Automorphic Spectral Analysis of a Self-Adjoint Operator Attached to a Triple-Product L-Function

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Dedication

To Karen

Abstract

The spectral theory of unbounded self-adjoint operators applied to invariant Laplacians on arithmetic quotients gives information about analytic behavior of L-functions. Given three cuspforms f_1, f_2, f on SL_2 and a strong subconvexity assumption on $L(s, f_1 \times f_2 \times f)$, we specify a natural Hilbert space of automorphic forms and a self-adjoint operator T such that the discrete spectrum (*if any*) of T is a subset of values $s(s-1)$ for $L(s, f_1 \times f_2 \times f) = 0$. Self-adjointness of T implies real eigenvalues, which implies that any such s is on the critical line or in $\mathbb{R}$.

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Chapter 1

Background

1.1 Hilbert-Polya

Speculation on the Riemann Hypothesis and self-adjoint operators led Hilbert and Polya to independently comment around 1910 about a hypothetical correspondence between zeroes of the Riemann zeta function and eigenvalues of an unknown self-adjoint operator (see [Odlyzko]). While largely off-the-cuff, the remarks of Hilbert and Polya have become iconic in the history of the Riemann zeta function and give some context for this project. One version of the Hilbert-Polya idea is that a self-adjoint operator T with eigenvalues $\lambda = s(s-1)$ has $\mathrm{Re}(s) = 1/2$ or $s \in \mathbb{R}$. The task would be to arrange that T is related to $\zeta(s)$ or other L-functions.

Research in the late 19th and early 20th century expanded to generalizations of the Riemann zeta function to L-functions of various types, including those attached to modular forms on the quotient $\Gamma\backslash\mathfrak{H}$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\mathfrak{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$, and more generally, reductive algebraic groups. The Hilbert-Polya idea, and related, are in principle applicable.

1.2 Haas Computation and Consequences

In 1977, Haas attempted a numerical investigation of the eigenvalues $\lambda = s(s-1)$ of the invariant Laplacian on $\Gamma\backslash\mathfrak{H}$

$$\Delta := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

He sent the list of spectral parameters s to A. Terras, who showed it to H. Stark and D. Hejhal. Stark noticed zeros of the Riemann zeta function and Hejhal observed zeros of an L-function with character modulo three among the spectral parameters. Hejhal, on further investigation of Haas' method, determined that, in effect, Haas was solving the *inhomogeneous* differential equation

$$(\Delta - \lambda)u = \delta_\omega^{\mathrm{afc}} \tag{1.1}$$

where $\delta_\omega^{\mathrm{afc}}$ is the automorphic Dirac delta at $\omega = e^{2\pi i/6}$. The numbers $\lambda = s(s-1)$ are not necessarily eigenvalues of a self-adjoint operator, so we cannot conclude that $s(s-1)$ is real.

The automorphic Dirac delta at ω, on the right side of equation (1.1) produces a *fundamental solution* as studied by Neunhöffer (cf. [Neunhöffer 1973]), Fay (cf. [Fay 1976]), and Niebur (cf. [Niebur 1973]) in the 1970's. The inhomogeneity means that even when $\zeta(s)L(s,\chi_{-3}) = 0$ the solution u is not necessarily a cuspform, but only a *pseudo-cuspform*, characterized by exponential decay in y, eventually vanishing constant term, and being *locally* an eigenfunction of the homogeneous equation when $y >> 1$. In particular, λ need not be an eigenvalue of a self-adjoint operator, so need not be *real*. A more general form of equation (1.1) involves a compactly supported distribution θ on $\Gamma\backslash\mathfrak{H}$:

$$(\Delta - \lambda)u = \theta \tag{1.2}$$

Compactly-supported distributions θ can be spectrally decomposed in a *global* automorphic Sobolev space (cf. [Garrett 2017] Sections 12.3, 12.4) as

$$\theta = \sum_F \langle \theta, F\rangle \cdot F + \frac{\langle \theta, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \langle \theta, E_s\rangle \cdot E_s ds \tag{1.3}$$

where the sum is over an orthonormal basis of cuspforms. Note that this equality is not in an L^2 sense but in a suitable Sobolev space. None of the integrals in the pairings involving θ is a literal integral, but must be defined using extension by continuity as in Section 6.

Since equation (1.1) is not homogeneous, we apparently do not get eigenvalues λ of a self-adjoint operator, hence, no guarantee that such λ are real. This obstacle was investigated by Y. Colin de Verdière and Bombieri-Garrett (cf. [Colin de

Verdière 1982], [Colin de Verdière 1983], [Bombieri-Garrett 2017]). The idea is to use the Friedrichs self-adjoint extension of a symmetric, semi-bounded, densely-defined operator restricted to a *subspace* of $L^2(\Gamma\backslash\mathfrak{H})$ to get a *homogeneous* equation. We choose θ strategically so that the period $\langle\theta, E_s\rangle$ is related to ζ or other L-functions. For example, [Garrett 2011] recalls the following fact known to Hecke and Maaß. If $\langle\theta, f\rangle = \overline{f}(\omega)$, where $\omega = e^{2\pi i/6}$, then

$$\langle\theta, E_{1-s}\rangle = \left(\frac{\sqrt{3}}{2}\right)^s \frac{\zeta_k(s)}{\zeta(2s)}$$

where $k = \mathbb{Q}(\sqrt{-3})$. As another example, for H a suitable copy of $\mathrm{GL}_1(\mathbb{Q}(\sqrt{2}))$ inside $GL_2(\mathbb{Q})$, with

$$\langle\theta, f\rangle = \int_{\Gamma\cap H\backslash H} \overline{f}(h\cdot i)dh$$

then

$$\langle\theta, E_{1-s}\rangle = \text{const}\cdot\frac{\Gamma(\frac{s}{2})\zeta_k(s)}{\Gamma(\frac{s+1}{2})\zeta(2s)} \qquad \Big(\text{with } k = \mathbb{Q}(\sqrt{2})\Big)$$

In the Haas episode, θ was a distribution supported at a point, and

$$\langle\theta, E_{1-s}\rangle = \frac{\zeta_k(s)}{\zeta(2s)}$$

with $k = \mathbb{Q}(\sqrt{-3})$. For $\theta \in H^{-1}$ and $u \in H^{+1}$ (with Sobolev spaces explicated in section 6.1), the nature of the Friedrichs extension gives the following:

1.2.1 Theorem *For Friedrichs extension $\widetilde{S_\Theta}$ of a restriction S_Θ defined by (7.8) with $w \in \mathbb{C}$ and $\lambda_w = w(1-w) > 1/4$*

$$(S_\Theta - \lambda_w)u = \theta \quad \textit{if and only if} \quad (\widetilde{S_\Theta} - \lambda_w)u = 0 \textit{ and } \theta u = 0 \tag{1.4}$$

Proof. Proof in Section 7.2.5. □

From the spectral decomposition (1.3), we have

$$(\Delta - \lambda_w)u = \sum_F \langle \theta, F\rangle \cdot F + \frac{\langle \theta, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(1/2)} \langle \theta, E_s\rangle \cdot E_s \, ds$$

which is convergent in H^{-1} with suitable pairings $\langle \theta, F\rangle$ and $\langle \theta, E_s\rangle$. We solve the differential equation *by division*:

$$u = \sum_F \frac{\langle \theta, F\rangle \cdot F}{\lambda_F - \lambda_w} + \frac{\langle \theta, 1\rangle \cdot 1}{\lambda_w \langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(1/2)} \frac{\langle \theta, E_s\rangle \cdot E_s}{\lambda_s - \lambda_w} \, ds \quad (\text{for } \mathrm{Re}(w) > \frac{1}{2})$$

which is convergent in H^{+1}, where λ_F is the eigenvalue of Δ for F.

Remark: $\delta_\omega^{\mathrm{afc}} \notin H^{-1}$, thus failing to meet requirements for Theorem 1.2.1. [Colin de Verdière 1983] suggested resolving this difficulty by projecting onto the continuous spectrum. He cited Lindelöf-on-average second moment bounds of the Dedekind ζ of the quadratic field to show the continuous spectrum restriction of $\delta_\omega^{\mathrm{afc}}$ is in H^{-1}.

An investigation roughly suggested by [Colin de Verdière 1982], [Colin de Verdière 1983] and executed rigorously by [Bombieri-Garrett 2017] looks only at the non-cuspidal spectrum in equation (1.3). This projects to the orthogonal complement to cuspforms. This restriction of the distribution is in H^{-1}, so that the Friedrichs extension behaves as required.

1.3 Triple-product L-functions, Spectral Parameters of Friedrichs Extension

We investigate a situation similar to the one inadvertently arising in Haas' numerical analysis. However, instead of a compactly supported distribution, we consider an L-function-producing period of Eisenstein series recast as a distribution. We replace $\delta_\omega^{\mathrm{nc}}$ with a distribution θ restricting a function to a subspace and integrating against cuspforms on the subspace (section 8.1). The automorphic forms are on the symplectic group $\mathrm{Sp}_{4\times 4}$ over the rational numbers $k = \mathbb{Q}$. Section 2 describes the physical setting while Section 6 describes the structure of the function spaces and their spectral decomposition.

As in [Bombieri-Garrett 2017], we restrict the domain to a small subspace of the spectrum so the distribution θ is in the -1 index Sobolev space and the solution u is in the domain of the Friedrichs extension. In section 6.2 we describe how to accomplish this, by restricting the spectral component to one maximal parabolic (either Siegel or Klingen) and fixing the cuspform in the cuspidal-data Eisenstein series, thus specifying an *Eisenstein-Sobolev* space $\mathfrak{E}^r$.

Section 8.1.2 describes a strong subconvexity assumption on the triple-product L-function $\Lambda(s, f_1 \otimes f_2 \otimes f)$ required to obtain a distribution in $H^{-1}(\mathrm{Sp}_{4\times 4}(k)\backslash \mathrm{Sp}_{4\times 4}(\mathbb{A}))$ to satisfy Theorem 1.2.1.

In Section 8.1 we discuss how to recast the triple-product-producing period of Eisenstein series as a distribution. Sections 3.2 and 3.3 recall the necessary concepts.

As in the Colin de Verdière investigation, the idea is make an inhomogeneous differential equation homogeneous by modifying the operator and the function space being operated upon. One goal is to show, as in [Bombieri-Garrett 2017], that the discrete spectrum λ_s, *if any*, only can occur for s among the zeros of θE_s.

Our main goal is to investigate the spectrum of a self-adjoint operator attached to the integral representation of triple-product L-function recast as a distribution θ with support on a smaller group H. We first restrict the unbounded operator descended from Casimir Ω to a small fragment of $L^2(G_k\backslash G_{\mathbb{A}}/K)$ and make a strong subconvexity assumption to get θ into $H^{-1}(G_k\backslash G_{\mathbb{A}}/K)$, and then take the Friedrichs extension. This allows us to solve an inhomogeneous differential equation as in Theorem 1.2.1. That is, for $f_1 \otimes f_2$ a fixed cuspform on $H = \mathrm{SL}_2 \times \mathrm{SL}_2$, let

$$\langle \theta, E^P_{s,F} \rangle = \int_{H_k\backslash H_{\mathbb{A}}} E^P_{s,F}(h) \cdot\ f_1 \otimes f_2(h) dh \tag{1.5}$$

We will address the nature of the pairings $\langle \theta, E^P_{s,f} \rangle$ to verify that the spectral synthesis is legitimate in each case, ie., that the function

$$s \to \langle \theta, E^P_{s,F} \rangle$$

is in a suitable weighted L^2-space. Importantly, while $E_{s,f}$ is smooth of moderate growth, it is not in any Sobolev space, so we cannot allow arbitrary $\theta \in H^{-1}(G_k\backslash G_{\mathbb{A}}/K)$. The bilinear pairing makes sense and θ can be spectrally decomposed as above when θ is in the space of *compactly supported* distributions $\mathcal{E}_{s,F}(G_k\backslash G_{\mathbb{A}}/K)^*$, the dual space to the space space of smooth functions $C^\infty(G_k\backslash G_{\mathbb{A}}/K)$, which is inside $H^{-\infty}(G_k\backslash G_{\mathbb{A}}/K)$.

However, (integrations against) cuspforms are *not* compactly supported but only of rapid decay. Fortunately, the topological vector space dual to smooth, moderate growth functions $C^{\infty}_{\mathrm{Mod}}(G_k\backslash G_{\mathbb{A}}/K)$, including Eisenstein series, is the space of rapid decay distributions $D_{\mathrm{Rap}}(G_k\backslash G_{\mathbb{A}}/K)$ (ref. Proposition 3.3.2), which includes our θ.

Chapter 2

Physical Setting

We will consider automorphic forms on a quotient of $G = \mathrm{Sp}_{4\times 4}(\mathbb{A})$ for the rational number field $k = \mathbb{Q}$. That is,

$$G = \mathrm{Sp}_{4\times 4} := \{g \in \mathrm{GL}_4 | g^t J g = J, \text{ where } J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}\} \tag{2.1}$$

To produce the triple-product L-function let

$$H = (\mathrm{SL}_2 \times \mathrm{SL}_2) := \{(g_1, g_2) : g_1, g_2 \in \mathrm{SL}_2\} \tag{2.2}$$

with the inclusion $H \hookrightarrow G$ defined by

$$j : \begin{pmatrix} a & b \\ c & c \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \to \left(\begin{array}{cc|cc} a & & b & \\ & a' & & b' \\ \hline c & & d & \\ & c' & & d' \end{array}\right) \tag{2.3}$$

Let $k = \mathbb{Q}$ the rational numbers with adeles $\mathbb{A}$, discrete subgroup $G_k = \mathrm{Sp}_{4\times 4}(k)$, and standard maximal compact subgroup K of $G_{\mathbb{A}}$. Consider the K_∞-invariants $L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}})^{K_\infty}$ in $L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}})$, where K_∞ is the standard maximal compact in the archimedean points G_∞.

2.1 Bruhat and Iwasawa Decompositions, Parabolics, and Weyl Group

The standard maximal torus in $\mathrm{Sp}_{4\times 4}$ is the subgroup

$$A = \left\{ \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & m_1^{-1} & \\ & & & m_2^{-1} \end{pmatrix} \right\}$$

The Weyl group acting on the maximal torus is the Coxeter group

$$\langle s, t | (st)^4 = 1, s^2 = 1 = t^2 \rangle$$

where

$$s = \left(\begin{array}{cc|cc} 1 & & & \\ & & & -1 \\ \hline & & 1 & \\ & 1 & & \end{array} \right), \quad t = \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array} \right)$$

There are two conjugacy classes of maximal proper parabolics, both of which are self-associate and not associate to any other parabolic: the Siegel parabolics, which stabilize two-dimensional isotropic vector subspaces, and the Klingen parabolics, which stabilize one-dimensional subspaces. The standard Siegel parabolic P^{Sieg} stabilizes the two-dimensional vector space $V_1 = \{e_1, e_2\}$. Its unipotent radical N^{Sieg} consists of the elements of P^{Sieg} that are pointwise trivial on V_1. Subgroups P^{Sieg} and N^{Sieg} are

$$P^{\text{Sieg}} := \left\{ \begin{pmatrix} A & * \\ 0 & (A^T)^{-1} \end{pmatrix} \right\} \quad N^{\text{Sieg}} := \left\{ \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \right\}$$

where $A \in \mathrm{GL}_2$ and $B = B^T$. A standard Levi subgroup of P^{Sieg} is

$$M^{\text{Sieg}} := \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \right\}$$

The standard Klingen parabolic $P^{\text{Kling}} \subset G$ stabilizes the isotropic flag $\{e_1\} \subset V$.

Its unipotent radical N^{Kling} consists of elements of P^{Kling} that induce identity maps on $\{e_1\}$ and on $(\mathbb{Q}e_1 \oplus \mathbb{Q}e_2)/\mathbb{Q}e_1$. P^{Kling} and N^{Kling} are

$$P^{\text{Kling}} := \left\{ p = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ 0 & a_{22} & b_{21} & b_{22} \\ 0 & 0 & d_{11} & 0 \\ 0 & c_{22} & d_{21} & d_{22} \end{pmatrix} \right\} \tag{2.4}$$

where $a_{11}d_{11} = a_{22}d_{22} - b_{22}c_{22} = 1$, $b_{21}c_{22} = a_{12}d_{11} + a_{22}d_{21}$, $b_{21}d_{22} = d_{11}b_{12} + d_{21}b_{22}$ and

$$N^{\text{Kling}} := \left\{ n = \begin{pmatrix} 1 & a_{12} & b_{11} & b_{12} \\ 0 & 1 & b_{21} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a_{12} & 1 \end{pmatrix} \right\} \tag{2.5}$$

where $b_{21} = b_{12}$. The standard Levi subgroup of the Klingen parabolic is

$$M^{\text{Kling}} := \left\{ m = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & b_{22} \\ 0 & 0 & d_{11} & 0 \\ 0 & c_{22} & 0 & d_{44} \end{pmatrix} \right\} \tag{2.6}$$

where $a_{11}d_{11} = a_{22}d_{22} - b_{22}c_{22} = 1$ and $d_{11} = a_{11}^{-1}$. The standard split component is

$$A^{\text{Kling}} := \left\{ a = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d_{11} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \tag{2.7}$$

where $a_{11}d_{11} = 1$.

