Symplectomorphism Group of Rational 4-Manifolds

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Abstract

We develop techniques for studying the symplectomorphism group of rational 4-manifolds.

We study the space of tamed almost complex structures $\mathcal{J}_\omega$ using a fine decomposition via smooth rational curves and a relative version of the infinite dimensional Alexander duality. This decomposition provides new understandings of both the variation and stability of the symplectomorphism group $Symp(X, \omega)$ when deforming ω. In particular, we compute the rank of $\pi_0(Symp(X, \omega))$, with Euler number less than 8 in terms of the number N of -2 symplectic sphere classes.

In addition, using the above decomposition and coarse moduli of rational surfaces with a given symplectic form, we are able to determine $\pi_0(Symp(X, \omega))$, the symplectic mapping class group (SMC). Our results can be uniformly presented regarding Dynkin diagrams of type $\mathbb{A}$ and type $\mathbb{D}$ Lie algebras.

Applications of $\pi_0(Symp(X, \omega))$ and $\pi_0(Symp(X, \omega))$ includes the classification of symplectic spheres and Lagrangian spheres up to Hamiltionian isotopy and a possible approach to determine the full rational homotopy type $Symp(X, \omega)$.

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Chapter 1

Introduction

A symplectic manifold (X, ω) is an even dimensional manifold X with a closed, nondegenerate two form ω. A symplectic submanifold $S \in (X, \omega)$ is a submanifold such that $\omega|_S$ is a symplectic form. A Lagrangian submanifold $L \in (X, \omega)$ is a submanifold such that $\omega|_L = 0$.

Let (X, ω) be a closed simply connected symplectic manifold. the symplectomorphism group with the standard C^∞-topology, denoted as $Symp(X, \omega)$, is an infinite dimensional Fréchet Lie group. Understanding the homotopy type of $Symp(X, \omega)$ is a classical problem in symplectic topology initiated by [20]. Let $\mathcal{J}_\omega$ be the space of $\omega-$tame almost complex structures. It is known that the stratification structure of $\mathcal{J}_\omega$ is closely related to the topology of $Symp(X)$ when $dim(X) = 4$ [20, 2, 4, 26] and [7], etc. However, the study of the whole stratification of $\mathcal{J}_\omega$ is usually formidable even when X is relatively simple [5][7].

Among all homotopy groups of $Symp(X)$, $\pi_0(Symp(X))$ and $\pi_1(Symp(X))$ have more direct geometric meaning. $\pi_0(Symp(X))$, which we also call the *symplectic mapping class group* (SMCG), is closely related to isotopy problems of symplectic/Lagrangian submanifolds in X. $\pi_1(Symp(X))$ is tied to Hofer geometry of $Symp(X)$ (cf. [51]) and quantum cohomology (cf.[53]). Also, generator of $\pi_1(Symp(X, \omega))$ is also the generator of the rational homotopy groups of $Symp(X, \omega)$, for some rational surfaces with small Euler number, as shown in [7, 6] Hence it is essential in determining the full homotopy type of $Symp(X, \omega)$.

This dissertation is a summary of a series [31], [32] and [29], which studies the relation between $\pi_i(Symp(X_k))$ for $X_k = \mathbb{C}P^2\#k\overline{\mathbb{C}P}^2$ and $i = 0, 1$, the lower stratification of J_ω and negative symplectic curves in symplectic rational manifolds.

1.1 A fine decomposition of the space of almost complex structures

When (X, ω) is a rational symplectic 4-manifold, we consider the following natural decomposition of J_ω via smooth $\omega-$symplectic spheres of negative self-intersection.

Let $\mathcal{S}^{<0}_\omega(\mathcal{S}^{\leq -2}_\omega$ respectively) denote the set of homology classes of embedded ω-symplectic sphere with negative self-intersection (square less than -1 respectively).

Definition 1.1.1. A subset $\mathcal{C} \subset \mathcal{S}^{\leq -2}_\omega$ is called admissible if

$$\mathcal{C} = \{A_1, \cdots, A_i, \cdots, A_n | A_i \cdot A_j \geq 0, \forall i \neq j\},$$

Given an admissible subset $\mathcal{C}$, we define the real codimension of the label set $\mathcal{C}$ as $cod_{\mathbb{R}} = 2\sum_{A_i \in \mathcal{C}}(-A_i \cdot A_i - 1)$. And we define the **prime subset**

$$\mathcal{J}_\mathcal{C} = \{J \in \mathcal{J}_\omega | A \in \mathcal{S} \text{ has an embedded } J\text{−hol representative if and only if } A \in \mathcal{C}\}.$$

In particular, for $\mathcal{C} = \emptyset$, it has codimension zero, and the corresponding $\mathcal{J}_\emptyset$ is often called $\mathcal{J}_{open}$.

In section 2 (see Proposition 3.1.13 and Remark 3.1.15 for details), we will show that a prime subset is either empty or a submanifold with real codimension of its labeling set under a reasonable Condition 3.1.9. Notice that these prime subsets are disjoint. Clearly, we have the decomposition: $\mathcal{J}_\omega = \amalg_\mathcal{C} J_\mathcal{C}$. Hence we can define a filtration according to the codimension of these prime subsets:

$$\emptyset = \mathcal{X}_{2n+1} \subset \mathcal{X}_{2n}(= \mathcal{X}_{2n-1}) \subset \mathcal{X}_{2n-2} \ldots \subset \mathcal{X}_2(= \mathcal{X}_1) \subset \mathcal{X}_0 = \mathcal{J}_\omega,$$

where $\mathcal{X}_{2i} := \amalg_{Cod_{\mathbb{R}} \geq 2i}\mathcal{J}_\mathcal{C}$ is the union of all prime subsets having codimension no less than $2i$. In [30] we prove the filtration is actually a stratification for a symplectic rational 4 manifold with Euler number $\chi(X) \leq 8$, see Remark 3.4.10. And in Section 2.1 of this paper we will give the proof of the following specific theorem focusing on $\mathcal{X}_0, \mathcal{X}_2$, and $\mathcal{X}_4$, which suffices for applications in this paper:

Proposition 1.1.2. *For a rational 4 manifold having Euler number $\chi(X) \leq 8$ and any symplectic form, $\mathcal{X}_4 = \cup_{cod(\mathcal{C})\geq 4}\mathcal{J}_{\mathcal{C}}$ and $\mathcal{X}_2 = \cup_{cod(\mathcal{C})\geq 2}\mathcal{J}_{\mathcal{C}}$ are closed subsets in $\mathcal{X}_0 = \mathcal{J}_\omega$. Consequently,*

- *1. $\mathcal{X}_0 - \mathcal{X}_4$ is a manifold.*
- *2. $\mathcal{X}_2 - \mathcal{X}_4$ is a manifold.*
- *3. $\mathcal{X}_2 - \mathcal{X}_4$ is a submanifold of $\mathcal{X}_0 - \mathcal{X}_4$*
- *4. $\mathcal{X}_2 - \mathcal{X}_4$ is closed in $\mathcal{X}_0 - \mathcal{X}_4$.*

By applying a relative version of the Alexander duality in [13], we get the following computation of $H^1(\mathcal{J}_{open})$ regarding $\mathcal{S}_\omega^{-2}$, which is the set of symplectic -2 sphere classes.

Corollary 1.1.3. *For a symplectic rational 4 manifold with Euler number $\chi(X) \leq 8$, $H^1(\mathcal{J}_{open}) = \oplus_{A_i \in \mathcal{S}_\omega^{-2}} H^0(\mathcal{J}_{A_i})$.*

1.2 Application to symplectomorphism group

Take a basis of $H_2(X_k, \mathbb{Z})$ as $\{H, E_1, \ldots, E_k\}$, where H is the line class, and E_i the exceptional classes. Any symplectic form on a rational 4 manifold $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ is diffeomorphic to a reduced form (Definition 2.1). And diffeomorphic symplectic forms have homeomorphic symplctomorphism groups. Hence it suffices to understand the symplectomorphism group $Symp(X, \omega)$ for an arbitrary reduced form ω. By normalizing the symplectic form ω to be reduced with $\omega(H) = 1$, we can identify ω as a vector $(1|, c_1, c_2, \cdots, c_k) \in \mathbb{R}^k$, We'll describe the combinatorial structure and Lie theory aspect of the cone of normalized reduced forms on X in section 2.1. For $k \leq 8$, such a cone is a k-dimensional polyhedron P^k with the point M_k of monotone symplectic form as a vertex. And we show in Lemma 2.1.10 that the collection of Lagrangian spheres is a root system for $(X_k, \omega), k \leq 8$, which we call the Lagrangian root system. The set $\mathcal{R}$ of edges of P^k through the monotone point M_k one-to-one corresponds to the set of simple roots of the Lagrangian root system of (X_k, ω_{mon}), where ω_{mon} is the monotone symplectic form on X_k. Hence we call an element in $\mathcal{R}$ a **simple root edge** or **root**

edge. And further, in section 2 we give the details of the following fact: the Lagrangian root system of (X_k, ω) determines the simplicial structure where ω belongs to in P^k and vise versa.

Note that any symplectic form on a rational 4 manifold $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ is diffeomorphic to a reduced form. And diffeomorphic symplectic forms have homeomorphic symplctomorphism groups. Hence it suffices to understand the symplectomorphism group $Symp(X, \omega)$ for an arbitrary reduced form ω. We can further normalize the reduced form ω such that the line class $H \in H_2(X, \mathbb{Z})$ has area one, still denoted as ω. And we can identify ω as a vector $(1|c_1, c_2, \cdots, c_k) \in \mathbb{R}^k$, Specifically, we'll describe the combinatorial structure and Lie theory aspect of the cone of normalized reduced forms on X in section 2.1. For $k \leq 8$, such a cone is a k-dimensional polyhedron P^k with the point M_k of monotone symplectic form as a vertex. The set $\mathcal{R}$ of edges of P^k through the monotone point M_k one to one corresponds to the set of simple roots of the Lagrangian root system of (X_k, ω_{mon}), where ω_{mon} is the monotone symplectic form on X_k. Hence we call an element in $\mathcal{R}$ a **simple root edge** or **root edge**.

In Sections 4.2 and 5.1 we study the topology of $Symp(X, \omega)$, where X is a rational 4 manifold with Euler number $\chi(X) \leq 8$. Note that $Symp(X, \omega) = Symp_h(X, \omega) \rtimes \Gamma(X, \omega)$, where $Symp_h(X, \omega)$ is the homological trivial part of $Symp(X, \omega)$, also called the Torelli part. And $\Gamma(X, \omega)$ is called the non-Torelli part of $Symp(X, \omega)$, which is the image of the induced map from $Symp(X, \omega)$ to $Aut[H^2(X, \mathbb{Z})]$.

The following diagram of homotopy fibrations, formulated in [15] (in the monotone case) and adopted in [31] for a general ω, relates $\mathcal{J}_{open}$ and $Symp_h(X, \omega)$:

$$\begin{array}{ccccccc}
Symp_c(U) & & & & & & \\
\downarrow & & & & & & \\
Stab^1(C) & \longrightarrow & Stab^0(C) & \longrightarrow & Stab(C) & \longrightarrow & Symp_h(X, \omega) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathcal{G}(C) & & Symp(C) & & \mathcal{C}_0 \simeq \mathcal{J}_{open}
\end{array} \tag{1.1}$$

Each term above is a topological group except $\mathcal{C}_0 \simeq \mathcal{J}_{open}$. We will carefully explain each term in Chapter 4 and here we only introduce the right end of diagram 1.1:

$$Stab(C) \to Symp_h(X,\omega) \to \mathcal{C}_0 \simeq \mathcal{J}_{open}. \tag{1.2}$$

Here $\mathcal{C}_0$ is the space of a full stable standard configuration of fixed homological type. Every other term in diagram (1.1) is a group associated to $C \in \mathcal{C}_0$, and $U = X \setminus C$. Now we give the definition of stable standard spherical configurations and the groups will be discussed later in section 4.1.1.

Definition 1.2.1. Given a symplectic 4-manifold (X,ω), we call an ordered finite collection of symplectic spheres $\{C_i, i = 1, ..., n\}$ a spherical symplectic configuration, or simply a **configuration** if

1. for any pair i, j with $i \neq j$, $[C_i] \neq [C_j]$ and $[C_i] \cdot [C_j] = 0$ or 1.
2. they are simultaneously $J-$holomorphic for some $J \in \mathcal{J}_\omega$.
3. $C = \bigcup C_i$ is connected.

We will often use C to denote the configuration. The homological type of C refers to the set of homology classes $\{[C_i]\}$.

Further, a configuration is called

- **standard** if the components intersect ω-orthogonally at every intersection point of the configuration. Denote by $\mathcal{C}_0$ the space of standard configurations having the same homology type as C.
- **stable** if $[C_i] \cdot [C_i] \geq -1$ for each i.
- **full** if $H^2(X, C; \mathbb{R}) = 0$.

$\mathcal{C}_0$, the space of such configurations whose components intersect symplectic orthogonally, is indeed homotopic to $\mathcal{J}_{open}$, and admit a transitive action of $Symp_h(X,\omega)$. Therefore we have the above homotopy fibration 1.2, where $Stab(C)$ is the stablizer of the transitive action. Moreover, the homotopy type of $Stab(C)$ can often be explicitly computed using the terms of the other parts of diagram 1.1. Hence if we can further reveal the homotopy type of $\mathcal{J}_{open}$, which is very sensitive to the symplectic structure ω, we may probe, at least partially, the homotopy type of $Symp_h(X,\omega)$ via the homotopy fibration 1.2.

Following this route, the full homotopy type of $Symp_h(X, \omega)$ in the monotone case is determined in [15] for $k = 3, 4, 5$ (the smaller k cases follow from [20] and [2, 26]), and π_0 for a general ω is shown to be trivial in [31] for $k = 4$ (The smaller k cases follow from [2, 4, 26, 7]). In addition, the non-compact cases [21] are very similar in idea. In this paper we continue to follow this route and systematically analyze the persistence and change of the topology of $Symp(X, \omega)$ under deformation of symplectic forms (such phenomena were also discussed in [52, 54] and [42]).

1.2.1 Symplectic Mapping Class Group(SMC)

Recall that $\pi_0(Symp(X, \omega))$ is called the Symplectic mapping class group(SMC), and it admits the short exact sequence

$$1 \to \pi_0(Symp_h(X, \omega)) \to \pi_0(Symp(X, \omega)) \to \Gamma(X, \omega) \to 1, \tag{1.3}$$

where $\pi_0(Symp_h(X, \omega))$ is the Torelli Symplectic mapping class group(TSMC).

Theorem 1.2.2 (Main Theorem 1)**.** *Let (X, ω) be a symplectic rational 4-manifold with Euler number no larger than 8. Then its Lagrangian -2 spheres form a root system Γ_L which is a sublattice of $\mathbb{D}_5$. There are 32 sub-systems, out of which 30 are of type $\mathbb{A}$ (which is type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, or their direct product), and the other two of type $\mathbb{D}_4$ or $\mathbb{D}_5$. We completely describe π_0 of $Symp(X, \omega)$ in terms of Γ_L as follows:*

- *When Γ_L is of type $\mathbb{A}$, the above sequence 1.3 is*

$$1 \to 1 \to \pi_0(Symp(X, \omega)) \to W(\Gamma_L) \to 1,$$

  where $W(\Gamma_L)$ is the Weyl group of the root system Γ_L. In other word, $\pi_0(Symp(X, \omega))$ is isomorphic to $W(\Gamma_L)$;

- *And when Γ_L is of type $\mathbb{D}_n$, sequence 1.3 is*

$$1 \to \pi_0(Diff^+(S^2, n)) \to \pi_0(Symp(X, \omega)) \to W(\Gamma_L) \to 1,$$

  where $\pi_0(Diff^+(S^2, n))$ is the mapping class group of n-punctured sphere.

The case of $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ is particularly interesting. It is the first time that the **forget strands phenomena** is discovered for $SMC(M^{2n}), n > 1$. And it is closely related to question 2.4 in [57] which asked about the (co)kernel of the representation of coarse moduli of projective hypersurfaces using their SMC, see the following discussion for details. By the previous discussion, it suffices to consider only reduced forms. And we list them in Table 5.1, together with the number of symplectic -2 sphere classes N and the root system Γ_L, where there are 30 cases Γ_L is of type $\mathbb{A}$ and the other two cases of type $\mathbb{D}_4$ or $\mathbb{D}_5$.

For $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$, we choose the configuration C of 6 symplectic spheres in classes $\{E_1, E_2, \cdots E_5, Q = 2H - \sum_{i=1}^{5} E_i\}$. Clearly the first five spheres are disjoint and they each intersect the last one at a single point. The term $Symp(C)$ in diagram 1.1, which is the product of symplectomorphism group of each marked sphere component, is homotopic to $\mathrm{Diff}^+(S^2, 5) \times (S^1)^5$. In the monotone case, [55] amounts to show there is a subgroup $\pi_0(\mathrm{Diff}^+(S^2, 5)) \subset \pi_0(Stab(C))$, which injects to $\pi_0(Symp_h(X, \omega_{mon}))$; and [15] showed $Stab(C)$ is homotopic to $\mathrm{Diff}^+(S^2, 5)$ and the above injection is indeed an isomorphism. Hence TSMC is $\pi_0(\mathrm{Diff}^+(S^2, 5))$. Note this is the case where $\Gamma_L = \mathbb{D}_5$, see Lemma 2.1.10.

When a symplectic form $\omega = (1|c_1, c_2, \cdots, c_5)$ has $c_i < \frac{1}{2}$, we have the fibration 1.2 still being

$$\mathrm{Diff}^+(S^2, 5) \cong Stab(C) \to Symp_h(X, \omega) \to \mathcal{C}_0.$$

To deal with the TSMC, we consider the following portion of the long exact sequence of fibration 1.2:

$$\pi_1(\mathcal{C}_0) \xrightarrow{\phi} \pi_0(Stab(C)) \xrightarrow{\psi} \pi_0(Symp_h(X, \omega)) \to 1. \tag{1.4}$$

And $\pi_0(Stab(C)) = \pi_0(Symp(X, \omega_{mon})) = \pi_0(\mathrm{Diff}^+(S^2, 5))$ can be identified with $P_5(S^2)/\mathbb{Z}_2$ where $P_5(S^2)$ is the 5-strand pure braid group on sphere. It admit a standard generating set where each element A_{ij} is the twist of the $j-$th point around the $i-$th point, see Lemma 5.1.9. And when a normalized reduced form has $c_i < \frac{1}{2}$, we can use an explicitly constructed **Semi-toric ball swapping model** in Figure 5.2 to analyze $Im(\phi)$, and show that when $c_i \neq c_j$, the ball swapping symplectomorphism corresponding to braid generator A_{ij} is in $Im(\phi)$.

We show in Section 5.1 that the Torelli symplectic mapping class group(TSMC) behaves in the way of "forgetting strands" as for the braid group on spheres when deforming the symplectic form: One can think the curves in classes $\{E_1, E_2, \cdots E_5\}$ are 5 strands on Q and $\pi_0(Stab(C))$ acting on them by the braid group. Recall (cf.[9]) there is the forget one strand map

$$1 \to \pi_1(S^2 - 4 \text{ points}) \to \mathrm{Diff}^+(S^2, 5) \xrightarrow{f_1} \mathrm{Diff}^+(S^2, 4) \to 1$$

and the forget two strands map $\pi_0(\mathrm{Diff}^+(S^2, 5)) \xrightarrow{f_2} \pi_0(\mathrm{Diff}^+(S^2, 3))$, which is actually the homomorphism to the trivial group since $\pi_0(\mathrm{Diff}^+(S^2, 3)) = \{1\}$. And we find that map ψ in sequence (1.4) is the analogue of the forget strands map of $\pi_0(Symp_h(X, \omega))$:

- The form for which symplectic -2 sphere classes is minimal(8 classes) other than the monotone point is a one dimensional family (a root edge) in Polyhedron P^5, and they are all diffeomorphic to a normalized form having $c_i < \frac{1}{2}$. On the one hand, using Semi-toric ball swapping model we show that $Im(\phi)$ of 1.4 contains $\pi_1(S^2 - 4 \text{ points})$; on the other hand, using coarse moduli of equal blow up of Hirzebruch surface and overcoming difficulty of holomorphic bubbling, we extend the argument in [55] to show $\pi_0(Symp_h(X, \omega))$ surjects onto $\pi_0(\mathrm{Diff}^+(S^2, 4))$ in Proposition 5.3.1. And because $\pi_0(\mathrm{Diff}^+(S^2, k))$ is Hopfian, the map ψ is exactly the forget one strand map f_1. Note in this case, $\Gamma_L = \mathbb{D}_4$.

- Further when the form admit more symplectic -2 sphere classes, we have proposition 5.2.4 to deal with forms that are diffeomorphic to normalized forms with $c_1 < \frac{1}{2}$. Note that as far as the form has more than 8 symplectic -2 sphere classes, there are enough ball swapping symplectimorphism isotopic to identity, such that $Im(\phi)$ contains a generating set of $\pi_0(\mathrm{Diff}^+(S^2, 5))$. We further have Lemma 5.1.11 which use Cremona transform, to show that the above results hold for any balanced reduced symplectic form, where the non-balanced forms are in a subset of the open chamber and lower (1 or 2) codimension walls. To deal with the remaining cases, we have Lemma 5.4.1 where we project a non-balanced form to a codimension 2 wall and then use deformation type argument in [26]. Packing these together we have Proposition 5.4.2, saying TSMC is trivial if and only if there are more than 8 symplectic -2 sphere classes. And this time the map ψ in

1.4 is homomorphism to the trivial group, which is the same as map f_2. Note this covers the 30 cases when Γ_L is of type $\mathbb{A}$.

For less or equal to 4-point blow up of the complex project plane, we gave a uniform approach for the following result due to [20, 2, 3, 26, 7, 31]

Proposition 1.2.3. *$Symp_h(X,\omega)$ is connected for a rational surface with Euler number smaller than 8, with arbitrary symplectic form ω.*

It could provide information of Symplectic/Lagrangian spheres together with the the following proposition shown in [10] that:

Proposition 1.2.4. *Suppose (X^4,ω) is a symplectic rational manifold. Then $Symp_h(X,\omega)$ acts transitively on the space of*

- *homologous Lagrangian spheres*
- *homologous symplectic -2-spheres*
- *$\mathbb{Z}_2$-homologous Lagrangian $\mathbb{R}P^2$'s and homologous symplectic -4-spheres if $b_2^-(X)\leq 8$*

Hence we have the following corollary:

Corollary 1.2.5. *For a rational manifold with Euler number up to 7, the space of*
- *homologous Lagrangian spheres,*
- *$\mathbb{Z}_2$-homologous Lagrangian $\mathbb{R}P^2$,*
- *homologous -2 symplectic spheres,*
- *homologous -4 symplectic spheres,*

is connected.

1.2.2 $\pi_1(Symp_h(X,\omega))$ and Topological Persistence

In the mean while, we are able to relate the fundamental group of $Symp(X,\omega)$ with symplectic -2 sphere classes: On the one hand, for $X=\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}, k\leq 5$, we compute $H^1(\mathcal{J}_{open})$ by counting -2 symplectic sphere classes as in Corollary 3.4.7. On the other hand, for $X=\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}, k\leq 4$ with arbitrary ω, Lemma 4.3.4 guarantees that

$H^1(\mathcal{J}_{open})$ is isomorphic to $\pi_1(\mathcal{J}_{open}) = \pi_1(\mathcal{C}_0)$. Hence symplectic -2 sphere classes determine $\pi_1(\mathcal{C}_0)$. And we further look at the following portion of long exact sequence of fibration 1.2:

$$\pi_1(Stab(C)) \to \pi_1(Symp_h(X,\omega)) \to \pi_1(\mathcal{C}_0) \to 1. \tag{1.5}$$

Note that in the cases we deal with, $\pi_1(Symp(X,\omega_{mon})) = \pi_1(Stab(C))$ and it always injects into $\pi_1(Symp_h(X,\omega))$. Hence $\pi_1(Symp_h(X,\omega))$ is determined as follows:

Theorem 1.2.6 (Main Theorem 2)**.** *If (X,ω) is a symplectic rational 4 manifold with Euler number $\chi(X) \leq 7$, and N is the number of -2 $\omega-$symplectic sphere classes, then*

$$\pi_1(Symp(X,\omega)) = \mathbb{Z}^N \oplus \pi_1(Symp(X,\omega_{mon})).$$

This means that $\pi_1(Symp(X,\omega))$ is persistent on the each open chamber or the interior of each wall. And we observe the following amusing consequence.

Corollary 1.2.7. *For any rational 4-manifold (X,ω) with Euler number $\chi(X) \leq 7$, the integer*

$$PR[\Gamma(X,\omega)] + Rank[\pi_1(Symp(X,\omega))]$$

is a constant only depending on the topology of X, where $PR[\Gamma(X,\omega)]$ is the number of positive root of the reflection group $\Gamma(X,\omega)$.

In addition, for $X = \mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$, we use abelianization of sequence 1.5 to derive a lower bound of the rank of $\pi_1(Symp(X,\omega))$ in Lemma 5.5.1 and Remark 5.5.2. Together with Corollary 6.9 in [42], we can obtain the precise rank of $\pi_1(Symp_h(X,\omega)$ for most cases:

Proposition 1.2.8. *Let X be $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ with a reduced symplectic form ω. If $c_i < 1/2$, and TSMC is connected, then rank of $\pi_1(Symp_h(X,\omega)) = N - 5$, where N is the number of symplectic -2 sphere classes.*

Remark 1.2.9. Note that $\pi_1(Symp(X,\omega))$ is an Abelian group. In terms of Γ_L, its free rank can be often be precisely computed if Γ_L is of type $\mathbb{A}$ and $\mathbb{D}_5$. And when Γ_L is of type $\mathbb{D}_4$, we have a fine estimate of the free rank. And hence we conjecture the persistence type result analogous to Corollary 1.2.7 will also apply here:

$$PR[\Gamma(X,\omega)] + Rank[\pi_1(Symp(X,\omega))] - Rank[\pi_0(Symp_h(X,\omega))]$$

is a constant for $X = (\mathbb{C}P^2\#5\overline{\mathbb{C}P^2},\omega)$, where ω is any symplectic form. Here rank of $\pi_0(Symp_h(X,\omega))$ means the rank of its abelianization.

Finally we combine the analysis of π_1 and π_0 of $Symp(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2},\omega)$ to obtain the following conclusion on -2 symplectic spheres:

Corollary 1.2.10. *Homologous -2 symplectic spheres in $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ are symplectically isotopic for any symplectic form.*

Chapter 2

The normalized reduced symplectic cone

In this chapter, we provide a comprehensive description of the structure of the Normalized reduced symplectic cone a rational 4-manifold, with both its combinatorics and Lie theoretical aspects. Any symplectic form on the rational 4 manifold is diffeomorphic to a reduced one. We show that the cone of normalized reduced symplectic forms is convexly generated by the set of root edges $\mathcal{R}$, which is also the set of simple roots of the Lagrangian root system.

2.1 Normalized reduced symplectic cone:

2.1.1 Reduced symplectic forms

It is convenient to introduce the notion of reduced symplectic forms. For $X = \mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$, let $\{H, E_1, \cdots, E_n\}$ be a standard basis of $H_2(\mathbb{C}P^2\#n\overline{\mathbb{C}P^2};\mathbb{Z})$ where$H$ is the line class and E_is are the exceptional classes. We often identify the degree 2 homology with degree 2 cohomology using Poincaré duality.

Definition 2.1.1. Let X be $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ with a standard basis $H, E_1, E_2, \cdots, E_n$ of $H_2(X;\mathbb{Z})$. Given a symplectic form ω such that each class $H, E_1, \cdots, E_n$ has $\omega-$area

$\nu, c_1, \cdots, c_n$, then ω is called **reduced** (with respect to the basis)if

$$\nu > c_1 \geq c_2 \geq \cdots \geq c_n > 0 \quad \text{and} \quad \nu \geq c_i + c_j + c_k. \tag{2.1}$$

Remark 2.1.2. For $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, the reduced condition is $\nu > c_1 > 0$; and for $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$, the reduced condition is $c_1 \geq c_2, c_1 + c_2 < \nu$.

The cohomology class of ω is $\nu H - c_1E_1 - c_2E_2 - \cdots - c_nE_n$. And with any $J \in \mathcal{J}_\omega$ on $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$, the first Chern class $c_1 := c_1(J) \in H^2(\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}; \mathbb{Z})$ is $K := 3H - \sum_i E_i$. Let $\mathcal{K}$ be the symplectic cone of $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$, i.e.

$$\mathcal{K} = \{A \in H^2(\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}; \mathbb{Z}) | A = [\omega] \text{ for some symplectic form } \quad \omega\}.$$

Because of the uniqueness of symplectic blowup Theorem in [39], the diffeomorphism class of the form only depends on its cohomology class and we only need to consider the fundamental domain for the action of $\text{Diff}^+(X) \times R^*$ on $\mathcal{K}$. Further, [33] shows that the canonical class K is unique up to $\text{Diff}^+(X)$ action, we only need to describe the action of the subgroup $\text{Diff}_K \subset \text{Diff}^+(X)$ of diffeomorphisms fixing K on $\mathcal{K}_K = \{A \in H^2(\mathbb{C}P^2\#n\overline{\mathbb{C}P^2} | A = [\omega] \text{ for some } \omega \in \Omega_K\}$, where Ω_K is the subset of symplectic cone with K as the symplectic canonical class. Now we recall [33] that

Theorem 2.1.3. *The fundamental domain of* Diff_K *acting on* $\mathcal{K}_K$ *is the set of reduced classes* $\nu H - c_1E_1 - c_2E_2 - \cdots - c_nE_n$.

We give the following change of basis in $H^2(X, \mathbb{Z})$ in preparation for section 3. Note that $X = S^2 \times S^2\#k\overline{\mathbb{C}P^2}, k \geq 1$ can be naturally identified with $\mathbb{C}P^2\#(k+1)\overline{\mathbb{C}P^2}$. Denote the basis of H_2 by $B, F, E_1', \cdots, E_k'$ and $H, E_1, \cdots, E_k, E_{k+1}$ respectively. Then the transition on the basis is explicitly given by

$$\begin{aligned} B &= H - E_2, \\ F &= H - E_1, \\ E_1' &= H - E_1 - E_2, \\ E_i' &= E_{i+1}, \forall i \geq 2, \end{aligned} \tag{2.2}$$

with the inverse transition given by:

$$\begin{aligned} H &= B + F - E_1', \\ E_1 &= B - E_1', \\ E_2 &= F - E_1', \\ E_j &= E_{j-1}', \forall j > 2. \end{aligned} \tag{2.3}$$

$\nu H - c_1 E_1 - c_2 E_2 - \cdots - c_k E_k = \mu B + F - a_1 E_1' - a_2 E_2' - \cdots - a_{k-1} E_{k-1}'$ if and only if

$$\mu = (\nu - c_2)/(\nu - c_1), a_1 = (\nu - c_1 - c_2)/(\nu - c_1), a_2 = c_3/(\nu - c_1), \cdots, a_{k-1} = c_k/(\nu - c_1). \tag{2.4}$$

Hence

Lemma 2.1.4. *For $X = S^2 \times S^2 \# n\overline{\mathbb{C}P^2}$, any symplectic form ω is diffeomorphic to a reduced form and it can be further normalized to have area:*

$$\omega(B) = \mu, \omega(F) = 1, \omega(E_1') = a_1, \omega(E_2') = a_2, \cdots, \omega(E_n') = a_n$$

such that

$$\mu \geq 1 \geq a_1 \geq a_2 \geq \cdots \geq a_n \quad \text{and} \quad a_i + a_j \leq 1. \tag{2.5}$$

We also have the adjunction formula for embedded symplectic spheres:

Let A be the homology class of an embedded symplectic sphere and K the canonical class, then we have

$$K \cdot A + A \cdot A + 2 = 0 \tag{2.6}$$

Note that the canonical class K for a reduced form can be written down as $K = -2B - 2F + \sum_{i=1}^{n+1} E_i'$ or $K = -3H + \sum_{i=1}^{n+1} E_i$.

And we also observe the useful fact, which will be applied in section 5.1:

Lemma 2.1.5. *Let X be $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ with a reduced symplectic form ω, and ω is represented using a vector $(1|c_1, c_2, \cdots, c_n)$. Then E_n has the smallest area among exceptional sphere classes in X.*

Proof. One can explicitly write down the exceptional spheres using basis $H, E_1, \cdots, E_n$, Then because the form is reduced, the canonical class can be written as $K = -3H + E_1 + \cdots + E_n$. Then by Adjunction formula 2.6, for any exceptional sphere class $A = dH - \sum_i a_i E_i$, $K \cdot A = -1$. By the reduced assumption, $\omega(E_1) \geq \cdots \geq \omega(E_i) \geq \cdots \geq \omega(E_n)$. Because the form is reduced, then $\omega(H - E_i - E_j - E_k) \geq 0$ for any i, j, k if there is no repeated E_1 or E_2. We now verify that the curve $A = dH - \sum_i a_i E_i$ can always be rearranged such that it is the sum of d curves: $d-1$ of them in homology class $H - E_i - E_j - E_k$ with no repeated E_1 or E_2 and one curve in $H - E_i - E_j$ which is neither $H - 2E_1$ nor $H - 2E_2$.

Firstly, for the coefficient of H, choose any positive partition of d into $d-1$ parts, each as the coefficient one of $d-2$ curves of type $H - E_i - E_j - E_k$ and one $H - E_i - E_j$. Note that the genus of A is larger than 0, which means any coefficient a_i cannot exceed $d-1$, in particular, $a_1, a_2 \geq d-1$.

Hence for the coefficient of E_1, it is always possible to choose a positive integral partition of the a_1 into a_1 parts, which means that there are a_1 different curves of type $H - E_i - E_j - E_k$ having E_1 component. And the same can be done for E_2, such that there are a_2 different curves having the form $H - E_i - E_j - E_k$ where $H - E_i - E_j - E_k \cdot E_2 = 1$. After the two steps, we can rearrange the rest E_i's arbitrarily. Then it is easy to see that we have the desired rearrangement of $A = dH - \sum_i a_i E_i$. And clearly, $\omega(A) = \omega(dH - \sum_i a_i E_i) \geq \omega(E_k), \forall k$. This argument means any exceptional sphere class A has an area no less than c_n.

□

2.1.2 Combinatorics: Normalized Reduced symplectic cone as polyhedron

For a rational 4 manifolds X, the space of normalized reduced symplectic form is called **Normalized Reduced symplectic cone** (see section 2.2.2 for its relation with the symplectic cone). When $\chi(X) < 12$, it is a strongly convex polyhedron generated by its edges, as defined below:

Definition 2.1.6. A **closed convex polyhedron** is defined as $P_c := \{\vec{x} \in \mathbb{R}^n | A(\vec{x} -$

$\vec{x}_0) \leq 0\}$, where $\vec{x}$ is a n-dimensional column vector and $A = [\vec{a}_1, \cdots, \vec{a}_n]^t$ is a non-singular matrix. And a facet F_m is a codimension 1 subset of P_c, where $F_m := \{\vec{x} \in \mathbb{R}^n | \vec{a}_n^t \cdot \vec{x} = 0\}$.

Remark 2.1.7. Similarly, we can define a **open convex polyhedron** $P_o := \{\vec{x} \in \mathbb{R}^n | A(\vec{x} - \vec{x}_0) < 0\}$, which can be realized as removing all facets from the corresponding closed convex polyhedron. For simplicity, we call a subset of a closed convex polyhedron P_c a **polyhedron** P, if P is the complement of a union(possibly null) of facets. A **k-face**, $0 \leq k \leq n$ is a subset of P given by $n-k$ equations: $\{\vec{x} | \vec{a}_{r_i}^t = 0, r_i = 1, \cdots, n-k\}$, where $a_{r_i}^t$ is a row vector of matrix A. In particular, we use the following names interchangeably: a **vertex** is a 0-face and obviously $\vec{x}_0$ is a vertex point; a **edge** is a 1-face; and a **facet** is a $(n-1)-$face. an **open chamber** is an n-face; and a **wall** is a k-face where $k < n$. And we can compare this with remark 2.2.5

Proposition 2.1.8. *For $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, 3 \leq k \leq 8$, the normalized reduced symplectic cone is defined as the space of reduced symplectic forms having area 1 on H, the line class. It is a polyhedron with a vertex being the monotone form.*

Proof. For $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, 3 \leq k \leq 8$, we normalize the form into the vector $(1|c_1, \cdots, c_k)$. There is a form $\omega_{mon} = (1|\frac{1}{3}, \cdots, \frac{1}{3})$, called the monotone form. We consider the form $(c_1, \cdots, c_k) \in \mathbb{R}^k$. There is a linear translation moving $M_k = (\frac{1}{3}, \cdots, \frac{1}{3})$ to 0. And under this linear translation, it is easy to see from 2.1 that the image of all reduced symplectic form is a polyhedron P, i.e., if $(1|c_1, \cdots, c_k)$ is a reduced form, then for $x = (c_1 - \frac{1}{3}, \cdots, c_k - \frac{1}{3})$, the reduced condition 2.1 can be written as subset of $\{x \in \mathbb{R}^k, Ax \leq 0\}$, for some matrix $A \in GL_k(\mathbb{Z})$, with one or two facet removed. And further, $P \cap (-P) = \{0\}$, because $\{x \in \mathbb{R}^k, Ax \leq 0\} \cap \{-x \in \mathbb{R}^k, Ax \leq 0\} = 0$. This means the space of reduced form is a strictly convex polyhedron, with a vertex being the monotone form. □

And for the above manifolds X and their reduced cone P, any $k-$face is convexly generated by their root edge $\mathcal{R}$, denoted by $P_{\mathcal{S}}$. The interior of $P_{\mathcal{S}}$ is called a wall of the reduced cone.

