand we have

$$(A.23)\ R_2 = \frac{2^{1/2}}{\pi}\int_0^{\pi/(\beta/2+1)}(\cos\theta - \cos\phi)^{-1/2}\sin\phi\ \exp[s\cos(\tfrac{\beta}{2}+1)\phi]d\phi.$$

We easily see that

$$(A.24)\quad R_2 = \frac{2^{1/2}}{\pi}\int_\theta^{\theta+\varepsilon}(\cos\theta - \cos\phi)^{-1/2}\sin\phi\ \exp[s\cos(\tfrac{\beta}{2}+1)\phi]d\phi$$

$$+\ O(\exp[s\cos(\tfrac{\beta}{2}+1)(\theta+\varepsilon)])$$

as $s \to \infty$, where $0 < \varepsilon < \pi/(\beta/2+1) - \theta$. Denote the integral in (A.24) by $S$. Then we have by changing the variable $\phi$ to $u = \cos(\beta/2+1)\theta - \cos(\beta/2+1)\phi$ that

$$S = \exp[s\cos(\tfrac{\beta}{2}+1)\theta]\int_0^\delta(\cos\theta - \cos\phi)^{-1/2}\sin\phi$$

$$\times\ [(\beta/2+1)\sin(\beta/2+1)\phi]^{-1}\exp(-su)du\ ,$$

where $\delta = \cos(\beta/2+1) - \cos(\beta/2+1)(\theta+\varepsilon)$. Noting that

$$\cos\theta - \cos\phi = \sin\theta[(\beta/2+1)\sin(\beta/2+1)\theta]u + O(u^2)$$

as $u \to 0$, we obtain that

$$\text{(A.25)} \qquad S = \exp[s\cos(\tfrac{\beta}{2}+1)\theta]\left[\frac{\sin\theta}{(\beta/2+1)\sin(\beta/2+1)\theta}\right]^{1/2}$$

$$\times\ s^{-1/2}\int_0^\infty v^{-1/2}e^{-v}dv(1+O(s^{-1}))$$

as $s \to \infty$. (A.21) ~ (A.25) yeild

$$\text{(A.26)} \quad Q = \Gamma(\tfrac{n}{2})^{-1}\left[\frac{\sin\theta}{\sin(\beta/2+1)\theta}\right]^{1/2}$$

$$\times\ [(\tfrac{\beta}{2}+1)\tfrac{s}{2}]^{n/2-1}\exp[s\cos(\tfrac{\beta}{2}+1)\theta](1+O(s^{-1}))$$

as $s \to \infty$. This together with (A.16) shows (4.38) for $n$ odd and $\theta$ with $0 < \theta < \min(\pi, \pi/(\beta/2+1))$.

Next we treat the case where $n$ is odd and $\theta = 0$. The same argument as above shows (A.21) with

$$\text{(A.27)} \qquad R = \frac{2^{1/2}}{\pi}\int_0^\varepsilon (1-\cos\phi)^{-1/2}\sin\phi\ \exp[s\cos(\tfrac{\beta}{2}+1)\phi]d\phi$$

$$+\ O(\exp[s\cos(\tfrac{\beta}{2}+1)\varepsilon])$$

as $s \to \infty$, where $\varepsilon$ is a sufficiently small positive number. Denote the integral in (A.27) by $S$. By changing the variable

$\phi$ in $S$ to $u = [1 - \cos(\beta/2+1)\phi]^{1/2}$ we then have

$$S = e^s\int_0^\delta (1-\cos\phi)^{-1/2}\sin\phi\ \exp[-su^2]\frac{d\phi}{du}\,du,$$

where $\delta = [1-\cos(\beta/2+1)\varepsilon]^{1/2}$. Noting that $(1-\cos\phi)^{-1/2} = (\beta/2+1)u^{-1} + O(1)$, $\sin\phi = 2^{1/2}(\beta/2+1)^{-1}u + O(u^2)$, and $d\phi/du = 2^{1/2}(\beta/2+1)^{-1} + O(u)$, we have that

$$S = e^s\int_0^\delta [2/(\beta/2+1) + O(u)]e^{-su^2}du$$

$$= [\pi^{1/2}/(\beta/2+1)]s^{-1/2}e^s(1 + O(s^{-1}))$$

as $s \to \infty$. This yields (A.26) with $\theta = 0$, which implies (4.38) for $n$ odd and $\theta = 0$.