The standard minimal parabolic P^{min} is

$$P^{\text{min}} := \left\{ p = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ 0 & a_{22} & b_{21} & b_{22} \\ 0 & 0 & d_{11} & 0 \\ 0 & 0 & d_{21} & d_{22} \end{pmatrix} \right\} \tag{2.8}$$

with unipotent radical $N^{\min}$

$$N^{\min} := \left\{ n = \begin{pmatrix} 1 & a_{12} & b_{11} & b_{12} \\ 0 & 1 & b_{21} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & d_{21} & 1 \end{pmatrix} \right\} \tag{2.9}$$

where $a_{12} = -d_{12}$ and $b_{21} = b_{12}$. The standard Levi component $M^{\min}$

$$M^{\min} := \left\{ m = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & d_{11} & \\ & & & d_{22} \end{pmatrix} \right\} \tag{2.10}$$

where $a_{11}d_{11} = 1$ and $a_{22}d_{22} = 1$.

2.2 Lie Algebra, Roots

The Lie algebra $\mathfrak{g}$ of $\mathrm{Sp}_{4\times 4}(\mathbb{R})$ is $\mathfrak{g} = \{X \in M_4 : X^T J + JX = 0\}$, where

$$J = \begin{pmatrix} & -1_2 \\ 1_2 & \end{pmatrix}$$

From this

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \implies C = C^T,\ B = B^T,\ A = -D^T$$

For $a \in A = \mathrm{diag}\{m_1, m_2, m_1^{-1}, m_2^{-1}\}$ a maximal torus, the simple positive roots Δ are:

$$\alpha(a) = m_1/m_2, \quad \beta(a) = m_2^2$$

The simple positive and negative root vectors are:

$$X_\alpha = \left(\begin{array}{cc|cc} & 1 & & \\ & & & \\ \hline & & & \\ & & -1 & \end{array}\right), \quad X_{-\alpha} = \left(\begin{array}{cc|cc} & & & \\ 1 & & & \\ \hline & & & -1 \\ & & & \end{array}\right),$$

$$X_\beta = \left(\begin{array}{cc|cc} & & & \\ & & & 1 \\ \hline & & & \\ & & & \end{array}\right), \quad X_{-\beta} = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & & \\ & 1 & & \end{array}\right)$$

The corresponding coroots are:

$$H_\alpha = \left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & 1 \end{array}\right), \quad H_\beta = \left(\begin{array}{cc|cc} & & & \\ & 1 & & \\ \hline & & & \\ & & & -1 \end{array}\right) \tag{2.11}$$

The other root spaces are generated by

$$[X_\alpha, X_\beta] = \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & & & \\ & & & \end{array}\right) = X_{\alpha+\beta} \in \mathrm{Lie} N^{\min},$$

$$X_{-(\alpha+\beta)} = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & 1 & & \\ 1 & & & \end{array}\right),$$

$$[X_\alpha, X_{\alpha+\beta}] = 2 \cdot \left(\begin{array}{cc|cc} & & 1 & \\ & & & \\ \hline & & & \\ & & & \end{array}\right) = X_{2\alpha+\beta} \in \mathrm{Lie} N^{\min},$$

$$X_{-(2\alpha+\beta)} = 2\left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline 1 & & & \\ & & & \end{array}\right)$$

filling out the set of positive roots $\Sigma^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$ and coroots $\check{\Sigma}^+ = \{H_\alpha, H_\beta, H_{\alpha+2\beta}, H_{\alpha+\beta}\}$. The fundamental weights are $\frac{1}{2}(2\alpha + \beta)$ and $\alpha + \beta$.

2.3 Maximal Compact Subgroup

The Lie algebra of the standard maximal compact subgroup of $\mathrm{Sp}_{4\times 4}(\mathbb{R})$ is

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathfrak{sp}_{4\times 4} : \ A \text{ skew symmetric, } B \text{ symmetric} \right\}$$

Since

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longrightarrow A + Bi$$

is an isomorphism $\mathfrak{k} \to \mathfrak{u}(2)$, the maximal compact subgroup at the archimedean place is $K \cong U(2)$.

2.3.1 Lemma *At non-archimedean places $v < \infty$, let $K_v = G(\mathbb{Z}_v)$. Then K_v is a maximal compact subgroup of $G_v = G(k_v)$.* $\square$

Chapter 3

Function Spaces Related to Moderate Growth

In this section we establish characteristics of spaces of moderate-growth functions and spaces dual to moderate-growth functions. Both spaces are important for understanding distributions created by integrating against cuspforms, which we recast in Section 8.1 from *triple-product L-functions as periods of Eisenstein series.*

Verifying that moderate-growth (in Siegel sets) implies moderate-growth (in Siegel sets) on subquotients is expressed in terms of Siegel sets. Let α, β be the standard simple roots. From Sections 2.1 and 4.1 with

$$\left(\begin{array}{cc|cc} a & & & \\ & b & & \\ \hline & & a^{-1} & \\ & & & b^{-1} \end{array}\right) \left(\begin{array}{cc|cc} & \alpha & & \\ & & & \beta \\ \hline & & & \\ & & -\alpha & \end{array}\right) \left(\begin{array}{cc|cc} a^{-1} & & & \\ & b^{-1} & & \\ \hline & & a & \\ & & & b \end{array}\right)$$

$\alpha, \beta : M^{min} \to k^\times$ are

$$\alpha(m_{a,b}) = \frac{a}{b}, \text{ and } \beta(m_{a,b}) = b^2$$

Using Iwasawa decomposition $G = NMK$ for the minimal parabolic, a *height function* $\eta_G : G \to (0, +\infty)$ can be defined by

$$\eta_G(nmk) = \min\{|\alpha(m)|, |\beta(m)|\} \tag{3.1}$$

3.0.1 Definition Define standard Siegel sets for $t >> 0$ and compact $C \subset N^{min}$:

$$\mathfrak{S}_t^C = \{g = nm_ak \in CM^{\min}K : \eta_G(g) \geq t\} \tag{3.2}$$

Reduction theory asserts the existence of a sufficiently large Siegel set $\mathfrak{S}_t^C$ that covers the quotient: surject to $G_k \backslash G_{\mathbb{A}}$. Lemma 3.1.2 proves that functions of moderate-growth in Siegel sets on G (defined below) restrict to functions of moderate-growth in Siegel sets on H (2.2).

3.1 Moderate Growth

We show that functions of moderate-growth in standard Siegel sets on the quotient $G_k \backslash G_{\mathbb{A}}/K$ are of moderate-growth in standard Siegel sets on H.

3.1.1 Definition Let the height function η_G be as in (3.1) and $\mathfrak{S}_t^C$ as in (3.2) for fixed Iwasawa decomposition $G = NMK$ with standard minimal parabolic P and Levi decomposition $P = NM$. A function $f : G_k \backslash G_{\mathbb{A}} \to \mathbb{C}$ is of *moderate-growth* in standard Siegel sets when there is $\lambda \in \mathbb{R}$ such that for every standard Siegel set $\mathfrak{S}_t^C$

$$|f(g)| \ll_{\mathfrak{S}_t^C} \eta_G(g)^\lambda \quad (\text{for } g \in \mathfrak{S}_t^C)$$

3.1.2 Lemma *f of moderate-growth on G implies $f \circ j$ is of moderate-growth on H.*

Proof. It suffices to show that $\eta_H(h) = O(\eta_G(h))$ for $h \in H$. Define the height function $\eta_H : H \to \mathbb{R}_{\geq 0}$ as follows. For $H_1 = \mathrm{SL}_2$, fix an Iwasawa decomposition $H_1 = PK$ for fixed standard parabolic and $P = NM$ with standard Levi component. The standard positive simple root for SL_2 gives a character for $h = nmk$ with $h \in H_1$

$$\sigma\left(n \begin{pmatrix} m & \\ & m^{-1} \end{pmatrix} k\right) = |m|^2 \tag{3.3}$$

For $H = \mathrm{SL}_2 \times \mathrm{SL}_2$ let

$$\sigma_1\left(n\begin{pmatrix} m_1 & \\ & m_1^{-1}\end{pmatrix}k \times n\begin{pmatrix} m_2 & \\ & m_2^{-1}\end{pmatrix}k\right) = |m_1|^2$$

and

$$\sigma_2\left(n\begin{pmatrix} m_1 & \\ & m_1^{-1}\end{pmatrix}k \times n\begin{pmatrix} m_2 & \\ & m_2^{-1}\end{pmatrix}k\right) = |m_2|^2$$

With j defined by (2.3), for $h \in H$, by Iwasawa let $j^{-1}(h) = nmk$ and define η_H by

$$\eta_H(h) = \min\{\sigma_1(h), \sigma_2(h)\} \tag{3.4}$$

Fix $0 < t_0 \leq 1$. For m in the Levi component of the standard minimal parabolic of G, and assume $t_0 < \min\{\sigma_1(j^{-1}(m)), \sigma_2(j^{-1}(m))\}$. For convenience of notation let $m' = j^{-1}(m)$. Then

$$\begin{aligned}\eta_G(m) &= \min\{|\alpha(m)|, |\beta(m)|\} = \min\{|m_1/m_2|, |m_2|^2\} \\ &= \min\left\{\left(\frac{\sigma_1(m')}{\sigma_2(m')}\right)^{\frac{1}{2}}, \sigma_2(m')\right\}\end{aligned}$$

If $m_1 \geq m_2$, then

$$\begin{aligned}\min\left\{\left(\frac{\sigma_1(m')}{\sigma_2(m')}\right)^{\frac{1}{2}}, \sigma_2(m')\right\} &\leq \min\left\{\frac{\sigma_1(m')}{\sigma_2(m')}, \sigma_2(m')\right\} \\ &\leq \min\{\sigma_1(m'), \sigma_2(m')\} \\ &= \eta_H(m')\end{aligned}$$

If $m_1 < m_2$, then

$$\begin{aligned}\min\left\{\left(\frac{\sigma_1(m')}{\sigma_2(m')}\right)^{\frac{1}{2}}, \sigma_2(m')\right\} &\ll \min\{\sigma_1(m') \cdot t_0, \sigma_2(m')\} \\ &\leq \min\{\sigma_1(m'), \sigma_2(m')\} \\ &= \eta_H(m')\end{aligned}$$

Therefore, if $f(g) \ll \eta_G(g)^\lambda$ for some $\lambda \geq 0$, then for h in a standard Siegel set of H, $f(h) \ll \eta_H(h)$. $\square$

We will verify that the distribution of *restrict to the diagonal embedding of* $SL_2 \times SL_2$ *and integrate against (given fixed, strong-sense) cuspforms* is a rapid decay distribution in the sense of being in the continuous dual of moderate-growth smooth functions on G (ref 3.3.2).

Remark: The topology of the space of moderate-growth functions is slightly subtle, as illustrated below.

3.1.3 Example

Even on $\mathbb{R}$, the topology on the space of smooth, moderate-growth functions

$$C^\infty_{\mathrm{Mod}}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \exists N \text{ such that } \sup_{x\in\mathbb{R}}\{|f(x)|(1+|x|^2)^{-N}\} < \infty\}$$

needs to be defined with care as we would like the action of the group (right translation) to be continuous. For fixed r, consider two possible topologies

$$A_n = \{f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}}\{|f(x)|(1+|x|^2)^{-n}\} < \infty\}$$

with norm

$$\sup_{x\in\mathbb{R}}\{|f(x)|(1+|x|^2)^{-n}\}$$

and

$$\begin{aligned} B_n = \{f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}}\{|f(x)|(1+|x|^2)^{-n}\} < \infty \\ \text{and } |f(x)|(1+|x|^2)^{-n} \longrightarrow 0 \text{ as } x \to +\infty\} \end{aligned}$$

with norm

$$\sup_{x\in\mathbb{R}}\{|f(x)|(1+|x|^2)^{-n}\} \tag{3.5}$$

The function $\sin(x^2) \in A_0$, but right translation is not continuous at $\sin(x^2)$, in this topology, because the oscillations are ever-increasing in frequency. That is, $\sin(x^2) \notin B_0$ but $\sin(x^2) \in B_1$ and the translation action is continuous. We can see

$$\cdots \subsetneq B_n \subsetneq A_n \subsetneq B_{n+1} \subsetneq A_{n+1} \subsetneq \cdots$$

$B_n \subsetneq A_n$ is obvious, and $A_n \subsetneq B_{n+1}$ by dividing by $(1+|x|^2)$, and by considering for example functions like $f(x) = x$, which is not in A_0, but is in B_1.

The space of test functions $C_c^\infty(\mathbb{R})$ is dense in every B_n and not dense in any A_n. For example $f(x) = (1+|x|^2)^n \in A_n$, but for any $\varphi \in C_c^\infty(\mathbb{R})$, $\sup\{(f-\varphi)(x)(1+|x|^2)\} \geq 1$.

Nevertheless, the distinction is obliterated in the colimit:

$$\bigcup_n A_n = \bigcup_n B_n = \varinjlim B_n = C^\infty_{\mathrm{Mod}}(\mathbb{R})$$

3.2 Uniform Moderate-Growth

An Eisenstein series is smooth of moderate-growth and all of its derivatives are also smooth of moderate-growth (Lemma 5.1.1). We are considering distributions defined by integrating against a domain that includes Eisenstein series. By shrinking the space of functions from smooth, moderate-growth functions to smooth, *uniform* moderate-growth, which includes Eisenstein series (Corollary 5.1.2), we effectively enlarge the dual space and can reach stronger conclusions.

We show that functions of uniform moderate-growth in standard Siegel sets are of uniform moderate-growth in standard Siegel sets when restricted to H.

3.2.1 Definition Let η_G be as in (3.1) and $\mathfrak{S}_t^C$ as in (3.2) for fixed Iwasawa decomposition $G = NMK$ with standard minimal parabolic P and Levi decomposition $P = NM$. Then, $f : G_k\backslash G_{\mathbb{A}} \to \mathbb{C}$ is of *uniform moderate-growth of exponent* n in standard Siegel sets when for every standard Siegel set $\mathfrak{S}_t^C$

$$|Tf(g)| \ll_{T,\mathfrak{S}_t^C} \eta_G(g)^n \quad (\text{for } g \in \mathfrak{S}_t^C \text{ and every } T \in U\mathfrak{g}) \tag{3.6}$$

Fix a standard Siegel set $\mathfrak{S}_t^C$ and $n \geq 0$. Let

$$\begin{aligned} B_{n,r} = \{f \in C^\infty(G_k \backslash G_\mathbb{A}) : \forall g \in \mathfrak{S}_t^C,\ \sup_{g \in \mathfrak{S}_t^C} \{|T_r f(g)| \cdot \eta_G(g)^{-n}\} < \infty \\ \text{and } |T_r f(g)| \cdot \eta_G(g)^{-n} \to 0 \text{ as } \eta_G(g) \to +\infty\} \end{aligned} \tag{3.7}$$

for any $T_r \in U\mathfrak{g}$ of degree less than or equal to r. For fixed $n \geq 0$, the topology of $B_{n,r}$ is defined by the norm

$$\nu_{n,r}(f) = \sup_{g \in \mathfrak{S}_t^C} \{|T_r f(g)| \cdot \eta_G(g)^{-n}\} \quad (T_r \in U\mathfrak{g} \text{ of degree} \leq r)$$

Each $B_{n,r}$ is a Banach space i.e., it is complete with the topology induced by the norm. Taking the projective limit with respect to r is a limit of Banach spaces. The locally convex topological vector space $\mathcal{B}_n = \bigcap_r B_{n,r}(G_k \backslash G_\mathbb{A}) = \lim_r B_{n,r}(G_k \backslash G_\mathbb{A})$ is a Fréchet space. The vector space $\mathcal{B}_n$ is the the space of functions of *uniform moderate-growth of exponent* n with the topology based on seminorms

$$\nu_n(f) = \sup_{g \in \mathfrak{S}_t^C} \left\{|Tf(g)|\eta_G(g)^{-n}\right\} \quad (T \in U\mathfrak{g}) \tag{3.8}$$

3.2.2 Definition Taking the colimit of $\mathcal{B}_n$ with respect to n gives the space of *uniform moderate-growth* functions

$$\begin{tikzcd} \mathcal{B}_0 \arrow[r, hook, "i_{0,1}"] \arrow[rrrrr, bend left, "i_0"] & \cdots \arrow[r, hook, "i_{n-1,n}"] & \mathcal{B}_n \arrow[r, hook, "i_{n,n+1}"] \arrow[rrr, bend left, "i_n"] & \mathcal{B}_{n+1} \arrow[r, hook, "i_{n+1,n+2}"] & \cdots\cdots & \operatorname{colim}_n \mathcal{B}_n = \mathcal{B} \end{tikzcd}$$

with the family of seminorms (3.8)

3.2.3 Lemma On standard Siegel sets $\mathfrak{S}_t^C$, $C_c^\infty(G_k \backslash G_\mathbb{A})$ is dense in $\mathcal{B}$ in the topology of $\mathcal{B}$.