Remark 2.1.9. For smaller rational manifolds, the cone is easier to describe, as illustrated in section 2.2.2.

2.1.3 Lie theory: Wall and chambers labeled by Dynkin diagram

This part is to review some Lie theoretic aspects of rational 4-manifolds. And in the next section, we will identify the root of a rational 4-manifolds (as defined below) with the edge of its normalized reduced symplecic cone.

Firstly, Lagrangian -2 sphere classes in rational manifolds generate a Root system. And we start with a reformulation of facts in [37], giving the concept of **monotone Lagrangian root system** in 2.1.11.

Note that for a rational 4-manifold with Euler number $\chi(X) < 12$ equipped with a reduced symplectic form, the number of Lagrangian sphere classes is finite and can be described as root system correspond to a simple laced Dynkin diagram in the following way: Now let X be a Del Pezzo surface of degree d which is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Define $r := 9 - d$, d is the degree of X. There exists a basis of $PicV : H, E_1 \cdots E_r$. For an integer r we define $(N_r, K_r, \langle \cdot, \cdot \rangle)$ and subsets $R_r, I_r \subset N_r$.

- $N_r := \bigoplus_{0 \leq i \leq r} \mathbb{Z} \cdot A_i$ where A_i is the above basis.
- $K_r := (-3, 1, 1, \dots, 1) \in N_r$.
- $\langle \cdot, \cdot \rangle$ is a bilinear form $N_r \times N_r \to \mathbb{Z}$ given on the basis by

$$
\begin{aligned}
\langle H, H \rangle &= 1 \\
\langle E_i, E_i \rangle &= -1 \quad \text{for } i \geq 1, \\
\langle E_i, E_j \rangle &= 0 \quad \text{if } i \neq j.
\end{aligned}
$$

- $R_r := \{A \in N_r \mid \langle A, K_r \rangle = 0, \ \langle A, A \rangle = -2\}$,
- $I_r := \{A \in N_r \mid \langle A, K_r \rangle = \langle A, A \rangle = -1\}$.

Then it is well known(see [37]) that one can describe these root systems R_r in terms of the standard irreducible root systems:

Lemma 2.1.10. *The -2 classes in R_r of a Del Pezzo surface X_r of degree $9 - r, 2 \leq r \leq 8$ form a root system $\mathbb{E}_r$:*

r	2	3	4	5	6	7	8
R_r	$\mathbb{A}_1$	$\mathbb{A}_1 \times \mathbb{A}_2$	$\mathbb{A}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\lvert R_r\rvert$	2	8	20	40	72	126	240

Remark 2.1.11. In the above Lemma 2.1.10, the root system only depends on the smooth topology of the ambient manifold, and we denote it by $\Gamma(X_r)$, and name it as the monotone Lagrangian root system of X_r.

Recall from Lie algebra that the set of roots $R_r = R(\Gamma(X_r)) = R^+(X_r) \cup -R^+(X_r)$ where $R^+(X_r)$ is the set of positive roots, which is defined here as $R^+(X_r) := \{A \in R(\Gamma(X_r))|A \cdot [\omega] > 0, [\omega]$ is the class of a reduced form $\omega\}$. And $R^+(X_r)$ is positive integrally spanned by a set of simple roots, which 1-1 corresponds to the vertices in the Dynkin diagram of Γ_L.

Denote $\mathbb{E}_0 = \mathbb{E}_1 = \emptyset, \mathbb{E}_2 = \mathbb{A}_1$, $\mathbb{E}_3 = \mathbb{A}_1 \times \mathbb{A}_2, \mathbb{E}_4 = \mathbb{A}_4, \mathbb{E}_5 = \mathbb{D}_5$, then $X_k := \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, 0 \leq k \leq 8$ has root system $\mathbb{E}_k$.

2.1.4 Identifying edges with roots

We know from Proportion 2.1.8 that for $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, 3 \leq k \leq 8$, the normalized reduced cone is convexly generated by its edges. And the edges are labeled using M the monotone point and another vertex, e.g. $O, A, B, \cdots$. Also, any k-face contains M as a vertex, and any vertex other than M is 1-1 corresponding to a edge. On the other hand, Lie theory together with Lemma 2.1.10 tells us that the -2 sphere classes form a set of positive roots of the manifolds above, where the reduced condition gives us a canonical set of **simple roots** by fixing a chamber.

The fundamental observation is that an edge appearing in the closure of the k-face $P_{\mathcal{S}}$ corresponds to a simple root of the lagrangian system. And hence we often call the edge in the normalized reduced cone a **root edge**. This fact can be thought in this way: the equation of an edge in the polyhedron is equivalent to fixing one " $\geq$ " as " $=$ " and every other " $\geq$ " as " $>$ " in equation (2.1); and this is equivalent to the existence of a unique Lagrangian simple root in Lemma 2.1.10. For $X_k = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, 3 \leq k \leq 8$, there is a standard choice of simple roots given in this way: $MO = H - E_1 - E_2 - E_3$, $MA = E_1 - E_2$, $MB = E_2 - E_3, \cdots$, $ML_{k-1} = E_{k-1} - E_k, \cdots$, $MG = E_7 - E_8$.

Note that they being symplectic or Lagrangian corresponds to the sign being " $>$ "

or " = " respectively in the reduced condition 2.1: $\lambda := c_1 + c_2 + c_1 \leq 1$, $c_1 \geq c_2, \dots$ $c_{k-1} \geq c_k$.

Remark 2.1.12. From the Lie theoretic aspect, there is a polyhedron P_L^k constructed from the Dynkin diagram. And we can compare it with P^k, which is the normalized reduced symplectic cone given combinatorially as in Proposition 2.1.8.

• $P_L^k \cong P^k$, as subsets in $\mathbb{R}^k$.

Further, in the monotone point of the normalized reduced cone, all simple roots are Lagrangian, and the root edges form the root lattice of the manifold Γ_X. On each wall, the set of edges not in the closure of the wall form a set of simple roots of the Lagrangian sublattice Γ_L, whose Weyl group will be shown to be the homological action of $Symp(X,\omega), \omega \in P_{\mathcal{S}}$. And indeed,

• Denote the wall of sublattice in P_L^k by W_L and the wall of corresponding vertices in P^k by W_V. Then $W_L \cong W_V$, as subsets in $\mathbb{R}^k$.

As discussed above, each wall(or chamber) is labeled by Lagrangian root system(Dynkin diagram). And the set of Symplectic -2 sphere classes, which is also labeled on the wall. Here we give the Lie algebraic way to observe this fact:

$$SS(\Gamma_L) = R^+(X) \setminus R^+(\Gamma_L), \tag{2.7}$$

where $SS(\Gamma_L)$ is the set of Symplectic -2 sphere classes of the wall labeled by Γ_L, while $R^+(X)$(and $R^+(\Gamma_L)$) means the set of positive roots of the manifold X (and Γ_L respectively).

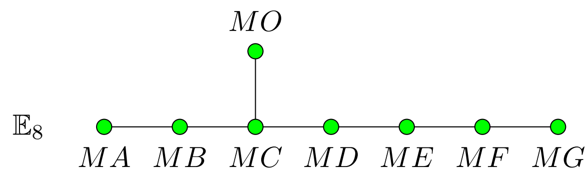
2.1.5 A uniform description for reduced cone of M when $\chi(M) < 12$

For $X_k = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, 3 \leq k \leq 8$, the normalized reduced cone P^k can be described uniformly in both ways:

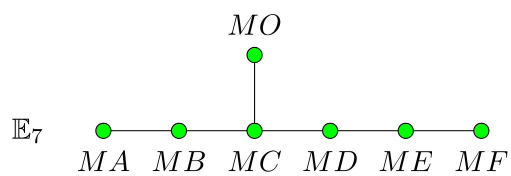
Combinatorially, it is obtained by the polyhedron using the reduced condition 2.1, as in Proposition 2.1.8. The effect of the blowdown process on the cone can also be described explicitly. For any rational 4 manifold with $\chi(X) < 12$ the reduced cone of them are unified in this way: Take the closure of P^8 and obtain P_c^8. $P_c^8 - P^8$ is the closure of P^7, denoted by P_c^7. while projecting P_c^8 to the plane $c_8 = 0$, one get closure of P_c^7. This operation is to blow down along E_8. And the monotone point of P_c^7 is obtained by projecting P_c^8 to plane $c_8 = 0$. And for $k \geq 4$, one can do the same by induction: projecting P_c^k to the plane $c_k = 0$ and obtain P_c^{k-1}, where the monotone point is preserved under this blowdown operation. And start from P_c^3, which will be illustrated and remarked in 2.2.3, 2.2.2,2.2.1, projecting to plane $c_3 = 0$, one get the P_c^2, but the monotone point is not preserved this time. Further, from P_c^2 one obtain the normalized reduced cone of $S^2 \times S^2$ and one point blow up.

Correspondingly, Lie theoretic approach goes this way:

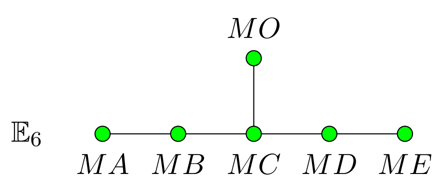
Start from P_c^8 labeled by $\mathbb{E}_8$ with all simple roots $MO, MA, \cdots, MG:$



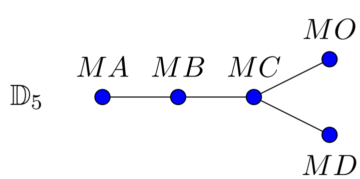
Blowing down one get P_c^7 labeled by $\mathbb{E}_7$ with simple roots $MO, MA, \cdots, MF:$



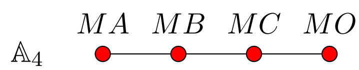
Blowing down one more time one get P_c^6 labeled by $\mathbb{E}_6$ simple roots MO, MA, $\cdots, ME:$



Blowing down one more time one get P_c^5 labeled by $\mathbb{D}_5$ simple roots $MO, MA, \cdots, MD$. Note this is $X_5 = \mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$.



Blowing down one more time one get P_c^5 labeled by $\mathbb{A}_4$ simple roots $MO, MA, \cdots, MC$. Note this is $X_4 = \mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$.



2.2 Examples: $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k = 1, 2, 3$ and remarks for $k > 9$

To make the above general discussion clear, we give an example using $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k = 1, 2, 3$,:

2.2.1 Examples: $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k = 1, 2, 3$

Explicit cone structure and illustration will be given for $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k = 1, 2, 3$:

The case of $\mathbb{C}\mathbf{P^2} \# 3\overline{\mathbb{C}\mathbf{P^2}}$

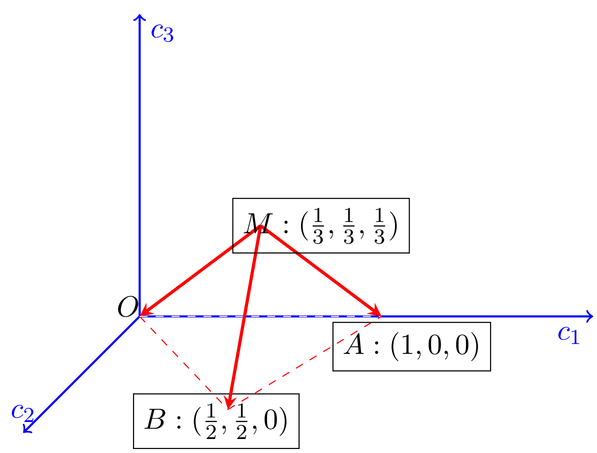


Figure 2.1: Normalized Reduced cone of $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$

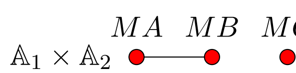
In picture 2.1, the tetrahedron $MOAB$ is the normalized reduced symplectic cone, which further described using table 2.1. And the open chamber is the space of reduced forms $\lambda := c_1 + c_2 + c_3 < 1; c_1 > c_2 > c_3$. A wall of codimension k is a the connected subsets of the closure of this open chamber where k number of " $>$ " made into " $=$ ". Also, N and N_L are the number of Symplectic (or Lagrangian respectively) -2 sphere

k-Face	N_L	N	$\omega-$area and position in the cone
Point M,	4	0	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$: monotone
Edge MO:	3	1	$\lambda < 1; c_1 = c_2 = c_3$
Edge MA:	2	2	$\lambda = 1; c_1 > c_2 = c_3$
Edge MB:	2	2	$\lambda = 1; c_1 = c_2 > c_3$
ΔMOA:	1	3	$\lambda < 1; c_1 > c_2 = c_3$
ΔMOB:	1	3	$\lambda < 1; c_1 = c_2 > c_3$
ΔMAB:	1	3	$\lambda = 1; c_1 > c_2 > c_3$
T_{MOAB}:	0	4	$\lambda < 1; c_1 > c_2 > c_3$

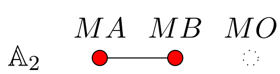
Table 2.1: Reduced cone of $X_3 = \mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$

classes associated with each wall, where in this case $N = N_L = 4$ because there are 4 smooth -2 sphere classes. And we can describe the reduced cone as follows:

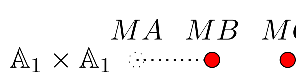
- The monotone case (point M) where there is no symplectic -2 sphere classes. And 3 Lagrangian simple roots $MO = H-E_1-E_2-E_3$, $MA = E_1-E_2$, $MB = E_2-E_3$ form Γ_{X_3} with the Dynkin diagram:



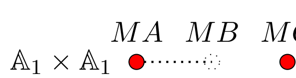
- one walls MO, corresponding to $\Gamma_L = \mathbb{A}_2$, which is obtained from Γ_{X_3} removing MO:



- MA, corresponding to $\Gamma_L = \mathbb{A}_1 \times \mathbb{A}_1$, which is obtained from Γ_{X_3} removing MA:



- MB, corresponding to $\Gamma_L = \mathbb{A}_1 \times \mathbb{A}_1$ which is obtained from Γ_{X_3} removing MB:



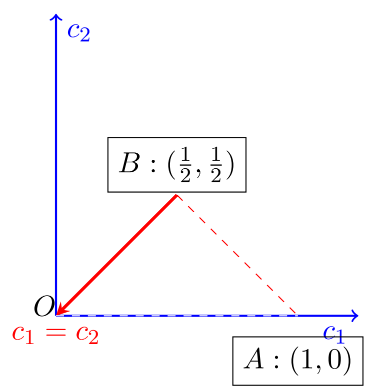
- 3 walls of codimension 1: 3 facets of the tetrahedron, triangle OAM, OBM, MAB, each corresponding to $\Gamma_L = \mathbb{A}_1$ lattice of node MB, MA and MO respectively(by removing all other vertices). Note that the horizontal dashed triangle OAB is not a root hyperplane. Instead, it is the blowdown of X, see Remark 2.2.1

- The open chamber of reduced forms, which is the interior of the tetrahedron $MOAB$, denoted by T_{MOAB} where 4 spherical -2 class are all symplectic and the Lagrangian lattice Γ_L is $\emptyset$.

Remark 2.2.1. Note that the projection onto plane $c_3 = 0$ is the closure of the normalized reduced cone of $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$, which is performing a blow down along E_3.

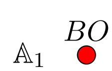
The case of $\mathbb{C}\mathbf{P^2}\#\mathbf{2}\overline{\mathbb{C}\mathbf{P^2}}$

The picture below is the reduced cone of $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$, which is a closed polyhedron with two facets removed:



To obtain the whole symplectic cone, one just apply the reflection of $\mathbb{A}_1$ to the reduced cone. And we can describe the reduced cone as follows:

- One wall OB where the symplectic form is on the line $c_1 = c_2$, and the edge BO is a Lagrangian -2 sphere forming Γ_L with the Dynkin diagram



- One open chamber, the interior of ΔBOA, there's no Lagrangian -2 sphere which means Γ_L is null. And symplectic -2 spheres is the whole $R^+ = \{E_1 - E_2\}$.

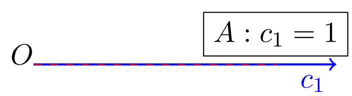
Remark 2.2.2. Note that the edge OA and AB are not in the reduced cone of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$.

OA is the normalized reduced cone of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ as the previous case. Hence projecting to the $c_2 = 0$ axis is to blow down along E_2.

AB is the normalized reduced cone of $S^2 \times S^2$. The reason is follows. Think $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ as $S^2 \times S^2\#\overline{\mathbb{C}P^2}$ with basis B, F, E. Take the base change as in equation 2.2, we have on AB, $\omega(B) = 1 - c_2, \omega(F) = 1 - c_1$ and $\omega(E) = 1 - c_1 - c_2 = 0$. Hence on AB we actually have $S^2 \times S^2$ and the ratio of the area of the two spheres can be any real number no less than 1. Hence projecting to the $c_1 + c_2 = 1$ direction is the same as blowing down $H - E_1 - E_2$.

The case of $\mathbb{C}\mathbf{P^2}\#\overline{\mathbb{C}\mathbf{P^2}}$

The picture below is the reduced cone of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, it is slightly different from the latter cases because monotone form is not a vertex:



Remark 2.2.3. Note that the point O is not in the reduced cone. Instead, it stands for $\mathbb{C}P^2$ with the Fubuni-Study form. Hence projection toward O direction means blowing down along E_1.

2.2.2 Symplectic cone and Normalized reduced cone, and discussion for general cases

This section is a discussion, which contains no result needed.

First, we discuss the relation of the normalized reduced cone with the symplectic cone as following for $X = \mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$, following notation in [27].

We start from the positive cone $\mathcal{P} = \{e \in H^2(X;\mathbb{R}) | e \cdot e > 0\}$. This is a subset of $\mathbb{R}^{n+1}$, which is called the positive cone. A corollary of [28] is that the action of Diff(X) on $H^2(X, \mathbb{Z})$ is transitive on the positive cone $\mathcal{P}$. When $\chi(X) < 12$, reflection along elements I_r (defined in front of Lemma 2.1.10) is finite, denoting D_{-1}, and the fundamental domain of this action is called the P-Cell. For the basis and canonical class K given in 2.1, when Euler number is less than 12, the **P-Cell** is $PC = \{e \in \mathcal{P} | K \cdot e > 0\}$.

The P-Cell can be thought as the fundamental domain under the reflection along hyperplanes of -1 curves D_{-1}. For the P-Cell of rational manifold with larger Euler number, see Remark 2.2.4.

And we can apply Cremona transform on the P-Cell, which is the reflection along hyperplane of -2 curves. And the Cremona transform is the Weyl group of the root system given in Lemma 2.1.10. The fundamental domain of $\mathrm{Diff}_K(X)$ as in Theorem 2.1.3 is the space of reduced forms $\mathcal{P}_r$. In a Lie theoretic point view, this is to take the intersection of the P-Cell with the chamber of reduced form the Weyl arrangement, we obtain $\mathcal{P}_r$. Precisely, start with Euclidean space $\mathbb{R}^{n+1}$ with an arrangement of hyperplanes called the root hyperplane of the root system of X as given in Lemma 2.1.10, the connected component of the complement of the union of the root hyperplane is called the Weyl chamber.

And we normalize symplectic forms in $\mathcal{P}_r$ such that any form has area 1 on H class. Then we obtain the corresponding so-called **normalized reduced cone** of M. We'll see immediately this normalized reduced cone is a polyhedron of dimension n and we denote it by P^n for $X_n = \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$.

More explicitly, there is an open chamber and many walls(open part of root hyperplane): Suppose we blow up $\mathbb{C}P^2$ with Fubini-Study form ω_{FS} k times to obtain X, i.e. the line class H has area 1. And suppose the blow up sizes are $c_1, c_2, \cdots, c_n$(one can denote the form by $(1|, c_1, c_2, \cdots, c_n)$,) satisfying the reduced condition 2.1. Then the open chamber is given by the forms $\omega = (1|, c_1, c_2, \cdots, c_n)$ such that any $\geq$ in 2.1 replaced by $>$. And the walls are given by the forms $\omega = (1|, c_1, c_2, \cdots, c_n)$ such that some $\geq$ in 2.1 replaced by $=$.

Remark 2.2.4. When $n \geq 9$, the P-Cell needs to be cut by one more quadratic equation and a K-positive equation. Further, the Cremona transform group is infinite, guided by Kac-Moody algebra, see [60] for details.

Then we compare our Polyhedron with cone as in the next remark:

Remark 2.2.5. Let $\mathcal{C} \in \mathbb{R}^n$ be a subset of a finite dimensional real vector space. We say that C is a **cone** (respectively convex subset) if whenever α and $\beta \in \mathcal{C}$ then $\lambda\alpha + \mu\beta \in C$ for all $\lambda \geq 0$, $\mu \geq 0$, (respectively such that $\lambda + \mu = 1$). We say that $\mathcal{C}$ is **strictly convex** if $\mathcal{C}$ contains no positive dimensional linear subspaces, or equivalently,

$\mathcal{C} \cap (-\mathcal{C}) = \{0\}$. We say that $R \in \mathcal{C}$ is a ray of a cone $\mathcal{C}$ if $R = \mathbb{R}^+\alpha$, for some nonzero vector $\alpha \in \mathcal{C}$. We say that R is an **extremal ray** if whenever $\beta + \gamma \in R$, where β and $\gamma \in \mathcal{C}$, then β and $\gamma \in R$.

One find that the closed convex polyhedron with the origin as a vertex can be obtained by taking the intersection of the cone $\mathcal{C}$ with the half space $H = \{\vec{x} \in \mathbb{R}^n | a^t\vec{x} \leq 0\}$, which contains the origin.

By [16], section 1.2 (13), strictly convex cone is generated by its rays. And indeed, we can choose the extremal ray as generators of the cone of reduced forms. Because of the above Remark, we often call it the **normalized reduced symplectic cone** or **reduced cone**. And a **root edge** of the cone refers to the edges of the polyhedron while a **wall** of the cone refers to the interior of the k-face of polyhedron P, $2 \leq k < n = dimP$.

Finally, we give another equivalent (combinatorial) way to interpret symplectic spheres and Lagrangian spheres as in 2.7:

Remark 2.2.6. A generic symplectic form ω corresponds to an open chamber of the Weyl arrangement of $\mathbb{E}_n$. And fix such generic form is the same as choosing a polarization of the root system, and hence determining a set of positive roots R_r^+, each of them can be represented by a smooth embedded symplectic -2 sphere. And on each root hyperplane(wall), as the form deformed, some of them become Lagrangian -2 sphere classes while the rest classes in R_r^+ remain symplectic. And any -2 symplectic sphere in $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k \leq 5$ are in classes $E_i - E_j$ or $H - E_i - E_j - E_k$. We will perform a base change and we can list the set of positive roots R_{k+1}^+ of $S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, k \leq 4$ w.r.t this form explicitly in Remark 3.4.4. Hence we have a another description of $SS_{\mathcal{S}}$ of $\omega-$symplectic sphere classes and the set $LS_{\mathcal{S}}$ of $\omega-$Lagrangian sphere classes of each wall: $LS_{\mathcal{S}}$ is the set of positive roots, and $SS_{\mathcal{S}}$ is the complement of $LS_{\mathcal{S}}$ in $R^+(X)$. And the wall is naturally labeled with the set $SS_{\mathcal{S}}$ of $\omega-$symplectic sphere classes and the set $LS_{\mathcal{S}}$ of $\omega-$Lagrangian sphere classes for $\omega \in int(P_{\mathcal{S}})$. Specifically, each root edge R_i is labeled with SS_{R_i} and LS_{R_i}. And there are simple relations about the sets on the wall $SS_{\mathcal{S}}$, $LS_{\mathcal{S}}$ and the sets on the root edges SS_{R_i} and LS_{R_i}:

$$SS_{\mathcal{S}} = \cup_{R_i \in \mathcal{S}} SS_{R_i}; \quad LS_{\mathcal{S}} = \cap_{R_i \in \mathcal{S}} LS_{R_i}.$$

And we denote the cardinality of $SS_{\mathcal{S}}$, $LS_{\mathcal{S}}$ by N and N_L respectively.

Chapter 3

The space of almost complex structures

In this chapter, we study the space of tamed almost complex structures $\mathcal{J}_\omega$ of a symplectic 4 manifold X with given symplectic form ω, and define a decomposition of $\mathcal{J}_\omega$ via smooth rational curves into prime submanifolds. For $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}, k \leq 5$ with arbitrary symplectic form ω, we directly prove that such decomposition is indeed a stratification in the sense that taking closure agrees with the codimension at certain levels, see Proposition 3.4.1.

Throughout this section we identify $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$ with $S^2 \times S^2\#(k-1)\overline{\mathbb{C}P^2}$ and apply base change 2.2 to use the basis $B, F, E_1, \cdots, E_{k-1}$ as basis of $H_2(X)$. The purpose is to make the notation compatible with [7], where Lemma 3.2.1 can be think as generalization of Lemma 2.10 in [7].

3.1 Decomposition of $\mathcal{J}_\omega$ via smooth rational curves

3.1.1 General facts of J-holomorphic curves and symplectic spheres

We review general facts of J-holomorphic curves for symplectic 4-manifold. Firstly the local properties due to Gromov [20], McDuff [39]:

Theorem 3.1.1 (Positivity)**.** *In a given closed symplectic 4-manifold (X, ω), let C, C' be two closed J-holomorphic curves. Then the contribution k_p of each point of intersection* *of C and C′ to the intersection number $C \cdot C'$ is a strictly positive integer. It is equal to 1 if and only if C and C' are both regular at p and meet transversally at that point.*

Theorem 3.1.2 (Adjunction Inequality)**.** *Let(X^4, J) be an almost complex manifold and $u : \Sigma \to M$ is a J-holomorphic curve which is not a multiple covering. Then the virtual genus of the image $C = Im(u)$ is defined as $g_v(C) = (C \cdot C - c_1(C))/2 + 1$, which is always no less than the genus of Σ, where equality holds if and only if the map is an embedding.*

Then we need to recall Gromov Compactness Theorem (cf [20],[45]) and Fredholm framework:

Theorem 3.1.3 (Gromov Compactness Theorem)**.** *Let (X, ω) be a compact closed, connected symplectic manifold, and let $(J_n) \in \mathcal{J}_\omega$ be a sequence which converges to J_0 in C^∞-topology. Let Σ be a compact, connected Riemann surface without boundary, and let (j_n) be a sequence of complex structures on Σ. Suppose $u_n : \Sigma \to M$ is a sequence of $(J_n, j_n)-$holomorphic curves such that*

$$u_n^*[\Sigma] = [A] \in H_2(X; \mathbb{Z}), [A] \neq 0.$$

Then up to a re-parametrization of each u_n, there are

- *finitely many simple closed loops γ_i in Σ,*
- *a finite union of Riemann surfaces $\Sigma' = \cup_\alpha \Sigma_\alpha$, which is obtained by collapsing each of the simple closed curves γ_i on Σ to a point,*
- *a continuous map $u : \Sigma_0 \to M$ such that $u|\Sigma_\alpha$ is a (J, j_0)-holomorphic curve, where j_0 is the complex structure on each component of Σ_0,*

such that

- *a subsequence of $\{u_n\}$, converges to u and in the complement of any fixed open neighborhood of $\cup_i \gamma_i$, j_n converges to j_0 in C^∞-topology;*
- $\sum_\alpha u_*([\Sigma_\alpha]) = [A] \in H_2(X; \mathbb{Z})$.

Following Gromov, we call the limiting curve $u : \Sigma_0 \to M$ a cusp curve. We are particularly interested in the case when the J-holomorphic curve has a sphere as its domain.

Let $\mathcal{S}_\omega$ denote the set of homology classes of embedded ω-symplectic sphere with negative self-intersection. Let

$$\mathcal{S}_\omega^{<0}, \quad \mathcal{S}_\omega^{\geq 0}, \quad \mathcal{S}_\omega^{-1}, \quad \mathcal{S}_\omega^{\leq -2}$$

be the subsets of classes with self-intersection negative, non-negative, -1 and less than -1 respectively. Meanwhile we call the classes in $\mathcal{S}_\omega^{\leq -2}$ **K-nef** classes, because they are sphere classes that are numerical effective on the canonical class K.

If $A \cdot A = -k, k \in \mathbb{Z}^+$, we define $cod_A = 2k_i - 2$ the codimension of the curve class. And if $A \cdot A \geq -1$, the codimension of the curve class is defined as 0.

Proposition 3.1.4 (Exceptional sphere and Non-negative sphere)**.** *Let(X^4, ω) be a closed symplectic 4-manifold and A a class in $\mathcal{S}_\omega^{-1}$ or $\mathcal{S}_\omega^{\geq 0}$. Then for a generic $J \in \mathcal{J}_\omega$, there is an embedded $J-$holomorphic curve in the class A.*

Using the Fredholm framework the following is proved in [7] Appendix B.1:

Proposition 3.1.5. *Let (X, ω) be a 4-dimensional symplectic manifold. Suppose $U_\mathcal{C} \subset \mathcal{J}_\omega$ is a subset characterized by the existence of a configuration of J-holomorphic embedded spheres $C_1 \cup C_2 \cup \cdots \cup C_N$ of negative self-intersection whose classes $\{[C_1], [C_2], \cdots, [C_N]\} = \mathcal{C}$. Then $U_\mathcal{C}$ is a cooriented Fréchet submanifold of $\mathcal{J}_\omega$ of (real) codimension $codim_\mathbb{R}(U_\mathcal{C}) = 2N - 2c_1([C_1] + \cdots + [C_N])$.*

3.1.2 Prime submanifolds

Now we give conditions to well define the decomposition of $\mathcal{J}_\omega$ of a general symplectic 4-manifold (X, ω) via smooth rational curves into prime subsets. And we further give a condition when prime subsets are sub manifolds.

Definition 3.1.6. Given a finite subset $\mathcal{C} \subset \mathcal{S}_\omega^{<0}$,

$$\mathcal{C} = \{A_1, \cdots, A_i, \cdots, A_n | A_i \cdot A_j \geq 0 \text{ if } i \neq j\},$$

define **prime subsets**

$$\mathcal{J}_{\mathcal{C}} = \{J \in \mathcal{J}_\omega | A \in \mathcal{S} \text{ has an embedded } J\text{−hol representative if and only if } A \in \mathcal{C}\}.$$

And we define the codimension of the label set $\mathcal{C}$ as the sum of the codimension of each curve class, i.e. $cod_{\mathcal{C}} = \sum_{A_i \in \mathcal{C}} cod_{A_i}$.

Clearly, we have the decomposition: $\mathcal{J}_\omega = \amalg_{\mathcal{C}} J_{\mathcal{C}}$. We will show in Proposition 3.1.13 that, under certain conditions, $\mathcal{J}_{\mathcal{C}}$ is a submanifold of $\mathcal{J}_\omega$ of real codimension $cod_{\mathcal{C}} = \sum_{A_i \in \mathcal{C}} cod_{A_i}$.

Remark 3.1.7. Note that in [7] Lemma 2.10, there is a decomposition of $\mathcal{J}_\omega$ for $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ where each stratum is characterized by the existence of a certain negative curve. Their decomposition is shown to be a stratification with finite codimension submanifolds as strata. We point out that our decomposition is finer in the sense that each stratum in [7] is a union of prime submanifolds in our decomposition. In particular, our decomposition being a stratification as in definition 3.4.11 implies their decomposition is a stratification. And to compute higher cohomology of $\mathcal{J}_{open}$ and generalize Proposition 3.4.7, we needs this decomposition of $\mathcal{J}_\omega$.

Further, note that an arbitrary set $\mathcal{C} \subset \mathcal{S}_\omega^{<0}$,

$$\mathcal{C} = \{A_1, \cdots, A_i, \cdots, A_n | A_i \cdot A_j \geq 0 \text{ if } i \neq j\}$$

dose not necessarily define a nonempty submanifold $\mathcal{J}_{\mathcal{C}}$.

Lemma 3.1.8. *There is an action of $Symp_h$ on each prime subset defined as above.*

Proof. This simply follows from the fact that $Symp_h$ acting on $\mathcal{J}_\omega$ preserves the class of J-holomorphic curve.

□

We assume the following for the symplectic manifold (X, ω):

Condiction 3.1.9. *If A is a homology class in $H^2(X;\mathbb{Z})$ with negative self-intersection, which is represented by a simple J−holomorphic map $u : \mathbb{C}P^1 \to M$ for some tamed J, then u is an embedding.*

And by [59] Proposition 4.2, we have

Lemma 3.1.10. *Condition 3.1.9 holds true for* $S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, 0 \leq k < 8$.

Note that under this assumption, $\mathcal{S}_\omega^{<0}$ is the same as the set of homology classes with negative self-intersection and having a simple rational pseudo-holomorphic representative.

Lemma 3.1.11. *Assume condition 3.1.9. Then* $\overline{\mathcal{J}_\mathcal{C}} \cap \mathcal{J}_{\mathcal{C}'} \neq \emptyset$ *only if we can find subset* $\mathcal{C}_{deg} = \{A^1, A^2, \cdots, A^n\} \subset \mathcal{C}$, *such that* $\mathcal{C} \setminus \mathcal{C}_{deg} \subset \mathcal{C}'$; *and a corresponding subset*

$$\mathcal{C}'_{bubble} = \{A_1^1 \cdots A_{j_1}^1, A_1^2 \cdots A_{j_2}^2, \cdots, A_1^n \cdots A_{j_n}^n\} \subset \mathcal{C}',$$

such that there are simultaneous decompositions of homology classes:

$$A^i = \sum_{j_i} \alpha_i A^i_{j_i} + \sum_{k_i} \beta_i A^i_{k_i}, \alpha_i, \beta_i \geq 0, \tag{3.1}$$

where $A^i_{j_i} \in \mathcal{S}_\omega^{<0}$ *and* $A^i_{k_i}$*s are square non-negative classes which has a simple J-holomorphic representative.*

Proof. In definition 3.1.6, for a given $A_i \in \mathcal{C}$, we use C_i to denote the embedded $J-$holomorphic sphere for $J \in \mathcal{J}_\mathcal{C}$. And we give a description of taking closure of $\mathcal{J}_\mathcal{C}$ in terms of $\mathcal{C}$: Suppose $\mathcal{J}_\mathcal{C}$ and $\mathcal{J}_{\mathcal{C}'}$ are two prime subsets, $\overline{\mathcal{J}_\mathcal{C}} \cap \mathcal{J}_{\mathcal{C}'} \neq \emptyset$.

Then there is a convergent sequence of $\{J_n\} \subset \mathcal{J}_\mathcal{C}$ such that $\{J_n\} \to J_0 \in \mathcal{J}_{\mathcal{C}'}$. For J_0, take all the elements in $\mathcal{C}$ that are not irreducibly J_0-holomorphic, and denote the subset by $\mathcal{C}_{deg} = \{A^1, A^2, \cdots, A^n\}$. It follows that $\mathcal{C} \setminus \mathcal{C}_{deg} \subset \mathcal{C}'$. For the collection of cusp curves for $\mathcal{C}_{deg}$, we take their irreducible component with negative self-intersection.

$$\mathcal{C}'_{bubble} = \{A_1^1 \cdots A_{j_1}^1, A_1^2 \cdots A_{j_2}^2, \cdots, A_1^n \cdots A_{j_n}^n\} \subset \mathcal{C}',$$

Because $J_0 \in \mathcal{J}_{\mathcal{C}'}$, by Condition 3.1.9, their homology classes must belong to $\mathcal{C}'$. And Gromov compactness theorem gives us the desired homology decompositions:

$$A^i = \sum_{j_i} \alpha_i A^i_{j_i} + \sum_{k_i} \beta_i A^i_{k_i}, \alpha_i, \beta_i \geq 0$$

∎

Lemma 3.1.12. *Assuming Condition 3.1.9. If $\mathcal{C} \subsetneqq \mathcal{C}'$, then $\overline{\mathcal{J}}_{\mathcal{C}'} \cap \mathcal{J}_{\mathcal{C}} = \emptyset$.*

Proof. We argue by contradiction. Suppose there exists some $J \in \overline{\mathcal{J}}_{\mathcal{C}'} \cap \mathcal{J}_{\mathcal{C}}$. It follows from equation (3.1) that for some $A \in \mathcal{C}' \setminus \mathcal{C}$ there is a decomposition of homology class

$$A = \sum_{\alpha} r_\alpha [C_\alpha] + \sum_{\beta} r_\beta [C_\beta],$$

where each C_β is a simple $J-$holomorphic curve with non-negative self-intersection, and each C_β is a simple $J-$holomorphic curve with negative self-intersection.

By Condition 3.1.9, we have $[C_\alpha] \in \mathcal{C}$. Therefore $A, [C_\alpha] \in \mathcal{C}'$, which implies $A \cdot [C_\alpha] \geq 0$.

We claim that $A \cdot [C_\beta] \geq 0$ for each β as well. First of all, since C_α's and C_β's are simple $J-$holomorphic curves, by positivity of intersection, they pair each other non-negatively. Moreover, for each β, $[C_\beta] \cdot [C_\beta] \geq 0$. Now the claim $A \cdot [C_\beta] \geq 0$ follows from pairing $[C_\beta]$ with the equation above.