Third let us treat the case where $n$ is even and $\theta < \min(\pi,\pi/(\beta/2+1))$. We easily see that the formula (A.19) is also valid for $n$ even. By integration by parts we have

$$\text{(A.28)}\qquad Q = \Gamma(n/2)^{-1}[(\beta/2+1)s/2]^{n/2-1}(\beta/2+1)R,$$

$$R = \frac{1}{2\pi i}\int_{\gamma_1+\gamma_2+\gamma_3}(z-2t+z^{-1})^{-1}(z^{-2}-1)\exp[\tfrac{s}{2}\,(z^{\beta/2+1}+z^{-\beta/2-1})]dz,$$

where $\gamma_j$ are the contours (A.20). By making use of the standard limiting procedure we then have by symmetry of the integrand in the above integral that

$$R = \exp[s\cos(\beta/2+1)\theta](1 + O(s^{-1}))$$

as $s \to \infty$. Thus

$$(A.29) \qquad Q = \Gamma(n/2)^{-1}[(\beta/2+1)s/2]^{n/2-1}$$

$$\times\ (\beta/2+1)\exp[s\cos(\beta/2+1)\theta](1+O(s^{-1}))$$

as $s \to \infty$, which shows (4.38) for $n$ even and $0 < \theta < \min(\pi, \pi/(\beta/2+1))$.

The asymptotic formula of $\partial P/\partial\theta$ can be obtained similarly. Q.E.D.

Proof of Theorem 4.10. We see that $G(0,y) = c_n f_0(|y|)$ with

$$f_0(|y|) = cY^{(2-n)/(\beta+2)} K_{(n-2)/(\beta+2)}(Y)$$

for some positive constant $c$, where $K_\nu$ is the modified Bessel function of the second kind. Thus

$$G(0,y) = C|y|^{-(n-1)/2-\beta/4} e^{-Y}(1 + O(Y^{-1}))$$

as $y \to \infty$. This together with the Harnack inequality shows (4.40) for $x$ and $y$ with $X \le 1$ and $Y \ge 2$. The same argument shows (4.40) for $X \ge 2$ and $Y \le 1$. Since (4.40) ~ (4.43) clearly hold for $x$ and $y$ with $|x| + |y|$ bounded, it suffices to show them for $x$ and $y$ with $X$ and $Y$ sufficiently large.

From now on we assume that $x \ge 2$ and $y \ge 2$. It follows from (3.31), (3.32) and the proof of Theorem 4.8 that

$$G = c_n \sum_{j=0}^{\infty} g_j(|x|)f_j(|y|)P_j(t)$$

for $x$ and $y$ with $|x| \leq |y|$, where

$$g_j(|x|) = C_j X^{(2-n)/(\beta+2)} I_{(n-2+2j)/(\beta+2)}(X),$$

$$f_j(|y|) = D_j Y^{(2-n)/(\beta+2)} K_{(n-2+2j)/(\beta+2)}(Y).$$

Now let us determine the constants $D_j$. Since

$$(g_j'f_j - g_jf_j')(r) = \max(2j+n-2,1)r^{1-n},$$

we have by making use of the asymptotic expansion as $r \to \infty$ of $f_j(r)$ and $g_j(r)$ that

$$\text{(A.30)} \qquad C_jD_j = (\tfrac{\beta}{2}+1)^{-1}(\frac{a^{1/2}}{\beta/2+1})^{(n-2)/(\beta/2+1)} \max(2j+n-2,1).$$

Thus we obtain that for $X$ and $Y$ with $X \leq Y$

$$\text{(A.31)} \quad G = B(XY)^{(2-n)/(\beta+2)} \sum_{j=0}^{\infty} \max(2j+n-2,1) I_{m_j}(x)K_{m_j}(Y)P_j(t),$$

where $B = c_n(\beta/2+1)^{-1}[a^{1/2}/(\beta/2+1)]^{(n-2)/(\beta/2+1)}$ and $m_j = (2j+n-2)/(\beta+2)$. By the formula (38) in [6, p. 53],

$$\text{(A.32)} \qquad I_\nu(X)K_\nu(Y) = \frac{1}{2}\int_0^\infty \exp[-\frac{v}{2} - \frac{X^2+Y^2}{2v}] I_\nu(\frac{XY}{v})\frac{dv}{v}.$$