Proof. Since $\mathcal{B}$ is a colimit, $f \in \mathcal{B}$ implies $f \in \mathcal{B}_n$ for some n. Let $\varphi_i : \mathfrak{S}_t^C \longrightarrow \mathbb{R}$ be a smooth, compactly supported *approximate identity*:

$$\int_{\mathfrak{S}_t^C} \varphi_i = 1 \text{ for all } i \text{ and } \mathrm{spt}(\varphi_i) \to 1_{\mathfrak{S}_t^C} \text{ as } i \to \infty$$

For $i \in \mathbb{N}$, let $f_i : \mathfrak{S}_t^C \to \mathbb{R}$ be

$$f_i(g) = \int_{\mathfrak{S}_t^C} \varphi_i(g^{-1}n) \cdot f(n)\, dn$$

By construction $f_i \in C_c^\infty(\mathfrak{S}_t^C)$ for all i, and $f_i \to f$ as $i \to \infty$. □

3.2.4 Corollary $C_c^\infty(G_k \backslash G_{\mathbb{A}})$ is dense in $\mathcal{B}_i$ on standard Siegel sets $\mathfrak{S}_t^C$ for $i = 1, 2, \ldots$. □

3.2.5 Proposition f of uniform moderate-growth on G implies $f \circ j$ is of uniform moderate-growth on H (ref. (2.2)).

Proof. This follows from Lemma 3.1.2 directly, using induction on the order of $T \in U\mathfrak{g}$. □

3.3 Dual to Uniform Moderate-Growth

The dual $\mathcal{B}^*$ to the space of smooth uniform moderate-growth functions

$$\mathcal{B} = \operatorname*{colim}_n \mathcal{B}_n = \operatorname*{colim}_n \lim_r (B_{n,r})$$

is the space of continuous linear maps $\lambda : \mathcal{B} \to \mathbb{C}$:

$$\begin{aligned}
\mathcal{B}^* &= (\operatorname*{colim}_n \mathcal{B}_n)^* = (\operatorname*{colim}_n \lim_r (B_{n,r}))^* \\
&= \lim_n ((\lim_r B_{n,r})^*) \quad \text{(by a general categorical argument)} \\
&= \lim_n \operatorname*{colim}_r (B_{n,r}^*) \\
&= \lim_n \mathcal{B}_n^*
\end{aligned}$$

That is, taking the colimit with respect to r, we have the following commutative diagram

$$\begin{tikzcd}
\hom(B_{n,0},\mathbb{C}) \arrow[r, hook, "i_{0,1}"] \arrow[rrrr, bend left, "i_0"] & \cdots \arrow[r, hook, "i_{n-1,n}"] & \hom(B_{n,k},\mathbb{C}) \arrow[r, hook, "i_{k,k+1}"] \arrow[rr, bend left, "i_k"] & \cdots & \operatorname*{colim}_r \hom(B_{n,r},\mathbb{C}) = \mathcal{B}_n^*
\end{tikzcd}$$

Then, taking the limit with respect to n leads to

$$\begin{tikzcd}
\mathcal{B}^* := \lim\limits_n \mathcal{B}_n^* \cdots \arrow[r, "j_{k+1,k}"] \arrow[r, bend left, "j_k"] \arrow[rrr, bend left, "j_0"] & \mathcal{B}_k^* \arrow[r, "j_{k,k-1}"] & \cdots \arrow[r, "j_{1,0}"] & \mathcal{B}_0^*
\end{tikzcd} \tag{3.9}$$

or

$$\begin{tikzcd}
\mathcal{B}^* := \lim\limits_n \hom_{\mathbb{R}}(\mathcal{B}_n,\mathbb{C}) \cdots \arrow[r, "j_{k+1,k}"] \arrow[r, bend left, "j_k"] \arrow[rrr, bend left, "j_0"] & \hom_{\mathbb{R}}(\mathcal{B}_k,\mathbb{C}) \arrow[r, "j_{k,k-1}"] & \cdots \arrow[r, "j_{1,0}"] & \hom_{\mathbb{R}}(\mathcal{B}_0,\mathbb{C})
\end{tikzcd} \tag{3.10}$$

The maps $j_{n+1,n} : \mathcal{B}_{n+1}^* \to \mathcal{B}_n^*$ between limitands $\mathcal{B}_n^*$ are induced by the maps $i_{n,n+1} : \mathcal{B}_n \to \mathcal{B}_{n+1}$ between colimitands making the following diagram commute:

$$\begin{tikzcd}
\mathcal{B}_{k+1} \arrow[dr, "\lambda_{k+1}"'] & & \mathcal{B}_k \arrow[ll, hook', "i_{k,k+1}"'] \arrow[dl, "\lambda_k"] \\
& \mathbb{C} &
\end{tikzcd}$$

That is, $(j_{k+1,k} \circ \lambda_{k+1})(u_k) = \lambda_{k+1}(i_{k,k+1}(u_k))$.

All of the spaces $B_{n,r}^*$, $\mathcal{B}_n^*$, and $\mathcal{B}^*$ are given the weak dual topology.

Since $\mathcal{B}$ is a Fréchet space and $\mathbb{C}$ is quasi-complete, [Garrett 2017] Theorem [13.12.1] implies $\mathcal{B}_n^*$ is quasi-complete. Each dense injection $i_{n,n+1} : \mathcal{B}_n \to \mathcal{B}_{n+1}$ induces a mapping between the dual spaces $j : \mathcal{B}_{n+1}^* \to \mathcal{B}_n^*$.

Remark: $\mathcal{D}(G_k\backslash G_{\mathbb{A}})$ being dense in $\mathcal{B}$ implies $\mathcal{B}^*$ injects to $\mathcal{D}(G_k\backslash G_{\mathbb{A}})^*$

Remark: We need the following theorem to see that the *restrict to the diagonal embedding of $SL_2 \times SL_2$ and integrate against (given, fixed, strong-sense) cuspforms* distribution in Section 8.1 (ref. 8.2) is an element of $\mathcal{B}^*$. First, we need

3.3.1 Lemma

For $f \in \mathcal{B}$, the integral

$$\int_{H_k \backslash H_{\mathbb{A}}} f(h) \cdot F(h)\, dh \qquad \text{(for } F \text{ a fixed strong-sense cuspform on } \mathrm{SL}_2 \times \mathrm{SL}_2)$$

is absolutely convergent.

Proof. A cuspform F is *rapidly decreasing* on standard Siegel sets $\mathfrak{S}_H$ of H when

$$|F(g)| \ll_{\mathfrak{S}_H} \eta_H(g)^\lambda \quad (\text{for } g \in \mathfrak{S}_H, \text{ for every } \mathfrak{S}_H, \text{ for every } \lambda \in \mathbb{R}) \tag{3.11}$$

By construction, the standard Siegel set $\mathfrak{S}_H$ covers $H_k \backslash H_{\mathbb{A}}$. Since f is in one of the limitands in $\operatorname{colim}_n \mathcal{B}_n$ we have for some λ_1, for fixed F,

$$\begin{aligned}
\left| \int_{H_k \backslash H_{\mathbb{A}}} f(h) \cdot F(h)\, dh \right| &\leq \left| \int_{\mathfrak{S}_H} f(h) \cdot F(h)\, dh \right| \\
&\ll_f \left| \int_{\mathfrak{S}_H} \eta_H(h)^{\lambda_1} \cdot F(h)\, dh \right| \\
&\ll_F \int_{\mathfrak{S}_H} \eta_H(h)^{\lambda_1} \cdot \eta_H(h)^{\lambda_2}\, dh \quad (\text{for all } \lambda_2) \\
&\leq \int_{\mathfrak{S}_H} \eta_H(h)^{\lambda_1 + \lambda_2}\, dh \quad (\text{for all } \lambda_2)
\end{aligned}$$

For λ_2 sufficiently negative,

$$\leq \int_0^\infty (|m|^2)^{\lambda_1 + \lambda_2} \frac{dm}{m} < \infty$$

□

3.3.2 Theorem

Given $f \in \mathcal{B}$, for F a fixed strong-sense cuspform on $\mathrm{SL}_2 \times \mathrm{SL}_2$, the functional $\theta : \mathcal{B} \to \mathbb{C}$ defined by

$$f \to \langle \theta, f \rangle = \int_{H_k \backslash H_{\mathbb{A}}} f(h) \cdot F(h)\, dh \tag{3.12}$$

is in $\mathcal{B}^*$.

Proof. Lemma 3.3.1 proves that the integral in (3.12) is absolutely convergent. Hence, we have left to show that the functional (3.12) is continuous. Since $\mathcal{B}$ is a topological vector space, it suffices to show continuity at 0. Since the seminorm on $\mathcal{B}_n$ separates points, we can consider a sequence $f_i \to 0$ as $i \to \infty$, $f_i \not\equiv 0$ for $i = 1, 2, \ldots$. For any $\varepsilon > 0$ there is an N and a seminorm ν_n such that $i > N$ implies $0 < \nu_n(f_i) < \varepsilon$. Fix a strong-sense cuspform F on $\mathrm{SL}_2 \times \mathrm{SL}_2$ and standard Siegel set $\mathfrak{S}_H$ of H. Then,

$$\begin{aligned}
\left| \int_{H_k \backslash H_{\mathbb{A}}} f(h) \cdot F(h)\, dh \right| &\le \left| \int_{\mathfrak{S}_H} f(h) \cdot F(h)\, dh \right| \\
&\le \left| \int_{\mathfrak{S}_H} \nu_n(f) \eta_H(h)^n \cdot F(h)\, dh \right| \\
&\ll_{\lambda, F} \int_{\mathfrak{S}_H} \nu_n(f) \eta_H(h)^n \cdot \eta_H(h)^\lambda\, dh
\end{aligned}$$

for all λ because F is a strong-sense cuspform. Hence,

$$\begin{aligned}
&\ll_{n,\lambda} \int_{\mathfrak{S}_H} \nu_n(f) \eta_H(h)^\lambda\, dh \quad (\text{for all } \lambda) \\
&= \varepsilon \int_{\mathfrak{S}_H} \eta_H(h)^\lambda\, dh \quad (\text{for all } \lambda)
\end{aligned}$$

For λ sufficiently negative, the integral is absolutely convergent. Standard Siegel sets (3.2) bound the height function η_H (3.4) away from 0. This puts an upper bound on the values the integral can attain for any f_i. Therefore, using the definition of rapid-decay and the height function η_H in (3.4), this is

$$\begin{aligned}
&\le \varepsilon \int_{\alpha_{\mathfrak{S}_H}}^{\infty} (|m|^2)^\lambda \, \frac{dm}{m} \quad (\text{for some } \alpha_{\mathfrak{S}_H} \gg 0) \\
&< \varepsilon M_{\mathfrak{S}_H} \quad (\text{for some } M_{\mathfrak{S}_H} \ge 0)
\end{aligned}$$

Since $M_{\mathfrak{S}_H}$ is fixed, let $\varepsilon' = \varepsilon / M_{\mathfrak{S}_H}$ and adjust N accordingly. This gives the result. □

Chapter 4

Invariant Operators

We use spectral decomposition with respect to unbounded differential operators at the archimedean place and integral operators at non-archimedean places. To show that *the discrete spectrum, if any, is among the zeros of the L-function*, we use the Friedrichs extension of a restriction of the invariant Laplacian. Hecke algebras of integral operators at non-archimedean places ease the computation of the gamma factors and produce Euler products in spectral decompositions.

4.1 Differential Operators

As unbounded operators, certain restrictions of differential operators are defined on dense subsets of the Hilbert space $L^2(G_k\backslash G_{\mathbb{A}}/K_\infty)$ and are symmetric but not always self-adjoint. The Friedrichs extension discussed in Section 7 extends semibounded symmetric operators to self-adjoint but no longer precisely differential operators. These are extensions of restrictions of the invariant Laplacian Δ on $C_c^\infty(G_k\backslash G_{\mathbb{A}}/K_\infty)$, all of which descend from the Casimir element of the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$.

The Casimir element Ω of the universal enveloping algebra $U\mathfrak{g}$ is invariant under the action by G on both sides. By invariance, it descends to any quotient $G_k\backslash G_{\mathbb{A}}/K_\infty$. We will limit ourselves to the spherical case, ie., the space of functions right K_v-invariant for K_v the standard maximal compact subgroup of G_v at all $v \le \infty$. To characterize Ω, consider

$$\zeta : \mathrm{End}(\mathfrak{g}) \longrightarrow U\mathfrak{g}$$

given by

$$\mathrm{End}(\mathfrak{g}) \xrightarrow{\cong} \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\cong} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\text{inc}} \otimes^\bullet \mathfrak{g} \xrightarrow{\text{quot}} U\mathfrak{g}$$

The first map is the canonical isomorphism from the Lie algebra tensored with its dual space to endomorphisms of the Lie algebra. The second map is the natural isomorphism induced by the ideal generated by the non-degenerate Killing form, the third is inclusion, and the fourth is the quotient by

$$a \otimes b - b \otimes a - [a, b] \quad (\text{for } a, b \in \mathfrak{g})$$

The Casimir element is $\Omega := \zeta(Id_{\mathfrak{g}})$.

In coordinates, the invariant differential operator at the archimedean place v on the space $L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}} / K_\infty)$ is descended from the Casimir element. Therefore, letting

$$H_\alpha = \left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & 1 \end{array}\right), \quad H_\beta = \left(\begin{array}{cc|cc} & & & \\ & 1 & & \\ \hline & & & \\ & & & -1 \end{array}\right)$$

the Casimir operator at v is

$$\begin{aligned} \Omega = {} & H_\alpha H_\alpha^* + H_\beta H_\beta^* + X_\alpha X_\alpha^* + X_{-\alpha} X_{-\alpha}^* + X_{2\alpha+\beta} X_{2\alpha+\beta}^* + X_{-(2\alpha+\beta)} X_{-(2\alpha+\beta)}^* \\ & + X_\beta X_\beta^* + X_{-\beta} X_{-\beta}^* + X_{\alpha+\beta} X_{\alpha+\beta}^* + X_{-(\alpha+\beta)} X_{\alpha+\beta}^* \end{aligned}$$

From the non-degenerate symmetric bilinear pairing $\langle X, Y \rangle = \mathrm{tr}(XY)$, which is a constant times the Killing form $\mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$,

$$\begin{array}{ll} \langle H_\alpha, H_\alpha \rangle = 4 & \langle H_\beta, H_{-\beta} \rangle = 2 \\ \langle X_\alpha, X_{-\alpha} \rangle = 2 & \langle X_\beta, X_{-\beta} \rangle = 1 \\ \langle X_{\alpha+\beta}, X_{-(\alpha+\beta)} \rangle = 2 & \langle X_{2\alpha+\beta}, X_{-(2\alpha+\beta)} \rangle = 1 \end{array}$$

the dual basis is

$$\begin{array}{ll} H_\alpha^* = \frac{1}{4}H_\alpha & H_\beta^* = \frac{1}{2}H_\beta \\ X_\alpha^* = \frac{1}{2}X_{-\alpha} & X_{-\alpha}^* = \frac{1}{2}X_\alpha \\ X_\beta^* = X_{-\beta} & X_{-\beta}^* = X_\beta \\ X_{\alpha+\beta}^* = \frac{1}{2}X_{-(\alpha+\beta)} & X_{-(\alpha+\beta)}^* = \frac{1}{2}X_{\alpha+\beta} \\ X_{2\alpha+\beta}^* = X_{-(2\alpha+\beta)} & X_{-(2\alpha+\beta)}^* = X_{2\alpha+\beta} \end{array}$$

Substituting

$$\begin{aligned}
\Omega = {} & \frac{1}{4}H_\alpha^2 + \frac{1}{2}H_\beta^2 + \frac{1}{2}X_\alpha X_{-\alpha} + \frac{1}{2}X_{-\alpha}X_\alpha + X_{2\alpha+\beta}X_{-(2\alpha+\beta)} + X_{-(2\alpha+\beta)}X_{2\alpha+\beta} \\
& + X_\beta X_{-\beta} + X_{-\beta}X_\beta + \frac{1}{2}\left(X_{\alpha+\beta}X_{-(\alpha+\beta)} + X_{-(\alpha+\beta)}X_{\alpha+\beta}\right) \\
= {} & \frac{1}{4}H_\alpha^2 + \frac{1}{2}H_\beta^2 + X_\alpha X_{-\alpha} + \frac{1}{2}\left(X_{-\alpha}X_\alpha - X_\alpha X_{-\alpha}\right) \\
& + 2X_{2\alpha+\beta}X_{-(2\alpha+\beta)} + \left(X_{-(2\alpha+\beta)}X_{2\alpha+\beta} - X_{2\alpha+\beta}X_{-(2\alpha+\beta)}\right) \\
& + 2X_\beta X_{-\beta+} + \left(X_{-\beta}X_\beta - X_\beta X_{-\beta}\right) \\
& + X_{\alpha+\beta}X_{-(\alpha+\beta)} + \frac{1}{2}\left(X_{-(\alpha+\beta)}X_{\alpha+\beta} - X_{\alpha+\beta}X_{-(\alpha+\beta)}\right) \\
= {} & \frac{1}{4}H_\alpha^2 + \frac{1}{2}H_\beta^2 + X_\alpha X_{-\alpha} + \frac{1}{2}\left[X_{-\alpha}, X_\alpha\right] \\
& + 2X_{2\alpha+\beta}X_{-(2\alpha+\beta)} + \left[X_{-(2\alpha+\beta)}, X_{2\alpha+\beta}\right] \\
& + 2X_\beta X_{-\beta+} + \left[X_{-\beta}, X_\beta\right] \\
& + X_{\alpha+\beta}X_{-(\alpha+\beta)} + \frac{1}{2}\left[X_{-(\alpha+\beta)}, X_{\alpha+\beta}\right] \\
= {} & \frac{1}{4}H_\alpha^2 + \frac{1}{2}H_\beta^2 + X_\alpha X_{-\alpha} - \frac{1}{2}H_\alpha \\
& + 2X_{2\alpha+\beta}X_{-(2\alpha+\beta)} - 4\left(H_\alpha + H_\beta\right) \\
& + 2X_\beta X_{-\beta} - H_\beta \\
& + X_{\alpha+\beta}X_{-(\alpha+\beta)} - \frac{1}{2}\left(H_\alpha + 2H_\beta\right) \\
= {} & \frac{1}{4}H_\alpha^2 + \frac{1}{2}H_\beta^2 - 5H_\alpha - 6H_\beta + X_\alpha X_{-\alpha} + 2X_\beta X_{-\beta} + X_{\alpha+\beta}X_{-(\alpha+\beta)} \\
& + 2X_{2\alpha+\beta}X_{-(2\alpha+\beta)}
\end{aligned}$$

Since right K-invariant functions on $(G_k\backslash G_{\mathbb{A}})$ are annihilated by $\mathfrak{k} = \mathrm{Lie}K$, the right

K-invariant differential operator descended from Casimir is:

$$\Delta = \Omega|_{\substack{\text{right}\\ K\text{-invariant}\\ \text{form}}} = \frac{1}{4}H_\alpha^2 + \frac{1}{2}H_\beta^2 - 5H_\alpha - 6H_\beta + X_\alpha X_{-\alpha} + 2X_\beta X_{-\beta} + X_{\alpha+\beta}X_{-(\alpha+\beta)} + 2X_{2\alpha+\beta}X_{-(2\alpha+\beta)} \tag{4.1}$$

We document for future reference:

4.1.1 Lemma *The differential operator descended from Casimir*

$$\Delta : L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \to L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$$

is symmetric and negative semi-definite.