Finally, pairing the equation with A. The left-hand side is negative, while the right-hand side is non-negative. This is a contradiction and hence $\overline{\mathcal{J}}_{\mathcal{C}'} \cap \mathcal{J}_{\mathcal{C}} = \emptyset$. □

Hence we verify the prime subsets in Definition 3.1.6 are actually submanifolds:

Proposition 3.1.13. *If (X, ω) is a 4-dimensional symplectic manifold, assuming Condition 3.1.9, we verify that prime submanifold is well defined. Further, $cod(\mathcal{J}_{\mathcal{C}}) = cod_{\mathcal{C}} = \sum_{C_i \in \mathcal{C}} cod_{C_i}$.*

Proof. Empty set is a submanifold of $\mathcal{J}_\omega$, and we then assume that $\mathcal{J}_{\mathcal{C}}$ is non-empty. First note that $\mathcal{J}_{\mathcal{C}}$ is a subset of $U_{\mathcal{C}}$, which is a submanifold of $\mathcal{J}_\omega$ whose codimension is $d = \sum_{i \in I} cod_{C_i}$ by Proposition 3.1.5. Then we look at $U_{\mathcal{C}} \setminus \mathcal{J}_{\mathcal{C}}$. $U_{\mathcal{C}}$ is a disjoint union of $\mathcal{J}_{\mathcal{S}_i}$ where each $\mathcal{S}_i$ is a curve set which contains $\mathcal{C}$ as a proper subset. And the union of these $\mathcal{J}_{\mathcal{S}_i}$s is relatively closed in $U_{\mathcal{C}}$ by lemma 3.1.12 And hence $\mathcal{J}_{\mathcal{C}}$ is itself a submanifold of codimension $d = \sum_{i \in I} cod_{C_i}$. □

And we address a special case of Proposition 3.1.13:

Lemma 3.1.14. *Let X be any symplectic 4 manifold with given symplectic form ω, for any $K-nef$ class $A \in \mathcal{S}_\omega^{\leq -2}$ with $A^2 = k$ the set $\mathcal{J}_A$ where A is the only K-nef curve in the label set is a codimension $2k-2$ stratum.*

Remark 3.1.15. Notice that to label the prime submanifolds using subsets of $\mathcal{S}_\omega^{<0}$ spherical classes is equivalent to labeling them using $\mathcal{S}_\omega^{\leq -2}$, under a certain assumption, see the next lemma. And we call the subset of $\mathcal{S}_\omega^{\leq -2}$ whose elements intersecting pairwise non-negative an **admissible subset**. By the adjunction formula, any class in $\mathcal{S}_\omega^{\leq -2}$ is **K-nef**.

Also note that not any admissible subset defines a non-empty prime submanifold, for example, the class of a single -4 curve in a rational surface defines an admissible subset whose prime submanifold is empty.

This is because of the lemma below:

Lemma 3.1.16. *If Condition 3.1.9 holds, $\mathcal{C}$ is completely determined by its subset of K-nef curves and by positivity of intersection.*

Proof. For any J, the -1 class A must have a $J-$holomorphic representative. This means either A has an embedded $J-$holomorphic sphere representative or A is represented by a cusp curve. In the latter case we look at the homology class of this cusp curve:

$$A = \sum_\alpha r_\alpha [C_\alpha] + \sum_\beta r_\beta [C_\beta] + \sum_\gamma r_\gamma [C_\gamma].$$

By Condition 3.1.9, we have the negative self-intersecting classes $[C_\alpha], [C_\beta] \in \mathcal{C}$, where $[C_\alpha]^2 \leq -2, [C_\beta]^2 = -1$. $[C_\gamma]$'s are symplectic sphere classes with non-negative self-intersection. We now show if $A \cdot [C_\alpha]$ is nonnegative for any$[C_\alpha]$, then there cannot be homology decomposition $A = \sum_\alpha r_\alpha [C_\alpha] + \sum_\beta r_\beta [C_\beta]$: If A pair any of $[C_\alpha]$ is nonnegative, we multiply $\sum_\gamma [C_\gamma]$ on both sides of the decomposition, it is clear that $A \cdot \sum_\gamma [C_\gamma]$ is positive. Also, $A \cdot \sum_\beta [C_\beta]$ is nonnegative for all $[C_\beta]$ that are -1 classes having simultaneously holomorphic representatives a generic J. we compute the product with A on both sides of the decomposition equation: the left-hand side is $A^2 < 0$ and the right-hand side by positivity of intersection is nonnegative. This is a contradiction and hence if A pair any of $[C_\alpha]$ is nonnegative, A has an embedded $J-$holomorphic representative. ∎

3.2 Constraints on simple $J-$holomorphic curves for a reduced form

We make the following elementary but crucial observation under the assumption $n \leq 4$: for a reduced form,

$$\sum_{k=1}^{n}(a_k)^2 \leq 1, \quad \text{if } n \leq 4. \tag{3.2}$$

If we consider the extreme value of the function $\sum_{k=1}^{n}(a_k)^2$ under the constrain given by the reduced condition 2.5, $a_i \in [0,1], a_i + a_j \leq 1$, then the extreme value can only appear at $(1, 0, \cdots, 0)$ or $(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ meaning that $\sum_{i=1}^{n}(a_i)^2 \leq max(1, n/4)$ and given $n \leq 4$, $\sum_{k=1}^{n}(a_k)^2 \leq 1$.

3.2.1 The key lemma

Lemma 3.2.1. *Suppose $X = S^2 \times S^2 \# n\overline{\mathbb{C}P^2}, n \leq 4$, and ω is a reduced symplectic form in the class $\mu B + F - \sum_{i=1}^{n} a_i E_i$ as in Lemma 2.1.4. If $A = pB + qF - \sum r_i E_i \in H_2(X; \mathbb{Z})$ is a class with a simple connected J-holomorphic representative for some $\omega-$tamed J, then $p \geq 0$.*

And if $p = 0$, then $q = 0$ or 1.

If $p = 1$, then $r_i \in \{0, 1\}$.

If $p > 1$, then $q \geq 1$.

Proof. We start by stating three inequalities: area, adjunction, r_i integer.

The area of the curve class A is positive and hence

$$\omega(A) = p\mu + q - \sum a_i r_i > 0. \tag{3.3}$$

Since ω is reduced, $K_\omega = -2B - 2F + E_1 + \cdots E_n$ is the canonical class, we have the following adjunction inequality for simple $J-$holomorphic curves:

$$0 \leq 2g_\omega(A) := A \cdot A + K \cdot A + 2 = 2(p-1)(q-1) - \sum_{i=1}^{n} r_i(r_i - 1). \tag{3.4}$$

In many cases we will estimate the sum $-\sum_{i=1}^{n} r_i(r_i - 1)$. Since each r_i is an integer, it is easy to see that

$$-\sum_{i=1}^{n} r_i(r_i-1) \le 0, \tag{3.5}$$

and $-\sum_{i=1}^{n} r_i(r_i-1)=0$ if and only if $r_i=0$ or 1 for each i.

In particular, if $p=1$, then $-\sum_{i=1}^{n} r_i(r_i-1)=2g(A)\ge 0$. It follows from 3.5 that $r_i(r_i-1)$ has to be 0 and hence $r_i\in\{0,1\}$.

And if we assume that $p>1$ and $q\le 0$, then $-\sum_{i=1}^{n} r_i(r_i-1)=2g(A)-2(p-1)(q-1)\ge 0-2(p-1)(q-1)>0$. This is impossible. Therefore $q\ge 1$ if $p>1$.

Now let us assume $p\le 0$ and we divide into three cases:

(i) $p<0, q\ge 1$, (ii) $p<0, q\le 0$, (iii) $p=0$.

Case (i). $p<0$ and $q\ge 1$.

We show this case is impossible. Because $p\le -1$, the adjunction inequality 3.4 implies that

$$0\ge -2g_\omega(A)\ge 4(q-1)+\sum_{i=1}^{n} r_i(r_i-1)\ge (q-1)+\sum_{i=1}^{n} r_i(r_i-1).$$

Applying the area equation 3.3, we have

$$(q-1)+\sum_{i=1}^{n} r_i(r_i-1)>(\sum_{i=1}^{n} a_ir_i-\mu p-1)+\sum_{i=1}^{n} r_i(r_i-1).$$

Since $-\mu p-1\ge 0$,

$$(\sum_{i=1}^{n} a_ir_i-\mu p-1)+\sum_{i=1}^{n} r_i(r_i-1)\ge(\sum_{i=1}^{n} a_ir_i)+\sum_{i=1}^{n} r_i(r_i-1)=\sum_{i=1}^{n} r_i(r_i-1+a_i).$$

For any integer r_i we have $r_i(r_i-1+a_i)\ge 0$, which is because $r_i(r_i-1+a_i)\ge 0$ except on interval $r_i\in[0,1-a_i]$ and $1-a_i\in[0,1]$ since the form is reduced. And therefore we would have $-2g_\omega(A)>0$, which is a contradiction.

Case (ii). $p<0, q\le 0$

We show this case is also impossible. This will follow from the following estimate, under a slightly general assumption:

$$0 \le 2g_\omega(A) \le 1 + |p| + |q| - p^2 - q^2, \qquad \text{if } p \le 0, q \le 0. \tag{3.6}$$

Before proving this inequality, we note that a direct consequence of this inequality is that it is impossible to have $p \le -2, q \le 0$, or $p \le 0, q \le -2$: If $|p| > 1$, $|p|+|q|+1-(p^2+q^2)$ is clearly negative since $q^2 \ge |q|, p^2 > |p|+1$; it is the same if $|q| > 1$.

So the inequality 3.6 leaves only two cases to analyze: $p = q = -1$, or $p = -1, q = 0$.

• $p = -1$ and $q = 0$

In this case, we have $2g = 4 - \sum r_i(r_i - 1)$ so $\sum_{k=1}^u r_k^2 - \sum_{k=1}^u r_k \le 4$. Also by the area inequality 3.3, $\sum_{k=1}^u r_k < p + q = -1$, and hence $\sum_{k=1}^u r_k \le -1$. It is easy to see that $\{r_k\} = \{-1\}$ or $\{-1, -1\}$. But these possibilities are excluded by the reduced condition $a_i + a_j \le 1 \le \mu$ for any pair i, j and the area inequality.

• $p = q = -1$

In this case, we have $2g = 8 - \sum r_i(r_i - 1)$ so $\sum_{k=1}^u r_k^2 - \sum_{k=1}^u r_k \le 8$. Also by the area inequality 3.3, $\sum_{k=1}^u r_k < p + q = -2$, and hence $\sum_{k=1}^u r_k \le -2$. It is easy to see that $\{r_k\} = \{-1, -1, -1\}, \{-1, -1, -1, -1\}$ or $\{-1, -2\}$. Again these possibilities are excluded by the reduced condition $a_i + a_j \le 1 \le \mu$ for any pair i, j and the area inequality.

Now we set out to prove the inequality 3.6. In order to estimate $-\sum_{i=1}^n r_i(r_i - 1)$ we rewrite the sum

$$\sum_{i=1}^n r_i = \sum_{k=1}^u r_k + \sum_{l=u+1}^n r_l, \tag{3.7}$$

where each r_k is negative and each r_l is non-negative.

Since $p \le 0, q \le 0$, the area inequality 3.3 takes the following form:

$$-\sum a_i r_i > (|p| + |q|) \ge 1 + (|p| + |q|). \tag{3.8}$$

Note that there exists at least one negative r_i term, ie. $u \ge 1$ in 3.7. An important consequence is

$$\sum_{k=1}^u a_k r_k \le \sum_{i=1}^n a_i r_i < 0, \quad (\sum_{k=1}^u a_k r_k)^2 \ge (\sum_{i=1}^n a_i r_i)^2 \tag{3.9}$$

We first observe that, by the Cauchy-Schwarz inequality and 3.2, we have

$$(\sum_{k=1}^{u} a_k r_k)^2 \leq \sum_{k=1}^{u}(r_k)^2 \times \sum_{k=1}^{u}(a_k)^2 \leq \sum_{k=1}^{u}(r_k)^2. \tag{3.10}$$

Then we do the estimate:

$$\begin{aligned}
\sum_{i=1}^{n} r_i(r_i-1) = \sum_{i=1}^{n} r_i^2 - \sum_{i=1}^{n} r_i &= \sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k + (\sum_{l=u+1}^{n} r_l^2 - \sum_{l=u+1}^{n} r_l) \\
&\geq \sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k \quad \text{(since } x^2 - x \geq 0 \text{ for any integer)} \\
&\geq (\sum_{k=1}^{u} a_k r_k)^2 - \sum_{k=1}^{u} a_k r_k \quad \text{(follows from the two inequalities:} \qquad (3.11) \\
&\quad -\textstyle\sum_{k=1}^{u} r_k > -\sum_{k=1}^{u} a_k r_k \text{ and } \sum_{k=1}^{u} r_k^2 \geq (\sum_{k=1}^{u} a_k r_k)^2) \\
&\geq (\sum_{i=1}^{n} a_i r_i)^2 - \sum_{i=1}^{n} a_i r_i \quad \text{(is crucial and it comes from 3.9)} \\
&> |p| + |q| + (|p| + |q|)^2.
\end{aligned}$$

Because $\sum_{i=1}^{n} r_i(r_i-1)$ is an integer, we actually have

$$\sum_{i=1}^{n} r_i(r_i-1) \geq 1 + |p| + |q| + (|p| + |q|)^2.$$

Now the inequality 3.6 follows from the inequality 3.11 and the adjunction 3.4:

$$2g(A) = 2pq - 2(p+q) + 2 - [\sum_{i=1}^{n}(r_i)^2 - \sum_{i=1}^{n}(r_i)] \leq |p| + |q| + 1 - (p^2 + q^2).$$

Case (iii). $p = 0$.

In this case the adjunction is of the form $-2(q-1) - \sum_{i=1}^{n} r_i(r_i-1) \geq 0$. Since $-\sum_{i=1}^{n} r_i(r_i-1) \leq 0$ we must have $q \leq 1$.

If $p = 0, q \leq 0$, then we can apply the inequality 3.6 and conclude that $q = 0$.

In conclusion, we must have $q = 0$ or 1 if $p = 0$.

□

3.3 Negative square classes and their decompositions

For $X = S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, k \geq 0$, with basis $B, F, E_1, \cdots, E_k \in H_2(X, \mathbb{Z})$ and a given symplectic form ω such that $\omega(B) = \mu \geq 1$, $\omega(F) = 1$, $\omega(E_i) = a_i$, there are three possible K-nef classes appear in the label set of the prime submanifolds:

• **E-class** $\mathcal{E}$: integer combination of E_i;

• **F-class** $\mathcal{F}$: integer combination of F and E_i where coefficient of F is nonzero;

• **B-class** $\mathcal{B}$: integer combination where coefficient of B is nonzero.

And in the following proposition list all possible negative sphere in the above three sets for number of blow up points $k \leq 4$:

Proposition 3.3.1. *Let $X = S^2 \times S^2 \# n\overline{\mathbb{C}P^2}, n \leq 4$ with a reduced symplectic form. Suppose a class $A = pB + qF - \sum r_i E_i \in H_2(X; \mathbb{Z})$ has a simple J-holomorphic spherical representative such that J is tamed by ω. Then $p = 0, 1$. And we can further classify spherical classes with negative square as follows:*

$$\mathcal{B} = \{B - kF - \sum r_i E_i, k \geq -1, r_i \in \{0, 1\}\};$$

$$\mathcal{F} = \{F - \sum r_i E_i, r_i \in \{0, 1\}\};$$

$$\mathcal{E} = \{E_j - \sum r_i E_i, j < i, r_i \in \{0, 1\}\}.$$

Proof. • $p \geq 2$

In this case $q \geq 1$ by Lemma 3.2.1. We need to exclude this case using the fact that $g_\omega(A) = 0$ and $A^2 < 0$. Observe that by the adjunction we have

$$\sum_{i=1}^{n} r_i(r_i - 1) = 2(p - 1)(q - 1). \tag{3.12}$$

Since $g_\omega(A) = 0$ and $2g_\omega(A) - 2 = K_\omega \cdot A + A^2$, we have $-1 \leq K_\omega \cdot A = \sum r_i - 2p - 2q$, namely,

$$\sum r_i = 2p + 2q + k, k \geq -1. \tag{3.13}$$

Now if $p > 1, q > 1$, since $n \leq 4$, by Cauchy Schwartz and 3.13,

$$\sum r_i^2 \geq [\sum r_i]^2/4 \geq (2p + 2q + k)^2/4. \tag{3.14}$$

It follows from 3.13 and 3.14, $p \geq 2, q \geq 1, k \geq -1$ that

$$\begin{aligned}\sum r_i^2 - \sum r_i &\geq (2p+2q+k)^2/4 - (2p+2q+k) \\ &= (p+q)^2 + (p+q)k + \frac{k^2}{4} - k - 2(p+q) \\ &= [2pq + 2 - 2(p+q)] + (p^2 + pk - 2) + q^2 + \frac{k^2}{4} + (qk - k) \\ &> 2(p-1)(q-1).\end{aligned}$$

Notice the last 3 terms are all non-negative and cannot be zero simultaneously.

Hence a spherical class has $p = 0, 1$.

• $p = 1$.

If $p = 1$, then $r_i = 0$ or 1 as shown in Lemma 3.2.1. So

$$A = B + qF - \sum r_i E_i, r_i \in \{0, 1\}.$$

And the condition $A^2 < 0$ and $n \leq 4$ implies that $q \leq 1$.

• $p = 0$.

In this case, we have shown that $q = 0$ or 1 in Lemma 3.2.1.

If $p = 0, q = 0$, the adjunction inequality 3.4 is of the form $2 - \sum_{i=1}^{n} r_i(r_i - 1) \geq 0$. Let x be an integer. Notice that $x(x-1) \geq 0$, and $x(x-1) = 0$ if $x = 0$ or 1. Notice also that if $x(x-1) > 0$ then $x(x-1) \geq 2$, and $x(x-1) = 2$ if $x = 2$ or $x = -1$. We see there is at most one j such that $r_j \neq 0$ or $\neq 1$, and for this j, $r_j = -1$ or 2. By considering the area of such a class, we must have $r_j = -1$, and $j < i$ for any $r_i = 1$. Therefore, in this case, A can only be of the form

$$E_j - \sum r_i E_i, i > j, \quad r_i \in \{0, 1\}.$$

We are left with $q = 1$. In this case, the adjunction inequality 3.4 is of the form $-\sum_{i=1}^{n} r_i(r_i - 1) \geq 0$. So we must have $r_i = 0$ or 1. Namely,

$$A = F - \sum r_i E_i, \quad r_i \in \{0, 1\}.$$

□

Remark 3.3.2. A similar analysis leads to the classification of square zero classes. for $X = S^2 \times S^2 \# n\overline{\mathbb{C}P^2}, n \leq 4$ with a reduced symplectic form. Suppose a class A has a simple J-holomorphic spherical representative such that J is tamed by ω. Then A is one of the following classes:

$$2B + F - E_1 - E_2 - E_3 - E_4, B + 2F - E_1 - E_2 - E_3 - E_4, B + F - E_i - E_j, B, F.$$

In particular, A is the class of an embedded symplectic sphere. We do not need this fact in the paper.

Here is an important consequence of Lemma 3.2.1 and Proposition 3.3.1.

Proposition 3.3.3. *Let $X = S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, k \leq 4$ with an arbitrary symplectic form. Let A be a K-nef class which has an embedded representative for some J, Then for any simple $J'-$holomorphic representative of A for some J', there is no component whose class has a positive square. Moreover, if the symplectic form is reduced,* • *any square zero class in the decomposition is of the form B or $kF, k \in \mathbb{Z}^+$,*

• *any negative square class is from the list in Proposition 3.3.1, in particular, a class of an embedded symplectic sphere.*

Proof. Without loss of generality, we can assume the symplectic form is reduced.

Let $C = pB + qF - \sum_i r_i E_i$ be a K-nef class on the left hand side of (3.1). Then by Lemma 3.2.1, $p = 0$ or $p = 1$, and if $p = 1$ then $q \leq 1$. We argue by contradiction to show that there's no square positive class.

Suppose on the right hand side there is a square positive class $C' = p'B + q'F - \sum_i r_i' E_i$. Denote the decomposition by

$$C = C' + C(B, F, E_k).$$

Here $C(B, F, E_k)$ is a sum of curves that have simple J-holomorphic representative.

Let us first inspect the B coefficients.By Lemma 3.2.1, the B coefficient of any class in the decomposition is non-negative. This implies that $p' \geq 0$ and $p - p' \geq 0$. Since C' is assumed to have positive square, $p' \neq 0$. Since $p = 0$ or 1, we must have both $p' = 1$

and $p = 1$. And because the B coefficient of each class is non-negative, we conclude that the B coefficient of each class in $C(B, F, E_k)$ is 0.

Now let us inspect the F coefficients. For the class C', since $p'q' \geq 1$ we have $q' \geq 1$. For any class in $C(B, F, E_k)$, since the B coefficient is zero, by Lemma 3.2.1, the F coefficient is 0 or 1. Hence $q \geq q' \geq 1$, and by Proposition 3.3.1, $q \leq 1$. Hence we conclude that both $q = 1$ and $q' = 1$.

And this means $C(B, F, E_k)$ is a sum of curves where each of them having 0 as the coefficients on B and F. In addition, because $C^2 \leq -2$, $C = B + F - E_1 - E_2 - E_3 - E_4$. Recall that by Lemma 3.2.1, since $C' = B + F - \sum_i r_i' E_i$ have coefficient '1' on B, we have $r_i' \in \{0, 1\}$.

This means $C(B, F, E_k) = -E_1 - E_2 - E_3 - E_4 - E_i + \sum_i r_i' E_i$, having coefficient 0 or -1 on each E_i, and hence it has negative symplectic area. This has a contradiction against the fact $C(B, F, E_k)$ is the sum of J-holomorphic homology classes. Hence there is no positive squared curve in decomposition (3.1) of a K-nef curve.

Next, we analyze the possible square 0 classes in the decomposition. From the analysis above, we only need to deal with the case that either $p' = 0$ or $q' = 0$.

For the case $p' = 0$, the only square zero classes can be $kF, k \in \mathbb{Z}^+$. And for the case $q' = 0$, the only square zero class can be B. □

3.4 Codimension 2 prime submanifolds

3.4.1 Level 2 stratification

The next theorem holds for any symplectic rational 4 manifold having Euler number $\chi(X) \leq 8$ with a given reduced symplectic form:

Theorem 3.4.1. *For a symplectic rational 4 manifold having Euler number $\chi(X) \leq 8$ and any symplectic form, $\mathcal{X}_4 = \cup_{cod(\mathcal{C}) \geq 4} \mathcal{J}_{\mathcal{C}}$ and $\mathcal{X}_2 = \cup_{cod(\mathcal{C}) \geq 2} \mathcal{J}_{\mathcal{C}}$, are closed subsets in $\mathcal{X}_0 = \mathcal{J}_\omega$. Consequently,*

(i). $\mathcal{X}_0 - \mathcal{X}_4$ is a manifold.

(ii). $\mathcal{X}_2 - \mathcal{X}_4$ is closed in $\mathcal{X}_0 - \mathcal{X}_4$.

(iii). $\mathcal{X}_2 - \mathcal{X}_4$ is a manifold.

(iv). $\mathcal{X}_2 - \mathcal{X}_4$ is a submanifold of $\mathcal{X}_0 - \mathcal{X}_4$.

Proof. We first show that $\mathcal{X}_2$ is closed in $\mathcal{X}_0$, namely, $\overline{\mathcal{X}_2} \cap (\mathcal{X}_0 - \mathcal{X}_2) = \emptyset$.

We will argue by contradiction. For each $J \in \mathcal{X}_2$ there is at least one embedded $J-$holomorphic sphere with square at most -2. And by Lemma 3.1.10, Condition 3.1.9 applies here. Hence for each $J' \in \mathcal{X}_0 - \mathcal{X}_2$, every simple $J'-$holomorphic sphere has square at least -1.

Thus, if $\overline{\mathcal{X}_2} \cap (\mathcal{X}_0 - \mathcal{X}_2) \neq \emptyset$, then by Lemma 3.1.11 there is a square at most -2 symplectic sphere C whose class $[C]$ admits a decomposition as in equation (3.1), with no class having square less than -1. Moreover, by Proposition 3.3.3, the decompositions has the form

$$[C] = \sum a_i F + \sum_j b_j D_j + rB,$$

where a_i, b_j are non-negative integers, $r \in \{0, 1\}$, and $D_j \in \mathcal{S}_\omega^{-1}$. By pairing with K_ω on both sides. The left hand side is $[C] \cdot K_\omega \geq 0$. And the right hand side is $\sum a_i F \cdot K_\omega + \sum_j b_j D_i \cdot K_\omega + rB \cdot K_\omega$. Since $D_i \cdot K_\omega = -1, F \cdot K_\omega = B \cdot K_\omega = -2$. There is a contradiction since the righthand side is strictly negative.

We next show that $\mathcal{X}_4$ is closed in X, namely, $\overline{\mathcal{X}_4} \cap (\mathcal{X}_0 - \mathcal{X}_4) = \emptyset$. Since $\mathcal{X}_4 \subset \mathcal{X}_2$ and $\mathcal{X}_2$ is closed in X, it suffices to show that $\mathcal{X}_4$ is closed in $\mathcal{X}_2$.

For each $J \in \mathcal{X}_4$ there is either one embedded $J-$holomorphic sphere with square at most -3, or there are at least two embedded $J-$holomorphic sphere with square -2. And by Lemma 3.1.10, Condition 3.1.9 holds here and for each $J' \in \mathcal{X}_2 - \mathcal{X}_4$ every simple $J'-$holomorphic sphere has square at least -2.

Suppose $\overline{\mathcal{X}_4} \cap (\mathcal{X}_2 - \mathcal{X}_4) \neq \emptyset$. Then

1) either there is a curve class $\bar{C} \in \mathcal{S}^{<-2}$ such that $\bar{C} = \sum c_i \bar{C}_i$ with $\bar{C}_i \in \mathcal{S}^{\geq -2}$;

2) or there is a curve class $\bar{C} \in \mathcal{S}^{-2}$ such that $\bar{C} = \bar{C}' + \sum c_i \bar{C}_i$ with $\bar{C}_i \in \mathcal{S}^{-2}$ and $\emptyset \neq \{\bar{C}_i\} \subset \mathcal{S}^{>-2}$.

For the both cases, by Proposition 3.3.3, the decomposition can only have four types of classes: the zero class B, kF, $D_j \in \mathcal{S}_\omega^{-1}$, $G_k \in \mathcal{S}_\omega^{-2}$. Since either $a \neq 0$ or some $b_j \neq 0$, we have the contradiction $0 = [\tilde{C}] \cdot K_\omega = rB \cdot K_\omega + \sum aF \cdot K_\omega + \sum_j b_j D_i \cdot K_\omega + c_k G_k \cdot K_\omega < 0$. A contradiction.

Next let us establish the claims (i)-(iv).

(i). $\mathcal{X}_0 - \mathcal{X}_4$ is a manifold. This statement is true since $\mathcal{X}_4$ is closed in $\mathcal{X}_0$ and $\mathcal{X}_0$ is a manifold. Similarly, $\mathcal{X}_0 - \mathcal{X}_2$ is a manifold since $\mathcal{X}_2$ is also closed in $\mathcal{X}_0$. And both $\mathcal{X}_0 - \mathcal{X}_4$ and $\mathcal{X}_0 - \mathcal{X}_2$ are open submanifolds of $\mathcal{X}_0$.

(ii). $\mathcal{X}_2 - \mathcal{X}_4$ is closed in $\mathcal{X}_0 - \mathcal{X}_4$. This follows from the fact that $\mathcal{X}_2$ is closed in $\mathcal{X}_0$.

(iii). $\mathcal{X}_2 - \mathcal{X}_4$ is a manifold. This statement follows from the fact that $\mathcal{X}_2 - \mathcal{X}_4$ is a submanifold of $\mathcal{X}_0$. This latter fact follows from the fact that $\mathcal{X}_2 - \mathcal{X}_4$ is the disjoint union of cod 2 prime sets $\mathcal{J}_A$ over $\mathcal{S}_\omega^{-2}$, and Lemma 3.1.13.

(iv). $\mathcal{X}_2 - \mathcal{X}_4$ is a closed submanifold of $\mathcal{X}_0 - \mathcal{X}_4$. Since $\mathcal{X}_0 - \mathcal{X}_4$ is an open submanifold of $\mathcal{X}_0$, $\mathcal{X}_2 - \mathcal{X}_4$ is also a submanifold of $\mathcal{X}_0 - \mathcal{X}_4$.

Hence this proves that $\emptyset = \mathcal{X}_5 \subset \mathcal{X}_4 (= \mathcal{X}_3) \subset \mathcal{X}_2 (= \mathcal{X}_1) \subset \mathcal{X}_0 = \mathcal{J}_\omega$, is a level 2 stratification.

□

3.4.2 Enumerating the components by -2 symplectic sphere classes

We use the following lemma as stated in [31] to further describe each $\mathcal{J}_A$:

Lemma 3.4.2. *For a rational manifold $X = S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, k \geq 0$, the group $Symp_h$ acts transitively on the space of homologous -2 symplectic spheres.*

Proof. Here we give a proof follows steps sketched in in [35] and [10]: Without loss of generality, we can do base change to make a symplectic sphere S_i in the homology class $[S_i] = B - F$. For each pair (X, S_i), by [43], there is a set $\mathcal{C}_i$ of disjoint (-1) symplectic spheres C_i^l for $l = 1, \cdots, k$ such that

$$[C_i^l] = E_l, \text{for} \quad l = 1, \cdots k.$$

Blowing down the set $\{C_i^1, \cdots C_i^k\}$ separately, results in $(X_i, \tilde{S}_i, \mathcal{B}_i)$ where X_i is a symplectic $S^2 \times S^2$ with equal symplectic areas admitting from the original symplectic form of X on factor B and F, $\tilde{S}_i$ a symplectic sphere in X_i, and $\mathcal{B}_i = \{B_i^2, \cdots, B_i^k\}$ is a symplectic ball packing in $X_i \setminus \tilde{S}_i$ corresponding to $\mathcal{C}_i$. For any two pairs, since the symplectic forms are homologous, by [25], there is a symplecotomorphism Φ from $(X_1, \tilde{S}_1)$ to $(X_2, \tilde{S}_2)$, such that for fixed l, $Vol(\Phi(B_1^l)) = Vol(B_2^l)$. Then according to [4], we can choose this Φ such that the two symplectic spheres are isotopic, i.e. $\Phi(\tilde{S}_1) = \tilde{S}_2$.

Then apply Theorem 1.1 in [10], there is a compactly supported Hamiltonian isotopy ι of $(X_2, \tilde{S}_2)$ such that the symplectic ball packing $\Phi(\mathcal{B}_1))$ and $\mathcal{B}_2$ is connected by ι in $(X_2, \tilde{S}_2)$. Then $\iota \circ \Phi$ is a symplecotomorphism between the tuples $(X_i, \tilde{S}_i, \mathcal{B}_i)$ and hence blowing up induces a symplecotomorphism $\psi : (X_1, \tilde{S}_1, \mathcal{B}_1) \to (X_2, \tilde{S}_2, \mathcal{B}_2)$. Further note that ψ preserve homology classes $B, F, E_1, E_2, \cdots, E_k$ and hence $\psi \in Symp_h(X, \omega)$. □

Hence we have the following corollary about codimension 2 stratum in the stratification of $\mathcal{J}_\omega$:

Corollary 3.4.3. *If the group $Symp_h$ is itself connected, which holds true for $X = S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, k = 0, 1, 2, 3$ as in Theorem 1.1 in [31], then homologous symplectic -2 spheres are Hamiltonian isotopic. This means the stratum $\mathcal{J}_{\{A\}}$ is connected if A is a -2 symplectic sphere.*

Remark 3.4.4. Here we list the set R^+_{k+1} as defined in Remark 2.1.11 for $X = S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, k \leq 4$, which is the set of all possible homology classes of symplectic or Lagrangian square (-2) spheres for some reduced symplectic form:

- $S^2 \times S^2$: $R^+_1 = \{B - F\}$.
- $S^2 \times S^2 \# \overline{\mathbb{C}P^2}$: $R^+_2 = \{B - F\}$.
- $S^2 \times S^2 \# 2\overline{\mathbb{C}P^2}$: $R^+_3 = \{B - F, E_1 - E_2, B - E_1 - E_2, F - E_1 - E_2\}$.
- $S^2 \times S^2 \# 3\overline{\mathbb{C}P^2}$:$R^+_4 = \{B - E_i - E_j, F - E_i - E_j, where \quad i > j, i, j \in \{1, 2, 3\}, B - F\}$.
- $S^2 \times S^2 \# 4\overline{\mathbb{C}P^2}$: $R^+_5 = \{B + F - E_1 - E_2 - E_3 - E_4, B - E_i - E_j, F - E_i - E_j, E_i - E_j, where \quad i > j, i, j \in \{1, 2, 3, 4\}, B - F\}$.

In particular, for a rational manifold $X = S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, n \leq 4$ with a reduced form, $\mathcal{S}^{-2}_\omega$ is a subset of classes listed above, as described in section 2.1.

Remark 3.4.5. The following observation will be used in Lemma 4.3.1: for a symplectic rational 4 manifold of Euler number $\chi(X) \leq 7$, any symplectic -2 sphere S is an edge of a toric moment polytope. Consequently, there is a semi-free circle action having S as a component of the fixed locus.

Relative Alexander duality for regular Fréchet stratification

We have the following relative version of Alexander duality proved in [13]:

Theorem 3.4.6. *Let $\mathcal{X}$ be a Hausdorff space, $\mathcal{Z} \subset \mathcal{Y}$ a closed subset of $\mathcal{X}$ such that $\mathcal{X} - \mathcal{Z}, \mathcal{Y} - \mathcal{Z}$ are manifolds modeled by topological linear spaces. Suppose $\mathcal{Y} - \mathcal{Z}$ is a closed submanifold of $\mathcal{X} - \mathcal{Z}$ of codimension p, then we say $(\mathcal{Y}, \mathcal{Z})$ is a closed relative submanifold of $(\mathcal{X}, \mathcal{Z})$ of codimension p.*

And we have the isomorphism $H^i(\mathcal{X}-\mathcal{Z}, \mathcal{X}-\mathcal{Y}) = H^{i-p}(\mathcal{Y}-\mathcal{Z})$ for a given coefficient sheaf.

Further, we have the following sequence

$$\cdots \to H^{i-1}(\mathcal{X} - \mathcal{Y}) \to H^{i-p}(\mathcal{Y} - \mathcal{Z}) \to H^i(\mathcal{X} - \mathcal{Z}) \to H^i(\mathcal{X} - \mathcal{Y}) \to \cdots$$

This duality in Theorem 3.4.6 together with Lemma 3.1.13 and Theorem 3.4.1 gives the following:

Corollary 3.4.7. *When the decomposition $\mathcal{J}_\omega$ is a stratification at the first two level as in Theorem 3.4.1 with top stratum $\mathcal{J}_{open}$, then $H^1(\mathcal{J}_{open}) = \bigoplus_{A_i \in \mathcal{S}_\omega^{-2}} H^0(\mathcal{J}_{A_i})$.*

Proof. In Theorem 3.4.1, let $\mathcal{X}_0 = \mathcal{X}, \mathcal{X}_2 = \mathcal{Y}, \mathcal{X}_4 = \mathcal{Z}$. It is easy to check the condition holds in Theorem 3.4.6. The the conclusion easily follows from the sequence in Theorem 3.4.6.

□

In the next lemma, we give a characterization of $\mathcal{J}_{open}$ using a configuration C of -1 spheres:

Lemma 3.4.8. *Let X be $S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, k \leq 4$ with a reduced symplectic form and configuration $\mathcal{C}$ of exceptional spheres containing there is a subset, of cardinality $\leq k+1$(see remark 3.4.9), such that $U_{\mathcal{C}} = \mathcal{J}_{open}$. And we give a proper choice of subsets as follows for later use:*

- $S^2 \times S^2 \# \overline{\mathbb{C}P^2}$, $\mathcal{C} = \{B - E\}$,
- $S^2 \times S^2 \# 2\overline{\mathbb{C}P^2}$, $\mathcal{C} = \{E_1, B - E_1, F - E_1\}$,

- $S^2\times S^2\#3\overline{\mathbb{C}P^2}$, $\mathcal{C}=\{F-E_1,E_2,B-E_1,B+F-E_1-E_2-E_3\}$,
- $S^2\times S^2\#4\overline{\mathbb{C}P^2}$, $\mathcal{C}=\{B+F-E_2-E_3-E_4,B-E_1,F-E_1,E_2,E_3\}$,

Proof. By [36], for any $S^2\times S^2\#n\overline{\mathbb{C}P^2}$, $\mathcal{J}_{open}$ is characterized by the existence of all exceptional spheres and the absences of embedded negative square spheres. Since in the configuration $\mathcal{C}$ we have only -1 sphere components, for each $J\in\mathcal{J}_{open}$, there is a unique $J-$holomorphic configuration in $\mathcal{C}$. Then we can define a natural map $\mathcal{J}_{open}\to\mathcal{C}$, sending the almost complex structure to the unique $J-$holomorphic configuration in $\mathcal{C}$. We can check that for each given small rational manifold as listed above, any negative curve as in Lemma 3.3.1 intersects at least one curve in the configuration negatively:

- $S^2\times S^2\#k\overline{\mathbb{C}P^2}, k=1,2$, it is easy to check the curves with square -2 in section 3.4.2.