This together with (A.15), (A.26), and (A.29) implies that for $X < Y$

$$\sum_{j=0}^{\infty} \max(2j+n-2,1) I_{m_j}(X) K_{m_j}(Y) |P_j(t)|$$

$$\leq \frac{\max(n-2,1)}{2} \int_0^{\infty} \exp[-\frac{v}{2} - \frac{X^2+Y^2}{2v}] Q(\frac{XY}{v}, 1) \frac{dv}{v}$$

$$\leq M \int_0^{\infty} \exp[-\frac{v}{2} - \frac{X^2+Y^2-2XY}{2v}] (\frac{XY}{v} + 1)^{n/2-1} \frac{dv}{v}$$

$$< \infty,$$

where $M$ is a positive constant. Thus Lebesuge's dominated convergence theorem shows that

$$\text{(A.33)} \quad G = C(XY)^{(2-n)/(\beta+2)} \int_0^{\infty} \exp[-\frac{v}{2} - \frac{X^2+Y^2}{2v}] Q(\frac{XY}{v}, t) \frac{dv}{v},$$

where $C = B \max(n/2-1, 1/2)$. Since (A.16) implies that for $v \geq XY$

$$Q(XY/v, t) \leq M(XY/v)^{(n-2)/(\beta+2)},$$

we have

$$(XY)^{(2-n)/(\beta+2)} \int_{XY}^{\infty} \exp[-\frac{v}{2} - \frac{X^2+Y^2}{2v}] Q(\frac{XY}{v}, t) \frac{dv}{v}$$

$$\leq M \int_{XY}^{\infty} \exp[-\frac{v}{2} - \frac{X^2+Y^2}{2v}] v^{(2-n)/(\beta+2)} \frac{dv}{v}$$

$$= MZ^{(2-n)/(\beta+2)} \int_{XY/Z}^{\infty} \exp[-\frac{Z}{2}(u + \frac{1}{u})] u^{(2-n)/(\beta+2)} \frac{du}{u},$$

where $Z = (X^2 + Y^2)^{1/2}$. Since $XY/Z \geq 2^{1/2}$ when $X,Y \geq 2$, the above integral estimated from above by

$$M'\exp[-(21/20)Z]$$

for some constant $M'$. Thus in order to prove (4.40) ~ (4.43) we have only to estimate the integral

$$\text{(A.34)} \qquad F \equiv \int_0^{XY} \exp[-\frac{v}{2} - \frac{X^2+Y^2}{2v}]Q(\frac{XY}{v},t)\frac{dv}{v}.$$

Fixing $\theta$ in $[0,\min(\pi,\pi/(\beta/2+1)))$ we have by (A.26) and (A.29)

$$Q(s,t) = As^{n/2-1}\exp[s\cos(\frac{\beta}{2}+1)\theta](1+O(s^{-1}))$$

as $s \to \infty$, where $A$ is a positive constant. Thus

$$F = A(XY)^{n/2-1}\int_0^{XY} \exp[-\frac{v}{2} - \frac{d^2}{2v}]v^{-n/2}(1+O(\frac{v}{XY}))dv,$$

where $d = (X^2 + Y^2 - XY\cos(\beta/2+1)\theta)^{1/2}$. Change of variable $v$ to $u = v/d$ yields

$$\text{(A.35)} \qquad F = A(\frac{XY}{d})^{n/2-1}\int_0^{XY/d} \exp[-\frac{d}{2}(u+\frac{1}{u})]u^{-n/2}(1+O(\frac{ud}{XY}))du.$$

Elementary Calculations now show that

$$\text{(A.36)} \qquad F = A_1(XY)^{n/2-1}d^{-(n-1)/2}e^{-d}(1+O(d^{-1})) \quad \text{as } d \to \infty,$$

(A.37) $\quad F = A_2(XY)^{n/2-1}d^{-n+2}(1+O(d)) \quad \text{as} \quad d \to 0, \quad n \geq 3,$

$$= A_3(\log d^{-1} + O(d)) \quad \text{as} \quad d \to 0, \quad n = 2,$$

where $A_1$ , $A_2$ and $A_3$ are positive constants. Hence we obtain that

(A.38) $\quad G = D_1(XY)^{(n/2-1)\beta/(\beta+2)}d^{-(n-1)/2}e^{-d}(1 + O(d^{-1}))$

as $d \to \infty$, and that

(A.39) $\quad G = (XY)^{(n/2-1)\beta/(\beta+2)}d^{-n+2}(1+O(d)) \quad \text{as} \quad d \to 0, \ n \geq 3,$

$$= D_3(\log d^{-1} + O(d)) \quad \text{as} \quad d \to 0, \quad n = 2$$

This proves the theorem. Q.E.D.