Proof. Given $f \in C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ for

$$X \in \{H_{\pm\alpha}, H_{\pm\beta}, X_{\pm\alpha}, X_{\pm\beta}, X_{\pm(\alpha+\beta)}, X_{\pm(2\alpha+\beta)}\},$$

$$\langle Xf, f\rangle = \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K} \frac{\partial}{\partial t}\bigg|_{t=0} f(ge^{tX}\, f(g)\, dg$$

Since evaluation of the derivative at $t = 0$ is a limit as $t \to 0$ of the derivative and is continuous in t, we can move the evaluation outside the integral by Gelfand-Pettis. Since we are taking the limit as $t \to 0$, consider a t in a small interval around 0. The derivative is continuous and compactly supported in t in the small interval, hence also commutes with the integral by Gelfand-Pettis. Then, by substitution $g \to ge^{-tX}$ and applying Gelfand-Pettis twice in the reverse order, we have:

$$\begin{aligned}
\langle Xf, f\rangle &= \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K} \frac{\partial}{\partial t}\bigg|_{t=0} f(ge^{tX}\, f(g)\, dg \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K} f(ge^{tX}\, f(g)\, dg \\
&= -\frac{\partial}{\partial t}\bigg|_{t=0} \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K} f(g)\, f(ge^{-tX}\, dg \\
&= -\langle f, Xf\rangle
\end{aligned}$$

From (4.1):

$$
\begin{aligned}
\langle \Delta f, f\rangle &= \langle\Big(\frac{1}{4}H_\alpha^2 + \frac{1}{2}H_\beta^2 - 5H_\alpha - 6H_\beta + X_\alpha X_{-\alpha} + 2X_\beta X_{-\beta} \\
&\qquad + X_{\alpha+\beta}X_{-(\alpha+\beta)} + 2X_{2\alpha+\beta}X_{-(2\alpha+\beta)}\Big) f, f\rangle \\
&= -\langle\Big(\frac{1}{4}H_\alpha + \frac{1}{2}H_\beta - 5H_\alpha - 6H_\beta + X_\alpha + 2X_\beta \\
&\qquad + X_{\alpha+\beta}X_{-(\alpha+\beta)} + 2X_{2\alpha+\beta}X_{-(2\alpha+\beta)}\Big) f, \\
&\Big(H_\alpha + H_\beta + X_{-\alpha} + X_{-\beta} + X_{-(\alpha+\beta)} + X_{-(2\alpha+\beta)}\Big) f\rangle \\
&= -\langle \nabla f, \widetilde{\nabla} f\rangle \qquad \text{(i.e., negative, semi-definite)} \\
&= \langle f, \Big(\frac{1}{4}H_\alpha^2 + \frac{1}{2}H_\beta^2 - 5H_\alpha - 6H_\beta + X_\alpha X_{-\alpha} + 2X_\beta X_{-\beta} \\
&\qquad + X_{\alpha+\beta}X_{-(\alpha+\beta)} + 2X_{2\alpha+\beta}X_{-(2\alpha+\beta)}\Big) f\rangle \\
&= \langle f, \Delta f\rangle
\end{aligned}
$$

Hence, Δ is symmetric and negative semi-definite for $f \in C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$. Extension by continuity for dense $C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \subset L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ proves the lemma. □

4.2 Hecke Operators

Hecke operators by an integral operator at non-archimedean places. For rational numbers $\mathbb{Q}$ with adeles $\mathbb{A}$ let v be a non-archimedean place of $k = \mathbb{Q}$, k_v the completion of k at v, and $\mathbb{Z}_v$ the ring of integers of k_v. For $f \in L^2(Z^+G_k\backslash G_{\mathbb{A}}/K_{\mathbb{A}})$, at non-archimedean place $v < \infty$, the Hecke operators ϕ_v are

$$\phi_v \cdot f = \int_{G_v} \phi_v(g)\, g \cdot f \; dg \quad (\text{for } \phi \in C_c^\infty(G_v)) \tag{4.2}$$

where $(g \cdot f)(x) = f(xg)$.

4.2.1 Proposition *Let $L^2_0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$ be the space of cuspforms. There exist Hecke operators that are bounded and self-adjoint on the space of spherical cuspforms $L^2_0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/G_k)$.*

Proof. Restrict ϕ_v to functions in $C_c^\infty(G_v)$ with $\phi_v(g^{-1}) = \overline{\phi_v}(g)$ for all $g \in G_v$, and ϕ_v left and right K_v-invariant.

Bounded: The operator attached to ϕ_v is bounded if there exists $M > 0$ such that for every $f \in L^2_0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$, $|\phi_v \cdot f|_{L^2} < M|f|_{L^2}$. For $v < \infty$ and fixed $\phi_v \in C_c^\infty(G_v)$, $|\phi_v(g)| < M$ for all $g \in G_v$ and some $M > 0$. Therefore,

$$\begin{aligned}
|\phi_v \cdot f|^2_{L^2} &= \int_{G_v} |\phi_v(g)g \cdot f|^2 \, dg \\
&\leq M \int_{G_v} |g \cdot f|^2 \, dg \\
&= M \int_{G_v} |f(g)|^2 \, dg \quad \text{(by change of variables)} \\
&= M|f|^2_{L^2}
\end{aligned}$$

Self-adjoint: For $\phi_v \in C_c^\infty(G_v)$ as above, we have

$$\begin{aligned}
\langle \phi_v \cdot f, F\rangle &= \left\langle \int_{G_v} \phi_v(g) g \cdot f \, dg, F \right\rangle \\
&= \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \int_{G_v} \phi_v(g) f(xg) \overline{F(x)} \, dg \, dx \\
&= \int_{G_v} \phi_v(g) \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} f(xg)\overline{F(x)} \, dx \, dg \text{ (Fubini for Gelfand-Pettis integrals)} \\
&= \int_{G_v} \phi_v(g) \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} f(x)\overline{F(xg^{-1})} \, dx \, dg \quad \text{(substituting } x \to xg^{-1}\text{)} \\
&= \int_{G_v} \phi_v(g^{-1}) \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} f(x)\overline{F(xg)} \, dx \, dg \quad \text{(substituting } g \to g^{-1}\text{)} \\
&= \int_{G_v} \overline{\phi_v(g)} \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} f(x)\overline{F(xg)} \, dx \, dg \quad \text{(substituting } g \to g^{-1}\text{)} \\
&= \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} f(x) \int_{G_v} \overline{\phi_v(g)} g \cdot \overline{F(x)} \, dg \, dx \quad \text{(by Fubini)} \\
&= \langle f, \phi_v \cdot F\rangle
\end{aligned}$$

□

4.2.2 Definition: A *spherical* Hecke operator is invariant from both the left and right with respect to the maximal compact subgroup K_v. The following lemma establishes basic characteristics for the algebra of spherical Hecke operators.

4.2.3 Lemma *For $v < \infty$, matrix transposition is an anti-automorphic involution that stabilizes $K_v = G_v(\mathbb{Z}_v)$ and fixes the Levi component M_v^P. The algebra of spherical Hecke operators is commutative with respect to composition, which is realized as operator convolution.*

Proof. The first assertion is clear because transpose is an involution, $K_v = G_v(\mathbb{Z}_v)$ is closed under transposition, and $\mathbb{Z}_v$ is a ring.

For the second assertion, compute using the transpose involution g^t

$$
\begin{aligned}
(\phi_v * \psi) \cdot f &= \int_{K_v\backslash G_v/K_v} \left(\int_{K_v\backslash G_v/K_v} \phi_v(g)\psi(g^{-1}h)\, dg \right) h \cdot f\, dh \\
&= \int_{K_v\backslash G_v/K_v} \left(\int_{K_v\backslash G_v/K_v} \phi_v(g)\psi(g^{-1}h^t)\, dg \right) h^t \cdot f\, dh \quad \text{(Cartan decomposition)} \\
&= \int_{K_v\backslash G_v/K_v} \left(\int_{K_v\backslash G_v/K_v} \phi_v(g^t)\psi((g^{-1}h^t)^t)\, dg \right) (h^t)^t \cdot f\, dh \ \text{(Cartan decomposition)} \\
&= \int_{K_v\backslash G_v/K_v} \left(\int_{K_v\backslash G_v/K_v} \phi_v(g^t)\psi(h(g^t)^{-1})\, dg \right) h \cdot f\, dh \\
&= \int_{K_v\backslash G_v/K_v} \left(\int_{K_v\backslash G_v/K_v} \phi_v(g)\psi(hg^{-1})\, dg \right) h \cdot f\, dh \quad \text{(Cartan decomposition)} \\
&= \int_{K_v\backslash G_v/K_v} \left(\int_{K_v\backslash G_v/K_v} \phi_v(g^{-1}h)\psi(g)\, dg \right) h \cdot f\, dh \quad (g \to g^{-1}h) \\
&= (\psi * \phi_v) \cdot f
\end{aligned}
$$

□

4.2.4 Corollary *Matrix transposition is an anti-automorphic involution that stabilizes $K_{\mathbb{A}}$, the maximal compact subgroup of $G_{\mathbb{A}}$, and fixes the Levi component $M_{\mathbb{A}}^P$. The algebra of spherical Hecke operators is commutative with respect to composition, which is realized as operator convolution.* □

4.2.5 Definition: For f a cuspform on $\mathrm{GL}_2(\mathbb{R})$, the corresponding *cuspidal-data Eisenstein series* are of the form

$$\varphi_{s,f}(g = nmk) = |a_{11}|^s\, f\begin{pmatrix} a_{22} & b_{22} \\ c_{22} & d_{22} \end{pmatrix} \tag{4.3}$$

$$E_{s,f}^{\mathrm{Kling}}(g) = \sum_{\gamma \in P_k^{\mathrm{Kling}} \backslash G_k} \varphi_{s,f}(\gamma g) \tag{4.4}$$

where the Klingen parabolic is defined by (2.4).

4.2.6 Proposition *Let f be a strong-sense cuspform on $GL_2(\mathbb{A})$ with trivial central character. Klingen parabolic Eisenstein series (Definition 4.2.5) with strong-sense cuspidal data f on the Levi component are eigenfunctions for spherical Hecke operators.*

Proof. For $v < \infty$, the cuspidal-data Eisenstein series $E_{s,f}^{\mathrm{Kling}}$ defined by (4.3) is right K_v invariant. Hence, for $\phi_v \in C_c^\infty(K_v\backslash G_v/K_v)$, the spherical Hecke operator ϕ_v defined by

$$(\phi_v \cdot E_{s,f}^{\mathrm{Kling}})(x) = \int_{G_v} \phi_v(g) E_{s,f}^{\mathrm{Kling}}(xg)\, dg$$

produces a right K_v-invariant image

$$\begin{aligned}
(\phi_v \cdot E_{s,f}^{\mathrm{Kling}})(xk) &= \int_{G_v} \phi_v(g) E_{s,f}^{\mathrm{Kling}}(xkg)\, dg \quad (\text{for } k \in K_v) \\
&= \int_{G_v} \phi_v(k^{-1}g) E_{s,f}^{\mathrm{Kling}}(xg)\, dg \quad (\text{change of variables } g \to k^{-1}g) \\
&= \int_{G_v} \phi_v(g) E_{s,f}^{\mathrm{Kling}}(xg)\, dg \quad (\text{for } \phi_v \in C_c^\infty(K_v\backslash G_v/K_v)) \\
&= (\phi_v \cdot E_{s,f}^{\mathrm{Kling}})(x)
\end{aligned}$$

Since the Hecke operator

$$(\phi_v \cdot E_{s,f}^{\mathrm{Kling}})(x) = \int_{G_v} \phi_v(g) \left(\sum_{\gamma \in P_k^{\mathrm{Kling}} \backslash G_k} \varphi_{s,f}(\gamma x g) \right) dg$$

is an action on the right and the Eisenstein series is defined by a sum where γ is an action on the left, the Eisenstein series is an eigenfunction of the Hecke operator if each term is invariant with respect to the action on the right. Using Iwasasa decomposition with modular function $\delta(g)$, the integral becomes

$$(\phi_v \cdot \varphi_{s,f})(n_0 m_0 k_0) = \int_{N_v} \int_{M_v} \int_{K_v} \phi_v(n_A m_B k) \varphi_{s,f}(n_0 m_0 n_A m_B k) \delta(nmk) \, dn_A \, dm_B \, dk$$

The right K_v invariance of ϕ_v and $\varphi_{s,f}$ give

$$(\phi_v \cdot \varphi_{s,f})(n_0 m_0 k_0) = \int_{N_v} \int_{M_v} \phi_v(n_A m_B) \varphi_{s,f}(n_0 m_0 n_A m_B) \delta(nm) \, dn_A \, dm_B$$

where dn_A is the Haar measure on N_v^{Kling}, and dm_B is the Haar measure on M_v^{Kling}. Consider the modular function for N_v^{Kling}: $\delta(n_A) = \delta(n_0) = 1$. The equality

$$\varphi_{s,f}(n_0 m_0 n_A m_B) = \varphi(n_0 m_0 n_A m_0^{-1} m_0 m_B) = \varphi(n_0 n_{A'} m_0 m_B) \tag{4.5}$$

shows that we have

$$(\phi_v \cdot \varphi_{s,f})(n_0 m_0 k_0) = \int_{N_v} \int_{M_v} \phi_v(n_A m_B) \varphi_{s,f}(n_0 n_{A'} m_0 m_B) \delta(m_0) \, dn_A \, dm_B \tag{4.6}$$

By Fubini-Tonelli and Gelfand-Pettis, we can reverse the order of integration. By left N_v^{Kling} invariance of vector $\varphi_{s,f}$

$$(\phi_v \cdot \varphi_{s,f})(n_0 m_0 k_0) = \int_{M_v} \left(\int_{N_v} \phi_v(n_A m_B) \, dn_A \right) \varphi_{s,f}(m_0 m_B) \delta(m_0) \, dm_B$$

The inner integral is a function on M_v^{Kling}. That ϕ_v is left and right K_v invariant implies it is left and right $K_v \cap M_v^{\text{Kling}}$ invariant, that is, for $\widetilde{\phi}_v \in C_c^\infty((K_v \cap M_v^{\text{Kling}}) \backslash M_v^{\text{Kling}} / (K_v \cap M_v^{\text{Kling}}))$ the inner integral is:

$$(\phi_v \cdot \varphi_{s,f})(n_0 m_0 k_0) = \int_{M_v} \widetilde{\phi}_v(m_B) \varphi_{s,f}(m_0 m_B) \delta(m_0) \, dm_B$$

By hypothesis, for $\varphi_{s,f}$ defined by (4.3), the cuspidal data f is an eigenfunction for the Hecke algebra on the Levi component M_v^{Kling}. The compact support of $\widetilde{\phi}_v$ implies

$$\int_{M_v} \widetilde{\phi}_v(m_B) \varphi_{s,f}(m_0 m_B) \delta(m_0) \, dm_B = \eta_s(\widetilde{\phi}_v) \varphi_{s,f}(m_0) \tag{4.7}$$

for some function $\eta_s(\widetilde{\phi_v})$ of the Hecke algebra depending holomorphically on s.

Therefore, by the previous argument and by the identity principle from complex analysis extended by Schwartz-Grothendieck, a strong-sense cuspidal-data Eisenstein series for the Klingen parabolic is an eigenfunction for Hecke operators. $\square$

4.2.7 Proposition *Speh forms, i.e., residues of Klingen parabolic Eisenstein series with strong-sense cuspidal data on the Levi component are eigenfunction for Hecke operators.*

Proof. Taking residues is a Gelfand-Pettis integral. Hence, by the theory of Gelfand-Pettis vector valued functions (cf. [Garrett 2017] Section 14.1), Hecke operators commute with taking residues. Then, by the previous proposition, since Eisenstein series are eigenfunctions of Hecke operators, the residues of cuspidal-data Eisenstein series are eigenfunctions of Hecke operators. This proves that Speh forms are eigenfunctions of Hecke operators. $\square$

Chapter 5

Spectral Components

We restrict to only a small part of the spectrum of $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K_\infty)$ associated with a fixed strong-sense cuspform on the Levi component of a fixed maximal parabolic. Recall that our motivation for looking at a small part of the spectrum is to project distributions to better Sobolev spaces. In this section, we describe the spectral components specific to this project.

5.1 Eisenstein Series with Cuspidal Data

Eisenstein series are at the heart of this project. They serve two related purposes. Eisenstein series are the eigenfunctions (of the invariant Laplacian) with which we express functions in the orthogonal complement to cuspforms.