  And any curve with square less than -2 can be written as $B-qF-r_iE_i, q\geq 1; i\leq k, r_i\in\{0,1\}$. And $[B-qF-r_iE_i]\cdot[B-E_1]=-q-r_1<0$.

- $S^2\times S^2\#3\overline{\mathbb{C}P^2}$, any curve with square -2 as listed in in section 3.4.2 is easy to check.

  Any class in $\mathcal{F},\mathcal{E}$ with square less than -2 can be $F-E_1-E_2-E_3$ or $E_1-E_2-E_3$, and each of them pair with $B+F-E_1-E_2-E_3$ is negative. And any class in $\mathcal{B}$ with square less than -2 can be written as either $B-E_1-E_2-E_3$ or $B-kF-r_iE_i, k\geq 1, r_i\in\{0,1\}$. And $[B-E_1-E_2-E_3]\cdot[B+F-E_1-E_2-E_3]<0$; $[B-kF-r_iE_i]\cdot[B-E_1]=-k-r_1<0$.

- And we can check all the curves with square -2 in $S^2\times S^2\#4\overline{\mathbb{C}P^2}$ pairing at least one of $\{B+F-E_2-E_3-E_4,B-E_1,F-E_1,E_2,E_3\}$ negatively.

  Additionally, any class in $\mathcal{F},\mathcal{E}$ with square less than -2 is one of the following $E_i-E_j-E_k, F-E_i-E_j-E_k, i>j>k\in\{1,2,3,4\}, E_1-E_2-E_3-E_4, F-E_1-E_2-E_3-E_4$; and each of them pair $B+F-E_2-E_3-E_4$ negatively. Any class in $\mathcal{B}$ with square less than -2 can be written as either $B-E_i-E_j-E_k, i>j>k\in\{1,2,3,4\}$, or $B-kF-r_iE_i, k\geq 1, r_i\in\{0,1\}$. And $[B-E_i-E_j-E_k]\cdot[B+F-E_2-E_3-E_4]<0$; $[B-kF-r_iE_i]\cdot[B-E_1]=-k-r_1<0$.

Therefore any sphere class with square less than -1 can not have simultaneous $J-$holomorphic representative with the set $\mathcal{C}$. $\square$

Remarks

Remark 3.4.9. In the above Lemma 3.4.8, the subsets has the minimal cardinality, but they are in general not unique. This means there are other choices of subsets S of -1 spheres such that $\mathcal{J}_S = \mathcal{J}_{open}$, having larger cardinality, but S does not necessarily contain the subsets we list.

And note that in H, E_i basis, the minimal subsets we choose can be written down in the following way: First the -1 curve with maximal area A_m. And if there's no -1 curve $A = nH - \sum a_i E_i$ such that $A_m \cdot A \geq 0$, then take the set to be $\{A_m, E_1, \cdots, E_{k_1}\}$; otherwise we take A_m and another $A_p = n_p H - \sum p_i E_i$ such that $A_m \cdot A_p$ being largest, then we take the set to be $\{A_m, E_1, \cdots, \hat{E_i} \cdots, \hat{E_j} \cdots, E_k\}$, where E_i, E_i are the minimal area class pairing A_m, A_p positive respectively.

And one can do a base change for the sets in Lemma 3.4.8 to obtain the following, which agrees with the above method:

- $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$, $C = \{E_1\}$.
- $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$, $C = \{E_1, E_2, H - E_2 - E_3\}$.
- $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$,$C = \{H - E_1 - E_2, H - E_3 - E_4, E_1, E_3, \}$.
- $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$, $C = \{2H - E_1 - E_2 - E_3 - E_4 - E_5, E_1, E_2, E_3, E_4\}$.

Remark 3.4.10. In a separate paper [30] we will prove a general result. Let $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with arbitrary ω. If $k \leq 8$, then each prime subset $\mathcal{J}_C$ is a submanifold. Moreover, if $k \leq 5$, then $\mathcal{X}_{2i}$ is relatively closed in the union $\cup_{j \geq i} \mathcal{X}_{2j}$.

Therefore, for $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k \leq 5$ with arbitrary ω, this filtration of $\mathcal{J}_\omega$ fits into the following notion of **stratification** in ∞-dimension (For finite dimension, see eg. [19]):

Definition 3.4.11. For an ∞-dimensional real Fréchet manifold X, a finite filtration of $\mathcal{X}$ is called an **even stratification** if it is a sequence of **closed** subspaces

$$\emptyset = \mathcal{X}_{2n+2} \subset \mathcal{X}_{2n} \subset \mathcal{X}_{2n-2} \ldots \subset \mathcal{X}_2 \subset \mathcal{X}_0 = \mathcal{X},$$

where $\mathcal{X}_{2i} \setminus \mathcal{X}_{2i+2}$ is a submanifold of real codimension $2i$.

Remark 3.4.12. An absolute version of Alexander duality in [13] was applied by Abreu to detect the topology of $\mathcal{J}_{open}$ for $S^2 \times S^2$ with a symplectic form with ratio within $(1, 2)$ in [2]. In the paper [30] we will establish an Alexander duality for stratifications as in Definition 3.4.11, generalizing [13]. The following special case can also be applied to compute the fundamental group of the symplectomorphism group of small rational 4-manifolds.

Theorem 3.4.13. *Let $\mathcal{X}$ be a contractible paracompact C^∞ smooth (in the Graves-Hildebrandt sense) manifold modeled by a complex Fréchet space. Suppose $\mathcal{X}$ is evenly stratified by $\{\mathcal{X}_{2i}\}_{i=0}^n$ as in Definition 3.4.11 at the first 2 levels. Then we have the duality on the integral cohomology of $\mathcal{X} \setminus \mathcal{X}_2$ and $\mathcal{X}_2 \setminus \mathcal{X}_4$ at certain level:*

$$H^1(\mathcal{X} \setminus \mathcal{X}_2) \cong H^0(\mathcal{X}_2 \setminus \mathcal{X}_4).$$

Chapter 4

Symplectic rational 4-manifold with Euler number less than 8

This chapter is devoted to the study of a rational surface with Euler number less than 8. We provide a uniform approach to prove the connectedness of Torelli SMC and to compute $\pi_1(Symp(X,\omega))$ when (X,ω) is a rational 4 manifold with Euler number $\chi(X) \leq 7$. We will further discuss the full homotopy type of $(Symp(X,\omega))$.

4.1 Strategy

We summarize the strategy using diagram (1.1),

$$\begin{array}{ccccccc} Symp_c(U) & & & & & & \\ \downarrow & & & & & & \\ Stab^1(C) & \longrightarrow & Stab^0(C) & \longrightarrow & Stab(C) & \longrightarrow & Symp_h(X,\omega) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathcal{G}(C) & & Symp(C) & & \mathcal{C}_0 \simeq \mathcal{J}_{open}, \end{array} \qquad (4.1)$$

Recall definition 1.2.1 that C is stable full configuration of symplectic spheres. We then analyze the diagram (1.1) and derive a criterion for the connectedness of $Symp_h(X,\omega)$ in Corollary 4.2.11.

4.1.1 Groups associated to a configuration

Let C be a configuration in X. We first introduce the groups appearing in (1.1):

Subgroups of $Symp_h(X, \omega)$

Recall that $Symp_h(X, \omega)$ is the group of symplectomorphisms of (X, ω) which acts trivially on $H_*(X, \mathbb{Z})$.

• $Stab(C) \subset Symp_h(X, \omega)$ is the subgroup of symplectomorphisms fixing C setwise, but not necessarily pointwise.

• $Stab^0(C) \subset Stab(C)$ is the subgroup the group fixing C pointwise.

• $Stab^1(C) \subset Stab^0(C)$ is subgroup fixing C pointwise and acting trivially on the normal bundles of its components.

$Symp_c(U)$ **for the complement** U

$Symp_c(U)$ is the group of compactly supported symplectomorphisms of $(U, \omega|_U)$, where $U = X \setminus C$ and the form $\omega|_U$ is the inherited form on U from X. It is topologised in this way: let (U, ω) be a non-compact symplectic manifold and let $\mathcal{K}$ be the set of compact subsets of U. For each $K \in \mathcal{K}$ let $Symp_K(W)$ denote the group of symplectomorphisms of U supported in K, with the topology of $\mathcal{C}^\infty$-convergence. The group $Symp_c(U, \omega)$ of compactly-supported symplectomorphisms of (U, ω) is topologised as the direct limit of $Symp_K(W)$ under inclusions.

$Symp(C)$ **and** $\mathcal{G}(C)$ **for the configuration** C

Given a configuration of embedded symplectic spheres $C = C_1 \cup \cdots \cup C_n \subset X$ in a 4-manifold, let I denote the set of intersection points amongst the components. Suppose that there is no triple intersection amongst components and that all intersections are transverse. Let k_i denote the cardinality of $I \cap C_i$, which is the number of intersection of points on C_i.

The group $Symp(C)$ of symplectomorphisms of C fixing the components of C is the product $\prod_{i=1}^n Symp(C_i, I \cap C_i)$. Here $Symp(C_i, I \cap C_i)$ denotes the group of symplectomorphisms of C_i fixing the intersection points $I \cap C_i$. Since each C_i is a $2-$sphere and $Symp(S^2)$ acts transitivity on $N-$tuples of distinct points in S^2, we can write

$Symp(C_i, I\cap C_i)$ as $Symp(S^2, k_i)$. Thus

$$Symp(C) \cong \prod_{i=1}^{n} Symp(S^2, k_i) \tag{4.2}$$

As shown in [15] we have:

$$Symp(S^2,1) \simeq S^1; \quad Symp(S^2,2) \simeq S^1; \quad Symp(S^2,3) \simeq \star; \tag{4.3}$$

where $\simeq$ means homotopy equivalence. And when $k = 1, 2$, the S^1 on the right can be taken to be the loop of a Hamiltonian circle action fixing the k points.

The symplectic gauge group $\mathcal{G}(C)$ is the product $\prod_{i=1}^{n} \mathcal{G}_{k_i}(C_i)$. Here $\mathcal{G}_{k_i}(C_i)$ denotes the group of symplectic gauge transformations of the symplectic normal bundle to $C_i \subset X$ which are equal to the identity at the k_i intersection points. Also shown in [15]:

$$\mathcal{G}_0(S^2) \simeq S^1; \quad \mathcal{G}_1(S^2) \simeq \star; \quad \mathcal{G}_k(S^2) \simeq \mathbb{Z}^{k-1},\ k > 1. \tag{4.4}$$

Since we assume the configuration is connected, each $k_i \geq 1$. Thus by (4.4), we have

$$\pi_0(\mathcal{G}(C)) = \oplus_{i=1}^{n} \pi_0(\mathcal{G}_{k_i}(S^2)) = \oplus_{i=1}^{n} \mathbb{Z}^{k_i - 1} \tag{4.5}$$

It is useful to describe a canonical set of k_i generators for $\mathcal{G}_{k_i}(C_i)$. For each intersection point $y \in I \cap C_i$, the evaluation map

$$ev_y : \mathcal{G}_{k_i}(C_i) \to SL(2, \mathbb{R})$$

is a homotopy fibration with fiber $\mathcal{G}_{k_i+1}(C_i)$ which is the gauge group fixing one more point. And hence it induces a map $\mathbb{Z} = \pi_1(SL(2,\mathbb{R})) \to \pi_0(\mathcal{G}_{k_i}(C_i))$. Let $g_{C_i}(y) \in \pi_0(\mathcal{G}_{k_i}(C_i))$ denote the image of $1 \in \mathbb{Z}$.

4.1.2 Choice of the configuration in each case

Now for an arbitrary symplectic form, by Lemma 2.1.4, it is diffeomorphic to a reduced form. Symplectomorphic symplectic forms have homeomorphic symplectomorphism groups. Hence it suffices to list the choice of standard stable configuration C for any rational 4-manifold with Euler number $\chi(X) \leq 7$, equipped with a reduced symplectic form:

- $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$, $C = \{E_1, E_2, H - E_1 - E_2\}$.
- $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$, $C = \{E_1, E_2, E_3, H - E_1 - E_2, H - E_2 - E_3\}$.
- $\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$, $C = \{H - E_1 - E_2, H - E_3 - E_4, E_1, E_2, E_3, E_4\}$.

For such configuration, we have the following lemma about the weak homotopy type of $\mathcal{C}, \mathcal{C}_0$:

Lemma 4.1.1. *For $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}, k = 2, 3, 4$, $\mathcal{C}_0$ is weakly homotopic to $\mathcal{C}$. Denote $\mathcal{J}_C$ the set of almost complex structure that making the configuration C J-holomoprhic, then $\mathcal{C}$ is weakly homotopic to $\mathcal{J}_C$.*

Proof. The first statement is proved in [26] and [15]. Then we show that $\mathcal{J}_{open}$ is weakly homotopic to $\mathcal{C}$. By Lemma 3.4.8, the set of almost complex structures making $C \in \mathcal{C}$ J-holomorphic is the set $\mathcal{J}_{open}$. In addition, $\mathcal{J}_{open} \to \mathcal{C}$ is a surjection and hence a submersion. And as shown in Proposition 4.8 in [26], this map is a fibration with contractible fiber; then we have the desired weak homotopy equivalence between $\mathcal{J}_{open}$ and $\mathcal{C}$. □

And we know from Lemma 3.4.8 that $\mathcal{J}_C = \mathcal{J}_{open}$, because each above configuration contains a minimal subset as in Lemma 3.4.8.

4.2 Connectedness of the Torelli symplectic mapping class group

We first focus on the symplectic mapping class group, and show that

Theorem 4.2.1. *$Symp_h(X, \omega)$ is connected for $X = \mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$ with arbitrary symplectic form ω.*

4.2.1 Reduction to the connectedness of $Stab(C)$

The aim of this subsection is to show

Proposition 4.2.2. *$Symp_h(X,\omega)$ is connected if there is a full, stable, standard configuration C with connected $Stab(C)$.*

This is derived from the right end of diagram (1.1) for a full, stable, standard configuration C, that is, the fibration:

$$Stab(C) \to Symp_h(X,\omega) \to \mathcal{C}_0 \tag{4.6}$$

Recall that $\mathcal{C}_0$ is the space of standard configurations having the homology type of C. We will show (1.1) is a homotopy fibration and $\mathcal{C}_0$ is connected.

We first review certain general facts regarding these configurations which are well-known to experts. By [34], we have the following fact.

Lemma 4.2.3. *Let (M,ω) be a symplectic 4-manifold and C a stable configuration $\cup_i C_i$. Then there is a path connected Baire subset $\mathcal{T}_D$ of $\mathcal{J}_\omega \times M^{\times d(C_i)}$ such that a pair (J,Ω) lies in $\mathcal{T}_D$ if and only if there is a unique embedded $J-$holomorphic configuration having the same homological type as C with the $i-th$ component containing Ω_i.*

Lemma 4.2.4. *Assume C is a stable, standard configuration. The space $\mathcal{C}_0$ of standard configurations having the homology type of C is path connected.*

Proof. Consider $\mathcal{C}$, the space of configurations as in Definition 1.2.1. By Lemma 4.2.3 we see that the space $\mathcal{C}$ is connected. Using a Gompf isotopy argument, it is shown in [15] that the inclusion $\iota : \mathcal{C}_0 \to \mathcal{C}$ is a weak homotopy equivalence. Therefore, $\mathcal{C}_0$ is also connected.

□

With C being full, the following lemma holds:

Lemma 4.2.5. *If the stable, standard configuration C is also full, then $Symp_h(X,\omega)$ acts transitively on $\mathcal{C}_0$. In particular,* (4.3.1) *is a homotopy fibration.*

Proof. From Lemma 4.2.4 any $C_1, C_2 \in \mathcal{C}_0$ are isotopic through standard configurations. The property that the configurations are **symplectically orthogonal** where they intersect, together with the **vanishing** of $H^2(X, C; \mathbb{R})$, allows us to extend such an isotopy to a global homologically trivial symplectomorphism of X (by Banyaga's symplectic isotopy extension theorem, see [46], Theorem 3.19). So we have shown that the action of $Symp_h(X, \omega)$ on the connected space $\mathcal{C}_0$ is transitive by establishing the $1-$dimensional homotopy lifting property of the map $Symp_h(X, \omega) \to \mathcal{C}_0$. By a finite dimensional version of this argument (or Theorem A in [50]), we conclude that (4.3.1) is a homotopy fibration. □

Proof of Proposition 4.2.2

Since (4.3.1) is a homotopy fibration by Lemma 4.2.5, we have the associated homotopy long exact sequence. Because of the connectedness of $\mathcal{C}_0$ as shown in Lemma 4.2.4, the connectedness of $Stab(C)$ implies the connectedness of $Symp_h(X, \omega)$. Therefore, we have 4.2.2 as the reduction of our problem.

4.2.2 Reduction to the surjectivity of ψ: $\pi_1(Symp(C)) \to \pi_0(Stab^0(C))$

To investigate the connectedness of $Stab(C)$, considering the action of $Stab(C)$ on C and the following portion of diagram 1.1 which appeared in [15] and [7]:

$$Stab^0(C) \to Stab(C) \to Symp(C) \tag{4.7}$$

The following lemma already appeared in [15] and was explained to the authors by J. D. Evans[1]. We here include more details for readers' convenience.

Lemma 4.2.6. *This diagram* (4.7) *is a homotopy fibration when C is a simply-connected standard configuration.*

Proof. First we show $Stab(C) \to Symp(C)$ is surjective.

Recall that at each intersection point between two different components $\{x_{ij}\} = C_i \cap C_j$, the two components are symplectically orthogonal to each other in a Darboux chart containing x_{ij}. For convenience of exposition define the *level* of components as follows: let C_1 be the unique component of level 1, and the level-k components are defined as those intersects components in level $k-1$ but does not belong to any lower levels. This is well-defined again because of the simply-connectedness assumption.

[1]Private communications.

An element in $Symp(C)$ is the composition of Hamiltonian diffeomorphism ϕ_i on each component C_i, because of the simply connectedness of sphere. We start with endowing C_1 with a Hamiltonian function f_1 generating ϕ_1. Let C_i^2 be curves on level 2. Because C_i^2 intersects C_1 ω-orthogonally, we can find a symplectic neighborhood U_1 of C_1, identified as a neighborhood of zero section of the normal bundle, so that $U_1 \cap C_i$ consists of finitely many fibers. Pull-back f_1 by the projection π of the normal bundle and multiply a cut-off function $\rho(r), \rho(r) = 1, r \leq \epsilon \ll 1; \rho(r) = 0, r \geq 2\epsilon$. Here r is the radius in the fiber direction. Denote by $\bar{\phi}_1$ the symplectomorphism generated by this cut-off. Notice that $\bar{\phi}_1$ creates an extra Hamiltonian diffeomorphism ϵ_j on each component C_j of level 2, and we denote $\phi_j' = \phi_j \circ \epsilon_j^{-1}$ for C_j belonging to level 2.

One proceeds by induction on the level k. Notice one could always choose a Hamiltonian function f_i on a component C_i on level k which generates ϕ_i' with the property that $f_i(x_{il}) = 0$. Here C_l is the component of level $k-1$ intersecting C_i. We emphasize this can be done because the component C_l on level $k-1$ which intersects C_i is unique (and that the intersection is a single point) due to the simply connectedness assumption, and we do not restrict the value on any other intersections of C_i and components of level $k+1$. Therefore we only fix the value of f_i at a single point.

One then again use the pull-back on the symplectic neighborhood and cut-off along the fiber direction to get a Hamiltonian function H_i which generates a diffeomorphism $\bar{\phi}_i$ supported on the neighborhood of C_i. We note that $d(\pi^* f_1 \cdot \rho(r))|_{F_x} = 0$ whenever $f_1(x) = 0$, where F_x is the normal fiber over the point $x \in C_1$. Hence $dH_i|_{C_l} = 0$ since $f_i(x_{il}) = 0$ as prescribed earlier, which means action of $\bar{\phi}_i$ on C_l is trivial. Taking the composition ϕ of all these $\bar{\phi}_i{}'s$, ϕ is supported on a neighborhood of C and equals ϕ_i when restricted to C_i.

The transitivity of the action of $Stab(C)$ on $Symp(C)$ follows easily. For any two maps $\phi_1, \phi_2 \in Symp(C)$, $\phi_2\phi_1^{-1} \in Symp(C)$. We can extend $\phi_2\phi_1^{-1}$ to $Stab(C)$. Then this extended $\phi_2\phi_1^{-1}$ maps ϕ_1 to ϕ_2.

Now symplectic isotopy theorem (or Theorem A in [50]) for the surjective map $Stab(C) \to Symp(C)$ proves the diagram (4.7) is a fibration.

□

Now we can establish the connectedness of $Stab(C)$ under the following assumptions:

Proposition 4.2.7. *Let (X,ω) be a symplectic 4-manifold, and C a simply-connected, full, stable, standard configuration. If each component of C has no more than 3 intersection points, then the surjectivity of the connecting map ψ: $\pi_1(Symp(C)) \to \pi_0(Stab^0(C))$ implies the connectedness of $Stab(C)$.*

Proof. Since we assume that each component of C has no more than 3 intersection points, it follows from (4.3) and (4.2) that $\pi_0(Symp(C)) = 1$.

By Lemma 4.2.6 we have the homotopy long exact sequence associated to (4.7),

$$\cdots \to \pi_1(Symp(C)) \xrightarrow{\psi} \pi_0(Stab^0(C)) \to \pi_0(Stab(C)) \to \pi_0(Symp(C))$$

Then the surjectivity of ψ implies that $Stab(C)$ is connected.

□

4.2.3 Three types of configurations

Next we investigate when the map ψ: $\pi_1(Symp(C)) \to \pi_0(Stab^0(C))$ is surjective. For this purpose we observe that an element of $Stab^0(C)$ induces an automorphisms of the normal bundle of C. Thus we further have the following homotopy fibration appeared in [15] and [7]:

$$Stab^1(C) \to Stab^0(C) \to \mathcal{G}(C) \tag{4.8}$$

In particular, there is the associated map $\iota : \pi_0(Stab^0(C)) \to \pi_0(\mathcal{G})(C)$. Consider the composition map

$$\bar{\psi} = \iota \circ \psi : \pi_1(Symp(C)) \to \pi_0(Stab^0(C)) \to \pi_0(\mathcal{G}(C)).$$

Notice that $\pi_0(\mathcal{G}(C))$ inherits a group structure from $\mathcal{G}(C)$ and $\bar{\psi}$ is a group homomorphism. As shown in [15], $\bar{\psi}$ can be computed explicitly.

When $k_i = 3$, $\pi_1(Symp(S^2, k))$ is trivial by (4.3). When $k_i = 1, 2$, $Symp(C_i, I \cap C_i)$ is homotopic to the loop of a Hamiltonian circle action on C_i fixing the k_i points. Denote such a loop by $(\phi_i)_t$. Observe that $(\phi_i)_t$ is a generator of $\pi_1(Symp(C_i, I \cap C_i)) = \mathbb{Z}$. Recall that for each component C_j there is a canonical set of generators $\{g_{C_j}(y), y \in I \cap C_j\}$ for $\mathcal{G}_{k_j}(C_j)$, introduced at the end of 2.1. The following is Lemma 4.1 in [15]

Lemma 4.2.8. *Suppose C_i is a component with $k_i = 1, 2$. The image of $[(\phi_i)_t] \in \pi_1(Symp(C_i, I \cap C_i))$ under $\bar{\psi}$ is described as follows.*

- *if $k_i = 1$ and C_j is the only component intersecting C_i with $\{x\} = C_i \cap C_j$, then $(\phi_i)_{2\pi}$ is sent to*
$$g_{C_j}(x)$$
in the factor subgroup $\pi_0(\mathcal{G}_{k_j}(C_j))$ of $\pi_0(\mathcal{G}(C))$.

- *if $k_i = 2$ and $x \in C_i \cap C_j$, $y \in C_i \cap C_l$, then $(\phi_i)_{2\pi}$ is sent to*
$$(g_{C_j}(x), g_{C_l}(y))$$
in the factor subgroup $\pi_0(\mathcal{G}_{k_j}(C_j)) \times \pi_0(\mathcal{G}_{k_l}(C_l))$ of $\pi_0(\mathcal{G}(C))$.

Use Lemma 4.2.8 we will show that $\bar{\psi}$ is surjective for the following configurations.

Definition 4.2.9. Introduce three types of configurations (see Figure 1 for examples).

- (type I) $C = \bigcup_1^n C_i$ is called a chain, or a type I configuration, if $k_1 = k_n = 1$ and $k_j = 2$, $2 \leq j \leq n-1$.

- (type II) Suppose $C = \bigcup_1^n C_i$ is a chain. $C' = C \cup \overline{C_p}$ is called a type II configuration if the sphere $\overline{C_p}$ is attached to C_p at exactly one point for some p with $2 \leq p \leq n-1$.

- (type III) Suppose $C' = C \cup \overline{C_p}$ is a type II configuration. $C'' = C' \cup \overline{C_q}$ is called a type III configuration if the sphere $\overline{C_q}$ is attached to C_q at exactly one point for some q with $2 \leq q \leq n-1$ and $q \neq p$.

Lemma 4.2.10. *$\bar{\psi}$ is surjective for a type I or II configuration and an isomorphism for a type III configuration.*

Proof. We first prove the surjectivity for a type I configuration $C = \bigcup_1^n C_i$. In this case, there are $n-1$ intersection points $x_1, ..., x_{n-1}$ in total with

$$I \cap C_1 = \{x_1\}, \quad I \cap C_n = \{x_{n-1}\}, \quad I \cap C_i = \{x_{i-1}, x_i\}, \ i = 2, ..., n.$$

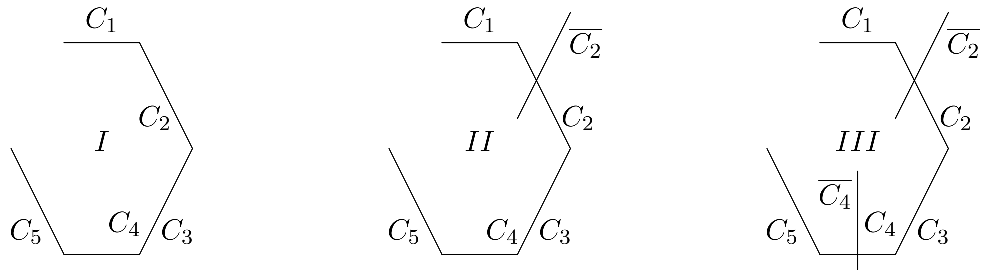


Figure 4.1: 3 types of configruations

Notice that $\pi_1(Symp(C_i, k_i)) = \mathbb{Z}$ for each $i = 1, ..., n$. Notice also that $\pi_0(\mathcal{G}_{k_i}(C_i)) = \mathbb{Z}$ for each i for $i = 2, ..., n-1$, and $\pi_0(\mathcal{G}_{k_1}(C_1))$ and $\pi_0(\mathcal{G}_{k_n}(C_n))$ are trivial. Thus the homomorphism $\bar{\psi}_C$ associated to C is of the form $\mathbb{Z}^n \to \mathbb{Z}^{n-2}$.

For each $i = 1, ..., n$, denote the generator $(\phi_i)_t$ of $\pi_1(Symp(C_i, k_i)) = \mathbb{Z}$ by $\mathrm{rot}(i)$. For each $i = 2, ..., n-1$, denote by $g_i(i-1)$ and $g_i(i)$ the generators $g_{C_i}(x_{i-1})$ and $g_{C_i}(x_i)$ of $\pi_0(\mathcal{G}_2(C_i)) = \mathbb{Z}$.

Then by Lemma 4.2.8 the homomorphism $\bar{\psi}_C$ is described by

$$\bar{\psi}_C: \begin{array}{lcl} \mathrm{rot}(1) & \to & g_2(1), \\ \mathrm{rot}(2) & \to & (0, g_3(2)), \\ \mathrm{rot}(j) & \to & (g_{j-1}(j-1), g_{j+1}(j)), \quad 3 \leq j \leq n-2 \\ \mathrm{rot}(n-1) & \to & (g_{n-2}(n-2), 0) \\ \mathrm{rot}(n) & \to & g_{n-1}(n-1) \end{array} \tag{4.9}$$

Choose the bases of $\pi_1(Symp(C_i))$ and $\pi_0(\mathcal{G}(C))$ to be

$$\{\mathrm{rot}(1), \cdots, \mathrm{rot}(n)\}$$

and

$$\{g_2(2), g_3(3), g_4(4), \cdots, g_{n-1}(n-1)\},$$

respectively. Notice that $g_i(i-1) = \pm g_i(i)$, then by (4.9), $\bar{\psi}_C$ is represented by the following $(n-2) \times n$ matrix if we drop the possible negative sign for each entry,

$$\begin{bmatrix} 1 & 0 & 1 & & & & & & \\ 0 & 1 & 0 & 1 & & & & & \\ 0 & 0 & 1 & 0 & 1 & 0 & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 1 & 0 & 1 & 0 & 0 \\ & & & & & 1 & 0 & 1 & 0 \\ & & & & & & 1 & 0 & 1 \end{bmatrix}$$

Observe that the first $n-2$ minor as a $(n-2)\times(n-2)$ is upper triangular matrix whose determinant is ± 1. This shows that $\bar{\psi}_C$ is surjective.

For a type II configuration $C' = C \cup \overline{C_p}$, let $\bar{x}_p$ be the intersection of C_p and $\overline{C_p}$. Notice that $\pi_1(Symp(C')) = \mathbb{Z}^n$ as in the case of C, with the $\mathbb{Z}$ summand from C_p replaced by a $\mathbb{Z}$ summand from $\overline{C_p}$. Notice also that $\pi_0(\mathcal{G}(C')) = \mathbb{Z}^{n-1}$ with the extra $\mathbb{Z}$ summand coming from the new intersection point $\bar{x}_p$ in C_p. Denote by $\mathrm{rot}(\bar{p})$ the generator of $\pi_1(Symp(\overline{C_p}, \bar{x}_p))$. Denote by $g'_p(p)$ the generator $g_{C_p}(\bar{x}_p)$ of $\pi_0(\mathcal{G}_3(C_p))$. By Lemma 4.2.8, the homomorphism $\bar{\psi}_{C'}$ is of the form $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$, and it differs from $\bar{\psi}_C$ as in (4.9) :

$$\begin{aligned} &\mathrm{rot}(p) = 0 \\ &\mathrm{rot}(\bar{p}) \to g'_p(p) \end{aligned} \tag{4.10}$$

It is not hard to see that $\bar{\psi}_{C'}$ is again surjective. We illustrate by the type II configuration in Figure 1. With respect to the bases

$$\{\mathrm{rot}(1), \mathrm{rot}(\bar{2}), \mathrm{rot}(3), \mathrm{rot}(4), \mathrm{rot}(5)\} \quad \text{and} \quad \{g_2(2), g'_2(2), g_3(3), g_4(4)\},$$

$\bar{\psi}_{C'}$ is represented by the following 4×5 matrix (if we drop the possible negative sign),

$$\begin{bmatrix} 1 & 0 & 1 & & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 1 \end{bmatrix}$$

For a type III configuration $C'' = C' \cup \overline{C_q} = C \cup \overline{C_p} \cup \overline{C_q}$, observe first that

$\pi_1(Symp(C'')) = \mathbb{Z}^n$ and $\pi_0(\mathcal{G}(C') = \mathbb{Z}^n$. By Lemma 4.2.8, we can describe $\bar{\psi}_{C''}$: $\mathbb{Z}^n \to \mathbb{Z}^n$ similar to the case of the type II configuration C'. Precisely, $\bar{\psi}_{C''}$ differs from $\bar{\psi}_C$ in (4.9) as follows:

$$\begin{aligned} \mathrm{rot}(p) &= \mathrm{rot}(q) = 0 \\ \mathrm{rot}(\bar{p}) &\to g'_p(p) \\ \mathrm{rot}(\bar{q}) &\to g'_q(q) \end{aligned} \tag{4.11}$$

It is easy to see that $\bar{\psi}_{C''}$ is an isomorphism in this case. We illustrate by the type III configuration in Figure 1. With respect to the bases

$$\{\mathrm{rot}(1), \mathrm{rot}(\bar{2}), \mathrm{rot}(3), \mathrm{rot}(\bar{4}), \mathrm{rot}(5)\} \quad \text{and} \quad \{g_2(2), g'_2(2), g_3(3), g'_4(4), g_4(4)\},$$

$\bar{\psi}_{C''}$ is represented by the following square matrix (if we drop the possible negative sign),

$$\begin{bmatrix} 1 & 0 & 1 & & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 1 \end{bmatrix}$$

□

4.2.4 Criterion

Finally, we arrive at the following criterion for the connectedness of $Symp_h(X, \omega)$.

Corollary 4.2.11. *Suppose a stable, standard configuration C is type I, II or III, and it is full. If $Symp_c(U)$ is connected, then $Symp_h(X, \omega)$ is connnected.*

Proof. By Lemma 5.2 in [15], $Symp_c(U)$ is weakly homotopy equivalent to $Stab^1(C)$. So by our assumption that $Symp_c(U)$ being connected, $Stab^1(C)$ is also connected. Therefore the map $\iota : \pi_0(Stab^0(C)) \to \pi_0(\mathcal{G})(C)$ associated to the homotopy fibration (4.8) is a group isomorphism. Now we have $\psi_C = \bar{\psi}_C$.

Since C is type I, II or III, by Lemma 4.2.10, ψ_C is surjective. Notice that any type I, II, or III configuration is simply-connected. By the assumption of C being full, we can apply Proposition 4.2.7 and Proposition 4.2.2 to conclude that $Symp_h(X, \omega)$ is connnected.

□

4.2.5 Contractibility of $Symp_c(U)$ and the proof in the case of $\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$

Let $X = \mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$ and ω an arbitrary symplectic form on X. We consider a configuration C in [15], consisting of symplectic spheres in homology classes $S_{12} = H - E_1 - E_2$, $S_{34} = H - E_3 - E_4$, E_1, E_2, E_3 and E_4. Here $\{H, E_i\}$ is the standard basis of $H_2(X; \mathbb{Z})$ with positive pairing with ω. In Figure 2 we label the spheres by their homology classes.

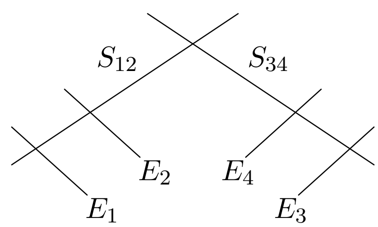


Figure 4.2: Configuration of 4-point blow up

To apply the criterion in Corollary 4.2.11, we need to check that we can always find a configuration C of such a homology type, so that

- C is stable.
- C is a type I, II or III configuration.
- C is full.
- $Symp_c(U)$ is connected.

Existence of such a configuration is a direct consequence of Gromov-Witten theory and the first three statements follows from definition. Note also that the actual choice of configuration will not affect the last statement because $Symp_h(X)$ acts transitively on $\mathcal{C}_0$, which means U is well-defined up to symplectomorphism for any choice of $C \in \mathcal{C}_0$.

It thus remains to prove the connectedness of $Symp_c(U)$.

Let us first recall the following result of Evans (Theorem 1.6 in [15]):

Theorem 4.2.12. *If* $\mathbb{C}^* \times \mathbb{C}$ *is equipped with the standard (product) symplectic form* ω_{std} *then* $Symp_c(\mathbb{C}^* \times \mathbb{C})$ *is weakly contractible.*

This is relevant since Evans observed in section 4.2 in his thesis [14] that, if (ω, J_0) is Kähler with ω monotone and C holomorphic, then (U, J_0) has a finite type Stein structure f with $\omega|_U = -dd^c f$, and there is a biholomorphism Ψ from (U, J_0) to $\mathbb{C}^* \times \mathbb{C}$ (In addition, Ψ satisfies $\Psi^* \omega_{std} = \omega|_U$). We will generalize and prove this ovservation in non-monotone case in Proposition 4.2.14.

Let us also recall the next result of Evans (Proposition 2.2 in [15]):

Proposition 4.2.13. *If* (W, J_0) *is a complex manifold with two finite type Stein structures* ϕ_1 *and* ϕ_2*, then* $Symp_c(W, -dd^c\phi_1)$ *and* $Symp_c(W, -dd^c\phi_2)$ *are weakly homotopy equivalent.*

Now we complete our proof of the connectedness of $Symp_h(\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}, \omega)$ for an arbitrary ω by proving the following

Proposition 4.2.14. $Symp_c(U, \omega|_U)$ *is weakly contractible.*

Proof. We first choose a specific configuration C convenient for our purpose (as we explained in the paragraph below Figure 4.2 this does not affect our result). According to [22] Proposition 4.8, we can always pick an integrable complex structure J_0 compatible with ω, so that (X, J_0) is biholomorphic to a generic blow up of 4 points on $\mathbb{C}P^2$ (the genericity here means that no 3 points lies on the same line, and indeed this can always be done for less than 9 point blow ups). For such a generic holomorphic blow up, there is a unique smooth rational curve in each class in the homology type of C. Thus we canonically obtain a configuration C associated to J_0. Observe that the complement $U = X \setminus C$ is biholomorphic to $\mathbb{C}^* \times \mathbb{C}$. That is because the configuration C is the total transformation of two lines blowing up at four points. Removing C gives us a biholomorphism from (U, J_0) to $\mathbb{C}P^2$ with two lines removed, which is $\mathbb{C}^* \times \mathbb{C}$.