Acknowledgment. This paper grew out of [16], which was written while the author was at the Institute for Mathematics and its Applications, University of Minnesota. The research for this paper was done during visits to the Institute and the University of Alberta. The author would like to express his hearty thanks to the Institute for Mathematics and its Applications, the School of Mathematics of University of Minnesota, and the Department of Mathematics of University of Alberta for their invitation and hospitality. He cordially thanks W. Littman, N. Garofalo, C. Kenig, H. Weinberger, S. Orey, B. Simon, W. Allegretto, J. Fields, and H. Tsuji for valuable conversations.

References

[1] S. Agmon, On positive solutions of elliptic equations with periodic coefficients in $R^n$, spectral results and extensions to elliptic operators on Riemanian manifolds, Proc. Internat. Conf. on Differential Equations, held at Univ. Alabama in Birmingham, 1983.

[2] W. Allegretto, On the equivalence of two types of oscillation for elliptic operators, Pacific J. Math. 55 (1974), 319-328.

[3] _____, Positive solutions and spectral properties of second order elliptic operators, Pacific. J. Math. 92 (1981), 15-25.

[4] M.T. Anderson and R. Schoen, Positive harmonic functions on complete manifolds of negative curvature, Annales of Math. 121 (1985), 429-461.

[5] L.A. Caffarelli and W. Littman, Representation formulas for solutions to $\Delta u-u=0$ in $R^n$, Studies in Partial Differential Equations, Math. Assoc. America, 23 (1982), 249-263.

[6] A. Erdelyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, Higher Transcendental functions, vol. II, McGraw-Hill, New York-Toronto-London, 1953.

[7] M. Granston, S. Orey, and U. Rösler, The Martin boundary of two dimensional Ornstein-Uhlenbeck processes, Probability, Statistics, and Analysis, 63-78. London Math. Soc. Lecture Note Ser., 79, Cambridge Univ. Press, Cambridge, 1983.

[8] P. Hartman, Ordinary Differential Equations, John Wiley & Sons, New York-London-Sydney, 1964.

[9] F.I. Karpelevic, The geometry of geodesics and the eigen-functions of the Beltrami-Laplace operator on symmetric spaces, Trudy Moskow. Math. Obsc. 14 (1965), 48-185; Trans. Moscow Math. Soc. 1965, 51-199.

[10] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin-Heidelberg-New York, 1966.

[11] R.S. Martin, Minimal positive harmonic functions, Trans. Amer. Math. Soc. 49 (1941), 137-172.

[12] W. Moss and J. Piepenbrink, Positive solutions of elliptic equations, Pacific. J. Math. 75 (1978), 219-226.

[13] M. Murata, Positive solutions and large time behavior of Schrödinger semigroups, Simon's problem, J. Fuct. Anal. 56 (1984), 300-310.

[14] _____, Isomorphism theorems for elliptic operators in $R^n$, Comm. in Partial Differential Equations, 9 (1984), 1085-1105.

[15] _____, Large time asymptotics for fundamental solutions of diffusion equations, Tôhoku Math. J. 37 (1985), 151-195.

[16] _____, Positive solutions of Schrödinger equations, IMA Preprint Ser. No. 124, 1985. Inst. Math. Applicat. Univ. Minnesota, Minneapolis.

[17] M.H. Protter and H.F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, Englewood Cliffs, N.J. 1967.

[18] B. Simon, Large time behavior of the $L^p$ norm of Schrödinger semigroups, J. Funat. Anal. 40 (1981), 66-83.

[19] _____, Schrödinger semi-groups, Bull. Amer. Math. Soc. 7 (1982), 447-526.

[20] G. Stampacchia, Le problème de Dirichlet pour les équations elliptique du second ordre à coefficients discontinuous, Ann. Inst. Fourier, 15 (1965), 189-257.

[21] M.G. Sur, The Martin boundary for a linear elliptic second-order operator, Izv. Acad. Nauk SSSR Ser. Math. 27 (1963), 45-60 ; Amer. Math. Translations Ser. 2, 56, 19-35.