More importantly for this project, Eisenstein series with cuspidal data on the Levi component are in the domain of the distribution created by repurposing a triple-product L-function as a rapid-decay distribution on $Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}$. We first restrict $-\Delta$ to a small fragment of $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ and make a strong subconvexity assumption to get θ into $H^{-1}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$, and then take the Friedrichs extension. This allows us to convert an inhomogeneous differential equation as in Theorem 1.2.1 to a homogeneous equation with boundary condition. That is, let

$$\langle \theta, E_{s,f}\rangle = \int_{H_k\backslash H_{\mathbb{A}}} E^P_{s,f}(h)\cdot F(h)dh \qquad \text{(for } F \text{ a fixed cuspform on } \mathrm{SL}_2\times\mathrm{SL}_2)$$

We will address the nature of the pairings $\langle \theta, E^P_{s,f}\rangle$ to verify that the spectral synthesis is legitimate in each case, ie., that the function

$$s \to \langle \theta, E^P_{s,f} \rangle$$

is in a suitable weighted L^2-space. While $E_{s,F}$ is smooth of moderate growth, it is not in any Sobolev space, so we cannot allow arbitrary $\theta \in H^{-1}(G_k \backslash G_{\mathbb{A}}/K)$. The bilinear pairing makes sense and θ can be spectrally decomposed as above when θ is in the space of *compactly supported* distributions $\mathcal{E}_{s,F}(G_k \backslash G_{\mathbb{A}}/K)^*$, the dual space to the space space of smooth functions $C^\infty(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)$, which is inside $H^{-\infty}(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)$. However, integration against cuspforms is *not* compactly supported but only of rapid decay. Fortunately, the topological vector space dual to smooth, moderate growth functions $C^\infty_{\text{Mod}}(G_k \backslash G_{\mathbb{A}}/K)$ is the domain of rapid decay distributions $D_{\text{Rap}}(G_k \backslash G_{\mathbb{A}}/K)$ (see Section 8.1).

The cuspidal data f for Eisenstein series for Klingen parabolics is a strong-sense cuspform on the segment of the standard Levi component (2.6) that is isomorphic to SL_2. By assumption, f is an eigenfunction of the invariant Laplacian at archimedean places and an eigenfunction for spherical Hecke operators at non-archimedean places $v < \infty$ in addition to having vanishing constant term. From Definition 4.2.5 and (2.4), the cupidal-data Eisenstein series relative to the Klingen parabolic is

$$\begin{aligned} \varphi_{s,f}(g = nmk) &= |a_{11}|^s \, f \begin{pmatrix} a_{22} & b_{22} \\ c_{22} & d_{22} \end{pmatrix} \\ E^{\text{Kling}}_{s,f}(g) &= \sum_{\gamma \in P^{\text{Kling}}_k \backslash G_k} \varphi_{s,f}(\gamma g) \end{aligned} \tag{5.1}$$

5.1.1 Proposition *On standard Siegel sets $\mathfrak{S}^C_t$, an Eisenstein series* (5.1) *is a smooth moderate-growth function and all of its derivatives are smooth moderate-growth functions when $s \in C' \subset \mathbb{C}$ a compact set.*

Proof. Since power functions and cuspforms are smooth, $\varphi_{s,f}$ is smooth, and the sum $E^{\text{Kling}}_{s,f}$ is smooth. Similarly for all of its derivatives.

For $s \in C'$, the modulus of s is bounded. Then, for g is a standard Siegel set $\mathfrak{S}^C_t$

$$|a_{11}|^s \ll_{C,C'} \eta_G(g)^\lambda$$

for some $\lambda \in \mathbb{R}$. Therefore, $|a_{11}|^s$ is a moderate-growth function.

Strong-sense cuspforms are rapid-decay. By general principles, the product of a moderate-growth function with a rapid-decay function is a rapid-decay function. Similarly for all of its derivatives because the derivative of a smooth rapid-decay function is rapid decay; the derivative of a smooth moderate growth function is moderate growth; and the derivative of any order of a product is a finite sum of products. By induction on the order of the derivative, a derivative (of any order) of a product of a smooth moderate-growth function and a smooth rapid-decay function is a smooth rapid-decay function. Therefore, $\varphi_{s,f}$ is a rapid-decay function and so are all of its derivatives.

$G_k \backslash G_{\mathbb{A}}$ is the union of a compact set with $\eta_G(g) \leq M$ for some $M > 0$ and a non-compact set. By reduction theory (cf. [Garrett 2017] Corollary 3.3.3) there are a finite number of $\gamma \in P_k^{\text{Kling}} \backslash G_k$ such that for any $g \in G_{\mathbb{A}}$, $\eta_G(\gamma g) > \eta_G(g)$. Since $\varphi_{s,f}$ is smooth, it is bounded on the compact set by some $N > 0$. Hence

$$\sum_{\substack{\gamma \in P_k^{\text{Kling}} \backslash G_k \text{ and} \\ \eta_G(\gamma g) \leq M}} \varphi_{s,f}(\gamma \cdot g) \leq \sum_{\substack{\gamma \in P_k^{\text{Kling}} \backslash G_k \text{ and} \\ \eta_G(\gamma g) \leq M}} N$$

A constant function is moderate growth. Since the number of terms with $\eta_G(\gamma \cdot g) > M$ is finite and each of those terms is rapidly decaying, $E_{s,f}^{\text{Kling}}$ is the sum of a moderate-growth function and a rapid-decay function. Therefore, $E_{s,f}^{\text{Kling}}$ is a moderate-growth function. The same argument applies to all of its derivatives. $\square$

5.1.2 Corollary *For s in compact C' and g in a standard Siegel set $\mathfrak{S}_t^C$, the Eisenstein series $E_{s,f}^{\text{Kling}}$ is a* uniform moderate-growth function *(ref. Definition 3.2.2).*

Proof. By general characteristics of colimit, it suffices to show that $E_{s,f}^{\text{Kling}} \in \mathcal{B}_n$ for some $n \in \mathbb{Z}_+$ when $s \in C'$.

For $T \in U\mathfrak{g}$ and $g \in \mathfrak{S}_t^C$, we have

$$\begin{aligned} \left|TE_{s,f}^{\text{Kling}}\right| &= \left|T\left(\sum_{\gamma \in P_k^{\text{Kling}}\backslash G_k} \varphi_{s,f}(\gamma \cdot g)\right)\right| \\ &= \left|\sum_{\gamma \in P_k^{\text{Kling}}\backslash G_k} T\varphi_{s,f}(\gamma \cdot g)\right| \\ &\leq \sum_{\gamma \in P_k^{\text{Kling}}\backslash G_k} |T\varphi_{s,f}(\gamma \cdot g)| \end{aligned}$$

By the Proposition 5.1.1, we can move the operator inside the sum because each summand is smooth, and we can move the absolute value inside the sum by Cauchy-Schwarz-Bunyakovsky.

For X a basis element of $U\mathfrak{g}$ and $\eta(g) = |a_{11}|$,

$$\begin{aligned} X\varphi_{s,f}(\gamma \cdot g) &= \frac{\partial}{\partial r}\varphi_{s,f}(\gamma \cdot ge^{rX})\bigg|_{r=0} \\ &= \frac{\partial}{\partial r}\eta(\gamma \cdot ge^{rX})^s\bigg|_{r=0} f(\gamma \cdot g) + \eta(\gamma \cdot g)^s \frac{\partial}{\partial r} f(\gamma \cdot ge^{rX})\bigg|_{r=0} \end{aligned}$$

Visibly, since C and C' are compact, Definition 3.2.2 shows,

$$|\eta(g)|^s \ll_{(\mathfrak{S}_t^C, C')} \eta_G(g)^n \text{ implies } \frac{\partial}{\partial r}\left|\eta(g)\right|^s\bigg|_{r=0} \ll_{(\mathfrak{S}_t^C, C')} \eta_G(g)^n$$

for some n. A basis element $X \in U\mathfrak{g}$ maps a rapid-decay function f to another rapid-decay function. Combining these observations with the product rule,

$$|\varphi_{s,f}| \ll_{(\mathfrak{S}_t^C, C')} \eta_G(g)^n \text{ implies } |X\varphi_{s,f}| \ll_{(\mathfrak{S}_t^C, C')} \eta_G(g)^n$$

By induction, powers of X do not change the moderate growth exponent n. Similarly, products and finite sums of products of basis elements of $U\mathfrak{g}$ do not change n.

$T \in U\mathfrak{g}$ is a finite sum of powers and products of basis elements $X \in U\mathfrak{g}$. Therefore, $T \in U\mathfrak{g}$ maps moderate growth function $\varphi_{s,f}$ of exponent n to a moderate growth function of exponent n. By the argument in the final paragraph of the previous proposition (Proposition 5.1.1), $E_{s,f}^{\text{Kling}} \in \mathcal{B}_n$, and the corollary follows. $\square$

Eisenstein series with *strong-sense* cuspidal data for the Klingen parabolic are eigenfunctions at archimedean places for the invariant differential operator Δ descended from Casimir. Not just satisfying the zero constant term implied by the Gelfand condition, strong sense cuspforms are eigenfunctions for the invariant Laplacian, which by the theory of the constant term implies the cuspforms are rapid decay.

5.1.3 Proposition *For strong-sense cuspforms f on the Levi component $SL_2(\mathbb{A})$, the Eisenstein series $E_{s,f}^{\text{Kling}}$ is an eigenfunction for the invariant differential operator descended from Casimir at the archimedean place.*

Proof. Let $\Omega = \Omega_1 + \Omega_2$ where

$$\begin{aligned}\Omega_1 =& \frac{1}{4}H_\alpha^2 + \frac{1}{2}H_\beta^2 + 2X_\beta X_{-\beta}\\ \Omega_2 =& \frac{1}{2}X_\alpha X_{-\alpha} + 2X_{2\alpha+\beta}X_{-(2\alpha+\beta)} + X_{\alpha+\beta}X_{-(\alpha+\beta)} - 5H_\alpha - 6H_\beta\end{aligned}$$

Notice that $X_\alpha, X_{\alpha+\beta}, X_{2\alpha+\beta}$ are in $\text{Lie}(N^{\text{Kling}})$, so they kill f. In addition, SL_2 has trivial central character, which implies that $H_\alpha \cdot f = H_\beta \cdot f = 0$.

Finishing the calculation with $\Omega_1 \cdot f = \lambda f$ implies

$$\Omega \cdot \varphi_{s,f} = \lambda \varphi_{s,f}$$

□

5.2 Speh Forms

By the theory of the constant term, an eigenfunction of the invariant Laplacian that is also of moderate growth is asymptotic to its constant terms. Certainly, poles of constant terms of an Eisenstein series are poles of the Eisenstein series. L^2 residues of poles of Eisenstein series are part of the spectral decomposition of $L^2(G_k\backslash G_{\mathbb{A}})$. While our goal is to study the restriction to a small subspace of the continuous spectrum for a Klingen parabolic, it turns out to be best to include residues of the Eisenstein series for the Klingen parabolic in the subspace to which we restrict the Laplacian.

[Langlands 1976]) established the fact that the constant terms of an Eisenstein series determine its poles. Then, expanding on the work of [Speh 1981], which explicated the dual representation of $\text{GL}_3(\mathbb{R})$ and $\text{GL}_4(\mathbb{R})$, [Jacquet 1983] described the residues but not their multiplicities for $\text{GL}_n(\mathbb{R})$ using the Langlands dual group. The resulting residues of Eisenstein series with cuspidal data are known as Speh forms.

The Levi component for the constant term of the Eisenstein series along the Klingen parabolic is $M = \text{GL}_1 \times \text{SL}_2$. The constant term is then an integral over the unipotent subgroup N, where $P^{\text{Kling}} = NM$ of functions on these groups. [Kim 1995] observed that these constant terms are Gelbart-Jacquet L-functions ([Gelbart-Jacquet 1978]) and used [Jacquet 1983] to show the residues are elements of $L^2(G_k\backslash G_{\mathbb{A}})$. In a more computational approach, [Garrett 2017] Section 3.14 (in particular see Corollary 3.14.4) gives a method using *Maaß-Selberg* relations to show Speh forms are square-integrable.

5.2.1 Proposition *Speh forms, i.e., residues of Klingen parabolic Eisenstein series with strong-sense cuspidal data on the Levi component are eigenfunctions for differential operators* Δ *descended from Casimir.*

Proof. Taking residues in the Schwartz-Grothendieck extension of Cauchy-Goursat theory is a Gelfand-Pettis integral. Hence, by the theory of Gelfand-Pettis vector-valued functions (cf. [Garrett 2017] Section 14.1), differential operators commute with taking residues. Then, by Proposition 5.1.3, since Eisenstein series are eigenfunctions of differential operators Δ, the residues of cuspidal-data Eisenstein series are eigenfunctions of differential operators. This proves that Speh forms are eigenfunctions for Δ. $\square$

5.3 Cuspidal-data Eisenstein Series Constant Term

The constant terms $c_P E_{s,F}$ of an Eisenstein series control its asymptotic behavior and are eigenfunctions of the invariant Laplacian descended from Casimir Δ. In the range of convergence, the constant term along $P = P^{\text{Kling}}$ of a cuspidal-data Eisenstein series for the Klingen parabolic P is:

$$\begin{aligned} c_P E_{s,f}(g) &= \int_{N_k\backslash N_{\mathbb{A}}} E_{s,f}(ng)\, dn \\ &= \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in P\backslash G_k} \varphi_{s,f}(\gamma\cdot ng)\, dn \end{aligned}$$

where φ is defined by (4.3). The constant terms of a cuspidal-data Eisenstein series for the Klingen parabolic along Siegel parabolics and minimal parabolics are identically zero. Unwinding using Bruhat decomposition with $P = P^{\mathrm{Kling}}$

$$\begin{aligned}
c_P E_{s,f}(g) &= \int_{N_k\backslash N_{\mathbb{A}}} \sum_{w\in P\backslash G_k/P} \sum_{\gamma\in(w^{-1}Pw\cap P)\backslash P} \varphi_{s,f}(w\gamma\cdot ng)\,dn \\
&= \sum_{w\in P\backslash G_k/P} \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in(w^{-1}Pw\cap P)\backslash P} \varphi_{s,f}(w\gamma\cdot ng)\,dn
\end{aligned}$$

The orbits for the Klingen parabolic are

$$P^{\mathrm{Kling}}\backslash G_k/P^{\mathrm{Kling}} \longleftrightarrow (P^{\mathrm{Kling}}\cap W)\backslash W/(P^{\mathrm{Kling}}\cap W) =$$

$$\left\{ \left\{ (1), \left(\begin{array}{cc|cc} 1 & & & \\ & & & -1 \\ \hline & & 1 & \\ & 1 & & \end{array}\right) \right\}, \right.$$

$$\left\{ \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right), \left(\begin{array}{cc|cc} & & & -1 \\ & & -1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right), \left(\begin{array}{cc|cc} & 1 & & \\ & & 1 & \\ \hline & & & 1 \\ -1 & & & \end{array}\right), \left(\begin{array}{cc|cc} & & & -1 \\ 1 & & & \\ \hline & 1 & & \\ & & 1 & \end{array}\right) \right\},$$

$$\left. \left\{ \left(\begin{array}{cc|cc} & & -1 & \\ & 1 & & \\ \hline 1 & & & \\ & & & 1 \end{array}\right), \left(\begin{array}{cc|cc} & & -1 & \\ & & & -1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right) \right\} \right\}$$

There are three orbits for w with representatives

$$1_4, \left(\begin{array}{cc|cc} & & -1 & \\ & & & -1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right), \text{ and } \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right)$$

5.3.1 In the case where w is represented by 1_4, we get

$$\begin{aligned} c_P E_{s,f}(g) &= \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in(P\cap P)\backslash P} \varphi_{s,f}(\gamma\cdot ng)\,dn \\ &= \int_{N_k\backslash N_{\mathbb{A}}} \varphi_{s,f}(ng)\,dn \\ &= \varphi_{s,f}(g)\int_{N_k\backslash N_{\mathbb{A}}} 1\,dn \quad (\varphi \text{ is left } N_{\mathbb{A}} \text{ invariant}) \\ &= \varphi_{s,f}(g) \quad (\text{normalization}) \end{aligned}$$

5.3.2 In the case where w is represented by $w = \left(\begin{array}{cc|cc} & & -1 & \\ & & & -1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right)$,

$$wPw^{-1} = \left(\begin{array}{cc|cc} * & 0 & 0 & 0 \\ * & * & 0 & * \\ \hline * & * & * & * \\ * & * & 0 & * \end{array}\right); \quad wPw^{-1}\cap P = \left(\begin{array}{cc|cc} * & 0 & 0 & 0 \\ 0 & * & 0 & * \\ \hline 0 & 0 & * & 0 \\ 0 & * & 0 & * \end{array}\right) \subset M_k$$

For w the integral then becomes

$$\begin{aligned} c_P E_{s,f}(g) &= \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in(wPw^{-1}\cap P)\backslash P} \varphi_{s,f}(w\gamma\cdot ng)\,dn \\ &= \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in M_k\backslash P} \varphi_{s,f}(w\gamma\cdot ng)\,dn \quad (\text{by calculation above}) \\ &= \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in N_k} \varphi_{s,f}(w\gamma\cdot ng)\,dn \\ &= \int_{N_{\mathbb{A}}} \varphi_{s,f}(wng)\,dn \end{aligned}$$

Let

$$m_t = \left(\begin{array}{cc|cc} t & & & \\ & 1 & & \\ \hline & & t^{-1} & \\ & & & 1 \end{array}\right)$$

Then

$$c_P E_{s,f}(m_t m) = \int_{N_{\mathbb{A}}} \varphi_{s,f}(wn \cdot m_t m)\, dn$$

Since $(m_t)^{-1} n m_t = \left(\begin{array}{cc|cc} 1 & t^{-1}* & t^{-2}* & t^{-1}* \\ & 1 & t^{-1}* & \\ \hline & & 1 & \\ & & -t^{-1}* & 1 \end{array}\right)$ replacing n by $(m_t)^{-1} n m_t$ with corresponding change of measure for the Klingen parabolic gives

$$\begin{aligned} c_P E_{s,f}(m_t m) &= \int_{N_{\mathbb{A}}} \varphi_{s,f}(wn \cdot m_t m)\, dn \\ &= |t| \int_{N_{\mathbb{A}}} \varphi_{s,f}(w m_t \cdot nm)\, dn \\ &= |t| \int_{N_{\mathbb{A}}} \varphi_{s,f}(w m_t w^{-1} \cdot wnm)\, dn \end{aligned}$$

Since $w m_t w^{-1} = \left(\begin{array}{cc|cc} t^{-1} & & & \\ & 1 & & \\ \hline & & t & \\ & & & 1 \end{array}\right)$ we have

$$\begin{aligned} c_P E_{s,f}(m_t g) &= |t| \int_{N_{\mathbb{A}}} \varphi_{s,f}(w m_t w^{-1} \cdot wng)\, dn \\ &= |t| |t|^{-s} \int_{N_{\mathbb{A}}} \varphi_{s,f}(wng)\, dn \end{aligned}$$

By Proposition 4.2.6 (4.7) a strong-sense cuspforms with the same eigenvalues at all finite primes are scalar multiples, holomorphic in s. Therefore, $c_P E_{s,f}(g) = c_{s,f}\varphi_{1-s,f}(g)$ for some constant $c_{s,f}$ that is holomorphic in s.