Now we construct a Stein structure ϕ on (U, J_0) with $-dd^c\phi = \omega|_U$, whenever ω is a rational symplectic form on $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$. Since (U, J_0) is biholomorphic to $\mathbb{C}^* \times \mathbb{C}$ equipped with the standard finite type Stein structure $(J_{std}, \omega_{std} = -dd^c|z|^2)$, we can then apply Proposition 4.2.13 and Theorem 4.2.12 in this case to conclude the weak contractibility of $Symp_c(U, \omega|_U)$.

So we assume that $[\omega] \in H^2(X;\mathbb{Q})$. Up to rescaling, we can write $PD([l\omega]) = aH - b_1E_1 - b_2E_2 - b_3E_3 - b_4E_4$ with $a, b_i \in \mathbb{Z}^{\geq 0}$. Further, we assume $b_1 \geq b_2, b_3 \geq b_4$. Since $H - E_1 - E_3$ is an exceptional class we also have $\omega(H - E_1 - E_3) > 0$. This means that $a > b_1 + b_3$, namely, $2a \geq 2b_1 + 2b_3 + 2$. Rewrite

$$PD([2l\omega]) = (2b_1+1)(H-E_1-E_2)+E_1+(2b_1-2b_2+1)E_2+(2a-2b_1-1)(H-E_3-E_4)$$

$$+(2a-1-2b_1-2b_3)E_3 + (2a-1-2b_1-b_4)E_4.$$

Notice that the coefficients are all in $\mathbb{Z}^{>0}$. In this way we represent $PD([2l\omega])$ as a positive integral combination of all elements in the set $\{H - E_1 - E_2, H - E_3 - E_4, E_1, E_2, E_3, E_4\}$, which is the homology type of C.

Denote the symplectic sphere with homology class E_i in C by C_{E_i}, and similarly for the two remaining spheres. Notice that each sphere is a smooth divisor. Consider the effective divisor

$$F = (2b_1 + 1)C_{H-E_1-E_2} + C_{E_1} + (2b_1 - 2b_2 + 1)C_{E_2} + (2a - 2b_1 - 1)C_{H-E_3-E_4}$$

$$+(2a - 1 - 2b_1 - 2b_3)C_{E_3} + (2a - 1 - 2b_1 - b_4)C_{E_4}.$$

There is a holomorphic line bundle $\mathcal{L}$ with a holomorphic section s whose zero divisor is exactly F. Notice that

$$c_1(\mathcal{L}) = [F] = [2l\omega].$$

By [1] section 1.2, we can take an hermitian metric $|\cdot|$ and a compatible connection on $\mathcal{L}$ such that the curvature form is just $2l\omega$. Moreover, for the holomorphic section s, the fuction $\phi = -log|s|^2$ is plurisubharmonic on the complement U with $-d(d\phi \circ J_0) = 2l\omega$. Notice that F and C have the same support so the complement of F is the same as U. Thus we have endowed (U, J_0) with a finite type Stein structure ϕ.

As argued above, this implies that $Symp_c(U, \omega|_U) = Symp_c(U, 2l\omega|_U)$ is weakly contractible when $[\omega] \in H^2(X, \mathbb{Q})$ by the biholomorphism from (U, J_0) to $(\mathcal{C}^* \times \mathcal{C}, J_{std})$.

Finally, suppose ω is not rational, but we assume $\omega(H) \in \mathbb{Q}$ without loss of generality by rescaling. We take a base point $\varphi_0 \in Symp_c(U, \omega|_U)$, and a $S^n (n \geq 0)$ family of symplectomorphisms determined by a based map $\iota : S^n \to Symp_c(U, \omega'|_U)$. Denote the union of support of this S^n family by V_ι, which is a compact subset of U.

Note the following fact:

Claim 4.2.15. *There exists an ω' symplectic on X such that:*

1. $[\omega'] \in H^2(X, \mathbb{Q})$,
2. $[\omega'](E_i) \geq [\omega](E_i), [\omega'](H) = [\omega](H)$
3. *the configuration C is $\omega'-$ symplectic*
4. $(X \setminus C, \omega') \hookrightarrow (X \setminus C, \omega)$ *in such a way that the image contains V_ι.*

Proof. Recall that to blow up an embeded ball B in a symplectic manifold (M, ω), one removes the ball and collapses the boundary by Hopf fibration which incurs an exceptional divisor. The reverse of this procedure is a blowdown.

Now take E_i in the configuration C and blow them down to get a disjoint union of balls B_i in the blown-down manifold, which is a symplectic $\mathbb{C}P^2$ with line area equal $\omega(H)$. One then enlarge B_i by a very small amount to B_i' so that the sizes of B_i' become rational numbers. After the enlargement, blow up B_i'. This produces a symplectic form on X which clearly satisfies (1) and (2). (3) can be achieved as long as the enlarged ball has boundary intersecting proper transformation of S_{12} and S_{34} on a big circle. This is always possible: perturb S_{12} and S_{34} slightly so that they are symplectically orthogonal to E_i before blow-down. Then in a neigbhorhoold of the resulting balls B_i, one has a Darboux chart where B_i is the standard ball, while the portion of S_{12} and S_{34} inside this chart is the $x_1 - x_2$ plane. This is guaranteed by symplectic neighborhood theorem near E_i. Hence the (3) is obtained when the enlargement stays inside the Darboux chart. For more details one is referred to [38].

To see (4), we note that from the above description, $(X \setminus C, \omega')$ is symplectomorphic to the complement of $\bigcup_i B_i'$ union two lines (the proper transforms of S_{12} and S_{34}) in the symplectic $\mathbb{C}P^2$ from blowing down. The same thus applies to $(X \setminus C, \omega)$, while B_i' are replaced by $B_i \subset B_i'$. Therefore, the statement regarding embedding holds in (4). Since V_ι is compact and embeds in $(X \setminus C, \omega)$, as long as the amount of enlargement from B_i to B_i' is small enough, the embedded image contains V_ι as claimed. $\square$

Therefore we can find an isotopy in $Symp_c(U, \omega'|_U) \hookrightarrow Symp_c(U, \omega|_U)$, from the S^n family of maps to the base point φ_0 by the proved case when ω is rational. We emphasize in the above proof, the choice of ω' depends on ι, but this is irrelevant for our purpose. This concludes that for arbitrary symplectic form ω on X, $Symp_c(U, \omega|_U)$ is weakly contractible and hence $Symp_h(\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2})$ is connected for any symplectic form.

□

4.3 The fundamental group of $Symp(X, \omega)$ when $\chi(X) \leq 7$

We apply Corollary 3.4.7, which provides topological information for $\mathcal{J}_{open}$, together with a very useful fact in Lemma 4.3.4 to study the fundamental group of $Symp(X, \omega)$.

4.3.1 Proof of Theorem 1.2.6

We can work out the homotopy type of $\mathcal{G}(C)$ and $Symp(C)$, and we are particularly interested in these cases:

Proposition 4.3.1. *For $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k = 2, 3, 4$, $Stab(C)$ is independent of the given symplectic form. In particular, we know the weak homotopy type of $Stab(C)$:*

- *For $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$, $Stab(C) \simeq \mathbb{T}^2$.*
- *For $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$, $Stab(C) \simeq \mathbb{T}^2$.*
- *For $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$, $Stab(C) \simeq \star$.*

Proof. The monotone case for $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k = 3, 4$ is computed in [15]. And for monotone $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$, $Stab(C) = \mathbb{T}^2$.

For the general case we consider the following portion of fibration:

$$\begin{array}{ccccccc} Symp_c(U) & \longrightarrow & Stab^1(C) & \longrightarrow & Stab^0(C) & \longrightarrow & Stab(C) \\ & & & & \downarrow & & \downarrow \\ & & & & \mathcal{G}(C) & & Symp(C). \end{array} \tag{4.12}$$

In [31], we show that for the given configuration in the above cases, is weakly homotopic to a point:

$$\star \cong Symp_c(U) \cong Stab^1$$

. And as in Lemma 4.4 and 4.5, the homotopy type of $\mathcal{G}(C), Symp(C)$ are the same as the monotone case. With the computation of $\mathcal{G}(C), Symp(C)$ given in equations 4.4and 4.2, we have the three fibrations for $k = 1, 2, 3$ respectively:

$$\mathbb{Z} \to Stab(C) \to (S^1)^3,$$

$$\mathbb{Z}^3 \to Stab(C) \to (S^1)^5,$$

$$\mathbb{Z}^4 \to Stab(C) \to (S^1)^4,$$

And we need to consider the connecting homomorphism $\pi_1(Symp(C)) \to \pi_0(Stab^0(C))$.

To do this we consider the composition map

$$\bar{\psi} = \iota \circ \psi : \pi_1(Symp(C)) \to \pi_0(Stab^0(C)) \to \pi_0(\mathcal{G}(C)).$$

And by Lemma 2.9 in [31], the composition map is surjective and hence the connecting homomorphism $\pi_1(Symp(C)) \to \pi_0(Stab^0(C))$ is surjective.

And hence we finished the computation of weak homotopy type of $Stab(C)$ in each case: $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, for $k = 2, 3, 4$ $Stab(C) \cong \mathbb{T}^2, \mathbb{T}^2, \star$ respectively. □

Remark 4.3.2. For a symplectic rational 4 manifold X with Euler number $\chi(X) < 4$, the same computation is given in Lemma 4.3.11, where the proof is the same as here and much easier.

Remark 4.3.3. For the $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$ case, we will show in Proposition 5.1.4 that diagram 1.1 is a homotopy fibration, and give the description of $Stab(C)$, generalizing the result of [15] in the monotone case.

The non-trivial fact leads to the final computation of $\pi_1(Symp_h(X, \omega)$:

Lemma 4.3.4. *And in the cases as the previous lemma, $\pi_1(Symp_h(X,\omega), \pi_1(\mathcal{C}_0)$ are both free Abelian groups. And the rank of $\pi_1(\mathcal{C}_0)$ equals $N =$ the number of -2 symplectic sphere classes. In addition we have the exact sequence*

$$0 \to \pi_1(Stab(C)) \to \pi_1(Symp_h) \to \mathbb{Z}^N \to 0.$$

Proof. We analyze the right end of the diagram to prove the second statement: For 4-point blow up, by proposition 4.3.1, we have

$$Stab(C) \simeq \star \longrightarrow Symp_h \longrightarrow \mathcal{C}_0. \tag{4.13}$$

Since in this fibration the fiber is weakly contractible, the base $\mathcal{C}_0$ is weakly equivalent to the total space $Symp_h$. And hence $\pi_1(\mathcal{C}_0) \cong \pi_1(Symp_h)$. Since $Symp_h$ is a Lie group, $\pi_1(Symp_h)$ is Abelian and hence $\pi_1(\mathcal{C}_0)$ is an Abelian group. Then $\pi_1(\mathcal{C}_0) = H^1(\mathcal{C}_0)$, which is free Abelian and whose number of generators equals the number of -2 symplectic spheres by Theorem 3.4.6 and Lemma 3.1.14. For $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}, k = 2, 3$, by proposition 4.3.1, the right end becomes

$$Stab(C) \simeq \mathbb{T}^2 \longrightarrow Symp_h \longrightarrow \mathcal{C}_0. \tag{4.14}$$

And we write down the homotopy exact sequence

$$\mathbb{Z}^2 \to \pi_1(Symp_h) \to \pi_1(\mathcal{C}_0) \to 1.$$

As a fundamental group of a topological group, $\pi_1(Symp_h)$ is Abelian, And $\pi_1(\mathcal{C}_0)$ must also be Abelian because it is the surjective image of an Abelian group. Note that let X be $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}, k = 2, 3$, it admits a torus action for any given symplectic form. And the homology classes of configuration C can be realized as the boundary of the moment polytope Now take a toric divisor C, then we have a torus action T on X fixing C, i.e. $T \subset Stab(C)$. There is a inclusion map $T \to Symp_h(X,\omega)$, and theorem 1.3 and theorem 1.25 in [49] shows that the induced map on fundamental group $\iota\pi_1(T) = \mathbb{Z}^2 \to \pi_1(Symp_h)$ is an injection. Now observe that this inclusion actually factor through $Stab(C)$. Namely, we have the composition

$$T \to Stab(C) \to Symp_h(X, \omega),$$

where the first map is the inclusion of T into $Stab(C)$ and the second map is the inclusion of isotropy $Stab(C)$ at C into $Symp_h(X, \omega)$. Consider the induced map on fundamental group of the composition:

$$\mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^2 \xrightarrow{g} \pi_1(Symp_h)$$

We have shown that $\iota = g \circ f$ is injective, which means $g(Im(f))$ is a rank 2 free abelian group. Indeed, the image of a free abelian group is either itself or has less free rank. Namely, suppose for $a \neq b$, $g(a) = g(b)$, then we have $g(a-b) = 0$ and hence $Im(g)$ has free rank less than 2. This is contradictory against the fact that $g(Im(f))$ has rank 2. And it follows that the map $Stab(C) \to Symp_h(X, \omega)$ induces injective map of the left arrow $\mathbb{Z}^2 \to \pi_1(Symp_h)$ of the homotopy exact sequence.

In summary we have the following short exact sequence of groups:

$$0 \to \pi_1(Stab(C)) \to \pi_1(Symp_h) \to \mathbb{Z}^N \to 0.$$

□

Hence we understand the fundamental group of $Symp_h(X, \omega)$ in the following theorem:

Theorem 4.3.5. *If (X, ω) is a symplectic rational 4 manifold with Euler number $\chi(X) \leq 7$, and $N = rank(H^1(\mathcal{J}_{open}))$ equals the number of -2 $\omega-$symplectic spheres ω, then*

$$\pi_1(Symp(X, \omega)) = \mathbb{Z}^N \oplus \pi_1(Symp(X, \omega_{mon})).$$

Proof. We first deal with cases $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k = 2, 3, 4,$: By Lemma 4.3.4, $\pi_1(Symp(X, \omega) = \pi_1(Stab(C)) \oplus H^1(\mathcal{J}_{open})$. Corollary 3.4.7 shows that $H^1(\mathcal{J}_{open}) = \oplus_{A_i} H^0(\mathcal{J}_{A_i})$, where each A_i is a symplectic -2 class. Corollary 3.4.3 shows that for $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k = 2, 3, 4$, the space of -2 symplectic sphere in a fixed homology class is connected, hence $H^0(\mathcal{J}_{A_i}) = \mathbb{Z}$ for any A_i. And on the other hand, $\pi_0(Stab(C))$

is trivial, hence, the map $\pi_1(Symp_h) \to \pi_1(\mathcal{C}_0)$ is surjective. Hence we show

$$\pi_1(Symp(X,\omega) = \pi_1(Stab(C)) \oplus \mathbb{Z}^N,$$

where N equals the number of -2 symplectic spheres. And hence the rank of $\pi_1(Symp_h(X,\omega)$ equals the rank of $Stab(C)$ plus the number of -2 symplectic spheres.

And further, in the monotone case, the space $\mathcal{C}_0$ is contractible, and $(Symp(X,\omega_{mon}) \simeq Stab(C)$. Hence

$$\pi_1(Symp(X,\omega)) = \mathbb{Z}^N \oplus \pi_1(Symp(X,\omega_{mon})).$$

For the cases $\mathbb{C}P^2, S^2\times S^2, \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, the above results directly follows from the computation in [20], [2], and [4]: For monotone $\mathbb{C}P^2, S^2\times S^2, \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, the $Symp_h(X,\omega)$ is weakly homotopic equivalent to $PU(3), SO(3)\times SO(3), U(2)$ respectively. In particular,

$$\pi_1(Symp_h(S^2\times S^2),\omega_{mono}) = \mathbb{Z}_2\oplus\mathbb{Z}_2; \quad \pi_1(Symp_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}),\omega_{mono}) = \mathbb{Z}.$$

And for non-monotone form of the latter 2 cases, Corollary 2.7 in [4] shows that

$$\pi_1(Symp_h(S^2\times S^2),\omega) = \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2; \pi_1(Symp_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}),\omega) = \mathbb{Z}.$$

This verifies our assentation here. And we can also give a discussion about the cases with small Euler number in the section 4.3.2. □

Combine the results in [35] and [31], let X be a symplectic rational 4 manifold with Euler number $\chi(X)\leq 7$, $Symp_h(X,\omega)$ is connected, then the homological action of $Symp(X,\omega)$ is generated by Dehn twist along Lagrangian -2 spheres. Hence:

Corollary 4.3.6. *The homological action, $\pi_0(Symp(X,\omega) = \Gamma(X,\omega)$, which is a finite Coxeter group generated by reflection along -2 Lagrangian spheres. $\Gamma(X,\omega)$ is the subgroup of the Coxeter group corresponding to the root system of Lagrangians in the manifold X as in section 2.1.5.*

By considering the chamber structure of the symplectic cone for each case, we have the following corollary:

Corollary 4.3.7. *Let X be a rational 4 manifold with Euler number $\chi(X) \leq 7$, with a given symplectic form ω. We have the following quantity*

$$Q = PR[\pi_0(Symp_h(X,\omega))] + Rank[\pi_1(Symp(X,\omega))] - rank[\pi_0(Symp_h(X,\omega))],$$

which is a constant only depends on the topological type of X. Here $PR[\pi_0(Symp(X,\omega)]$ is the number of positive roots of the reflection group $\pi_0(Symp(X,\omega)$ which usually equals the number of Lagrangian -2 spheres, and $Rank[\pi_1(Symp(X,\omega))]$ denote the number of generator of the abelian group $\pi_1(Symp(X,\omega))$.

Proof. One can verify the corollary directly from the above-mentioned computation in the case $\mathbb{C}P^2$, $S^2 \times S^2$.

And for the other rational manifolds $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k = 1,2,3,4$ the group $\pi_1(Symp_h(X,\omega)$ is free abelian. And this corollary follows from the fact that for a symplectic rational 4 manifold whose Euler number is small(less than 12), the square -2 sphere classes is a set of positive roots of certain simple laced root system. And for a given X , when deforming the symplectic form in the symplectic cone, the set of Lagrangian -2 spheres is a set of positive roots of a subsystem, which generates $\pi_0(Symp(X,\omega)$. And the set of symplectic -2 sphere classes, which generates $\pi_1(Symp(X,\omega)$, is a set a set of positive roots of the system that is complementary to the Lagrangian system.

□

And we list the number of this constant in table 4.1: for 1,2,3,4 points blow up of $\mathbb{C}P^2$, Q is constant of any form. And for 5 blow up, in most circumstances, Q is a constant 15 as in Corollary 5.5.11, and we further conjecture that this holds for any form, see Conjecture 5.5.13.

Remark 4.3.8. Also as noticed in [28], the generator of homological action of diffeomorphism group of rational manifolds can be realized as Coxeter reflections. And as shown in [56] Theorem 4', the group is a Coxeter reflection group whose Dynkin diagram is a subgraph of the root system of the manifold.

Remark 4.3.9. One can compare the table 4.1 with the upper bound given by McDuff in [42] Corollary 6.6 when $X = \mathbb{C}P^2$. And we will see from the next section that the

M	Q
$\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$	1
$\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$	3
$\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$	6
$\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$	10
$\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$	15 ?

Table 4.1: The quantity Q on the persistence of $Symp(X,\omega)$

upper-bounds she gave there can be realized when one blow up $X = \mathbb{C}P^2$ for most occasions.

4.3.2 Discussion in each case

We explicitly compute $\pi_1(Symp_h(X,\omega)$ for Theorem 1.2.6 in table 4.2,4.3,4.4. Through out this section, N denotes the number of Symplectic -2 sphere classes of a given form for a rational 4 manifold; Γ_L denotes the Lagrangian lattice of a wall, which is the same as $\Gamma(X,\omega)$ when ω is on the wall. And $\Gamma(X,\omega)$ is the homological action of the Symplectic mapping class group $\pi_0(Symp(X,\omega))$ on $H_2(X)$ as in Theorem 4.3.6.

The case of $\mathbb{C}\mathbf{P}^2\#2\overline{\mathbb{C}\mathbf{P}^2}$

For $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$, with any symplectic form, rank of $\pi_1(Symp(X,\omega))$ equals 2 plus the number of -2 Symplectic spheres, while $\pi_0(Symp(X,\omega)$ is a Coxeter group of a sublattice of $\mathbb{A}_1$.

We can summarize the above in table 4.2.

k-Face	Γ_L	N	$\pi_1(Symp_h(X,\omega)$	ω area
OB	$\mathbb{A}_1$	0	$\mathbb{Z}^2$	$c_1 = c_2$
ΔBOA	trivial	1	$\mathbb{Z}^3$	$c_1 \neq c_2$

Table 4.2: Γ_L and $\pi_1(Symp_h(\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}))$

The case of $\mathbb{C}\mathbf{P}^2\#3\overline{\mathbb{C}\mathbf{P}^2}$

For $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$, with any symplectic form, $Rank[\pi_1(Symp(X,\omega))]$ equals 2 plus the number of -2 Symplectic spheres, while $\pi_0(Symp(X,\omega)$ is a Coxeter group of a sublattice of $\mathbb{A}_1 \times \mathbb{A}_2$.

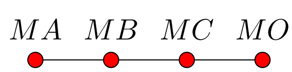
The Weyl arrangement of $\mathbb{E}_3 = \mathbb{A}_2 \times \mathbb{A}_1$ is illustrated using the picture 2.1. And we can fill the table 2.1 with Γ_L and $\pi_1(Symp(X,\omega))$ such that it becomes table 4.3.

k-Face	Γ_L	N	$\pi_1(Symp_h(X,\omega)$	$\omega-$area
Point M	$\mathbb{A}_1 \times \mathbb{A}_2$	0	$\mathbb{Z}^2$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$: monotone
Edge MO:	$\mathbb{A}_2$	1	$\mathbb{Z}^3$	$\lambda < 1; c_1 = c_2 = c_3$
Edge MA:	$\mathbb{A}_1 \times \mathbb{A}_1$	2	$\mathbb{Z}^4$	$\lambda = 1; c_1 > c_2 = c_3$
Edge MB:	$\mathbb{A}_1 \times \mathbb{A}_1$	2	$\mathbb{Z}^4$	$\lambda = 1; c_1 = c_2 > c_3$
ΔMOA:	$\mathbb{A}_1$	3	$\mathbb{Z}^5$	$\lambda < 1; c_1 > c_2 = c_3$
ΔMOB:	$\mathbb{A}_1$	3	$\mathbb{Z}^5$	$\lambda < 1; c_1 = c_2 > c_3$
ΔMAB:	$\mathbb{A}_1$	3	$\mathbb{Z}^5$	$\lambda = 1; c_1 > c_2 > c_3$
T_{MOAB}:	trivial	4	$\mathbb{Z}^6$	$\lambda < 1; c_1 > c_2 > c_3$

Table 4.3: Γ_L and $\pi_1(Symp_h(X,\omega)$ for $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$

The case of $\mathbb{C}\mathbf{P^2}\#4\overline{\mathbb{C}\mathbf{P^2}}$

For $\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$, as described in section 2.1.5: Combinatorially, the normalized reduced cone is convexly generated by 4 rays $\{MO, MA, MB, MC\}$, with 4 vertices $M = (\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3})$, $O = (0,0,0,0)$, $A = (1,0,0,0)$, $B = (\frac{1}{2},\frac{1}{2},0,0)$, $C = (\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$; and these root edges corresponding to Lagrangian simple roos as follows, $MO = H - E_1 - E_2 - E_3$, $MA = E_1 - E_2$, $MB = E_2 - E_3$, $MC = E_3 - E_4$,

$\mathbb{A}_4$ 

The open chamber in this case is a 4-dimensional polytope with the tetrahedron in 2.1 of the $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$ being a facet. $\pi_1(Symp_h(\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}))$ has 10 generators and the homology action is trivial. A wall of codimension k is the interior of a facet of the closure of open chamber, where k number of ">" made into "=". And the Lagrangian lattice of the wall W Γ_L is given by removing the generating rays of the wall W. Specifically, the walls are listed in the table 4.4.

Remark 4.3.10. In [7], the computation of $\pi_1(Symp_h(X,\omega)$ for any given form on 3-fold blow up of $\mathbb{C}P^2$ is given. There the strategy is counting torus (or circle) actions.

K-face	Γ_L	N	$\pi_1(Symp_h(X,\omega)$	ω area
Point M	$\mathbb{A}_4$	0	trivial	monotone, $\lambda = 1; c_1 = c_2 = c_3 = c_4$
MO	$\mathbb{A}_3$	4	$\mathbb{Z}^4$	$\lambda < 1; c_1 = c_2 = c_3 = c_4$
MA	$\mathbb{A}_3$	4	$\mathbb{Z}^4$	$\lambda = 1; c_1 > c_2 = c_3 = c_4$
MB	$\mathbb{A}_1 \times \mathbb{A}_2$	6	$\mathbb{Z}^6$	$\lambda = 1; c_1 = c_2 > c_3 = c_4$
MC	$\mathbb{A}_1 \times \mathbb{A}_2$	6	$\mathbb{Z}^6$	$\lambda = 1; c_1 = c_2 = c_3 > c_4$
MOA	$\mathbb{A}_2$	7	$\mathbb{Z}^7$	$\lambda < 1; c_1 > c_2 = c_3 = c_4$
MOB	$\mathbb{A}_1 \times \mathbb{A}_1$	8	$\mathbb{Z}^8$	$\lambda < 1; c_1 = c_2 > c_3 = c_4$
MOC	$\mathbb{A}_2$	7	$\mathbb{Z}^7$	$\lambda < 1; c_1 = c_2 = c_3 > c_4$
MAB	$\mathbb{A}_2$	7	$\mathbb{Z}^7$	$\lambda = 1; c_1 > c_2 > c_3 = c_4$
MAC	$\mathbb{A}_1 \times \mathbb{A}_1$	8	$\mathbb{Z}^7$	$\lambda = 1; c_1 > c_2 = c_3 > c_4$
MBC	$\mathbb{A}_1 \times \mathbb{A}_1$	8	$\mathbb{Z}^7$	$\lambda = 1; c_1 = c_2 > c_3 > c_4$
MOAB	$\mathbb{A}_1$	9	$\mathbb{Z}^8$	$\lambda < 1; c_1 > c_2 > c_3 = c_4$
MOAC	$\mathbb{A}_1$	9	$\mathbb{Z}^9$	$\lambda < 1; c_1 > c_2 = c_3 > c_4$
MOBC	$\mathbb{A}_1$	9	$\mathbb{Z}^9$	$\lambda < 1; c_1 = c_2 > c_3 > c_4$
MABC	$\mathbb{A}_1$	9	$\mathbb{Z}^9$	$\lambda = 1; c_1 > c_2 > c_3 > c_4$
MOABC	trivial	10	$\mathbb{Z}^{10}$	$\lambda < 1; c_1 > c_2 > c_3 > c_4$

Table 4.4: Γ_L and $\pi_1(Symp_h(X,\omega)$ for $\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$

And a generating set of $\pi_1(Symp_h(X,\omega)$ is given using circle action. Note that our approach gives another (minimal) set of $\pi_1(Symp_h(X,\omega)$. We give the correspondence of the two generating sets: By Remark 3.4.5, any -2 symplectic sphere in 3 fold blow up of $\mathbb{C}P^2$, there is a semi-free circle τ action having this -2 symplectic sphere as fixing locus, where τ is a generator of $\pi_1(Symp_h(X,\omega)$.

And if a rational 4 manifold X with Euler number $\chi(X) < 7$, then it is toric. We discuss the relation between our approach and counting torus or circle action in the next section 4.3.2:

The case of The cases $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ and counting torus or circle actions

From previous results [2],[4],[26], we know that

$$K = PR[\pi_0(Symp(X,\omega)] + Rank[\pi_1(Symp(X,\omega))]$$

is a constant for the cases mentioned in this section. We neither claim any originality nor provide any new results for these cases. Instead, we use our strategy can give a uniform description of this phenomenon. Specifically, we will show how one obtain a generating set of $\pi_1(Symp_h)$ using generators of $Stab(C)$ and $\pi_1(\mathcal{J}_{open})$.

In these cases, we need symplectic spheres with square 0 or 1. Note that in [31] for cases when $\chi(X) < 7$ we can choose an appropriate configuration such that it has a complement U whose compactly supported symplectomorphism group $Symp_c(U)$ is contractible. And the homotopy type of $Stab(C)$ in the monotone cases is computed in [20, 2, 4, 26].

Proof similar as proposition 4.3.1, it is easy to check the following theorem for those cases:

Lemma 4.3.11. *When $\chi(X) < 7$ with the configuration given below, the homotopy type of $Stab(C)$ for the non-monotone case is the same as the monotone case.*

Hence we can summarize the fibration $Stab(C) \to Symp_h \to \mathcal{C}$ for the cases below with any given form:

- *For $\mathbb{C}P^2$, $C = \{H\}$, the fibration is*

$$Stab(C) \simeq U(2) \to Symp_h \to \mathcal{C} \simeq \mathbb{C}P^2.$$

- *For $S^2 \times S^2$, $C = \{B, F\}$, the fibration is*

$$\mathbb{T}^2 \simeq Stab(C) \to Symp_h \to S^2 \times S^2 \times \mathcal{J}_{open}.$$

- *For $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $C = \{H, H-E, p \in H-E\}$, where p is a marked point on $H-E$, the fibration is*

$$Stab(C) \simeq S^1 \to Symp_h \to S^2 \times \mathcal{J}_{open}.$$

Note that the base $\mathcal{C} \simeq \mathbb{C}P^2$ comes from the fact that there is a fibration $\mathbb{C}P^2 \to \mathcal{M}(H, \mathcal{J}) \to \mathcal{J}$, where $\mathcal{M}(H, \mathcal{J})$ is the universal moduli of stable curve in class H.

In the following cases, we will use our approach to give a description how one can combine the generator of $Stab(C)$ and $\pi_1(\mathcal{J}_{open})$ to obtain a generating set of $\pi_1(Symp_h)$.

The argument in Lemma 4.3.4 can not directly apply since the torus action induced map on the fundamental group might be non-injective. Hence we deal with them separately:

- For $\mathbb{C}P^2$, the $Symp(X,\omega)$ is homotopic to $PU(3)$ and its fundamental group is $\mathbb{Z}_3$, which has one generator. We take the configuration to be $[z_0, 0, 0]$ and take the circle action to be $[z_0 : z_1 : z_2] \to [z_0 : t \cdot z_1 : t \cdot z_2]$. It is not hard to see this action is semi-free (note that if we change the weights of the action, it might be not semi-free, e.g. $[z_0 : z_1 : z_2] \to [z_0 : t \cdot z_1 : t^2 \cdot z_2]$ has isotropy $\mathbb{Z}_2$ at point $[1 : 0 : 1]$). And by Corollary 1.5 of [47], this action maps to a nontrivial element in $\pi_1(Symp_h)$. Further, this circle can be naturally included into $Stab(C)$ as a nontrivial loop. This means the image of the map $\pi_1(Stab(C)) = \mathbb{Z} \to \pi_1(Symp_h)$ contains the generator of $\pi_1(Symp_h)$.

- For $S^2 \times S^2$: Take a toric divisor $C = \{B, F\}$, take two circle actions C_B, C_F generated by the two factors of the torus, fixing spheres B and F respectively. Every effective circle action on $S^2 \times S^2$ is semi-free, and hence by Corollary 1.5 of [47], the inclusion of C_B, C_F each maps to non-trivial element in $\pi_1(Symp_h)$. In addition, by inclusion, we can map C_B, C_F to a non-trivial loop in the first and second factor in $\pi_1(Stab(C))$ respectively. (This is because the Seidel representation of the images of C_B, C_F are different in the quantum homology ring: $\mathcal{S}(C_B) = B \otimes qt^{\mu/2}, \mathcal{S}(C_F) = B \otimes qt^{1/2}$, see [48] Example 5.7. This means the image of the left arrow in the homotopy exact sequence

$$\mathbb{Z} \oplus \mathbb{Z} \to \pi_1(Symp_h) \to \pi_1(S^2 \times S^2 \times \mathcal{J}_{open}) \to 1$$

  has two generators in $\pi_1(Symp_h)$. On the other hand, $\pi_0(Symp_h(X,\omega))$ in Corollary 1.2.7 and the above cases are trivial, which means the right arrow is surjective.

- For $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, take a toric divisor $C = \{E, H-E\}$, If we choose the corresponding S^1 action with fixing locus E or $H - E$, it is clear that this action is semi-free, i.e. the isotropy is either trivial or the whole S^1. We denote them by C_E, C_H respectively. And by a theorem of [47], in the left arrows in the homotopy exact sequence

$$\mathbb{Z} \oplus \mathbb{Z} \to \pi_1(Symp_h) \to \pi_1(S^2 \times \mathcal{J}_{open}) \to 1,$$

the inclusion of C_E, C_H each maps to non-trivial element in $\pi_1(Symp_h)$. In addition, by inclusion, we can map C_E, C_H to a non-trivial loop in the first and second factor in $\pi_1(Stab(C))$ respectively. (This is because the Seidel representation of the images of C_E, C_H are different in the quantum homology ring: $\mathcal{S}(C_E) = B \otimes qt^{c_1}, \mathcal{S}(C_H) = B \otimes qt^{2c_1-1}$, see [48] Example 5.6. This means the image of the left arrow in the

$$\mathbb{Z} \oplus \mathbb{Z} \to \pi_1(Symp_h) \to \pi_1(S^2 \times \mathcal{J}_{open}) \to 1$$

is nonempty by the same argument as above. Also, [4] Corollary 2.7 shows that this generator has infinite order and hence the left arrow is injective. And the triviality of $\pi_0(Symp_h(X, \omega))$ apply here showing that the right arrow is surjective. And hence Lemma 4.3.4 still holds for these cases.

Chapter 5

Rational surfaces with Euler number grater or equal to 8

This Chapter is about the topology of $Symp_h(X,\omega)$ when X is a symplectic rational surface which is diffeomrophic to $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}, k \geq 5$. When $k=5$, we had a complete result except a 1-dimension family of form which is the equal blow up of the monotone Hirzburch surface.

5.1 Symplectic -2 spheres and $Symp(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2},\omega)$

In this section, we study the low-rank homotopy groups of $Symp_h(X,\omega)$, where X is $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ and ω is an arbitrary symplectic form. The π_0 is particularly interesting: In [55] and [15] for a monotone symplectic form, π_0 of $Symp_h(X,\omega_{mon})$ is shown to be $\pi_0\mathrm{Diff}^+(S^2,5)$, which is an infinite discrete group generated by square Lagrangian Dehn twists. In contrast, Dusa McDuff pointed out in [42] that for a certain symplectic form such that the blow-up size is small and there's no Lagrangian sphere, the group $Symp_h(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2},\omega)$ is connected, see Remark 1.11 in [42] and Remark 5.4.5 for details. We could now give a complete description of the symplectic mapping class group and discover the "forgetting strands" phenomena in Torelli SMC: as in the braid group on spheres when deforming the symplectic form:

Recall that the normalized reduced symplectic cone of $X=\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ is a 5-dimension Polyhedron P^5 with the monotone form as a vertex. And it is convexly spanned by five root edges, where each of them corresponds to a simple root in the Lagrangian root system Γ_L.

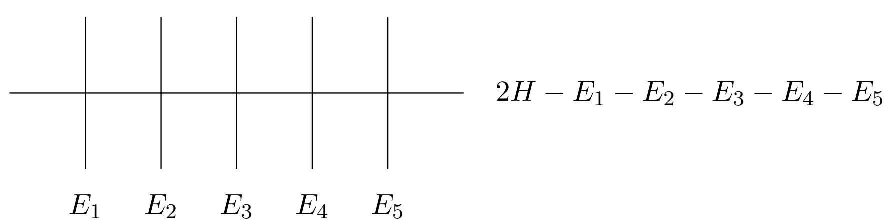
- On the monotone point, [55],[15] shows that $Symp_h(X,\omega_{mon})$ is weakly homotopic to $\text{Diff}^+(S^2,5)$ and hence the TSMC is $\pi_0(\text{Diff}^+(S^2,5))$.
- On a root edge MA where the number of symplectic sphere classes is minimal (8 classes) other than the monotone point, we show that TSMC "forgets one strand" and becomes $\pi_0(\text{Diff}^+(S^2,4))$ in Proposition 5.3.1.
- Further when the form admit more symplectic -2 sphere classes, namely, $\omega \in P^5 \setminus \overline{MA}$ where $\overline{MA}$ is the closure of MA, TSMC behaves like forgetting one more strand and become trivial(because $\pi_0(\text{Diff}^+(S^2,3))$ is trivial) in Proposition 5.4.2.

Further, for rank of fundamental group of $Symp_h(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$, we give a lower bound given by Lemma 5.5.1 and a upper-bound modified from [42] corollary 6.4 and 6.9. We achieve the following: for a reduced symplectic form ω, if $c_i < 1/2$, and TSMC is connected, then rank of $\pi_1(Symp_h(X,\omega)) = N-5$. This further imply the isotopy uniqueness up to symplectomorphism of homologous -2 symplectic spheres.

5.1.1 Basic set-up and pure braid groups on a sphere

The stragety

For $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$, we choose the configuration to be the following:



Identifying with $S^2\times S^2\#4\overline{\mathbb{C}P^2}$ by the base change in equation 2.2, the configuration is $C=\{B+F-E_2-E_3-E_4, B-E_1, F-E_1, E_2, E_3, E_4\}$. Let $\mathcal{C}_0$ denote the space of orthogonal configurations and by Lemma 4.1.1, $\mathcal{C}_0$ is weakly homotopic to $\mathcal{C}$, which is homotopic to $\mathcal{J}_C$. We know from Lemma 3.4.8 that $\mathcal{J}_C = \mathcal{J}_{open}$.

And we will study $Symp(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}, \omega)$ using the strategy described in 1.1 and the following symplectomorphism called **ball-swapping** as in [58]:

Definition 5.1.1. Suppose X is a symplectic manifold. And $\tilde{X}$ a blow up of X at a packing of n balls B_i of volume v_i. Consider the ball packing in X $\iota_0 : \coprod_{i=1}^n B(i) \to X$, with image K. Suppose there is a Hamiltonian family ι_t of X acting on this ball backing K such that $\iota_1(K) = K$ Then ι_1 defines a symplectomorphism on the complement of K in X. From the interpretation of blow-ups in the symplectic category [44], the blow-ups can be represented as

$$\tilde{X} = (X \backslash \iota_j(\coprod_{i=1}^n B_i))/\sim, \text{ for } j = 0, 1.$$

Here the equivalence relation $\sim$ collapses the natural S^1-action on $\partial B_i = S^3$. Hence this symplectomorphism on the complement defines a symplectomorphism on the blow up $\tilde{X}$.

First recall a fact about relative ball packing in $\mathbb{C}P^2$:

Lemma 5.1.2. *For $\mathbb{C}P^2$ with symplectic form ω, where $PD[\omega] = H$, suppose there are positive number $c_1, \cdots, c_5$ such that $max\{c_i\} \leq 1/2, \sum c_i < 2$, then there is a ball packing relative to a given $\mathbb{R}P^2$, denoted by $\iota : \coprod_{i=1}^5 B(i) \to \mathbb{C}P^2$, such that the symplectic area of exceptional curve E_i corresponding to B_i is c_i.*

Proof. By [10] Lemma 5.2, it suffices to pack 5 balls of given sizes c_i into $(S^2 \times S^2, \Omega_{1,\frac{1}{2}})$. Without lose of generality we assume that $c_1 \geq \cdots \geq c_5$. Since blowing up a ball of size c_1 (here by ball size we mean the area of the corresponding exceptional sphere) in $(S^2 \times S^2, \Omega_{1,\frac{1}{2}})$ leads to $(\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2, \omega')$ with ω' dual to the class $(\frac{3}{2} - c_1)H - (1 - c_1)E_1 - (\frac{1}{2} - c_1)E_2$, it suffices to prove that the vector

$$[(\frac{3}{2} - c_1)|(1 - c_1), c_2, c_3, c_4, c_5, (\frac{1}{2} - c_1)]$$

denoting the class

$$(\frac{3}{2}-c_1)H-(1-c_1)E_1-(\frac{1}{2}-c_1)E_6-\sum_{i=2}^{5}c_iE_i$$

is Poincaré dual to a symplectic form for $\mathbb{C}P^2\#6\overline{\mathbb{C}P}^2$.

It is a symplectic form ω', because it pair E_i, $H-E_i-E_j$ and $2H-E_1-\cdots-\check{E}_i-\cdots-E_6$ are positive:

- The minimal value of $\omega(H-E_i-E_j)$ is either

$$(\frac{3}{2}-c_1)-c_2-c_3>0;$$
$$or \quad (\frac{3}{2}-c_1)-c_2-(1-c_1)>0;$$

  this means each $H-E_i-E_j$ has positive area.

- The minimal value of $\omega(2H-E_1-\cdots-\check{E}_i-\cdots-E_6)$ is either

$$2(\frac{3}{2}-c_1)-(1-c_1)-c_2-c_3-c_4-(\frac{1}{2}-c_1)=2-c_2-c_3-c_4-(\frac{1}{2}-c_1)>0;$$

$$or \quad 2(\frac{3}{2}-c_1)-(1-c_1)-c_2-c_3-c_4-c_5=2-c_2-c_3-c_4-c_5>0;$$

  this means $2H-E_1-\cdots-\check{E}_i-\cdots-E_6$ has positive area.

And hence there is a ball packing $\iota:\coprod_{i=1}^{5}B(i)\to\mathbb{C}P^2$ relative to $\mathbb{R}P^2$, such that the symplectic area of exceptional curve E_i corresponding to B_i is c_i.

□

Then we review a fact about symplectomorphism of non compact surfaces:

Lemma 5.1.3. *Let $Symp(S^2,n)$ denote the group of symplectomrophism(indeed area preserving diffeomophism) of the n-punctured sphere. and $Symp(S^2,\coprod_{i=1}^{n}D_i)$ denote* *the group of symplectomrophism of the complement of n disjoint closed disk (with smooth boundary) in S^2. $Symp_0(S^2,n)$ and $Symp_0(S^2,\coprod_{i=1}^n D_i)$ are their identity component respectively. Then $Symp(S^2,n)$ is isomorphic to $Symp(S^2,\coprod_{i=1}^n D_i)$. Further,*

$$Symp(S^2,\coprod_{i=1}^n D_i)/Symp_0(S^2,\coprod_{i=1}^n D_i) = Symp(S^2,n)/Symp_0(S^2,n),$$

both can be identified with $\pi_0 Symp(S^2,n) = \pi_0 Diff^+(S^2,n)$.

Proof. The statements follow from the fact that n-punctured sphere is diffeomorphic to S^2 with n disjoint open disk removed. Indeed one can do this by local polar coordinate centered at the punctures to map the former to the latter. And because both domains (n-punctured sphere and S^2 with n open disk removed) have finite volume, the above diffeomorphism push the symplectic form on the former forward into a positive constant multiple of the form on the latter.

And the last statement is obvious because $Symp(S^2,n)$ is homotopic to $\text{Diff}^+(S^2,n)$ by Moser's theorem, see [8] Chapter 7. □

Now we review the strategy introduced in section 4.1 and verify the following:

Proposition 5.1.4. *For $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ with any symplectic form ω, the diagram 1.1 is a fibration. And if $PD[\omega] := H - c_1E_1 - c_2E_2 - c_3E_3 - c_4E_4 - c_5E_5$, also denoted by vector $(1|c_1,c_2,c_3,c_4,c_5)$, such that $c_i < 1/2, \forall i \in \{1,2,3,4,5\}$, then $Stab(C) \simeq Diff^+(S^2,5)$.*

Proof. Firstly, it suffices to verify that the following is a fibration:

$$\begin{array}{ccccccc} Symp_c(U) & \longrightarrow & Stab^1(C) & \longrightarrow & Stab^0(C) & \longrightarrow & Stab(C) \\ & & & & \downarrow & & \downarrow \\ & & & & \mathcal{G}(C) & & Symp(C). \end{array} \tag{5.1}$$

And indeed we only need to argue the restriction map $Stab(C) \to Symp(C)$ is surjective:

By Lemma 4.2, $Symp(C) = \Pi_i Symp(E_i) \times Symp(Q,5)$, where $Symp(E_i,p_i)$ is the symplectomorphism group of the sphere in class E_i fixing the intersection point $p_i = E_i \cap Q$, and $Symp(Q,5)$ is the symplectomorphism group of the sphere in class in

$2H - E_1 - \cdots E_5$ fixing 5 intersection points. The surjection to the group $Symp(E_i, p_i)$ is clear.

Then we need to prove the restriction map $Stab(C) \to Symp(C)$ is surjective on the factor $Symp(Q, 5)$. Note this means for any given $h^{(2)} \in Symp(Q, 5)$ we need to find a symplectomorphism $h^{(4)} \in Stab(C)$ which fixes the whole configuration C as a set, whose restriction on Q is $h^{(2)}$. To achieve this, we can blow down the exceptional spheres $E_1 \cdots E_5$, and obtain a $\mathbb{C}P^2 \setminus \coprod_{i=1}^5 B(i)$ with a conic S^2 in homology class $2H$ and five disjoint balls $\coprod_{i=1}^5 B(i)$ each centered on this conic and the intersections are 5 disjoint disks on this S^2. Note that by the above identification in Lemma 5.1.3, this blow down process sends $h^{(2)}$ in $Symp(Q, 5)$ to a unique $\overline{h^{(2)}}$ in $Symp(S^2, \coprod_{i=1}^5 D_i)$. It suffice to find a symplectomorphism $\overline{h^{(4)}}$ whose restriction is $\overline{h^{(2)}}$, and fixing the image of balls $\coprod_{i=1}^5 B(i)$. Because blowing the balls $\coprod_{i=1}^5 B(i)$ up and by definition 5.1.1, we obtain a symplectomorphism $h^{(4)} \in Stab(C)$ whose restriction is the given $h^{(2)} \in Symp(Q, 5)$.

Now for a given $h^{(2)} \in Symp(Q, 5)$, we will first consider its counterpart $\overline{h^{(2)}}$ in $Symp(S^2, \coprod_{i=1}^5 D_i)$. One can always find $f^{(4)} \in Symp(\mathbb{C}P^2, \omega)$ whose restriction on S^2 is $\overline{h^{(2)}}$ in $Symp(S^2, \coprod_{i=1}^5 D_i)$. We can construct $f^{(4)}$ using the method as in Lemma 2.5 in [31]: $\overline{h^{(2)}}$ in $Symp(S^2, \coprod_{i=1}^5 D_i)$. is a hamiltonian diffeomorphism on S^2 because S^2 is simply connected. Then we cut off in a symplectic neighborhood of S^2 to define the hamiltonian diffeomorphism $f^{(4)} \in Symp(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}, \omega)$, which fixing the 5 intersection disks $\coprod_{i=1}^5 D_i$. But the blow up of $f^{(4)}$ is not necessarily in $Stab(C)$ because there's no guarantee that $f^{(4)}$ will fix the image of 5 balls $\coprod_{i=1}^5 B(i)$. Then we need another symplectomorphism $g^{(4)} \in Symp(\mathbb{C}P^2, \omega)$ so that $g^{(4)}$ move the the five symplectic balls back to their original position in $\mathbb{C}P^2$. This can be done by connectedness of ball packing relative a divisor (the conic in class $2H$ in our case). Namely, by Lemma 4.3 and Lemma 4.4 in [58], there exists a symplectomorphism $g^{(4)} \in Symp(\mathbb{C}P^2, \omega)$ such that the composition $\overline{h^{(4)}} = g^{(4)} \circ f^{(4)}$ is a symplectomorphism fixing the five balls. And blowing up the 5 balls we obtain an element $h^{(4)}$ in $Stab(C)$, which is a ball swapping symplectomorphism whose restriction on $Symp(C)$ creates the group $Symp(Q, 5)$. Hence this restriction map $Stab(C) \to Symp(C)$ is surjective.

It is clearly that the action of $Stab(C)$ on $Symp(C)$ is transitive and by Theorem A in [50] $Stab(C) \to Symp(C)$ is a fibration. The rest parts of the diagram being a fibration is the same as the arguments in [31].

Then we verify that

Lemma 5.1.5. *If* $PD[\omega] := H - c_1E_1 - c_2E_2 - c_3E_3 - c_4E_4 - c_5E_5$, *also denoted by vector* $(1|c_1, c_2, c_3, c_4, c_5)$, *such that* $c_i < 1/2, \forall i \in \{1, 2, 3, 4, 5\}$, *then* $Stab(C) \simeq Diff^+(S^2, 5)$.

Proof. With the assumption $c_i < 1/2$, we can argue following [15] and show that $\pi_1(Symp(C))$ surjects onto $\pi_0(Symp_c(U))$: $c_i < 1/2$ here is because Lemma 36 in requires [15] the circle action to be away from the zero section.

Let μ be the moment map for the $SO(3)$-action on $T^*\mathbb{RP}^2$. Then $||\mu||$ generates a Hamiltonian circle action on $T^*\mathbb{RP}^2 \setminus \mathbb{RP}^2$ which commutes with the round cogeodesic flow. Symplectically cutting along a level set of $||\mu||$ gives $\mathbb{C}P^2$ and the reduced locus is a conic. Pick five points on the conic and $||\mu||$-equivariant balls of volume given by the symplectic form centered on them (this is always possible by Lemma 5.1.2 since the ball sizes $c_i < 1/2$ allow the packing to be away from the zero section). $(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}, \omega)$ is symplectomorphic to the blow up in these five balls and the circle action preserves the exceptional locus. Hence by Lemma 36 in [15], the diagonal element $(1, \ldots, 1) \in \pi_1(Symp(C)) = \mathbb{Z}^5$ maps to the generator of the Dehn twist of the zero section in $T^*\mathbb{RP}^2$, which is the generator in $\pi_0(Symp_c(U))$.

And here we also need Proposition 3.3 in [31], where the same argument work here after checking the following: Assume that $[\omega] \in H^2(X; \mathbb{Q})$. Up to rescaling, we can write $PD([l\omega]) = aH - b_1E_1 - b_2E_2 - b_3E_3 - b_4E_4 - b_5E_5$ with $a, b_i \in \mathbb{Z}^{\geq 0}$. Further, we assume $b_1 \geq \cdots \geq b_5$. Then we can represent $PD([l\omega])$ as a positive integral combination of all elements in the set $\{2H - E_1 - E_2 - E_3 - E_4 - E_5, E_1, E_2, E_3, E_4, E_5\}$, which is the homology type of C. And the proof is a direct computation:

$$\begin{aligned} PD([l\omega]) = aH - b_1E_1 - b_2E_2 - b_3E_3 - b_4E_4 - b_5E_5 \\ = \frac{a}{2}(2H - E_1 - E_2 - E_3 - E_4 - E_5) \\ +(\frac{a}{2} - b_1)E_1 \\ +\cdots \\ +(\frac{a}{2} - b_5)E_5. \end{aligned}$$

Therefore, assume $c_i < 1/2$, the homotopy type of each term and each connecting map is the same as the monotone case computed by [15] section 6.5. And we have the weak homotopy equivalence $Stab(C) \simeq \mathrm{Diff}^+(S^2, 5)$.

□

□

Remark 5.1.6. If some $c_i \geq 1/2$, we expect the diagonal element $(1, \dots, 1)$ in $\pi_1(Symp(C)) = \mathbb{Z}^5$ still maps to the generator of $\pi_0(Symp_c(U))$, but we won't give an explicit proof. Instead, there is an argument in Lemma 5.4.1 showing that the generator in $\pi_0(Symp_c(U))$ is isotopic to identity in $Symp_h$ for a given form without size restriction on $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$.

Pure braid group on sphere

By proposition 5.1.4, $Stab(C) \simeq \mathrm{Diff}^+(S^2, 5)$, which comes from the diffeomorphism of the base $2H - E_1 - E_2 - E_3 - E_4 - E_5$ sphere fixing the five intersection points of the exceptional spheres. Hence the right end of fibration (1.1) is:

$$Stab(C) \simeq \mathrm{Diff}^+(S^2, 5) \longrightarrow Symp_h \longrightarrow \mathcal{C}_0 \tag{5.2}$$

By Lemma 4.1.1, homotopically, $\mathcal{C}_0 \cong \mathcal{J}_{open}$, which is connected. And we can write down the homotopy long exact sequence of the fibration:

$$1 \to \pi_1(Symp_h(X, \omega)) \to \pi_1(\mathcal{C}_0) \xrightarrow{\phi} \pi_0(\mathrm{Diff}^+(S^2, 5)) \to \pi_0(Symp_h) \to 1 \tag{5.3}$$

Lemma 5.1.7. *There are isomorphisms*

$$\pi_0(Diff^+(S^2, 5)) \cong PB_5(S^2)/\langle \tau \rangle \cong PB_2(S^2 - \{x_1, x_2, x_3\}),$$

where $PB_5(S^2)$ and $PB_2(S^2 - \{x_1, x_2, x_3\}$ are the pure braid groups of 5 strings on S^2 and 2 strings on $S^2 - \{x_1, x_2, x_3\}$ respectively, and $\langle \tau \rangle = \mathbb{Z}_2$ is the center of the braid group $Br_5(S^2)$ generated by the full twist τ of order 2.

It follows that $Ab(\pi_0(\mathrm{Diff}(5, \mathrm{S}^2))) = \mathbb{Z}_5$.

Proof. The first identification comes from the homotopy fibration

$$\mathrm{Diff}^+(S^2,5) \to PSL(2,\mathbb{C}) \to Conf(S^2,5),$$

and the $\mathbb{Z}_2$ is the image of

$$\pi_1[PSL(2,\mathbb{C})] \to \pi_1[Conf(S^2,5)] = Br_5(S^2),$$

which is the center of $Br_5(S^2)$ generated by the full twist τ of order 2.

The second isomorphism follows from the direct sum decomposition (cf. the proof of Theorem 5 in [18]),

$$PB_n(S^2) \simeq PB_{n-3}(S^2 - \{x_1,x_2,x_3\}) \oplus \langle\tau\rangle.$$

Now we have $Ab(\pi_0(\mathrm{Diff}^+(S^2,5))) = \mathbb{Z}^5$ since $Ab(PB_2(S^2 - \{x_1,x_2,x_3\})) = \mathbb{Z}^5$ ([18] Theorem 5). □

We also recall the generating set and presentation of braid $B_n(S^2)$ and pure braid $PB_n(S^2)$ on the sphere. For details see [23] section 1.2 and 1.3.

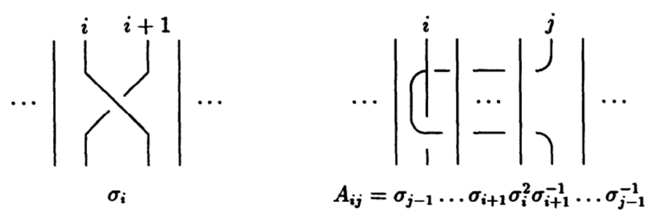


Figure 5.1: The Artin generator σ_i and the standard generator $A_{i,j}$

Lemma 5.1.8. *$B_n(S^2)$ admit the Artin presentation using Artin generators $\{\sigma_1,\cdots,\sigma_{n-1}\}$, where σ_i switches the ith point with (i+1)th point.*

$PB_n(S^2)$ admits a presentation using standard generators $A_{i,j}, 1 \le i < j \le n$. For $PB_{n-3}(S^2 - \{x_1,x_2,x_3\}) \simeq PB_n(S^2)/\mathbb{Z}_2$, the set $\{A_{i,j}, j \ge 4, 1 \le i < j\}$ is a generating

set. And further, by Theorem 5 in [18] there are relations $(\Pi_{i=1}^{j-1} A_{i,j})(\Pi_{k=n+1}^{j} A_{j,k}) = 1$, *ensuring that we can further remove the generators* $A_{1,j}$.

And the following equation gives the standard generator of Pure braid group:

$$A_{ij} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \tag{5.4}$$

Where one can think $A_{i,j}$ as the twist of the point i along the point j, which geometrically (see Figure 5.1) can be viewed as moving i around j through a loop separating j from all other points.

Lemma 5.1.9. $\pi_0(\mathit{Diff}^+(S^2, 5)) = P_5(S^2)/\mathbb{Z}_2$ *admit a minimal generating set*

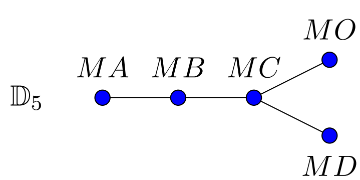
$$\{A_{24}, A_{25}, A_{34}, A_{35}, A_{45}\}.$$

And any permutation of $\{1, 2, 3, 4, 5\}$ *gives another minimal generating set. In particular, perform* $\{1, 2, 3, 4, 5\} \to \{5, 4, 3, 2, 1\}$, *we get*

$$\{A_{42}, A_{41}, A_{32}, A_{31}, A_{21}\}.$$

Reduced symplectic cone and walls

For $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$, as described in section 2.1.5: Combinatorially, the normalized reduced cone is convexly generated by 5 rays $\{MO, MA, MB, MC, MD\}$, with 5 vertices $M = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $O = (0, 0, 0, 0, 0)$, $A = (1, 0, 0, 0, 0)$, $B = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$, $C = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$,$D = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$; and these root edges corresponding to Lagrangian simple roos as follows, $MO = H - E_1 - E_2 - E_3$, $MA = E_1 - E_2$, $MB = E_2 - E_3$, $MC = E_3 - E_4$,$MD = E_4 - E_5$,



The open chamber in this case is a 5-dimensional polytope with the closure of reduced cone in 2.1 of the $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$ being a facet. A wall of codimension k is the interior of a facet of the closure of open chamber, where k many ">" made into "=". And the Lagrangian lattice of the wall W Γ_L is given by removing the generating rays of the wall W. Specifically, the k-faces are listed in the table 5.1.

Remark 5.1.10. Recall that any symplectic form ω is diffeomorphic to a reduced form ω_r and $Symp_h(X,\omega)$ is homeomorphic to $Symp_h(X,\omega_r)$.

And further note that we have the following Cremona transform showing that a reduced form satisfying some balanced condition is symplectormophic to a form that is obtained by blowing up ball packing relative to $\mathbb{R}P^2$ as in Lemma 5.1.2:

Lemma 5.1.11. *Given a reduced form ω on $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$, $PD[\omega] = H - c_1E_i - \cdots - c_5E_5$, with $c_1 \geq c_2 \geq c_3 \geq c_4 \geq c_5$, if $c_3 < c_4 + c_5$, then it is symplctomorphic to a form which admit a relative packing as in the previous Lemma 5.1.2.*

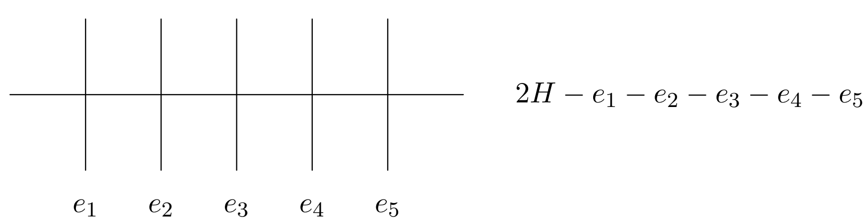
Proof. We do Cromena transform using $H - E_3 - E_4 - E_5$:

$$h = 2H - E_3 - E_4 - E_5, \quad e_1 = E_1, \quad e_2 = E_2,$$

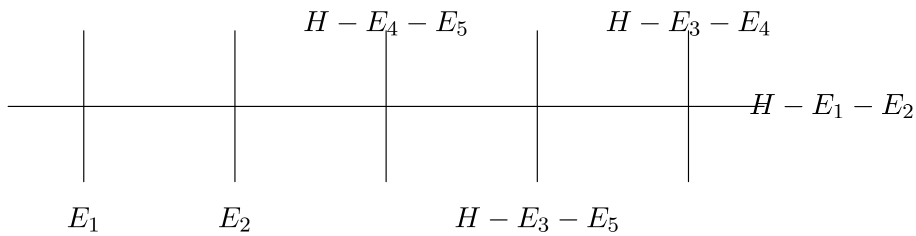
$$e_3 = H - E_4 - E_5, \quad e_4 = H - E_3 - E_5, \quad e_5 = H - E_3 - E_4.$$

We also have $2h - e_1 - \cdots - e_5 = H - E_1 - E_2$.

In the push forward manifold we choose the configuration



which is indeed the following configuration in the original manifold



Note that by [28], the above Cremona transform can be realized as a diffeomorphism on $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ denoted by Φ, with the push forward symplectic form $\Phi^*\omega$. Denote the basis of H_2 of the push forward manifold $(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}, \Phi^*\omega)$ by $\{h, e_1, \cdot, e_5\}$. And we can easily verify the assumption of Lemma 5.1.2 that$\Phi^*\omega(h) > 2\Phi^*\omega(e_i), \forall i$ by checking the following curves having positive area:

- $h - 2e_1 = 2H - 2E_1 - E_3 - E_4 - E_5 = (H - E_1 - E_3 - E_4) + H - E_1 - E_5$ has positive area since the old form is reduced;

- For $h - 2e_2$ we can apply the same argument as $h - 2e_1$;

- $h - 2e_3 = 2H - E_3 - E_4 - E_5 - 2(H - E_4 - E_5) = E_4 + E_5 - E_3$ has positive area from the assumption $c_3 < c_4 + c_5$;

- $h - 2e_4 = 2H - E_3 - E_4 - E_5 - 2(H - E_3 - E_5) = E_3 + E_5 - E_4$ has positive area because $c_3 \geq c_4$;

- For $h - 2e_5$ we can apply the same argument as $h - 2e_4$.

Hence the proof. □

Remark 5.1.12. In the previous Lemma 5.1.11, if one replace the assumption $c_3 < c_4 + c_5$ by $c_1 < c_2 + c_3$ or $c_2 < c_3 + c_4$, one may apply the corresponding Cremona transform(using $H-E_1-E_2-E_3, H-E_2-E_3-E_4$ respectively), to make the push forward admit a ball packing relative to $\mathbb{R}P^2$.

Hence one may state Lemma 5.1.11 as follows:

Lemma 5.1.13. *We call a reduce form* $\omega=(1|c_1,c_2,c_3,c_4,c_5)$ *on* $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ *with* $c_1\geq c_2\geq c_3\geq c_4\geq c_5$ **balanced** *if* $c_1\geq c_2+c_3$, $c_2\geq c_3+c_4$ *and* $c_3\geq c_4+c_5$ *not hold simultaneously.*

Any balanced reduced form ω_b *is diffeomorphic to a form admitting a ball packing relative to* $\mathbb{R}P^2$.

5.2 A semi-toric Ball-swapping model and the connecting homomorphism

In this section, we prove that:

Proposition 5.2.1. *Given* $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ *with symplectic form* ω*, such that* $PD[\omega]=H-c_1E_i-\cdots-c_5E_5$ *where there are at least 3 distinct values in* $\{c_1,\cdots,c_5\}$ *and* $max\{c_i\}<1/2,\sum_i c_i<2$*, then* $Symp_h(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2},\omega)$ *is connected.*

And because any balanced reduced symplectic form defined in Lemma 5.1.13 is Cremona equivalent to a form satisfying the condition $max\{c_i\}<1/2,\sum_i c_i<2$. As a corollary of 5.2.1, for any for in cases 3b to 5a in table 5.1, $Symp_h(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2},\omega)$ is connected. In addition, some form in cases 1 to 3a are also covered by Corollary 5.2.4, and the rest in cases 1 to 3a will be covered by Lemma 5.4.1 in the next section.

We first give a semi-toric model of ball-swapping relative to $\mathbb{R}P^2$: From the Biran decomposition we know $\mathbb{C}P^2=\mathbb{R}P^2\sqcup U$, where $U=H_4\setminus Z_\infty$, where H_4 is the 4th Hirzbruch surface with fiber area 1/2 and base area 2, and Z_∞ is the infinity section.

And if we have 5 balls with pairwise distinct sizes $a_1,a_2,\cdots,a_5$ such that

$$a_i<1/2,\sum_i a_i<2 \tag{5.5}$$

there is a toric packing as in Figure 5.2, by [40]. Note that each $B^4(a_i)\cap\{0\}\times C$ is a large disk in $B^4(a_i)$. Moreover, there is an ellipsoid $E_{ij}\subset U$, such that $B_i\cup B_j\subset E_{ij}$, and E_{ij} is disjoint from the rest of the ball packings. We call this an (i,j)-*standard packing*.

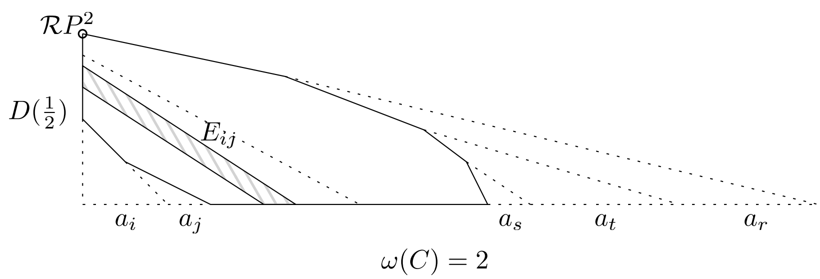


Figure 5.2: Standard toric packing and ball swapping in $\mathcal{O}(4)$

Next, we notice that when at least 2 elements from $\{a_r, a_s, a_t\} := \{a_1, a_2, \cdots, a_5\} \setminus \{a_i, a_j\}$ coincide, toric packing as in Figure 5.2 doesn't exist. Nonetheless, one could always slightly enlarge some of them to obtain distinct volumes satisfying equation (5.5), then pack the original balls into the enlarged ones to obtain a standard packing.

And there is a natural circle action induced from the toric action, rotating the base curve C, fixing the center of $B^4(a_i)$. Denote the Hamiltonian of the circle action H. This circle action clearly swaps the ball $B^4(a_i)$ and $B^4(a_j)$ and then place them back to their original positions. When these two balls are blown-up, the corresponding ball-swapping symplectomorphisms hence induce a pure braid generator A_{ij}.

To make it compactly supported, one multiplies to $H(r_1, r_2) = |r_2|^2$ a cut-off function $\eta(z_1, z_2)$ such that We can cutoff H using the function η defined as following:

$$\eta(x) = \begin{cases} 0, \quad x \in E_{ij} \setminus \{\dfrac{r_1^2}{2-\epsilon-a_r-a_s-a_t} + \dfrac{r_2^2}{\dfrac{1}{2}-\epsilon} \leq 1/\pi\}, \\ \\ 1, \quad x \in \{\dfrac{r_1^2}{a_i+a_j} + \dfrac{r_2^2}{\dfrac{1}{2}-2\epsilon} \leq 1/\pi\}, \end{cases} \tag{5.6}$$

The resulting Hamiltonian $\eta \circ H$ has time-one flow equal identity outside the ellipsoid in Figure 5.2 hence descends to a ball-swapping as in Definition 5.1.1. We call such a symplectomorphism an (i,j)-*model ball-swapping* in $\mathcal{O}(4)\#5\overline{\mathbb{C}}P^2$ when B_i and B_j are swapped. Consider the family of the compactly supported symplectomorphism, given by the Hamiltonian $t \circ \eta \circ H$ for $t \in [0,1]$, the following lemma is immediate.

Lemma 5.2.2. *The (i,j)-model ball-swapping is Hamiltonian isotopic to identity in the compactly supported symplectomorphism group of $\mathcal{O}(4)\#5\overline{\mathbb{C}P^2}$. Moreover, it induces a diffeomorphism on the proper transform of C, which is the generator A_{ij} on $\pi_0(Diff^+(S^2,5))$.*

Now we give the proof of

Proposition 5.2.3. *Given $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ with symplectic form ω, such that $PD[\omega] = H - c_1E_i - \cdots - c_5E_5$ where there are at least 3 distinct values in $\{c_1, \cdots c_5\}$ and $max\{c_i\} < 1/2, \sum_i c_i < 2$, then $Symp_h(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}, \omega)$ is connected.*

Proof. Fix a configuration $C_{std} \in \mathcal{C}_0$ in $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ with the given form ω. If we blow down the five exceptional spheres, we get a ball packing in $\mathbb{C}P^2$ with 5 balls, each B_i centered at the corresponding point P_i on the sphere S^2 of homology class $2H$, with the size determined by the form ω.

First note that there is a Lagrangian $\mathbb{R}P^2$ away from the curve Q in C_{std} of class $2H - E_1 - \cdots - E_5$ with the given symplectic form.

Blowing down the exceptional curves, we have a ball packing $\iota : B_l = B(c_l)$ relative to $\mathbb{R}P^2$. Suppose $c_i > c_j$, we have the semi-toric (i,j)-standard packing ι_s as defined above. One may further isotope ι to ι_s, by the connectedness of ball packing in [10] Theorem 1.1. Clearly, from Lemma 5.2.2, this is a Hamiltonian diffeomorphism, fixes all exceptional divisors from the 5 ball-packing, and induces the pure braid generator A_{ij} on C.

Take the isotopy ϖ_t from ϖ_{ij} to identity, then $\varpi_t(e_i)$ gives a loop in $\mathcal{C}_0$. This shows that the generator A_{ij} is in the image of ϕ for all i,j.

And finally, we verify that if there are no less than 3 distinct values in $\{c_1, \cdots c_5\}$, then a generating set as in Lemma 5.1.9 is contained in the image of ϕ and hence $Symp_h$ is connected: we can do a permutation on $\{1,2,3,4,5\}$ such that $c_1 > c_2 > c_3$. If $c_4 \notin \{c_1, c_2, c_3\}$, then clearly, $\{A_{12}, A_{13}, A_{14}, A_{23}, A_{24}\}$, which is a generating set in Lemma 5.1.9, is contained in the image of ϕ; and if $c_4 \in \{c_1, c_2, c_3\}$, suppose $c_4 = c_k$, then we do a permutation of set $\{1,2,3,4\}$ exchanging 3 with k, then we have the same generating set $\{A_{12}, A_{13}, A_{14}, A_{23}, A_{24}\}$ in the image of ϕ.

□

Indeed, one is already able to prove the following by performing a Cremona transform as in Lemma 5.1.11:

Proposition 5.2.4. *Given* $(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2},\omega)$*, let* ω *be a balanced reduced form. If there are more than 8 symplectic -2 sphere classes, then* $Symp_h$ *is connected.*

Proof. It suffices to consider only reduced forms, by Remark 5.1.10. We refer to table 5.1 for any reduced form $\omega=(1|c_1,\cdots,c_5),\quad \text{where}\quad c_1\geq c_2\geq c_3\geq c_4\geq c_5$. And the assumptions on the form means we only need to deal with balanced form on any $k-$face for $k\geq 2$ and edges MO, MB, MC, MD.

- For any $k-$face, $k\geq 3$: Because the form is balanced, by Lemma 5.1.13, it admit a ball packing relative to $\mathbb{R}P^2$ by Lemma 5.1.11 and 5.1.2. Note that the form has at least 3 district values in $c_1,\cdots,c_5$, then after pushing forward there are at least 3 distinct values among area of exceptional spheres. Hence by Lemma 5.2.1, $Symp_h$ is connected.

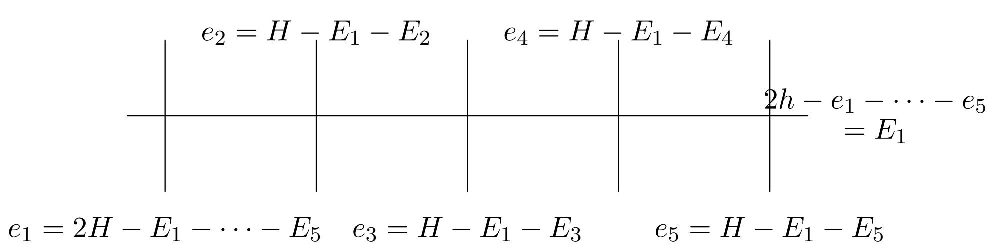
- For a 2$-$faces or edge where O is not a vertex of this 2$-$faces, then$\lambda=1$, and there is a composition of Cremona transform by $H-E_1-E_2-E_5$ and $H-E_1-E_3-E_4$ such that

$$h=3H-2E_1-E_2-E_3-E_4-E_5,\quad e_2=H-E_1-E_2,$$

$$e_3=H-E_1-E_3,\quad e_4=H-E_1-E_4,$$

$$e_5=H-E_1-E_5,\quad e_1=2H-E_1-\cdots-E_5.$$

We also have $2h-e_1-\cdots-e_5=E_1$ and obtain the following configuration:



And we can easily see that the push forward form satisfies the following: there are

3 distinct values for exceptional spheres, because

$$\omega(2H - E_1 - \cdots - E_5) > \omega(H - E_1 - E_5) > \omega(H - E_1 - E_2);$$

and each e_i has area less than $1/2$ of h, this is because $e_1 = 2H - E_1 - \cdots - E_5$ always has the largest area among $\{e_1, \cdots, e_5\}$. And the curve symplectic area of $h - 2e_1$ is :

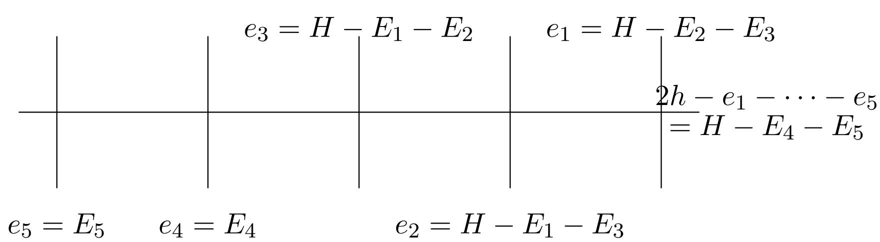
$$\omega(3H - 2E_1 - E_2 - E_3 - E_4 - E_5) - 2\omega(2H - E_1 - \cdots - E_5) > 0.$$

- For 2−faces where O is a vertex, then there there are 4 cases, MOA,MOB, MOC, MOD:
  - For MOA,MOB, C and D are not vertices: the form has $c_3 = c_4 = c_5$, and we can use the Cremona transform by $H - E_1 - E_2 - E_3$ such that

$$h = 2H - E_1 - E_2 - E_3, \quad e_4 = E_4, \quad e_5 = E_5,$$

$$e_3 = H - E_1 - E_2, \quad e_2 = H - E_1 - E_3, \quad e_1 = H - E_2 - E_3.$$

    We also have $2h - e_1 - \cdots - e_5 = H - E_4 - E_5$ and obtain the following configuration:



    And we can easily see that the push forward form satisfies the following: there are 3 distinct values for exceptional spheres, because

$$\omega(H - E_2 - E_3) > \omega(H - E_1 - E_2) > \omega(E_5);$$

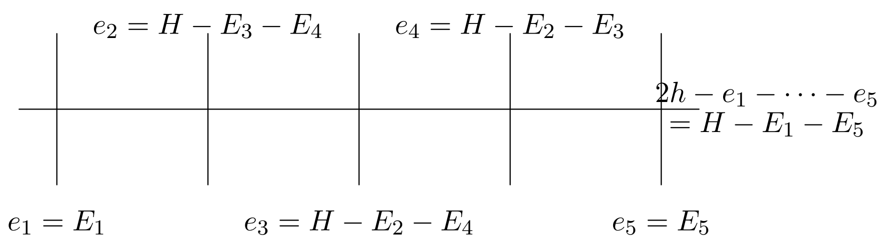
and each e_i has area less than 1/2 of h.