5.3.3 In the case where the orbit is represented by $w = \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right)$,

$$wPw^{-1} = \left(\begin{array}{cc|cc} * & & * & * \\ * & * & * & * \\ \hline * & & * & * \\ & & & * \end{array}\right); \quad wPw^{-1} \cap P = \left(\begin{array}{cc|cc} * & & * & * \\ & * & * & * \\ \hline & & * & \\ & & & * \end{array}\right)$$

Hence,

$$(wPw^{-1} \cap P)\backslash P = \left(\begin{array}{cc|cc} 1 & * & & \\ & 1 & & \\ \hline & & 1 & \\ & * & * & 1 \end{array}\right)$$

For w the integral then becomes

$$\begin{aligned} c_P E_{s,f}(g) &= \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in(wPw^{-1}\cap P)\backslash P} \varphi_{s,f}(w\gamma \cdot ng)\, dn \\ &= \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in(wPw^{-1}\cap P)\backslash P} \varphi_{s,f}(w\gamma n (w\gamma)^{-1} w\gamma g)\, dn \\ &= \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in(wPw^{-1}\cap P)\backslash P} \varphi_{s,f}(wnw^{-1} \cdot w\gamma g)\, dn \quad (\gamma n\gamma^{-1} \in N_{\mathbb{A}}) \end{aligned}$$

We have

$$wnw^{-1} = \left(\begin{array}{cc|cc} 1 & & & * \\ * & 1 & * & * \\ \hline & & 1 & * \\ & & & 1 \end{array}\right)$$

for $n \in N^{\text{Kling}}$. Iwasawa decomposition $w\gamma g = pk$ for some $p \in P^{\text{Kling}}$ and the right K invariance of $\varphi_{s,f}$ implies $\varphi_{s,f}(wnw^{-1} \cdot w\gamma g) = \varphi_{s,f}(wnw^{-1} \cdot p)$. Therefore, a calculation shows that the argument $wnw^{-1} \cdot w\gamma g$ has the form

$$wnw^{-1} \cap P^{\text{Kling}} = \left(\begin{array}{cc|cc} 1 & & & * \\ & 1 & * & * \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

Hence, for a fixed γ, using $\varphi_{s,f}$ in Definition 4.2.5 the integral has the form

$$\int_{N_k \backslash N_{\mathbb{A}}} |1|^s f \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$

Since the argument of f is an element of the unipotent radical of SL_2 in the lower right hand corner inside $N_k \backslash N_{\mathbb{A}}$ and f is a cuspform on SL_2, the integral is zero. Therefore, $c_P E_{s,f}(g) = 0$ in this case.

Combining the three cases we have

$$c_p E_{s,f} = \varphi_{s,f} + c_{s,f} \varphi_{1-s,f} \tag{5.2}$$

for some constant $c_{s,f}$ that is holomorphic in s.

5.4 Pseudo-Eisenstein Series

Orthogonal to the space of cuspforms $L^2_{\text{cfm}}(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)$ are spaces of pseudo-Eisenstein series. That is, the pseudo-Eisenstein series $\Psi_\psi = \Psi_\psi^P$ attached to the Klingen parabolic $P = P^{\text{Kling}} = NM$ and data ψ_F that is compactly supported on the split component of P is characterized by satisfying the adjunction

$$\int_{N_{\mathbb{A}}M_k\backslash G_{\mathbb{A}}} \psi_{s,F} \cdot c_P f = \int_{G_k\backslash G_{\mathbb{A}}} f \cdot \Psi_\psi \quad (\text{for } f \in L^2(G_k\backslash G_{\mathbb{A}})) \tag{5.3}$$

With this adjunction in mind, let

$$\Psi_\psi(g) = \sum_{\gamma \in P_k\backslash G_k} \psi_{s,F}(\gamma g) \tag{5.4}$$

where

$$\psi_{s,F}(g = nmk) = \eta_{\mathbb{R}}(a_{11}) F \begin{pmatrix} a_{22} & b_{22} \\ c_{22} & d_{22} \end{pmatrix} \quad (\text{ref. } (2.4)) \tag{5.5}$$

with F a strong-sense cuspform on the Levi component and $\eta_{\mathbb{R}} \in C_c^\infty(\mathbb{R})$ acting on the split component $A^{\text{Kling}} = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{11}^{-1} & \\ & & & a_{22}^{-1} \end{pmatrix}$ of P^{Kling}.

5.4.1 Lemma Ψ_ψ *is orthogonal to cuspforms.*

Proof. Fix a Klingen parabolic with Levi-Malcev decomposition $P = NM$. For ψ_F defined by (5.5), with possible cuspidal data F and $\psi_{s,F}$) is compactly supported on the split component A^{Kling} (2.7) by $\eta_{\mathbb{R}}(a_{11})$ and rapidly decaying on the Levi subgroup by $F \begin{pmatrix} a_{22} & b_{22} \\ c_{22} & d_{22} \end{pmatrix}$. The Gelfand condition $c_p f = 0$ is

$$0 = \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K} \psi_{s,F}(g) \cdot c_P f(g) dg = \int_{N_k M_k\backslash G_{\mathbb{A}}/K} \psi_{s,F}(g) \int_{N_k\backslash N_{\mathbb{A}}} f(ng) dn dg$$

Unwinding gives

$$\int_{N_k M_k\backslash G_{\mathbb{A}}/K} \psi_{s,F}(g) f(g) dg$$

Since $f \in L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)$ we can wind back up with respect to P

$$\begin{aligned} \int_{N_k M_k \backslash G_{\mathbb{A}}/K} \psi_{s,F}(g) f(g) dg &= \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K} \sum_{\gamma \in P_k \backslash G_k} \psi_{s,F}(\gamma g) f(\gamma g) dg \\ &= \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K} f(g) \sum_{\gamma \in P_k \backslash G_k} \psi_{s,F}(\gamma g) dg \\ &= \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K} f(g) \Psi_\psi(g) dg \end{aligned}$$

□

5.4.2 Theorem *A pseudo-Eisenstein Ψ_ψ series attached to the Klingen parabolic P^{Kling} can be spectrally decomposed in terms of Eisenstein series with cuspidal data on the Levi component of the Klingen parabolic and Speh forms Υ, which appear as residues of cuspidal data Eisenstein series*

$$\Psi_\psi = \sum_{\Upsilon \sim F} \langle \Psi_\psi, \Upsilon \rangle \cdot \Upsilon + \frac{1}{4\pi i} \int_{(1/2)} \mathcal{E}_{s,F}(\Psi_\psi) \cdot E^{\text{Kling}}_{s,F} ds \tag{5.6}$$

where the continuous spectral coefficients are

$$\mathcal{E}_{s,F}(\Psi_\psi) = \langle \Psi_\psi, E^{\text{Kling}}_{s,F} \rangle = \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \Psi_\psi \cdot \overline{E^{\text{Kling}}_{s,F}} \tag{5.7}$$

Proof. The inverse Fourier-Mellin transform for $\eta_{\mathbb{R}} \in C_c^\infty(\mathbb{R}_+)$ (ref. (5.5)) with $a \in A^{\text{Kling}}$ and $s = \sigma + it$ for $\sigma > 1$ is

$$\eta_{\mathbb{R}}(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}\eta_{\mathbb{R}}(\sigma + it)(a_{11})^{\sigma + it}\, dt$$

where $\sigma \in \mathbb{R}$

$$\mathcal{M}\eta_{\mathbb{R}}(s) = \int_0^\infty \eta_{\mathbb{R}}(y) y^{-s} \frac{dy}{y}$$

Hence,

$$\begin{aligned}
\Psi_\psi(g) &= \sum_{\gamma\in P_k\backslash G_k} \psi(\gamma namk) \\
&= \sum_{\gamma\in P_k\backslash G_k} \eta_{\mathbb{R}}(\gamma a) F(\gamma m) \\
&= \frac{1}{2\pi} \sum_{\gamma\in P_k\backslash G_k} F(\gamma\cdot m) \int_{-\infty}^{\infty} \mathcal{M}\eta_{\mathbb{R}}(\sigma+it)(\gamma\cdot a)_{a_{11}}^{\sigma+it}\,dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}\eta_{\mathbb{R}}(\sigma+it) \sum_{\gamma\in P_k\backslash G_k} (\gamma\cdot a)_{a_{11}}^{\sigma+it} F(\gamma\cdot m)\,dt \quad \text{(by Fubini-Tonelli)} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}\eta_{\mathbb{R}}(\sigma+it) E_{s,F}^{\mathrm{Kling}}(g)\,dt
\end{aligned}$$

By meromophic continuation and functional equation for the Eisenstein series, we can move the vertical bounds of integration to $\sigma = \dfrac{1}{2}$ and pick up finitely many residues of $E_{s,f}^{\mathrm{Kling}}$. Then,

$$\Psi_\psi(g) - (\text{residues}) = \frac{1}{2\pi i} \int_{\frac{1}{2}-\infty}^{\frac{1}{2}+\infty} \mathcal{M}\eta_{\mathbb{R}}(s) E_{s,F}^{\mathrm{Kling}}(g)\,ds$$

By meromophic continuation and functional equation for Klingen parabolic Eisenstein series, we use symmetry to average, giving

$$\Psi_\psi(g) - (\text{residues}) = \frac{1}{4\pi i} \int_{\frac{1}{2}-\infty}^{\frac{1}{2}+\infty} \Big(\mathcal{M}\eta_{\mathbb{R}}(s) E_{s,F}^{\mathrm{Kling}}(g) + \mathcal{M}\eta_{\mathbb{R}}(1-s) E_{1-s,F}^{\mathrm{Kling}}(g)\Big)\,ds \tag{5.8}$$

Working in the other direction, notice that the modular function of P^{Kling} (2.4) is $\delta(p) = a_{11}$ since $\det \begin{pmatrix} a_{22} & b_{22} \\ c_{22} & d_{22} \end{pmatrix} = 1$. Consider only functions on $Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}$ with trivial central character. Let Z^+ be scalar matrices whose entries are the diagonal embedding $(0,\infty) \to \mathbb{J}$, the ideles of $\mathbb{A}$. Then

$$\begin{aligned}
\langle \Psi_\psi, E_{s,F}^{\mathrm{Kling}} \rangle &= \int_{Z_{\mathbb{A}} G_k\backslash G_{\mathbb{A}}} \Psi_\psi(g) \cdot \overline{E_{s,F}^{\mathrm{Kling}}}(g)\,dg \\
&= \int_{Z^+ G_k\backslash G_{\mathbb{A}}} \Psi_\psi(g) \cdot \overline{E_{s,F}^{\mathrm{Kling}}}(g)\,dp
\end{aligned} \tag{5.9}$$

Following arguments from [Garrett 2017] Section 3.16 with $Z^+M_k\backslash M_{\mathbb{A}} \cong Z^+\backslash A^+ \times M_k\backslash M^1$, where A^+ is A^{Kling} restricted to entries with positive value at archimedean places and M^1 is $M_{\mathbb{A}}$ restricted to matrices with $|a_{11}| = \Pi_{v\leq\infty}|a_{11,v}|_v = 1$, unwind the above pairing (5.9)

$$\begin{aligned}\langle \Psi_\psi, E_{s,F}^{\text{Kling}}\rangle &= \int_{Z^+G_k\backslash G_{\mathbb{A}}} \Psi_\psi(g)\cdot \overline{E_{s,F}^{\text{Kling}}}(g)\,dg \\ &= \int_{Z^+P_k\backslash P_{\mathbb{A}}} \Psi_\psi(p)\cdot \overline{E_{s,F}^{\text{Kling}}}(p)\,\frac{dp}{\delta(p)}\end{aligned}$$

by Bruhat decomposition and right K invariance. So,

$$\begin{aligned}\langle \Psi_\psi, E_{s,F}^{\text{Kling}}\rangle &= \int_{Z^+M_k\backslash M_{\mathbb{A}}} \Psi_\psi(m)\int_{Z^+N_k\backslash N_{\mathbb{A}}} \overline{E_{s,F}^{\text{Kling}}}(nm)\,dn\,\frac{dm}{\delta(m)} \\ &= \int_{Z^+M_k\backslash M_{\mathbb{A}}} \Psi_\psi(m)c_P\overline{E_{s,F}^{\text{Kling}}}(m)\,\frac{dm}{\delta(m)} \\ &= \int_{Z^+M_k\backslash M_{\mathbb{A}}} \Psi_\psi(m)\overline{(\varphi_{s,f}(m)+c_{s,f}\varphi_{2-s,f}(m))}\,\frac{dm}{\delta(m)} \\ &= \int_{Z^+M_k\backslash M_{\mathbb{A}}} \psi(m)\overline{(\varphi_{s,f}(m)+c_{s,f}\varphi_{1-s,f}(m))}\,\frac{dm}{\delta(m)} \\ &= \int_{Z^+\backslash A^+\times M_k\backslash M^+} \eta_{\mathbb{R}}(a_{11})f(m)\overline{(|a_{11}|^s f(m)+c_{s,f}|a_{11}|^{1-s}f(m))}\,\frac{dm}{\delta(m)}\,da_{11} \\ &= \langle f,f\rangle \int_{Z^+\backslash A^+} \eta_{\mathbb{R}}(a_{11})\left(|a_{11}|^{-s}+\frac{1}{c_{s,f}}|a_{11}|^s\right)\frac{da_{11}}{|a_{11}|} \\ &= \langle f,f\rangle \int_0^\infty \eta_{\mathbb{R}}(t)\left((t)^{-s}+\frac{1}{c_{s,f}}(t)^s\right)\frac{dt}{t} \\ &= \langle f,f\rangle \left(\mathcal{M}\eta_{\mathbb{R}}(s)+\frac{1}{c_{s,f}}\mathcal{M}\eta_{\mathbb{R}}(1-s)\right)\end{aligned}$$

Normalizing $\langle f,f\rangle$ and subsituting into (5.8)

$$\Psi_\psi - (\text{residues}) = \frac{1}{4\pi i}\int_{\frac{1}{2}-\infty}^{\frac{1}{2}+\infty} \langle \Psi_\psi, E_{s,F}^{\text{Kling}}\rangle\, E_{s,F}^{\text{Kling}}\,ds$$

$\square$

While the pseudo-Eisenstein series are not eigenfunctions of the invariant operator descended from Casimir, pseudo-Eisenstein series are integrals of cuspidal data

Eisenstein series, which are eigenfunctions.

Chapter 6

Sobolev Spaces

The nature of the differential equation $(S_\Theta - \lambda_w)u = \theta$, the implications of Theorem 1.2.1, and the characteristics of θ defined in Section 8.1 suggest starting with an investigation of subspaces and adjoint extensions of $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$.

6.1 Global Sobolev Spaces

Restrictions and extensions of differential operators and solutions of differential equations are best examined in terms of global automorphic Sobolev spaces. The L^2 Sobolev spaces are Hilbert spaces and L^2-differentiation gives continuous maps between them. As a first definition of Sobolev spaces, we consider spaces indexed by a non-negative integer ℓ. Consider $Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K$ and the invariant differential operator $-\Delta$ (4.1) descended from the Casimir. Note that $-\Delta$ is a semi-bounded operator on $C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$. Let $T = \Delta\,|_{C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)}$. Applying $(1-T)$ to the Petersson inner product, then integration by parts twice for $u, v \in C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ gives:

$$\begin{aligned}\langle (1-T)u, v\rangle_{L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)} &= \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K} (1-T)u(g)\overline{v}(g)\,dg \\ &= \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K} u(g)\,(1-T)\overline{v}(g)\,dg\end{aligned}$$

because integration by parts has no boundary terms. Therfore,

$$\langle (1-T)u, v\rangle_{L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)} = \langle u, (1-T)v\rangle_{L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)}$$

Hence, $(1-T)$ is symmetric on $C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$. Let H^ℓ be the completion of $C_c^\infty(\Gamma\backslash G/K)$ with respect to the H^ℓ-norm

$$\|u\|_{H^\ell}^2 := \left|\langle(1-T)^\ell u, u\rangle_{L^2}\right|^2 \tag{6.1}$$

The differential operator $(1-T)$ maps

$$H^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \xrightarrow{(1-T)} H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) = L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$$

and

$$H^{\ell+2}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \xrightarrow{(1-T)} H^\ell(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$$

for $\ell \in \mathbb{Z} \geq 0$. Furthermore, the arguments in [Garrett 2017] Claim [12.3.9] apply directly to $Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K$ and imply the natural embeddings $H^{\ell+1} \hookrightarrow H^\ell$. Taking the projective limit of spaces $H^\ell(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$:

$$H^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)\cdots \hookrightarrow H^1(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \hookrightarrow H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) = L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \tag{6.2}$$

As a countable limit of Hilbert spaces,

$$H^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) = \bigcap_\ell H^\ell(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$$

is a Fréchet space. Since $C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ is dense in $H^\ell(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ for all $\ell \in \mathbb{Z} \geq 0$, all of the inclusion maps (6.2) are dense inclusions.