– For MOC, MOD, B is not a vertex: the form has $c_2 = c_3$ and we can do Cremona transform by $H - E_2 - E_3 - E_4$ such that

$$h = 2H - E_2 - E_3 - E_4, \quad e_1 = E_1, \quad e_5 = E_5,$$

$$e_2 = H - E_3 - E_4, \quad e_3 = H - E_2 - E_4, \quad e_4 = H - E_2 - E_3.$$

We also have $2h - e_1 - \cdots - e_5 = H - E_1 - E_5$ and obtain the following configuration:



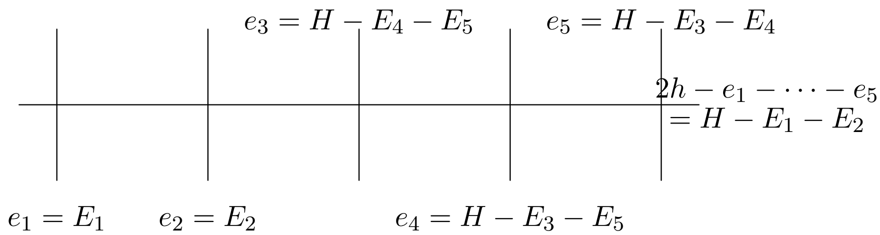
And we can easily see that the push forward form satisfies the following: there are 3 distinct values for exceptional spheres, because

$$\omega(H - E_3 - E_4) > \omega(H - E_2 - E_3) > \omega(E_5);$$

and each E_i has area less than 1/2 of h.

- For case MO, $\pi_0(Symp_h)$ is trivial:

  Firstly, we apply a Cremona transform for the two case to obtain the following:



  Then, in either root edges, we have two exceptional spheres with area a_1 and the other three with area a_2 where $a_1 \neq a_2$.

Hence up to a permutation of index in $\{1, \cdots, 5\}$, we obtain a set of braid generators $\{A_{14}, A_{24}, A_{34}, A_{15}, A_{25}, A_{35}\}$.

By Proposition 7 in [17], $\text{Diff}^+(S^2, 5)$ and $P_2(S^2 - 3 \text{ points})$ identified and there are surface relations

$$(\Pi_{i=1}^{j-1} A_{i,j})(\Pi_{k=j+1}^{m+n} A_{j,k}) = 1.$$

In our case $m = 2, n = 3$ hence let $j = 4$ we have $A_{14}A_{24}A_{34}A_{45} = 1$. This means the above set generators A_{45}. And hence we obtain a minimal generating set $\{A_{14}, A_{24}, A_{45}, A_{15}, A_{25}\}$.

It follows from Theorem 5.2.1 that $Symp_h$ is connected from cases MB to $MOABCD$ and case MO in table 5.1 with the balanced condition. □

5.3 Forget one strand map when $\Gamma_L = \mathbb{D}_4$

In this section we focus on the case when $\omega \in MA$, which is labeled by $\mathbb{D}_4$. One have the $\omega-$area of $H, E_1, \cdots, E_5$ being $1, c_1, \cdots, c_5$ and

$$c_1 > c_2 = c_3 = c_4 = c_5, \quad c_1 + c_2 + c_3 = 1. \tag{5.7}$$

The following Proposition 5.3.1 tells us in this case, $\pi_0(Symp_h)$ is $\pi_0(\text{Diff}^+(S^2, 4)) = P_4(S^2)/\mathbb{Z}_2$.

Proposition 5.3.1. *Let $X = \mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$ with a reduced symplectic form ω on MA where there are 8 symplectic -2 sphere classes, $\pi_0(Symp_h)$ is $\pi_0(\mathit{Diff}^+(S^2, 4)) = P_4(S^2)/\mathbb{Z}_2$.*

To prove this, we first find a family of $(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}, \omega), \omega \in MA$ whose fundamental group is $\pi_0(\text{Diff}^+(S^2, 4)) = P_4(S^2)/\mathbb{Z}_2$, see Lemma 5.3.5. Then using Lemma 5.3.6, 5.3.7, 5.3.8 we show that there's a surjection from $\pi_0(Symp_h)$ to $\pi_0(\text{Diff}^+(S^2, 4)) = P_4(S^2)/\mathbb{Z}_2$. On the other hand, Lemma 5.3.9 tells us that there's a surjection $\pi_0(\text{Diff}^+(S^2, 4)) \to \pi_0(Symp_h)$ on the opposite direction. And finally by the Hopfian property of $\pi_0(\text{Diff}^+(S^2, 4))$ as in Lemma 5.3.10, we complete the proof of Proposition 5.3.1 by showing that the above surjections are isomorphisms.

Firstly we prove a technical lemma about $J \in \mathcal{J}_\omega - \mathcal{X}_4$ and the curve configuration:

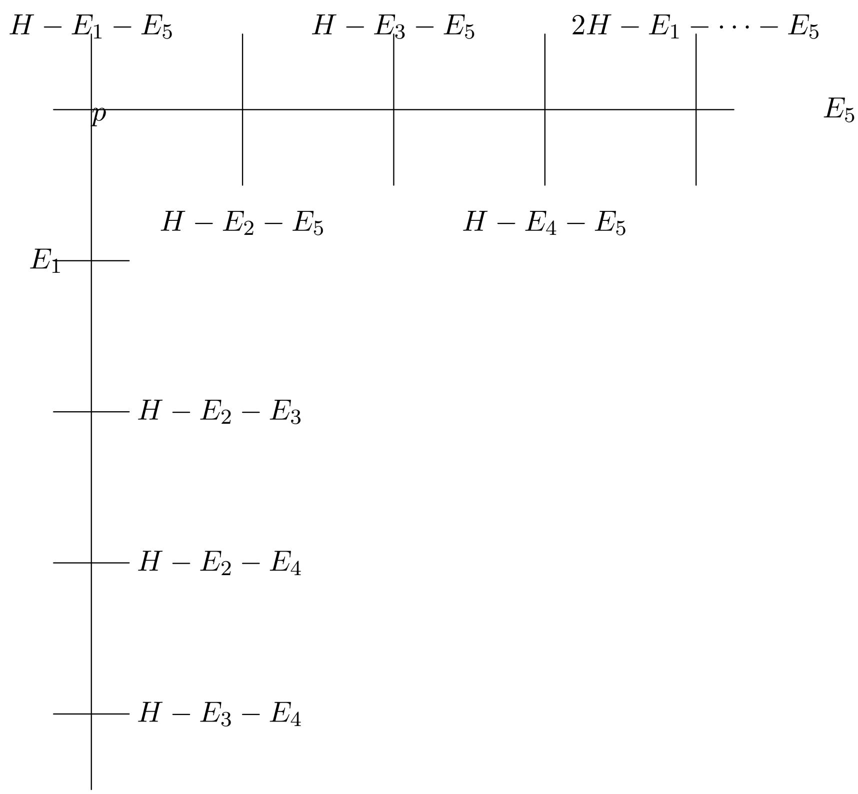


Figure 5.3: Configuration of two minimal area exceptional classes for $\omega \in MA$.

Lemma 5.3.2. *Choose a reduced form $\omega \in MA$, i.e. equation* (5.7) *holds. Take a configuration as in figure 5.3, and note that here each line is just a homology class. If$J \in \mathcal{J}_\omega - \mathcal{X}_4$, then the $J-$holomorphic representative of the vertical classes in figure 5.3 intersect the base curve Q once at a single point.*

Proof. We use $H, E_1, \cdots, E_5$, as the basis of $H_2(X, \mathbb{Z})$. And we have the identification $e_i = H - E_i - E_5, if \quad i = 2,3,4$ and $e_5 = 2H - \sum_{i=1}^5 E_i$. Note that If there's no component having self-intersection less than -2 in the stable curves of classes of each vertical line in figure 5.3. (i.e. $A = H - E_i - E_5, i \in 1,2,3,4$ or $A = 2H - \sum_{j=1}^5 E_j$)

By adjunction (or Lemma 3.2.1 performing a base change 2.3), any embedded sphere either has non-negative coefficient on H or is $(k+1)E_1 - kH - \sum_j E_j$. Further, any -2 sphere has non-negative coefficient on H, and has coefficient 0 or -1 on any class E_i.

We now prove the theorem by showing that if the components of the stable curve of A has square no less than -2, then there's exactly one embedded component intersecting Q once at a single point.

Assume $A = \sum_k A_k$. By Proposition 3.3.3, $\{A_k\}$ could have 4 type of classes: $B, kF, D_j \in \mathcal{S}^{-1}, G_k \in S^{-2}$. performing a base change 2.2, the $\{A_k\}$ could have $H - E_2, k(H - E_1), D_j \in \mathcal{S}^{-1}, G_k \in S^{-2}$, where each component D_j, G_k is embedded by Lemma 3.1.10.

Clearly, $H - E_2, k(H - E_1)$ does not intersect $Q = E_5$. Hence it suffice to consider the subset of $\{A_k\}$ consisting only $D_j \in \mathcal{S}^{-1}, G_k \in S^{-2}$, still denote it by $\{A_k\}$. Note that now any element in $\{A_k\}$ is embedded,hence each A_k has non-negative coefficient on H. Now we analyze possible $\{A_k\}$ for a choice of class A.

- For $A = H - E_i - E_5$, there must be at most one curve $A_1 = H - \sum_m E_m$ and other curves $E_i - \sum_j E_j$.

  On the one hand, if there are more than one curve have -1 on E_5, then either $A_p = E_{i_p} - \sum_{j_p} E_{j_p} - E_5, A_q = E_{i_q} - \sum_{j_q} E_{j_q} - E_5$ such that $A_p \cdot A_q \leq -1 < 0$, contradiction; or $A_1 = H - \sum_m E_m - E_5$, $A_r = E_{i_r} - \sum_{j_r} E_{j_r} - E_5$ and $A_1 \cdot A_r \geq 0$ means $A_1 \cdot E_{i_r} = 1$ where either $i_r \in 2, 3, 4$ such that $\omega(A_r) \leq 0$ or $i_r = 1$ such that $\omega(A_1) \leq 0$, contradiction. This means the stable curve has at most one component A_k such that $A_k \cdot E_5 = 1$.

  On the other hand, because $A = H - E_i - E_5$ and $A \cdot E_5 = 1$, there is at least one curve A_k such that $A_k \cdot E_5 = 1$. Hence the stable curve intersect Q exactly once at a single point.

- For $A = 2H - \sum_{j=1}^{5} E_j$, there must be at most two curves with positive H coefficient, denoting $A_1 = H - \sum_m E_m$, $A_2 = H - \sum_n E_n$, and other curves $E_i - \sum_j E_j$. From the above case we know if there are more than one curve have -1 on E_5, then must be $A_1 = H - \sum_m E_m - E_5$ $A_2 = H - \sum_n E_n - E_5$. $A = 2H - \sum_{i=1}^{5} E_i$ so that $A \cdot E_1 = 1$, at least one of A_1, A_2 intersect E_1 positively. Without loss of generality, we can assume $A_1 \cdot E_1 = 1$, then we have $\omega(A_1) \leq 0$, contradiction. This means that the stable curve has at most one component A_k

such that $A_k \cdot E_5 = 1$. On the other hand, because $A = 2H - \sum_{i=1}^{5} E_i$ so that $A \cdot E_5 = 1$, there is at least one curve A_k such that $A_k \cdot E_5 = 1$. Hence the stable curve intersect Q exactly once at a single point.

□

Remark 5.3.3. Noth that Lemma 5.3.2 holds for the horizontal curves too, the proof is the same.

And we also have

Lemma 5.3.4. *For a given form $\omega \in MA$, i.e. equation (5.7) holds, then the action of $Symp_h$ on $\mathcal{J}_\omega - \mathcal{X}_4$ is free. And hence $BSymp_h$ and $(\mathcal{J}_\omega - \mathcal{X}_4)/Symp_h$ have the same homotopy groups at least up to degree 3.*

Proof. By Lemma 3.1.8, there is an action of $Symp_h$ on $\mathcal{J}_\omega - \mathcal{X}_4$.

Look at configuration of homology classes as in 5.3

Both E_5 and $H - E_1 - E_5$ always have pseudo-holomorphic simple representatives because they both have minimal area and hence are embedded. For a given $J \in \mathcal{J}_\omega - \mathcal{X}_4$, the point p is geometric intersection of $J(E_5) \cap J(H - E_1 - E_5)$, where $J(A)$ means the $J-$holomorphic representative of class A.

We have Lemma 5.3.2, any exceptional sphere intersects E_5 or $H - E_1 - E_5$ at one single point if not empty. Indeed for a $J \in \mathcal{J}_\omega - \mathcal{X}_4$, we could explicitly write down the labelling set $\mathcal{C} \subset S^{\leq -2}$ for the prime submanifold $\mathcal{J}_{\mathcal{C}}$ where J belongs to. Indeed $\mathcal{C}$ is either empty or has a single square -2 class.

And we have the form

Suppose some element i in $Symp_h(X, \omega)$ fix some $J \in \mathcal{J}_\omega - \mathcal{X}_4$,, then it's an isometry. And this isometry i fixes 5 or 3 intersection points on sphere E_5 or $H - E_1 - E_5$ because the exceptional sphere(or their stable curve) are fixed as a set. Hence this action i restricting on sphere E_5 or $H - E_1 - E_5$ is identity because isometry of sphere fixing at least 3 points is an identity. Hence on the tangent space of $X = \mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$ at p, i is id. Then exponential map gives the action i itself is identity in $Symp_h(X, \omega)$. This means the action of $Symp_h(X, \omega)$ on $\mathcal{J}_\omega$ is free.

The last statement can be easily seen in the following diagram where the upper row is the pullback bundle of the lower row

$$\begin{array}{ccccc} Symp_h(X,\omega) & \longrightarrow & \mathcal{J}_\omega - \mathcal{X}_4 & \longrightarrow & (\mathcal{J}_\omega - \mathcal{X}_4)/Symp_h \\ \downarrow & & \downarrow & & \downarrow \\ Symp_h(X,\omega) & \longrightarrow & \mathcal{J}_\omega = ESymp_h & \longrightarrow & \mathcal{J}_\omega/Symp_h = BSymp_h \end{array} \tag{5.8}$$

□

Lemma 5.3.5. *There is a family of* $(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2},\omega), \omega \in MA$, *which can be identified with* $Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C})$. *This space is the ordered 4 points configuration space on* $\mathbb{C}P^1$ *modulo holomorphic automorphism, whose fundamental group is* $\pi_0(Diff^+(S^2,4)) = P_4(S^2)/\mathbb{Z}_2$.

Proof. Note that on MA one have the $\omega-$area of $H, E_1, \cdots, E_5$ being $1, c_1, \cdots, c_5$ and equation (5.7) holds.

We consider the following configuration of homology classes

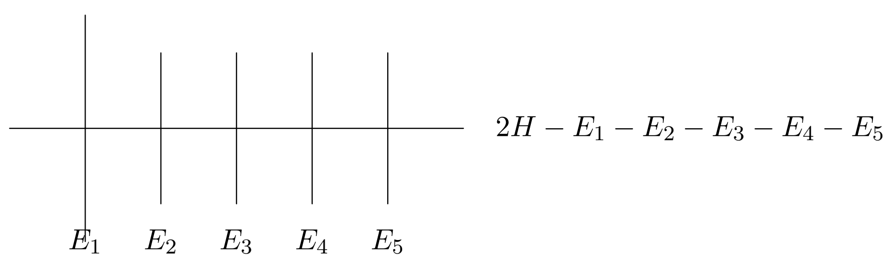


Figure 5.4: Configuration of exceptional classes for $\omega \in MA$.

We first blow up $\mathbb{C}P^2$ at one point with size c_1. Such configuration could be regarded as blowing up at 4 points in general position (no 2 collide, no 3 on a same line in class H) in $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, of size c_2. And the four points determines a unique embedded curve in class $2H - E_1$, we could identify this with the space $Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C})$.

□

By Picard-Lefschetz theory, the monodromy of the family of blowups over $Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C})$ gives the map:

$$\delta : \pi_1[Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C})] \quad \mapsto \pi_0(Symp_h). \tag{5.9}$$

Lemma 5.3.6. *For a rational point $\omega \in MA$, there exist a well defined continuous map*

$$\alpha : Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C}) \quad \mapsto (\mathcal{J}_\omega - \mathcal{X}_4)/Symp_h.$$

Proof. By choosing a rational point $\omega \in MA$, we assume that $[\omega] \in H^2(X;\mathbb{Q})$. Up to rescaling, we can write $PD([l\omega]) = aH + b_1E_1 + b_2E_2 + b_3E_3 + b_4E_4$ with $a, b_i \in \mathbb{Z}^{>0}$.

Each fiber over $\mathcal{B} = Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C})$ is an $X = \mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ with a standard complex structure J_0. And we consider the embedding of the ample line bundle given by the ample divisor $D = PD([l\omega]) = aH + b_1E_1 + b_2E_2 + b_3E_3 + b_4E_4$ in to some projective space. To get a symplectic structure we need a Fubini-Study form on $\mathbb{P}H^0(X;D)$. Such a form comes from a Euclidean metric on $H^0(X;D)$ and we can pick one from contractible space of choices of metric. This gives us a gives a symplectic structure on X, diffeomorphic to the standard one.

We can pull back the complex structure along this diffeomorphism, which gives a $J \in J_\omega$. And note that since we choose a 4 tuple in $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ such that no three are collinear, there's no curve with self-intersection less than -2 and there's at most one -2 curve in class $\{E_1 - E_2, E_1 - E_3, E_1 - E_4, E_1 - E_5\}$. So we further have $J \in \mathcal{J}_\omega - \mathcal{X}_4$. Two different pull back differ by a symplectomorphism in $Symp_h(X,\omega)$, and hence this gives a well-defined continuous map

$\alpha : \mathcal{B} = Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C}) \quad \mapsto (\mathcal{J}_\omega - \mathcal{X}_4)/Symp_h.$

And note that the induced map of α on the fundamental group is the same as the monodromy map as in (5.9).

□

On the other hand, we have a map β from $(\mathcal{J}_\omega - \mathcal{X}_4)/Symp_h$ to $\mathcal{B} = Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C})$:

Lemma 5.3.7. *Choose a given form $\omega \in MA$, i.e. equation* (5.7) *holds. The map β is well defined:*

$$\beta : (\mathcal{J}_\omega - \mathcal{X}_4)/Symp_h \mapsto Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C}),$$

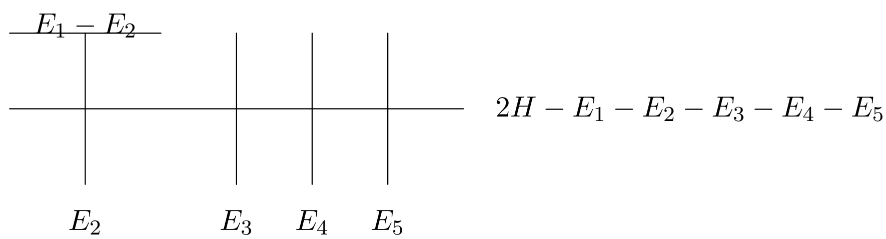
where $\mathcal{X}_4$ has codimension 4 in $\mathcal{J}_\omega$, see Lemma 3.4.1. And the composition of $\beta \circ \alpha$ always induces a surjective map $\beta^* : \pi_1(BSymp_h)$ to $\pi_1(Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C}))$, which is $P_4(S^2)/\mathbb{Z}_2$.

Proof. Let $M = (\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}, \omega)$ with a $\omega \in MA$. Recall that the classes E_2, E_3, E_4, E_5 always have pseudo-holomorphic simple representatives, because they are the minimal area exceptional classes.

Take any compatible almost complex structure $J \in \mathcal{J}_\omega - \mathcal{X}_4$. For a $J \in \mathcal{J}_{open} = \mathcal{J}_\omega - \mathcal{X}_2$, each class in figure 5.4 has an embedded representative.

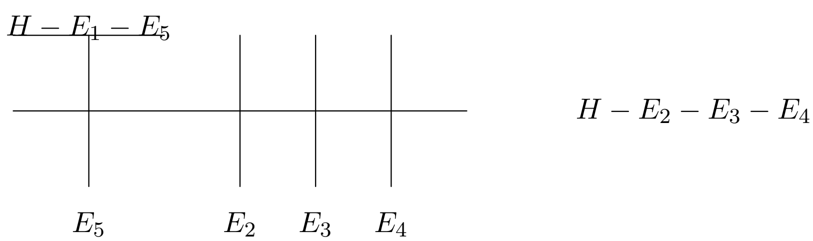
We denote $\mathcal{J}_{2H-E_1}$ to be the union of $\mathcal{J}_\mathcal{C}$ where $\mathcal{C}$ is $\{E_1 - E_5\}$, $\{E_1 - E_2\}$, $\{E_1 - E_3\}$, or $\{E_1 - E_4\}$.

Take $\mathcal{J}_{\{E_1-E_2\}}$ as an example, the simple embedded representative of 5.4 is



And we denote $\mathcal{J}_H$ to be to be the union of $\mathcal{J}_\mathcal{C}$ where $\mathcal{C}$ is $\{H - E_2 - E_3 - E_5\}$, $\{H - E_2 - E_4 - E_5\}$, $\{H - E_3 - E_4 - E_5\}$, or $\{H - E_2 - E_3 - E_4\}$.

Take $\mathcal{J}_{\{H-E_2-E_3-E_4\}}$ as an example, the simple embedded representative of 5.4 is



There is a continuous map from $\mathcal{J}_\omega - \mathcal{X}_4$ to $Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C})$ by projecting to the g.i.p (not including $J(E_1) \cap J(2H - E_1 - \cdots E_5)$) on the horizontal curve and identify it with a standard $\mathbb{C}P^1$.

To see this map is continuous can be verified in the following way: for any $J \in \mathcal{J}_{open} \amalg \mathcal{J}_{2H-E_1}$, the configuration 5.4 could be regarded as blowing up at 4 points in general position (no 2 collide, no 3 on a same line in class H), on the curve in class $2H - E_1$ in $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Meanwhile, for a $J \in \mathcal{J}_H$, the configuration 5.4 could be regarded as blowing up at 4 points in $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, where 3 are distinct points on a curve in class H and the other on the curve $H - E_1$. We could identify the last point with the intersection point of the curves H and $H-E_1$. And this identification does not change the continuity of the map from $\mathcal{J}_\omega - X_4$ to $Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C})$.

We define β by sending $J \in (\mathcal{J}_\omega - \mathcal{X}_4)$ to the points $\{e_2(J) \cap Q(J), e_3(J) \cap Q(J), e_4(J) \cap Q(J), e_5(J) \cap Q(J)\}$, where $e_i(J)$ denote the image of stable curve in homology class e_i for the given J; and $Q(J)$ denote the image of embedded J-holomorphic curve in the horizontal embedded curve Q.

Here if $J \in (\mathcal{J}_{2H-E_1}$, $Q = 2H - E_1 - \cdots - E_5$ and $e_i = E_i, i = 2,3,4,5$; and if $J \in \mathcal{J}_H$, let $\{p,q,r,s\} = \{2,3,4,5\}$, if $Q = H - E_r - E_p - E_q$, then $e_i = E_i$ for $i = p,q,r$ and $e_i = H - E_1 - E_s$ for $i = s$.

This gives the map as stated:

$$(\mathcal{J}_\omega - \mathcal{X}_4)/Symp_h \mapsto Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C}).$$

To obtain a surjective map $\beta^* : \pi_1(BSymp_h)$ to $\pi_1(Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C}))$, we just need to consider the composition of $\beta \circ \alpha$: it is an isomorphism of space $Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C})$. Hence the induced map on homotopy groups are isomorphic. This means the map $\beta^* : \pi_1(BSymp_h) \to \pi_1(Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C}))$ is surjective.

□

Lemma 5.3.8. *For a given form $\omega \in MA$, i.e. equation* (5.7) *holds,* $\pi_0(Symp_h) \twoheadrightarrow P_4(S^2)/\mathbb{Z}_2$.

Proof. We consider the action of $Symp_h$ on space of J-holomorphic curve Q in class $2h - e_1 - \cdots - e_5 = E_5$, which is contractible since it's homotopic to $\mathcal{J}_\omega$. We have a well defined (as in Lemma 5.3.7) forgetful map

$$\beta : (\mathcal{J}_\omega - \mathcal{X}_4)/Symp_h \mapsto Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C}),$$

where X_4 has codimension 4 in $\mathcal{J}_\omega$, see Lemma 3.4.1.

And because the composition of $\beta \circ \alpha$ gives an isomorphism of fundamental group $Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C})$. And hence there is a surjective map $\beta^* : \pi_1(BSymp_h)$ to $\pi_1(Conf_4^{ord}(\mathbb{C}P^1)/PSL_2(\mathbb{C}))$, which is $P_4(S^2)/\mathbb{Z}_2$. These will be proved in Lemma 5.3.7.

□

In addition, we prove that

Lemma 5.3.9. *For a given form $\omega \in MA$, i.e. equation* (5.7) *holds, $\pi_1(BSymp_h)$ is a quotient of $P_4(S^2)/\mathbb{Z}_2$, i.e. $P_4(S^2)/\mathbb{Z}_2 \twoheadrightarrow \pi_0(Symp_h)$.*

Proof. For case MA in table 5.1, Lemma 5.2.2 and 5.1.11 tells us that there is a set $\pi_1(S^2 - \{p_2, p_3, p_4, p_5\})$ in the image of the connecting homomorphism ϕ in sequence

$$1 \to \pi_1(Symp_h(X, \omega)) \to \pi_1(\mathcal{C}_0) \xrightarrow{\phi} \pi_0(\mathrm{Diff}^+(S^2, 5)) \to \pi_0(Symp_h) \to 1. \tag{5.10}$$

In [9], we have the short exact sequence of the forgetting one strand map:

$$0 \to \pi_1(S^2 - \{p_2, p_3, p_4, p_5\}) \to PB_5(S^2)/\mathbb{Z}_2 \to PB_4(S^2)/\mathbb{Z}_2 \to 0.$$

And hence we know that $\pi_0(Symp_h)$ is a quotient of $P_4(S^2)/\mathbb{Z}_2$ and there is a surjective homomorphism $\gamma : P_4(S^2)/\mathbb{Z}_2 \to \pi_0(Symp_h)$.

□

And (Pure or full)Braid groups (on disks or spheres) are Hopfian (cf. [23]) or see Lemma 5.3.10:

And we also write a proof of the fact known to experts of geometric group theory:

Lemma 5.3.10. *Pure and full Braid groups (on disks or spheres) are Hopfian, i.e. every epimorphism is an isomorphism.*

Proof. The disk case and sphere case can be dealt in the same way, and here we just need the sphere case:

- On disks: Bigelow and Krammer showed that (full) braid groups on disks are linear; and by a well-known result of Malćev, finitely generated linear groups are residually finite, and finitely generated residually finite groups are Hopfian. Residual finiteness is subgroup closed, hence pure braid group on disks as the subgroup of full braid is residually finite, and it is finitely generated hence Hopfian.
- On sphere: V. Bardakov shows sphere full braid groups and mapping class groups of the n-punctured sphere ($MCG(S^2, n)$) are linear. The rest argument is the same as above. In particular, $P_4/\mathbb{Z}_2 = \pi_0(\mathrm{Diff}(S^2, 4))$ is $MCG(S^2, 4)$ and in the meanwhile $P_4/\mathbb{Z}_2$ is finitely generated, hence $P_4/\mathbb{Z}_2$ is Hopfian.

□

Then for rational points on MA we have Proposition 5.3.1.

Lemma 5.3.11. *Proposition 5.3.1 holds true for a rational point $\omega \in MA$.*

Proof. Recall we need to deal with a given form $\omega \in MA$, where equation (5.7) holds.

Now let $G = P_4/\mathbb{Z}_2$, which is Hopfian, and $H = \pi_0(Symp_h, \omega)$ for the given symplectic form with 8 $\omega-$ symplectic -2 sphere classes. G and H are groups, and by Lemma 5.3.8, there is a surjective homomorphism $\beta^* : H \twoheadrightarrow G$; and by Lemma 5.3.8, a surjective homomorphism $\gamma : G \twoheadrightarrow H$. Then $\beta^* \circ \gamma : G \overset{\gamma}{\twoheadrightarrow} H \overset{\beta^*}{\twoheadrightarrow} G$ is a surjective homomorphism because it is the composition of two surjections. Then we have an epimorphism of G which has to be isomorphism because G is Hopfian. Then the map $\gamma : G \twoheadrightarrow H$ has to be injective. Hence it's both injective and surjective. And G and H are isomorphic, which means $\pi_0(Symp_h) = \pi_1(BSymp_h) = P_4(S^2)/\mathbb{Z}_2$.

□

To complete the proof of Proposition 5.3.1 as claimed, we need to prove a version stability of symplectomorphism group using inflation along pseudo-holomorphic curves. Under the base change (2.2), we can regard any X with $\omega \in MA$ as a 4 point equal size blow up of $S^2 \times S^2$ with base area μ, fiber area 1, and blow up size $\frac{1}{2}$. Let us denote the smooth manifold by X and the form by w_μ.

And we have the fibration

$$G_\mu \to \mathrm{Diff}_0(X) \to S_\mu,$$

where $G_\mu = Symp(X, \omega_\mu)$, $\mathrm{Diff}_0(X)$ is the identity component of the diffeomorphism group, and S_μ is the space of symplectic form of given size μ.

To do this we will use the idea introduced by McDuff in [41] and follow the notation of [7]. It consists in considering instead a larger space P_μ of pairs $P_\mu = \{(\omega, J) | S_\mu \times A_\mu : \omega \quad \text{is compatible with} \quad J\}$, where A_μ denotes the space of almost complex structures that are compatible some form in S_μ. Then both projection maps $P_\mu \to A_\mu, P_\mu \to S_\mu$ are fibrations with contractible fibers, and so are homotopy equivalences.

Therefore, to show that the groups G_μ and $G_{\mu'}$ are homotopy equivalent; it is sufficient to find a homotopy equivalence $A_\mu \to A_{\mu'}$ that commutes, up to homotopy, with the action of $\mathrm{Diff}_0(X)$.

Indeed we will use the following negative inflation Lemma as in [11]:

Lemma 5.3.12. *Let J be an ω_0-tame almost complex structure on a symplectic 4manifold (M, ω_0), that admits a Jholomorphic curve Z with $Z \cdot Z = -m, m \in \mathbb{N}$. Then for all $t > 0$, there is a family ω_t of symplectic forms, all taming J, which satisfy $[\omega_t] = [\omega_0] + tPD(Z)$ for all $0 \leq t \leq \frac{\omega_0(Z)}{m} - \epsilon$.*

to show that

Lemma 5.3.13. *Let $\mu = \frac{k}{2} + \lambda, k \in \mathbb{N}^{\geq 2}, \lambda \in (0, \frac{1}{2})$, then there exist some $\mu' > \mu$ such that $\mu' \in (\frac{k}{2}, \frac{k+1}{2}]$, and the spaces A_μ and $A_{\mu'}$ are equal.*

Proof. Firstly, if $\omega \in MA$, a simple computation as in Lemma 3.3.1 shows that the square negative curves in the $\mathcal{F}-$class can only be $F - E_i$, and in the $\mathcal{E}-$class can only be E_i. And let $\mu = \frac{k}{2} + \lambda, \lambda \in (0, \frac{1}{2}]$, for each prime submanifolds as in definition 3.1.6, there is exactly one curve whose square is less than -1, being $B - nF, n \geq 1$ $B - mF - \sum_i E_i$, or $B + F - \sum_{j=1}^{4} E_j$; or exactly two (-2) curves in $\mathcal{B}-$class pairing non-negatively. Note that in each prime submanifold, the curves in $B-$class only change when μ passes $\frac{k}{2}$, where k is an integer. Further, by Lemma 3.3.3 and the area restriction, the decomposition of type (3.1) for a given curve in $B-$class is the same if μ does not pass $\frac{k}{2}$. This means the spaces A_μ and $A_{\mu'}$ have the same decomposition into prime submanifolds as in definition 3.1.6.

And we show that $A_\mu = A_{\mu'}$ by the inclusion in both directions. Firstly, note that $PD[\omega_\mu] = B + \mu F - \frac{1}{2}E_1 - \frac{1}{2}E_2 - \frac{1}{2}E_3 - \frac{1}{2}E_4$, and we denote it by $PD[\omega_\mu]_B = [1, \mu, -\frac{1}{2}, \cdots, -\frac{1}{2}]$. In the HE basis (by equation (2.3)), $PD[\omega_\mu]_H = [\mu + \frac{1}{2}, \mu - \frac{1}{2}, -\frac{1}{2}, \cdots, -\frac{1}{2}]_H$.

$A_\mu \subset A_{\mu'}$: Since the class F pair any $J-$holomorphic embedded curve is non-negative, by the inflation lemma in [24], it is always possible to inflate the form ω_μ along some embedded Jholomorphic sphere representing the class F by $\mu' - \mu$. Then the class of form becomes $PD[\omega_{\mu'}]_B = [1, \mu', -\frac{1}{2}, \cdots, -\frac{1}{2}]_B$. In the HE basis (by equation (2.3)), this process is to inflate along $H - E_1$ by $\mu' - \mu$, making $PD[\omega_\mu]_H = [\mu + \frac{1}{2}, \mu - \frac{1}{2}, -\frac{1}{2}, \cdots, -\frac{1}{2}]_H$ into $PD[\omega_{\mu'}]_H = [\mu' + \frac{1}{2}, \mu' - \frac{1}{2}, -\frac{1}{2}, \cdots, -\frac{1}{2}]_H$.

Now we show the inverse, $A_{\mu'} \subset A_{\mu}$. We need to deal with different cases

- Case 1: $k = 2l, l \geq 2$ and $\mu' = \frac{k}{2}+\lambda, \lambda \in (0, \frac{1}{2}]$. The curve $B+\frac{k}{2}F-E_1-E_2-E_3-E_4$ is always represented by an embedded $J-$holomorphic sphere, because it pair any $J-$holomorphic embedded curve (note the square least curve is in class $B - lF$) is non-negative. By the inflation lemma in [24], it is always possible to inflate the form $\omega_{\mu'}$ along some embedded Jholomorphic sphere representing the class $B + \frac{k}{2}F - E_1 - E_2 - E_3 - E_4$ by $t \in \mathbb{R}^+$. This turns the $PD[\omega_{\mu'}] = [1, \mu', -\frac{1}{2}, \cdots, -\frac{1}{2}]_B$ into $PD[\omega_t] = [1 + t, \mu' + \frac{k}{2}t, -\frac{1}{2} - t, \cdots, -\frac{1}{2} - t]_B$.

  Then by the negative inflation Lemma 5.3.12, we can inflate the form ω_t along some embedded $J-$holomorphic sphere representing the classes E_1, E_2, E_3, E_4, each by size $\frac{t}{2}$. Then by multiplying a rescaling factor $\frac{1}{1+t}$, we have the desired form in $PD[\omega_\mu] = [1, \mu, -\frac{1}{2}, \cdots, -\frac{1}{2}]$. Note that as $t \to \infty$, the resulting $PD[\omega_\mu]$ has $\mu > \frac{k}{2} = l$.

- Case 2: $k = 2l + 1, l \geq 2$ If the curve $B + lF - E_1 - E_2 - E_3 - E_4$ is represented by an embedded $J-$holomorphic sphere. Then we could do the same as the above case.

  Otherwise, for any J there must be one of the classes $A_i = B + lF - E_1 - E_2 - E_3 - E_4 + E_i$ having an embedded $J-$holomorphic sphere representative, because the curve with least self-intersection is in class $B - lF - E_i$. We first inflate $\omega_{\mu'}$ along A_i by t, to obtain $PD[\omega_t] = [1 + t, \mu' + lt, \cdots, -\frac{1}{2}, \cdots - \frac{1}{2} - t]_B$. Secondly, we inflate ω_t along $F - E_i$ by $\frac{t}{2}$ along the embedded E_1, E_2, E_3, E_4, each by size $\frac{t}{2}$. Then by multiplying a rescaling factor $\frac{1}{1+t}$, we have the desired form in $PD[\omega_\mu] = [1, \mu, -\frac{1}{2}, \cdots, -\frac{1}{2}]_B$. Note that as $t \to \infty$, the resulting $PD[\omega_\mu]$ has $\mu > l + \frac{1}{2}$.

- Case 3: $k = 2$. For the prime submanifolds where $B + F - E_1 - E_2 - E_3 - E_4$ is embedded, we can first inflate along it by t and then along the classes E_1, E_2, E_3, E_4, each by size $\frac{t}{2}$.