We use the spectral transform and Plancherel to extend the global Sobolev space to $H^{-\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ for $-\ell \in \mathbb{Z} \leq 0$, then to $H^r(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K$ for $r \in \mathbb{R}$. [Mœglin-Waldspurger 1995] Chapter II shows how to decompose $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$ and [Garrett 2017] Sections 3.7-3.11 gives an example of spectral decomposition of automorphic functions as a linear combination of automorphic cuspforms and pseudo-cuspforms. Specifically for $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$, the spectrum consists of an orthonormal basis of cuspforms, periods of cuspidal data Eisenstein series, and residues of Eisenstein series.

For $f \in L^2(Z_{\mathbb{A}}G_k \backslash G_{\mathbb{A}}/K_{\mathbb{A}})$

$$
\begin{aligned}
f = \sum_{F \text{ cfm on } G} \langle f, F\rangle \cdot F + \frac{\langle f, 1\rangle \cdot 1}{\langle f, 1\rangle} + \frac{1}{4\pi i} \iint_{\rho + i\mathfrak{a}^*} \langle f, E_s^{\min}\rangle \cdot E_s^{\min} ds + \textit{[other degenerate]} \\
+ \sum_{\substack{F \text{ cfm} \\ \text{on } M^{\text{Sieg}}}} \left(\sum_{\Upsilon_F \sim F} \langle f, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i} \int_{(1/2)} \langle f, E_{s,F}^{\text{Sieg}}\rangle \cdot E_{s,F}^{\text{Sieg}} ds \right) \\
+ \sum_{\substack{F \text{ cfm} \\ \text{on } M^{\text{Kling}}}} \left(\sum_{\Upsilon_F \sim F} \langle f, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i} \int_{(1/2)} \langle f, E_{s,F}^{\text{Kling}}\rangle \cdot E_{s,F}^{\text{Kling}} ds \right)
\end{aligned}
\tag{6.3}
$$

where P^{Sieg} is the Siegel-type parabolic, P^{Kling} is the Klingen-type parabolic; $E^*_{s,F}$ are cuspidal-data Eisenstein series (Section 5.1); the inner sums are over the finite number of Speh forms for the minimal parabolic and corresponding to each cuspform on the Levi component of a maximum parabolic; and *[other degenerate]* refers to non-cuspidal-data Eisenstein series arising as residues of cuspidal-data Eisenstein series.

6.1.1 Definition For convenience, let Ξ be a *spectral parameter space* such that for $f \in C_c^\infty(Z_{\mathbb{A}}G_k \backslash G_{\mathbb{A}}/K)$, ξ a spectral parameter, eigenvalue λ_ξ for eigenvector F_ξ of Δ, and a suitable measure $d\xi$ on Ξ:

$$
F_\xi \in \begin{cases} \{\text{orthonormal basis of cuspforms}\} \\ \{\text{Speh forms attached to a maximal parabolic and cuspidal data}\} \\ \{\text{cuspidal-data Eisenstein series}\} \\ \{\text{other } L^2 \text{ residues of Eisenstein series}\} \end{cases} \tag{6.4}
$$

That is, $d\xi$ is counting measure for F_ξ a cuspform or Speh form, and $d\xi$ is a natural measure on $\rho + i\mathfrak{a}^*$ for cuspidal-data Eisenstein series, where $\mathfrak{a}^*$ is the dual of the Lie algebra $\mathfrak{a}$ of the split component of M^P, and ρ is the corresponding half-sum of roots. For $\ell \in \mathbb{Z}$, let

$$
V^\ell(\Xi) = \text{completion of } \{v \in C^\infty(\Xi) : \int_\Xi |v|^2 |(1-\lambda_\xi)|^\ell \, d\xi < \infty\} \tag{6.5}
$$

with respect to norm

$$
\|v\|^2_{V^\ell} = \int_\Xi |(1-\lambda_\xi)|^\ell |v|^2 \, d\xi \tag{6.6}
$$

with an inner product $\langle u, v\rangle = \int_\Xi u \cdot v$ for $u, v \in V^\ell(\Xi)$ and $\ell \geq 0$. This implies the dual $V^\ell(\Xi)^*$ is

$$\begin{aligned} V^\ell(\Xi)^* &= \text{completion of } \{v \in C^\infty(\Xi) : \int_\Xi u \cdot \overline{v} < \infty\} \\ &= \text{completion of } \{v \in C^\infty(\Xi) : \int_\Xi |v|^2 (1-\lambda_\xi)^{-\ell}\, d\xi < \infty\} \\ &= V^{-\ell} \end{aligned}$$

for all $u \in V^\ell(\Xi)$. The functional $\lambda_v \in V^{-\ell}$ is given by $\lambda_v(u) = \langle u, v\rangle_{V^\ell \times V^{-\ell}}$.

6.1.2 Definition

Let

$$\langle f, F_\xi\rangle = \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K} f(g) \overline{F}_\xi(g) dg \tag{6.7}$$

be the spectral transform $\mathcal{F} : C_c^\infty(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K) \to V(\Xi)$ literally for $f \in C_c^\infty(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)$, and then extending by continuity to $L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)$. Then, (6.3) can be written as

$$f = \int_\Xi \langle f, F_\xi\rangle \cdot F_\xi \, d\xi \tag{6.8}$$

From (6.7) the spectral transform $\mathcal{F} : H^\ell(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K) \longrightarrow V^\ell(\Xi)$ is defined by

$$\mathcal{F} f = \int_\Xi \langle f, F_\xi\rangle \, d\xi = \int_\Xi \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K} f(g) \overline{F}_\xi(g) \, dg \, d\xi$$

literally for $f \in C_c^\infty(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)$, and then extending by continuity to $L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)$.

6.1.3 Proposition

For $f \in C_c^\infty(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)$,
1) The spectral coefficients of Δf are

$$\langle \Delta f, F_\xi\rangle = \lambda_\xi \langle f, F_\xi\rangle$$

2) For $\ell \geq 0$, $\mathcal{F} : H^\ell(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K) \to V^\ell(\Xi)$ is an isometry, and $\mathcal{F} : H^\ell(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)^* \to V^{-\ell}(\Xi)$ is an isometry.

Proof. By Proposition 5.1.3, Corollary 5.2.1, and arguments in their proofs, maximal parabolic Speh forms and maximal parabolic Eisenstein series with strong-sense cuspidal data are eigenfunctions for Δ. The minimal parabolic Eisenstein series

$$E_s^{\min}(g) = \sum_{\gamma \in P_k^{\min}\backslash G_k} \varphi(\gamma g)$$

where

$$\varphi\left(n \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & d_{11} & \\ & & & d_{22} \end{pmatrix} k \right) = |a_{11}|^{2s_1} |a_{22}|^{2s_2}$$

where $s_1, s_2 \in \mathbb{C}$ is clearly an eigenfunction for Δ.

Arguments in [Garrett 2017] Chapter 7.1 apply to $Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K$ to show that the space of $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ cuspforms has a basis of eigenfunctions for Δ. Since Eisenstein series are eigenfunctions for Δ, elements of *[other]* are eigenfunctions for Δ.

Since the integrals in (6.7) are Gelfand-Pettis, the operator $\Delta : H^\ell(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \to H^{\ell-2}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ moves inside the summation and integral signs. That Δ is self-adjoint implies real eigenvalues. Spectral decomposition by eigenfunctions of Δ gives $\langle \Delta f, F_\xi \rangle = \lambda_\xi \langle f, F \rangle$ for $f \in C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$. Extension by continuity for dense $C^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \subset H^\ell(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \subset L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ justifies 1).

Plancherel gives an isometric embedding

$$\mathcal{F} : C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \to \left\{ v \in C^\infty(\Xi) : \int_\Xi |v|^2 (1-\lambda_\xi)^\ell \, d\xi < \infty \right\}$$

Since the space of test functions $C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ is dense in $H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) = L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$, extension by continuity gives Plancherel for $H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$. By Gelfand-Pettis for both $H^\ell(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ and $V^\ell(\Xi)$, Δ

and $\mathcal{F}$ commute, so the following diagram commutes for $\ell \geq 0$

$$\begin{array}{ccc} H^{\ell+2} & \xrightarrow{(1-\Delta)} & H^{\ell} \\ \cong\big\downarrow \mathcal{F} & & \cong\big\downarrow \mathcal{F} \\ V^{\ell+2} & \xrightarrow{(1-\lambda)} & V^{\ell} \end{array} \tag{6.9}$$

This proves 1) for even $\ell \geq 0$. Odd $\ell > 0$ follows by the same commuting argument after acknowledging that $H^1(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ injects into $H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ and $V^1(\Xi)$ injects into $V^0(\Xi)$.

Let $\Lambda : H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \xrightarrow{\sim} H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)^*$ be defined by

$$\Lambda(f)(g) = \langle g, \overline{f}\rangle_{H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)\times H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)}$$

Then by Riesz-Fréchet, Λ is a complex linear isomorphism, and by Riesz-Fréchet each $H^{\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ has (up to isomorphism) a unique Hilbert space dual $H^{\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)^* = H^{-\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ of continuous linear functionals $\lambda_v : H^{\ell}(G_k\backslash G_{\mathbb{A}}/K) \to \mathbb{R}$. That is, for each $v \in H^{\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ there is a unique dual $\lambda_v \in H^{\ell}(G_k\backslash G_{\mathbb{A}}/K)^*$ defined by the hermitian pairing on $H^{\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \times H^{\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)^*$:

$$\lambda_v(u) = \langle u, v\rangle_{H^{\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)\times H^{\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)^*} = \int_{H^{\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)} u(g)\overline{v(g)}\, dg$$

For $f \in C_c^{\infty}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$, the action of Δ on $\lambda \in H^{-\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ is by distributional differentiation

$$\Delta\lambda(f) = \lambda(\Delta \cdot f)$$

because there are no boundary terms. Then, by Gelfand-Pettis, the intertwining operators Δ and $\mathcal{F}$ give the commuting diagram (6.9) for $-\ell < 0$. Extension by continuity implies the isometry of $\mathcal{F}$ for all $\ell \in \mathbb{Z}$.

Since $\mathcal{F}$ is an isometry, $\mathcal{F}^{-1}$ exists and is an isometry. Then, since (6.5) and (6.6) can be defined for all $\ell \in \mathbb{R}$, $H^{\ell}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ is defined spectrally with Sobolev norm $\|u\|_{H^r} = \|\mathcal{F}u\|_{V^r}$ $\square$

6.1.4 Corollary *Taking the limit as $-\ell \to -\infty$ gives*

$$\begin{aligned} &H^{\infty}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)\cdots \xrightarrow{\Delta} H^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \xrightarrow{\Delta} H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \\ &\quad \xrightarrow{\Lambda} H^0(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)^* \xrightarrow{\Delta^*} H^{-2}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)^* \xrightarrow{\Delta^*} \cdots H^{-\infty}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \end{aligned} \tag{6.10}$$

□

6.2 Eisenstein-Sobolev Spaces

A starting point for an investigation of the spectral theory of the triple-product integral recast as a distribution is the spectral decomposition of the *whole* $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ and the determination in which weighted Sobolev-like spaces the spectral coefficients make sense. [Colin de Verdière 1983] and [Bombieri-Garrett 2017] show that restricting to a small subspace puts the Dirac delta $\delta_{\omega}^{\mathrm{afc}}$ in a -1 index Sobolev space and the solution u in a +1 index Sobolev space.

We use the same strategy here. The orthogonal complement to cuspforms is the subspace of pseudo-Eisenstein series $\Psi_{\eta,F}$ as in (5.4). Spectral decomposition of $\Psi_{\eta,F}$ by (6.3) allows us to restrict to a small subspace spanned by pseudo-Eisenstein series with strong-sense cuspidal data on the Levi component for the Klingen parabolic and Speh forms for the Klingen parabolic.

Restricting (5.7), the spectral transform $f \to (s \to \langle f, E_{s,F}\rangle)$ is

$$\mathcal{E}_{s,F}(f) = \langle f, E_{s,F}\rangle = \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K} f(g)\cdot \overline{E_{s,F}}(g)\,dg \tag{6.11}$$

For fixed cuspidal data F on the Levi component of the Klingen parabolic P^{Kling}, let $\mathfrak{E}_c^{\infty}$ be the set

$$\left\{ f \in C_c^{\infty}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) : f = \sum_{\Upsilon_F \sim F} \langle f, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} \langle f, E_{s,F}^{\mathrm{Kling}}\rangle \cdot E_{s,F}^{\mathrm{Kling}}\,ds \right\} \tag{6.12}$$

with norm

$$\|f\|_{\mathfrak{E}_c^\infty} = \sum_{\Upsilon_F \sim F} |\langle f, \Upsilon_F \rangle|^2 + \frac{1}{4\pi i} \int_{(1/2)} |\langle f, E_{s,F}^{\text{Kling}} \rangle|^2 \, ds$$

As a subspace of $H^\ell(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}/K)$, existence of $\mathfrak{E}_c^\infty$ is guaranteed by Proposition 6.1.3.

6.2.1 Definition Using results from the previous section, for $r \in \mathbb{R}$ and fixed cuspidal data F for the Klingen parabolic, let $\mathfrak{E}_F^r$ be the completion of $\mathfrak{E}_c^\infty$ with respect to the norm

$$\|f\|_{\mathfrak{E}_F^r} = \sum_{\Upsilon_F \sim F} |\langle f, \Upsilon_F \rangle|^2 (1 + |\lambda_F|)^r + \frac{1}{4\pi i} \int_{(1/2)} |\langle f, E_{s,F}^{\text{Kling}} \rangle|^2 (1 + |\lambda_{s,F}|)^r \, ds \quad (6.13)$$

The $\mathfrak{E}_F^r$ are *Eisenstein-Sobolev* spaces with respect to this norm.

The previous section implies the following ordering by inclusion:

$$\begin{aligned} \lim_r \mathfrak{E}_F^r = \mathfrak{E}_F^\infty \cdots \hookrightarrow \mathfrak{E}_F^r \hookrightarrow \mathfrak{E}_F^{r-1} \hookrightarrow \cdots \hookrightarrow \mathfrak{E}_F^0 \xrightarrow{\sim} (\mathfrak{E}^0)^* \hookrightarrow \\ \cdots \hookrightarrow \mathfrak{E}_F^{-r} \hookrightarrow \mathfrak{E}_F^{-r-1} \hookrightarrow \cdots \mathfrak{E}_F^{-\infty} = \operatorname*{colim}_r \mathfrak{E}_F^{-r} \end{aligned} \quad (6.14)$$

Chapter 7

Friedrichs Extensions of Restrictions

We will consider the Friedrichs extension of certain invariant differential operators descending from the Casimir operator Ω. First, [Garrett 2017] Sections 11.7-10 show how to use Friedrichs extension to meromorphically continue cuspidal data Eisenstein series. Second, by construction, Friedrichs extensions produce self-adjoint operators from symmetric semi-bounded operators (ref. [Friedrichs 1934], [Riesz-Sz.-Nagy 1955] Chapter VIII Section 124, [Garrett 2017] Theorem 9.2.1), which guarantees real eigenvalues as required in Theorem 1.2.1.

The differential operator Δ descended from Casimir (4.1) is a symmetric semi-bounded operator on $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ with a dense domain of test functions $C_c^\infty(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$.

7.1 General Case

More generally, consider non-negative (semi-bounded), symmetric, densely defined linear operator T on a Hilbert space V, and denote its domain by $\mathcal{D}(T)$. Let

$$\langle u, v\rangle_{V^1} = \langle (1+T)u, v\rangle_V \tag{7.1}$$

Let V^1 be the completion of $\mathcal{D}(T)$ with respect to the associated norm

$$|f|_{V^1}^2 = \langle (1+T)f, f\rangle_V \tag{7.2}$$

The vector subspace V^1 is a dense proper subspace of V. Let V^{-1} be the Hilbert space dual of V^1 with the *weak dual topology*, that is, given $\varepsilon > 0$, and fixed $u \in V^1$ open sets are of the form $\{\lambda : V^1 \to \mathbb{C} : \lambda(u) < \varepsilon\}$.

Consider a finite subspace $\Theta \subset (V^1)^*$.

7.1.1 Definition Let T_Θ be T restricted to $\mathcal{D}(T) \cap \ker \Theta$. We have:

$$\begin{tikzcd} \mathcal{D}(T_\Theta) \arrow[r, hook] \arrow[rr, bend left, "T_\Theta"] & V^1 \arrow[r, hook, "j"] & V \arrow[r, "\Lambda", "\sim"'] & V^* \arrow[r, "j^*"] & V^{-1} \end{tikzcd} \tag{7.3}$$

where complex linear $\Lambda : V \to V^*$ is defined by

$$\Lambda(f)(g) = \langle f, \overline{g} \rangle_{V \times V}$$

$j : V^1 \to V$ is the canonical injection into the coarser topology on V, and $j^* : V^* \to (V^1)^*$ is defined by

$$j^*(\lambda)(v) = \lambda(j(v))$$

with $\lambda \in V^*$ and $v \in V^1$.