  Otherwise, the prime submanifolds can be labeled by either one square (-2) curve in the set $\{B - E_p - E_q, B - F\}$, or two square (-2) curve being $\{B - E_p - E_q, B - E_u - E_v\}, \{p, q, u, v\} = \{1, 2, 3, 4\}$. In any case, there exist one class

$B+F-E_1-E_2-E_3-E_4+E_i$ pairing with any curve in the label set being non-negative. Hence $B+F-E_1-E_2-E_3-E_4+E_i$ has an embedded representative. Then we first inflate along $B+F-E_1-E_2-E_3-E_4+E_i$ by t, then along along $F-E_i$ by $\frac{t}{2}$, and finally along the classes E_1, E_2, E_3, E_4, each by size $\frac{t}{2}$.

- Case 4: $k=3$. If the curve $B+lF-E_1-E_2-E_3-E_4$ is represented by an embedded $J-$holomorphic sphere. Then we could do the same as the Case 1.

  If prime submanifolds can be labeled by either one square (-2) curve in the set $\{B-E_p-E_q, B-F\}$, or two square (-2) curve being $\{B-E_p-E_q, B-E_u-E_v\}, \{p,q,u,v\}=\{1,2,3,4\}$, then this is the same as Case 3.

  For the rest cases, the prime submanifolds is labeled by one single -3 curve being $B-E_1-E_2-E_3-E_4+E_s$ or $B-F-E_t$. In the former case, the class $B+F-E_1-E_2-E_3-E_4+E_s$ has and embedded representative, and the latter case $B+F-E_1-E_2-E_3-E_4+E_t$ is embedded. Hence there is always some class $B+F-E_1-E_2-E_3-E_4+E_i$ that has an embedded representative. Then we first inflate along $B+F-E_1-E_2-E_3-E_4+E_i$ by t, then along along $F-E_i$ by $\frac{t}{2}$, and finally along the classes E_1, E_2, E_3, E_4, each by size $\frac{t}{2}$.

□

Then we can **complete the proof of Proposition 5.3.1:**

By Lemma 5.3.13, when $\mu \in (\frac{k}{2}, \frac{k+1}{2}]$, the group $Symp(X, \omega_\mu)$ has the same homotopy type. And because the homological action is the same for any $\omega \in MA$, then the group $Symp_h(X, \omega_\mu)$ has the same homotopy type for the above interval. In particular, their π_0 is the same. And by Lemma 5.3.11, for any such interval $(\frac{k}{2}, \frac{k+1}{2}]$, there's some point such that $\pi_0(Symp_h(X, \omega_\mu)) = P_4(S^2/\mathbb{Z}_2)$. Then for any point $\omega \in MA$, we have $\pi_0(Symp_h(X, \omega_{mu})) = P_4(S^2/\mathbb{Z}_2)$.

Remark 5.3.14. One can give an alternative proof of Lemma 5.3.2 using Lemma 6.1 in [42].

Remark 5.3.15. Indeed, the results in section 5.2 can be interpreted using proof of Proposition 5.3.1: one need to forget more than two strands and hence the resulting $\pi_0(Symp_h(X, \omega))$ is the trivial group.

Remark 5.3.16. If the reduced form is not balanced, then $c_1 \geq c_2 + c_3$, $c_2 \geq c_3 + c_4$ and $c_3 \geq c_4 + c_5$, then this case is covered by the next Lemma 5.4.1. Note that non balanced reduced form can only appear in the 5-face, 4-faces or $MABC$ of the reduced cone as in table 5.1. And any 2-face and 3-face other than $MABC$ are all sets of balanced symplectic forms. The idea of dealing with non-balanced form is to look at their projection to a 2-face or a 3-face which does not contain C or D as a vertex.

Remark 5.3.17. Note that when a form is on MA, the manifold can be equipped with a symplectic G-conic bundle structure (see [12] section 2.1). The non-trivial minimal finite groups given in Theorem 1.7 of [12] which act symplectically and homological trivially can all be realized as finite subgroups of spherical pure braid groups. One may ask the following questions: Is each finite group action which is induced by a Hamiltonian action non-minimal (i.e. obtained by blowing up of some action)? And if the first question has positive answer, then can one use the SMC to classify all the minimal finite group actions?

5.4 Torelli Symplectic mapping class group for a general form

In this section, we deal with any symplectic form ω on $X = \mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$, and show that if there are more than eight symplectic -2 spheres, then $Symp_h(M, \omega)$ is connected.

The proof of this for arbitrary reduced form is **almost done**(see Remark 5.3.16) in the previous section, one only need to consider when $c_1 \geq c_2 + c_3$, $c_2 \geq c_3 + c_4$ and $c_3 \geq c_4 + c_5$, which is covered by the following Lemma 5.4.1:

Lemma 5.4.1. *Given reduced form* $\omega = PD[H - c_1E_1 - c_2E_2 - c_3E_3 - c_4E_4 - c_5E_5]$ *with* $c_1 \geq c_2 + c_3, c_2 > c_3 \geq c_4 \geq c_5$, *then* $Symp_h(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}, \omega)$ *is connected.*

Proof. $X = \mathbb{C}P^2$ with $[\omega] = PD[htH]$. Throughout the proof we compare two symplectic forms on $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$:

- $\omega_s = (1|c_1, c_2, c_3, c_4, c_5)$ where

$$PD[\omega_s] := H - c_1E_1 - c_2E_2 - c_3E_3 - c_4E_4 - c_5E_5,$$

- $\omega_l = (1|c_1, c_2, c_3, c_3, c_3)$ where

$$PD[\omega_l] = H - c_1E_1 - c_2E_2 - c_3E_3 - c_3E_4 - c_3E_5,$$

which can be obtained from ω_s by enlarging the area of E_4, E_5 to c_3.

Indeed ω_s is the projection of ω_l to a 2-face or a 3-face which does not contain C or D as a vertex. We verify that with the given assumption, the vector $(1|c_1, c_2, c_3, c_3, c_3)$ is reduced, and it is a symplectic form: because it pair E_i, $H - E_i - E_j$ and $2H - E_1 \cdots E_5$ are positive, for the last case, $2 - \sum_i c_i = (1 - c_1 - c_2 - c_3) + (1 - c_3 - c_3) > 0$.

And from reduced condition $1 \geq c_1 + c_2 + c_3$, $\omega_l(H - E_1 - E_2) \geq \omega_l(E_3)$; together with given the assumption $c_1 \geq c_2 + c_3$, we know $c_3 < 1/3$. And hence we know that $\omega_l(2H - E_1 - \cdots - E_5) - \omega_l(E_3) = (1 - c_1 - c_2 - c_3) + (1 - c_3 - c_5 - c_3) > 0$. Hence

$$min\{\omega_l(E_i), \omega_l(H - E_i - E_j), \omega_l(2H - E_1 - \cdots - E_5)\} = c_3 \qquad (5.11)$$

Hence given the form ω_l, c_3 is the smallest area of all 16 exceptional curves.

There is a ball packing $\iota_l : \coprod_{i=1}^5 B'(i) \to X$, with image K_l, such that $Vol(B_4') = Vol(B_5') = c_3, Vol(B_i') = c_i$ when $i = 1, 2, 3$.
Note that there is a packing $\iota_s : \coprod_{i=1}^5 B(i) \to X$, with image $K_s \subset K_l$, such that $VolB_i = B_i'$ when $i = 1, 2, 3$ and $B_4 \subset B_4', B_5 \subset B_5'$, with volume $Vol(B_4) = c_4, Vol(B_5) = c_5$.

Blowing up ι_s is the form ω_s which is Poincaré dual to $(1|c_1, c_2, c_3, c_4, c_5)$ with $c_1 > c_2 > c_3 \geq c_4 \geq c_5$ in our assumption, and blowing up ι_l one get the form ω_l Poincaré dual to $(1|c_1, c_2, c_3, c_3, c_3)$. This ω_l is a balanced reduced form. Hence by Lemma 5.2.4 and 5.1.11, $Symp_h(\omega_l)$ is connected.

Then we will derive the connectedness of $Symp_h(\omega_s)$ from $Symp_h(\omega_l)$: We first blow up the balls B_1', B_2', B_3', and denote the symplectic manifold $(X, \omega_{123}$ (refered to as X below). By Lemma 2.3 in [26], the two groups are homotopy equivalent:

$$Symp_h(X_l, E_4', E_5') \simeq Symp_h(X, B_4', B_5').$$

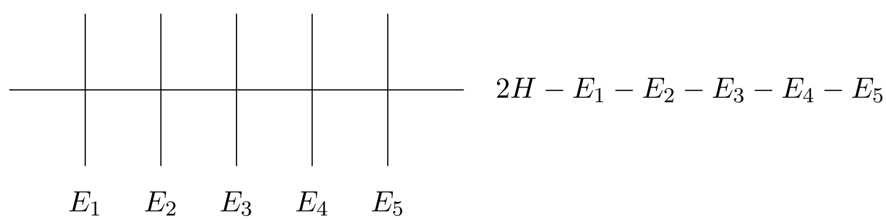
Here X_l is the blow up of X by B_4', B_5', and $Symp_h(X_l, E_4', E_5')$ is the subgroup of

$Symp_h(\omega_l)$ fixing E_4', E_5'. Note that we have a fibration(because of the transitive action, same as 1.1)

$$Symp_h(X_l, E_4', E_5') \to Symp_h(X_l, \omega_l) \to \mathcal{E}_l,$$

Where $\mathcal{E}_l$ is the space of exceptional spheres E_4', E_5'. Recall the area of E_4' and E_5' is c_3 and hence there is no exceptional curve with smaller area as computed in (5.11). There is no embedded pseudo-holomorphic curve with positive coefficient on E_4', E_5'. Hence the space $\mathcal{E}_l$ is contractible and this means $Symp_h(X_l, E_4', E_5')$ is connected. And $Symp_h(X, B_4', B_5')$ is connected.

Now we consider the blow up of ι_s and the configuration C_s



Because $B_4 \subset B_4', B_5 \subset B_5'$, $Symp_h(X, B_4', B_5')$ is a proper subgroup of $Symp_h(X, B_4, B_5)$. Because of Lemma 2.3 in [26], the following map given by blowing up B_4, B_5 is a homeomorphism: $Symp_h(X, B_4, B_5) \overset{Bl}{\to} Symp_h(X_s, E_4, E_5)$, where X_s is the blow up of X by B_4, B_5, and $Symp_h(X_s, E_4, E_5)$ is the subgroup of $Symp_h(\omega_s)$ fixing E_4, E_5.

Hence we proved that there is a proper subset $Symp_h'$ in $Symp_h(\omega_s)$ which is the image of $Symp_h(X, B_4', B_5')$ under the map $Symp_h(X, B_4, B_5) \overset{Bl}{\to} Symp_h(X_s, E_4, E_5)$. One can think this subset fixes small neighborhoods of E_4 and E_5 and acting freely in the complement of the neighborhoods. It is connected because it is a continuous image of a connected domain, and it contains identity because the image of identity element of $Symp_h(X, B_4', B_5')$ is identity in $Symp_h(\omega_s)$. And it can move around the intersection points of $p_i := E_i \cap 2H - E_1 - \cdots - E_5$.

In particular, in $Symp_h(X, B_4', B_5') \subset Symp_h(X, B_4, B_5)$, by Proposition 5.1.4, we can move B_1, B_2, B_3 around each other and we can move each B_1, B_2, B_3 around B_4, B_5, and hence the projection from this subset $Symp_h'$ to $\pi_0(\mathrm{Diff}^+(S^2, 5))$ contains the following: $\{A_{12}, A_{13}, A_{14}, A_{15}, A_{23}, A_{24}, A_{25}, A_{34}, A_{35}\}$, where A_{ij} means move p_i around p_j, as defined before. And this contains a minimal generating set $\{A_{12}, A_{13}, A_{14}, A_{23}, A_{24}\}$ or $\{A_{23}, A_{24}, A_{25}, A_{34}, A_{35}\}$ (the same as in 5.1.9). Hence we proved that in $Symp_h(X_s, \omega_s)$, we can find a subset $Symp'_h$ in the identity component, whose projection to $\text{Diff}^+(S^2, 5)$ contains a minimal generating set $\{A_{12}, A_{13}, A_{14}, A_{23}, A_{24}\}$. This means the generating set is contained in the image of the connecting homomorphism. And hence the whole group $\pi_0(\text{Diff}^+(S^2, 5))$ is in the image of the connecting homomorphism.

Finally, we deal with $Symp_c(U_s)$, where $U_s = X_s \setminus C_s$. Because both U_s and $U'_l = X_l \setminus (C_l)$ are biholomorphic to the complement U of a conic in $\mathbb{C}P^2$. Also note that (U_s, ω_s) is symplectomorphic to $(\mathbb{C}P^2 - 2H) \setminus K_s$ and (U_l, ω_l) is symplectomorphic to $(\mathbb{C}P^2 - 2H) \setminus K_l$. By [15] section 6.5, as a stein U domain, U has a symplectic completion which is $T^*\mathbb{R}P^2$ such that all critical points of the exhausting function are supported on U. And we can Consider U as a complex manifold, the form on U_s gives a finite type plurisubharmonic function whose critical points are in $(\mathbb{C}P^2 - 2H) \setminus K_s$. And the form on U_l gives a finite type Stein structure with critical points in $(\mathbb{C}P^2 - 2H) \setminus K_l$. The natural inclusion $(\mathbb{C}P^2 - 2H) \setminus K_l \hookrightarrow (\mathbb{C}P^2 - 2H) \setminus K_s$ induces a weak homotopy equivalence between $Symp_c(U_s)$ and $Symp_c(U_l)$, by [15] Proposition 15. This means the induced map is an isomorphism between $\pi_0[Symp_c(U_s)]$ and $\pi_0[Symp_c(U_l)]$. Hence pick any connected component of $Symp_c(U_s)$, there is an element ϕ_l supported on $(\mathbb{C}P^2 - 2H) \setminus K_l$. By connectedness of $Symp_h(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}, \omega_l)$, ϕ_l is isotopic to identity. This means the chosen connected component of $Symp_c(U_s)$ is in the identity component of $Symp_h(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}, \omega_l)$.

Hence we have any connected component of $Stab(C)$ and $Symp_c(U_s)$ are in the identity component of $Symp_h(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}, \omega_s)$. This means $Symp_h(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}, \omega_s)$ is connected.

□

And hence we have the concluding proposition 5.4.2 about the connectedness of $Symp_h$ for generic symplectic form:

Proposition 5.4.2. *Given $(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}, \omega)$, with any form ω, then if there are more than 8 symplectic -2 sphere classes, then $Symp_h$ is connected.*

Proof. If the form is reduced, then Proposition 5.2.4 and Lemma 5.4.1 cover it.

And for a non-reduced form which has more than eight symplectic -2 spheres, it is diffeomorphic to a reduced form having the same number of symplectic -2 spheres, by remark 5.1.10. And further, diffeomorphic forms have homeomorphic symplectomorphism group. Hence $Symp_h$ is connected for any form with more than eight symplectic -2 spheres.

□

Following from [35] that $Symp(X, \omega)$ can realize any homological action preserving the canonical class, we have

Corollary 5.4.3. *If there are more than 8 symplectic -2 sphere classes, then* $\pi_0(Symp(X, \omega))$ *is the homological action, which is a subgroup of* $W(\mathbb{D}_5)$*(the Weyl group of root system* $\mathbb{D}_5$*).*

Remark 5.4.4. As implicated in [55] and [15], in the monotone case(case 8 in tabel 5.1), any i, j, E_i has the same area as E_j, then in contrast against theorem 5.2.1, the swapping of two ball of the same size is not isotopic to identity, and it is a set of generator of $\pi_0(Symp(X, \omega))$ that satisfying the braid relation. One can further see from the fact that $\pi_0(Symp(X, \omega)) = \mathrm{Diff}^+(S^2, 5)$ that square Lagrangian Dehn twists provide another set of generators of $\pi_0(Symp(X, \omega))$ that satisfying the braid relation. For the relation of the two sets of generators: the ball swapping generator A_{ij} is compactly supported on a domain contain E_i and E_j, while the square Lagrangian Dehn twists along $E_i - E_j$ is compactly supported on a neighborhood of $E_i - E_j$. Hence one may expect the two generators to be isotopic.

Remark 5.4.5. In [42] Remark 1.11, an approach to establish the connectedness for $Symp_h$ by deforming Lagrangian Dehn twists to symplectic Dehn twists was outlined by Dusa McDuff, when the form has 5 distinct blow-up sizes and each slightly smaller than 1/3.

Corollary 5.4.6. *One can see that for any form* ω *except* ω *on edge* MA *and the monotone point* M *in table 5.1, any square Lagrangian Dehn twist is isotopic to identity because* $Symp_h$ *is connected.This fact can be applied to compute Quantum cohomology of the given form on* $X = \mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$*:*

Together with Corollary 2.8 in [55], we have $QH_*(X)/I_l$ is Frobenius for any Lagrangian L for a given form in the cases above, where I_l is the ideal of $QH_*(X)$ generated by the Lagrangian L.

5.5 Fundamental group and topological persistence of $Symp(X,\omega)$

Now we consider the rank of $\pi_1(Symp_h(X,\omega)$ when For a given form ω in table 5.1, more than 8 symplectic -2 spheres implies the connectedness of $Symp_h$, and we have the following:

Lemma 5.5.1. *Let $X = \mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$ with reduced symplectic form that can be normalized to be $\omega = (1|c_1, c_2, c_3, c_4, c_5)$ such that $c_i < 1/2$. If $\pi_0(Symp_h)$ is trivial, then there is a lower bound $N-5$ of the rank of $\pi_1(Symp_h(X,\omega)$, where N is the number of ω-symplectic spheres with self intersection -2.*

Proof. For a symplectic form that can be normalized to $(1|c_1, c_2, c_3, c_4, c_5)$ with $c_i < 1/2$, since $Symp_h$ is assumed to be connected, by Proposition 5.1.4, we have the exact sequence

$$1 \to \pi_1(Symp_h(X,\omega)) \to \pi_1(\mathcal{C}_0) \xrightarrow{\phi} \pi_0(\mathrm{Diff}^+(S^2,5)) \to 1 \qquad (5.12)$$

We consider the abelianization of this exact sequence. Since the abelianization functor is right exact and $\pi_1(Symp_h(X,\omega)$ is abelian, we have the induced exact sequence

$$\pi_1(Symp_h(X,\omega)) \to Ab(\pi_1(\mathcal{C}_0)) \xrightarrow{f} \mathbb{Z}^5 \to 1 \qquad (5.13)$$

Since $Symp_h$ is assumed to be connected, by Lemma 3.4.2, the number of generator of $H_1(J_{open}) = Ab(\pi_1(\mathcal{C}_0))$ is the same as the number N of -2 symplectic spheres listed as above. Statement a) follows immediately.

□

Remark 5.5.2.

- When the reduced form is not balanced, then push forward this form using any diffeomorphism to obtain a push forward form $(\lambda|c_1, c_2, c_3, c_4, c_5)$, there are always some $c_i \geq 1/2\lambda$. This case $Stab(C)$ might be homotopic to $\mathrm{Diff}^+(S^2,5)$ or the extension of $\mathrm{Diff}^+(S^2,5)$ by $\mathbb{Z}$, see Remarks 5.1.6. This case sequence 5.13 becomes

$$\pi_1(Symp_h(X,\omega)) \to Ab(\pi_1(\mathcal{C}_0)) \xrightarrow{f} \mathbb{Z}^6 \to 1 \tag{5.14}$$

where $\mathbb{Z}^6$ comes from abelianization of $0 \to \mathbb{Z} \to \pi_0(Stab(C)) \to \mathrm{Diff}^+(S^2,5) \to 0$. And we obtain $N-6$ as the lower-bound.

- If Torelli SMC is non-trivial for a non-monotone symplectic form, then it has to be on MA. We first assume $c_i < 1/2$ and denote this form ω_a: And using the same argument as in Lemma 5.5.1, we have

$$1 \to \pi_1(Symp_h(X,\omega_a)) \to \pi_1(\mathcal{C}_0) \to Im(\phi) = \pi_1(S^2 - \{4 \text{ points}\})) \to 1 \tag{5.15}$$

  We consider the abelianization of this exact sequence. Since the abelianization functor is right exact and $\pi_1(Symp_h(X,\omega_a)$ is abelian, we have the induced exact sequence

$$\pi_1(Symp_h(X,\omega_a)) \to Ab(\pi_1(\mathcal{C}_0)) \xrightarrow{f} \mathbb{Z}^3 \to 1. \tag{5.16}$$

  And hence we obtain a lower bound on for a form ω_a on MA, rank of $\pi_1(Symp_h(X,\omega_a)) \geq 5$. While without assumption we obtain $\pi_1(Symp_h(X,\omega_a) \geq 4$.

- And we believe that $Stab(C)$ can be made precise to be $\mathrm{Diff}^+(S^2,5)$ but could not do this currently due to a technicality. And the then lower bound could be strengthen as $N-5$ with the same proof, and hence it is the lower-bound of rank $\pi_1(Symp_h(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}))$ when $Symp_h$ is connected.

On the other hand, [42] gives approach to obtain the upper-bound of $\pi_1(X,\omega)$, where M is a symplectic rational 4 manifold. We can follow the route of Proposition 6.4 in [42] to give a proof of the following result, generalizing Dusa's Corollary 6.9:

Proposition 5.5.3. *Let (X,ω) be $\mathbb{C}P^2$ or its blow up at several points with a given reduced form, $(\tilde{X}_k,\tilde{\omega}_\epsilon)$ be the blow up of X at k points with different small size(less* *than any blow up size of X), then the rank of* $\pi_1(Symp_h(\tilde{X}_k))$ *can exceed* $\pi_1(Symp_h(X))$ *at most* $rk + k(k-1)/2$*, where* r *is the rank of* $\pi_2(X)$*.*

Proof. This is based on the proof of Corollary 6.9 and Proposition 6.4,6.5 in [42]: One need to argue that the exceptional sphere E_k in $\tilde{X}_k$ with smallest blow-up size always has an embedded representative. And this fact follows easily from the observation 2.1.5 we made in section 2.1.

Then the rest follows from 6.9 in [42] and counting Hamiltonian bundles in 6.4.

□

Remark 5.5.4. Note that for $S^2 \times S^2$ with size $(\mu, 1), \mu \geq 1$, a equal blow up of k points where the size $c < \frac{1}{2}$. One can easily check that the exceptional sphere E_k in $\tilde{X}_k$ with smallest blow up size always has an embedded representative. Hence by counting Hamiltonian bundle tech one obtain a upper-bound of rank of $\pi_1(Symp_h(S^2 \times S^2))$ plus $2k$. Note that rank of $\pi_1(Symp_h(S^2 \times S^2))$ means the free rank, where for monotone $S^2 \times S^2$ is 0 and non-monotone $S^2 \times S^2$ is 1.

Remark 5.5.5. Note that if we allow the blow up sizes to be all equal, then counting Hamiltonian bundle gives the following:

$$Rank[\pi_1(Symp(\tilde{X}_k, \tilde{\omega}_\epsilon)] \leq Rank[\pi_1(Symp(X, \omega))] + rk,$$

where where r is the rank of $\pi_2(X)$, and k is the number of points of blow up of $\tilde{X}_k$ from X.

Further, using the above argument together with Proposition 6.5 in [42], one can prove the following:

Lemma 5.5.6. *For* $(X, \omega) = (\mathbb{C}P^2, \omega_{FS})$*, let* $(\tilde{X}_k, \tilde{\omega}_\epsilon)$ *be the blow up of* (X, ω) k *times with area of* E_i *being* ϵ_i *and* $\tilde{\omega}_\epsilon$ *being a reduced form, then*

$$Rank[\pi_1(Symp(\tilde{X}_k, \tilde{\omega}_\epsilon)] \leq k + N_E,$$

where N_E *is the number of* -2 *spheres whose homology class is* $E_i - E_j$*.*

Hence Lemma 5.5.1 and Theorem 5.5.3 together give the precise rank of $\pi_1(Symp_h(X, \omega)$ in many cases:

Proposition 5.5.7. *The upper-bound given by 5.5.3 can be realized for 1,2,3,4,5 fold blow up of* $\mathbb{C}P^2$ *when the form is not on* MA.

Further, suppose the blow up sizes $c_1, \cdots, c_5$, *if* $c_i < 1/2$ *and the TSMC is connected (characteristized by existing more than 8 Symplectic -2 spheres), then the upper-bound given in Theorem 5.5.3 equals the lower-bound given in Lemma 5.5.1. Namely, if* $c_i < 1/2$, *and TSMC is connected, then rank of* $\pi_1(Symp_h(X, \omega)) = N - 5$.

Proof. For up to 4 fold blow up of $\mathbb{C}P^2$ with any form, rank of $\pi_1(Symp_h(X, \omega)$ is explicitly given in tables 4.2,4.3,4.4.

For $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$, we give the computation of each cases:

We give the optimal upper-bound using the following different methods, and show 1) and 2) are equal to the lower bound in each case:

- 1) For any k-face with vertex D, we have $c_4 > c_5$; then we use $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$ with sizes c_1, c_2, c_3, c_4 and find rank $R_4 = \pi_1(Symp_h(X, \omega)$ for $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$ in table 4.4. And by Theorem 5.5.3, the upper-bound for $\pi_1(Symp_h(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}))$ is no larger than $R_4 + 5$. Because E_5 is the only smallest area exceptional sphere, there are 10 symplectic -2 spheres pairing E_5 nonzero. Hence $N = R_4 + 10$, where by table 4.4, R_4 is the number of symplectic -2 spheres pairing E_5 equal 0. Hence we have the lower-bound $N - 5$ equals the upper-bound $R_4 + 5$.

- 2) For any k-face without vertex D but with C, we have $c_3 > c_4 = c_5$, then we use $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ with sizes c_1, c_2, c_3, and find rank $R_3 = \pi_1(Symp_h(X, \omega)$ for $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ in table 4.3. And by Theorem 5.5.3, the upper-bound for $\pi_1(Symp_h(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}))$ is no larger than $R_3 + 4 + 4 = R_3 + 8$. Because E_4, E_5 are the only two smallest area exceptional spheres, there are 15 (6 has only E_4, 6 has only E_5, 3 has both) symplectic -2 spheres pairing both E_4, E_5 nonzero. Hence $N = R_3 + 13$, where by table 4.3, R_3 is 2 plus the number of symplectic -2 spheres pairing both E_4, E_5 equal 0. Hence we have the lower-bound $N - 5$ equals the upper-bound $R_3 + 8$.

- 3)For any k-face without vertex D or C but with B, we have 4 cases, $MOAB$, MOB, MAB, MB. For $MOAB$, MOB, we have $c_2 > c_3 = c_4 = c_5$, and we use $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ with sizes c_1, c_2 in table 4.2. And we have the upper-bound equals $R_2 + 3 + 3 + 3 = 11$, where by 4.2, R_2 is 2 plus the number of symplectic -2 spheres pairing both E_4, E_5 equal 0. For MAB, perform base change 2.2, $B = 1 - c_2 >= F = 1 - c_1; E_1 = \cdots = E_4 = c_3$ then by Remark 5.5.4 $rk_1 \leq 1 + 2 + 2 + 2 + 2 = 9$, which coincide with the lower bound. For case MB, perform base change 2.2, $B = F = 1 - c_1; E_1 = \cdots = E_4 = c_3$ then by Remark 5.5.4 $rk_1 \leq 0 + 2 + 2 + 2 + 2 = 8$, which coincide with the lower bound.

- 4)For any k-face without vertex $B\ C\ D$, but with A, actually only MOA and MA: we have $c_1 > c_2 = c_3 = c_4 = c_5$. For this case, we use $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. And we have the upper-bound on equals $1 + 2 + 2 + 2 + 2 = 9$ and MA the same. (Note this method does not always give the precise rank, MA for instance, see Remark 5.5.8. But for MOA, π_0 is trivial and it does give the precise rank.)

- 5)MO: we use $\mathbb{C}P^2$ and the upper-bound equals 5.

More precisely, assuming $c_i < 1/2$, and let $G = \pi_1(Symp_h(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}))$

- For 5-face $MOABCD$, rank of G is 15.
- For each 4-face: from $MOABC$ to $MABCD$, rank of G is 14.
- For each 3-face with $N = 18$, MOBC,MOBD, MABC,MBCD: rank of G is 13.
- For the rest 3-faces, $N = 17$, rank of $G$is 12.
- For each 2-face with $N = 16$, MOB,MOC,MAC,MBC,MCD, rank of G is 11.
- For edge MC, the rank of G is 10.
- For each 2-face with $N = 14$ and containing vertex O, MOA,MOD,MAD,MAB: rank of G is 9.
- For case MB, perform base change 2.2, $B = F = 1 - c_1; E_1 = \cdots = E_4 = c_3$ then rank of $G \leq 0+2+2+2+2 = 8$, which coincide with the lower bound and hence rank is 8.

- For edge MO,MD, the rank of G is 5.

□

Remark 5.5.8. Again assuming after normalization, $c_i < 1/2$, denote rk_1 as rank of $\pi_1(Symp_h(\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}))$: For case MA, using method 1), $rk_1 \leq 9$. And together with Remark 5.5.2, we obtain $5 \leq rk_1 \leq 9$ in this case.

Corollary 5.5.9. *Homologous -2 symplectic spheres in 5 blowups are symplectically isotopic for any symplectic form.*

Proof. No less than 10 spheres, π_0 is trivial, then the conclusion follows naturally.

For the symplectic form with 8 $\omega-$ symplectic -2 sphere classes, homological action acts transitively in -2 classes because the ω area are the same. Hence the number of isotopy classes for each homology class is a constant $k, (k \in \mathbb{Z}^+ \cup \{\infty\})$. By the upper-bound in 5.5.3 and the argument in Lemma 5.5.1, rank $\pi_1(\mathcal{C}_0)$ is less than 12. If $k > 1$, then $8k \geq 16 > 12$, contradiction. This means homologous -2 symplectic spheres has to be symplectically isotopic. □

Remark 5.5.10. Note that in Theorem 5.5.3, using X to be blow up of several points of $\mathbb{C}P^2$ together with results in section 3.2, instead of $\mathbb{C}P^2$ itself, one get finer results on the upper-bound of rank $\pi_1(Symp_h(X,\omega)$, for example:

In case $MABCD$ in form 5.1, using $\mathbb{C}P^2$ one have 15 as the upper-bound, while using $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$ of sizes c_1, c_2, c_3, one have $5+4+5=14$ as the upper-bound. And Lemma 5.5.1 gives 14 as lower bound of rank $\pi_1(Symp_h(X,\omega)$ in this case. Hence 14 is the precise value of rank $\pi_1(Symp_h(X,\omega)$.

Note that on any k-face with B as a vertex, one have $c_1+c_2+c_3 = 1$, then Theorem 5.5.3 is needed for the computation of the precise rank of $\pi_1(Symp_h(X,\omega)$.

Corollary 5.5.11. *$\pi_0(Symp(X,\omega)$ is a reflection group, often denoted as $\Gamma(X,\omega)$. Assuming $c_i < 1/2$ and there are no less than 15 Symplectic -2 spheres, the number*

$$PR[\pi_0(Symp(X,\omega)] + Rank[\pi_1(Symp(X,\omega))] - rank[\pi_0(Symp_h(X,\omega))]$$

is a constant and is equal to $20 - Rank(PB_5/\mathbb{Z}^2) = 15$, where $PR[\pi_0(Symp(X,\omega)]$ is the number of positive roots of $\pi_0(Symp(X,\omega)$, and rank of $\pi_0(Symp_h(X,\omega))$ means the cardinality of the minimal generating set of $\pi_0(Symp_h(X,\omega))$.

Remark 5.5.12. Note that only when the closed manifold has a Lagrangian root lattice other than type $\mathbb{A}_n$ or its direct product, can one obtain a non-trivial Torelli SMC. Further, our results suggest that there is a coherent approach for the Symplectic mapping class group and $\pi_1(Symp_h(X,\omega)$ for a rational 4-manifold with Euler number up to 11.

And we also make the conjecture on the persistence type result analogous to Corollary 1.2.7 will also apply here:

Conjecture 5.5.13.

$$PR[\Gamma(X,\omega)] + Rank[\pi_1(Symp(X,\omega))] - Rank[\pi_0(Symp_h(X,\omega))]$$

is a constant equaling 15 for any symplectic form on $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$.

k-face	Γ_L	N	$\omega := (1\|c_1, c_2, c_3, c_4, c_5)$
Point M	$\mathbb{D}_5$	0	monotone
MO	$\mathbb{A}_4$	10	$1 > \lambda; c_1 = c_2 = c_3 = c_4 = c_5$
MA	$\mathbb{D}_4$	8	$\lambda = 1; c_1 > c_2 = c_3 = c_4 = c_5$
MB	$\mathbb{A}_1 \times \mathbb{A}_3$	13	$\lambda = 1; c_1 = c_2 > c_3 = c_4 = c_5$
MC	$\mathbb{A}_2 \times \mathbb{A}_2$	15	$\lambda = 1; c_1 = c_2 = c_3 > c_4 = c_5$
MD	$\mathbb{A}_4$	10	$\lambda = 1; c_1 = c_2 = c_3 = c_4 > c_5$
MOA	$\mathbb{A}_3$	14	$1 > \lambda; c_1 > c_2 = c_3 = c_4 = c_5$
MOB	$\mathbb{A}_1 \times \mathbb{A}_2$	16	$1 > \lambda; c_1 = c_2 > c_3 = c_4 = c_5$
MOC	$\mathbb{A}_1 \times \mathbb{A}_2$	16	$\lambda < 1; c_1 = c_2 = c_3 > c_4 = c_5$
MOD	$\mathbb{A}_3$	14	$1 > \lambda; c_1 = c_2 = c_3 = c_4 > c_5$
MAB	$\mathbb{A}_3$	14	$\lambda = 1 c_1 > c_2 > c_3 = c_4 = c_5$
MAC	$\mathbb{A}_1 \times \mathbb{A}_1 \times \mathbb{A}_1$	17	$\lambda = 1; c_1 = c_2 > c_3 > c_4 = c_5$
MAD	$\mathbb{A}_3$	14	$\lambda = 1;\ c_1 > c_2 = c_3 = c_4 > c_5$
MBC	$\mathbb{A}_1 \times \mathbb{A}_1 \times \mathbb{A}_1$	17	$\lambda = 1; c_1 > c_2 = c_3 > c_4 = c_5$
MBD	$\mathbb{A}_1 \times \mathbb{A}_2$	16	$\lambda = 1;\ c_1 = c_2 > c_3 = c_4 > c_5$
MCD	$\mathbb{A}_1 \times \mathbb{A}_2$	16	$\lambda = 1;\ c_1 = c_2 = c_3 > c_4 > c_5$
MOAB	$\mathbb{A}_2$	17	$\lambda < 1;\ c_1 > c_2 > c_3 = c_4 = c_5$
MOAC	$\mathbb{A}_1 \times \mathbb{A}_1$	18	$\lambda < 1;\ c_1 > c_2 = c_3 > c_4 = c_5$
MOAD	$\mathbb{A}_2$	17	$\lambda < 1;\ c_1 > c_2 = c_3 = c_4 > c_5$
MOBC	$\mathbb{A}_1 \times \mathbb{A}_1$	18	$\lambda < 1;\ c_1 = c_2 > c_3 > c_4 = c_5$
MOBD	$\mathbb{A}_1 \times \mathbb{A}_1$	18	$\lambda < 1;\ c_1 = c_2 > c_3 = c_4 > c_5$
MOCD	$\mathbb{A}_2$	17	$\lambda < 1;\ c_1 = c_2 = c_3 > c_4 > c_5$
MABC	$\mathbb{A}_1 \times \mathbb{A}_1$	18	$\lambda = 1;\ c_1 > c_2 > c_3 > c_4 = c_5$
MABD	$\mathbb{A}_2$	17	$\lambda = 1;\ c_1 > c_2 > c_3 = c_4 > c_5$
MACD	$\mathbb{A}_1 \times \mathbb{A}_1$	18	$\lambda = 1;\ c_1 > c_2 = c_3 > c_4 > c_5$
MBCD	$\mathbb{A}_1 \times \mathbb{A}_1$	18	$\lambda = 1;\ c_1 = c_2 > c_3 > c_4 > c_5$
MOABC	$\mathbb{A}_1$	19	$\lambda < 1;\ c_1 > c_2 > c_3 > c_4 = c_5$
MOABD	$\mathbb{A}_1$	19	$\lambda < 1;\ c_1 > c_2 > c_3 = c_4 > c_5$
MOACD	$\mathbb{A}_1$	19	$\lambda < 1;\ c_1 > c_2 = c_3 > c_4 > c_5$
MOBCD	$\mathbb{A}_1$	19	$\lambda < 1;\ c_1 = c_2 > c_3 > c_4 > c_5$
MABCD	$\mathbb{A}_1$	19	$\lambda = 1;\ c_1 > c_2 > c_3 > c_4 > c_5$
MOABCD	trivial	20	$\lambda < 1;\ c_1 > c_2 > c_3 > c_4 > c_5$

Table 5.1: Reduced symplectic form on $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$

element in $\mathcal{C}$	# of g.i.p. on E_5	# of g.i.p. on $H - E_1 - E_5$
$H - E_2 - E_3 - E_5$	3	5
$H - E_2 - E_4 - E_5$	3	5
$H - E_3 - E_4 - E_5$	3	5
$E_1 - E_5$	3	5
$H - E_2 - E_3 - E_4$	5	3
$E_1 - E_2$	5	3
$E_1 - E_3$	5	3
$E_1 - E_4$	5	3

Table 5.2: number of geometric intersection points (g.i.p.) for $J \in \mathcal{J}_{\mathcal{C}}$.

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