7.1.2 Lemma *Assume* $\Theta \cap (j^* \circ \Lambda)V = \{0\}$. *Then,* $\mathcal{D}(T_\Theta) = \mathcal{D}(T) \cap \ker(\Theta)$ *is dense in* V.

Proof. [Bombieri-Garrett 2017] Lemma 2. $\square$

7.1.3 Corollary T_Θ *has a Friedrichs extension* $\widetilde{T_\Theta}$ *characterized by*

$$\langle (1 - T_\Theta)u, (1 - \widetilde{T_\Theta})^{-1} v \rangle_{V \times V} = \langle u, v \rangle_{V \times V} \quad (\textit{for } u \in \mathcal{D}(T_\Theta), v \in V)$$

That is, we have:

$$\begin{tikzcd}
\mathcal{D}(\widetilde{T_\Theta}) \arrow[r, hook] \arrow[rr, bend left, "\widetilde{T_\Theta}"] & V^1 \arrow[r, hook, "j"] & V \arrow[r, "\Lambda", "\sim"'] & V^* \arrow[r, "j^*"] & V^{-1} \\
\mathcal{D}(T_\Theta) \arrow[u, hook, "inc"'] \arrow[ur] \arrow[urr, "T_\Theta"'] & & & &
\end{tikzcd} \tag{7.4}$$

where complex linear $\Lambda : V \to V^*$ *is defined by* $\Lambda(f)(g) = \langle f, \overline{g} \rangle_{V \times V}$, $j^*(\lambda)(v) = \lambda(j(v))$ *with* $\lambda \in V^*$ *and* $v \in V^1$*.*

Proof. By Lemma 4.1.1 T_Θ is non-negative. In light of Lemma 7.1.2 above, [Garrett 2017] Theorem 9.2.1 implies the characterization. □

Remark By construction of Friedrichs extension, the following subdiagram in (7.4):

$$\begin{tikzcd}
\mathcal{D}(\widetilde{T_\Theta}) \arrow[r, "\widetilde{T_\Theta}"] & V \\
\mathcal{D}(T_\Theta) \arrow[u, hook, "inc"'] \arrow[ur, "T_\Theta"'] &
\end{tikzcd}$$

is the only commutative subdiagram in (7.4).

While $\widetilde{T_\Theta}$ is not quite a differential operator, it is a self-adjoint operator by construction, which implies that its eigenvalues, *if any*, are *real*. This Friedrichs extension of a restriction discussed below satisfies Theorem 1.2.1, converting an inhomogeneous equation to a homogeneous equation with an added boundary condition.

7.2 Specific Case

We now consider the distribution $\theta : L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}} / K) \to \mathbb{C}$ specific to this project (ref. Section 8.1):

$$\langle v, \theta \rangle = \int_{H_k \backslash H_{\mathbb{A}}} v(h)\big|_{H_k \backslash H_{\mathbb{A}}} f_1 \otimes f_2(h) dh \quad \text{(with cuspform } f_1 \otimes f_2 \text{on } H_{\mathbb{A}} = \mathrm{SL}_2 \times \mathrm{SL}_2) \tag{7.5}$$

with domain $\mathcal{D}(\theta)$. Let Θ be the span of θ and $\mathcal{D}(\Theta)$, a subspace of $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K_{\mathbb{A}})$, be its domain.

To satisfy the strong subconvexity assumption (see Section 8.1.2), we will further restrict the domain of the operator to the span of pseudo-Eisenstein series with cuspidal data on the Levi component for the Klingen parabolic, as well as Speh forms for the Klingen parabolic. Then, as in Theorem 1.2.1, we use the Friedrichs extension to convert an inhomogeneous equation to a homogeneous equation. That is, our θ is an integral representation of a triple-product L-function projected to the fragment of the continuous spectrum generated by a fixed maximal parabolic and fixed cuspidal data. This is the space of Eisenstein-Sobolev spaces $\mathfrak{E}^r_F$ defined by Definition 6.2.1.

7.2.1 Definition Let S_Θ be the invariant Laplacian $\Delta_\Theta : L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K) \to L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ with domain $\mathcal{D}(\Delta) \cap \ker \Theta$ where S_Θ is further restricted to the Hilbert space $\mathfrak{E}^1_F$. Let $\mathcal{D}_\Theta = \mathcal{D}(S_\Theta) \subset \mathfrak{E}^1_F$ be the domain of S_Θ. As in (7.4), we have:

$$
\begin{tikzcd}
\widetilde{\mathcal{D}_\Theta} \arrow[r, hook] \arrow[rr, bend left, "\widetilde{S_\Theta}"] & \mathfrak{E}^1_F \arrow[r, hook, "j"] & \mathfrak{E}^0_F \arrow[r, "\Lambda", "\sim"'] & (\mathfrak{E}^0_F)^* \arrow[r, "j^*"] & \mathfrak{E}^{-1}_F \\
\mathcal{D}_\Theta \arrow[u, hook, "inc"'] \arrow[ur] \arrow[urr, "S_\Theta"'] & & & &
\end{tikzcd}
\tag{7.6}
$$

where $\Lambda : \mathfrak{E}^0_F \to (\mathfrak{E}^0_F)^*$ is pointwise complex conjugation $\Lambda(z) = \overline{z}$ (Hilbert space inner product is complex linear).

7.2.2 Proposition *$\mathcal{D}_\Theta$ is dense in $\mathfrak{E}^0_F$.*

Proof. This is a special case of Lemma 7.1.2 (cf. [Bombieri-Garrett 2017] Lemma 2) once we have shown:

7.2.3 Lemma *$\Theta \cap (j^* \circ \Lambda)(\mathfrak{E}^0_F) = \{0\}$.*

Proof. For $\theta \in \Theta$, Theorem 3.3.2 implies $\theta \in \mathcal{B}^*$ with domain $\mathcal{D}(\theta) \subset \mathcal{B}$ the space of *uniform moderate-growth* functions on $Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}$, which is not a subset of any positive index Sobolev space as required by Theorem 1.2.1. To put θ in an Eisenstein-Sobolev space, we restrict $\mathcal{D}(\theta)$ to the span of pseudo-Eisenstein series with cuspidal data on the Levi component for the Klingen parabolic and Speh forms for the Klingen parabolic. With the restricted domain, Theorem 5.4.2 puts θ in $H^r(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$ for some $r \in \mathbb{R}$, and the isometric isomorphism $\mathcal{E}_{s,F} : H^r(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}) \to \mathfrak{E}^r_F$ (6.11) puts θ in $\mathfrak{E}^r_F$ for the same $r \in \mathbb{R}$.

A strong convexity assumption with Proposition 8.1.3 implies that $\theta \in \Theta$ is in $\mathfrak{E}^{-1}_F$, but is not an element of of $\Lambda \circ \mathfrak{E}^0_F$. Injectivity of j^* implies θ is not in $(j^* \circ \Lambda)(\mathfrak{E}^0_F)$. Therefore, $\Theta \cap (j^* \circ \Lambda)(\mathfrak{E}^0_F) = \{0\}$. □

Proof of Proposition. The proposition follows. □

Having taken the Friedrichs extension, we will determine the domain $\widetilde{\mathcal{D}_\Theta}$. First, some background so we can apply [Bombieri-Garrett 2017] Theorem 8. By Friedrichs extension and Riesz-Fréchet for ψ, $\gamma \in \mathfrak{E}^1_F$, let $S^\# : \mathfrak{E}^1_F(S) \to (\mathfrak{E}^1_F)^*$ be:

$$\begin{aligned} S^\#(\psi)(\gamma) &= \langle \psi, \overline{\gamma} \rangle_{\mathfrak{E}^1_F \times \mathfrak{E}^1_F} \\ &= \langle j\psi, \widetilde{S}\overline{\gamma} \rangle_{\mathfrak{E}^0_F \times \mathfrak{E}^0_F} \end{aligned} \tag{7.7}$$

Similarly for ψ, $\gamma \in \mathfrak{E}^1_F \cap \ker\Theta$, let $S^\#_\Theta : \mathfrak{E}^1_F \cap \ker\Theta \longrightarrow (\mathfrak{E}^1_F \cap \ker\Theta)^*$ be:

$$\begin{aligned} S^\#_\Theta(\psi)(\gamma) &= \langle \psi, \overline{\gamma} \rangle_{(\mathfrak{E}^1_F \cap \ker\Theta) \times (\mathfrak{E}^1_F \cap \ker\Theta)} \\ &= \langle j\psi, \widetilde{S_\Theta}\overline{\gamma} \rangle_{(\mathfrak{E}^0_F \cap \ker\Theta) \times (\mathfrak{E}^0_F \cap \ker\Theta)} \end{aligned} \tag{7.8}$$

Let i_Θ, j be the injections with adjoints i^*_Θ, j^* in the following diagram:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, hook] \arrow[rrr, bend left, "S"] & \widetilde{\mathcal{D}} \arrow[r, hook] \arrow[rr, bend left, "\widetilde{S}"] & \mathfrak{E}^1_F \arrow[r, hook, "j"] \arrow[rrr, bend left, "S^\#"] & \mathfrak{E}^0_F \arrow[r, "\Lambda", "\sim"'] & (\mathfrak{E}^0_F)^* \arrow[r, "j^*"] & (\mathfrak{E}^1_F)^* \arrow[d, "i^*_\Theta"] \\
\mathcal{D}_\Theta \arrow[u, hook, "inc"] \arrow[r, hook] \arrow[rrr, bend right, "S_\Theta"'] & \widetilde{\mathcal{D}_\Theta} \arrow[u, hook, "inc"] \arrow[r, hook] \arrow[rr, bend right, "\widetilde{S_\Theta}"'] & \mathfrak{E}^1_F \cap \ker\Theta \arrow[u, hook, "i_\Theta"] \arrow[r, hook] \arrow[rrr, bend right, "S^\#_\Theta"'] & \mathfrak{E}^0_F \cap \ker\Theta \arrow[u, hook] & & (\mathfrak{E}^1_F \cap \ker\Theta)^*
\end{tikzcd} \tag{7.9}$$

The following theorem is essential for the proof of Theorem 1.2.1.

7.2.4 Theorem *Restricting $S^{\#}$ to $\widetilde{\mathcal{D}}$ and restricting $S^{\#}_{\Theta}$ to $\widetilde{\mathcal{D}_{\Theta}}$*

$$\widetilde{\mathcal{D}_{\Theta}} = \{\psi \in (\mathfrak{E}^1_F \cap \ker\Theta) : (S^{\#} \circ i_{\Theta})\psi \in (j^* \circ \Lambda)\mathfrak{E}^0_F + \Theta\}$$

Proof. Since $\mathcal{D}_{\Theta}$ is dense in $\mathfrak{E}^0_F$ by Proposition 7.2.2, the Friedrichs extension $\widetilde{\mathcal{D}_{\Theta}}$ is dense in $\mathfrak{E}^0_F$. Lemma 7.2.3 implies $\Theta \cap (j^* \circ \Lambda)(\mathfrak{E}^0_F) = \{0\}$ because $\mathfrak{E}^0_F$ is a subspace of $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$. We know by the construction of Friedrichs extension that for $\psi \in \mathcal{D}_{\Theta}$ and $\gamma \in \mathfrak{E}^0_F$

$$\langle \gamma, (\widetilde{S_{\Theta}})\psi\rangle_{\mathfrak{E}^0_F} = \langle \gamma, \psi\rangle_{\mathfrak{E}^0_F}$$

Let $\psi = \widetilde{S_{\Theta}}^{-1}(\psi')$ for $\psi' \in \mathfrak{E}^0_F \cap \ker\Theta$, $\gamma \in \mathfrak{E}^1_F \cap \ker\Theta$. Calculating

$$\begin{aligned}
(S^{\#}\psi)(\gamma) &= (S^{\#}\widetilde{S}^{-1}\psi')(\gamma) \quad (\mathfrak{E}^1_F \cap \ker\Theta \times \mathfrak{E}^1_F \cap \ker\Theta \to (\mathfrak{E}^1_F)^* : \text{substitution})\\
&= \langle \widetilde{S}^{-1}\psi', \overline{\gamma}\rangle_{\mathfrak{E}^1_F \times \mathfrak{E}^1_F} \quad (\mathfrak{E}^1_F \times \mathfrak{E}^1_F \to V_{-1} : \text{definition of } S^{\#})\\
&= \langle \widetilde{S}^{-1}\psi', \overline{\gamma}\rangle_{V_{-1} \times \mathfrak{E}^1_F} \quad (\text{Riesz-Fréchet})\\
&= \langle \psi, j\overline{\gamma}\rangle_{\mathfrak{E}^1_F \times (\mathfrak{E}^0_F)^*} \quad (\text{Friedrichs extension})\\
&= ((j^* \circ \Lambda)\psi')(\gamma) \quad (\text{Riesz-Fréchet})\\
&= \langle (j^* \circ \Lambda \circ \widetilde{S})\psi)(\gamma) \quad (\text{substitution})
\end{aligned}$$

Since this is true for any $\gamma \in \mathfrak{E}^1_F \cap \ker\Theta$, the theorem follows. □

7.2.5 Proof of Theorem 1.2.1

We recall the statement of the theorem. *For Friedrichs extension of a restriction with $w \in \mathbb{C}$ and $\lambda_w = w(1-w) > 1/4$*

$$(S_{\Theta} - \lambda_w)u = \theta \quad \textit{if and only if} \quad (\widetilde{S_{\Theta}} - \lambda_w)u = 0 \textit{ and } \theta u \ = 0 \tag{7.10}$$

Proof. Theorem 1.2.1 is a corollary of Theorem 7.2.4. Theorem 1.2.1 follows from the constuction of S_{Θ} and the density implied by Proposition 7.2.2. □

Chapter 8

Integral Representations of L-functions

Realizations of triple-product L-functions as periods of Eisenstein appeared in [Garrett 1987], [Piatetski-Shapiro-Rallis 1987], and [Harris-Kudla 1991]. They appeared as integrals of Eisenstein series over quotients of subgroups of $\mathrm{GSp}_{4\times 4}$ and $\mathrm{GSp}_{6\times 6}$.

Remark: Triple-product L-functions can also be produced by integrating against degenerate Eisenstein series, but degenerate Eisenstein series enter in a more complicated way in the spectral decomposition of $L^2(G_k\backslash G_{\mathbb{A}})$. Therefore, those integral representations will not be considered here.

Since we are projecting onto the spectral subspace spanned by the Levi component for the Klingen parabolic we use the following two $\mathrm{Sp}_{4\times 4} \hookrightarrow \mathrm{Sp}_{6\times 6}$:

$$\left(\begin{array}{cc|cc} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ \hline c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{array}\right) \mapsto \left(\begin{array}{ccc|ccc} a_{ii} & & & b_{ii} & & \\ & a_{11} & a_{12} & & b_{11} & b_{12} \\ & a_{21} & a_{22} & & b_{21} & b_{22} \\ \hline c_{ii} & & & d_{ii} & & \\ & c_{11} & c_{12} & & d_{11} & d_{12} \\ & c_{21} & c_{22} & & d_{21} & d_{22} \end{array}\right)$$

for $i = 1, 2$.

[Garrett 1987] and [Harris-Kudla 1991] give two different mechanisms by which

Eisenstein series on $\mathrm{Sp}_{4\times4}(\mathbb{A})$

$$E_{s,f}(g) = \sum_{\gamma\in P_\mathbb{Q}\backslash G_\mathbb{Q}} \varphi_{s,f}(\gamma g) \quad \text{(for cuspidal data } f)$$

restricted to $\mathrm{SL}_2(\mathbb{A}) \times \mathrm{SL}_2(\mathbb{A})$ and integrated against a cuspform $f_1 \otimes_k f_2$ on $\mathrm{SL}_2 \times \mathrm{SL}_2$ produces a triple-product L-function. For weight k, Iwasawa decomposition $G = NAK$ with maximal compact $K_\mathbb{A}$, cuspidal-data Eisenstein series $E_{s,f}$ for Klingen parabolic P of G, a holomorphic cuspform f on $Z_\mathbb{A}M_k\backslash M_\mathbb{A}$ with $M = M^P$, and φ_s locally right K_v-invariant at finite primes v, [Garrett 1987] Theorem 1.3 shows that in the holomorphic case

$$\begin{aligned}\int_{Z_\mathbb{A}H_\mathbb{Q}\backslash H_\mathbb{A}} & E_{2k,s,F}(h)(f_1\otimes f_2)(h)dh = L(s+4k-2, f_1\times f_2\times F)\\ &\times(-1)^k 2^{6-4s-10k}\pi^{3-s-4k}\zeta(2s+2k)^{-1}\zeta(4s+4k-2)^{-1}\frac{\Gamma(s+2k-1)^3\Gamma(s+4k-2)}{\Gamma(2s+4k-2)\Gamma(s+2k)}\end{aligned} \tag{8.1}$$

8.1 Recasting the Triple Product Integral

Viewing the L-function-producing procedure above from a different perspective, the procedure of *restrict and integrate against cuspforms* can also be seen as a distribution with support on the copy of $H_k\backslash H_\mathbb{A}$ inside $G_k\backslash G_\mathbb{A}$. As such, the triple-product distribution

$$\langle v, \theta\rangle = \int_{H_k\backslash H_\mathbb{A}} v(h)\big|_{H_k\backslash H_\mathbb{A}} f_1\otimes f_2(h)dh \quad \text{(with cuspform } f_1\otimes f_2 \text{on } H_\mathbb{A} = \mathrm{SL}_2\times\mathrm{SL}_2) \tag{8.2}$$

is a functional on a subspace of $L^2(G_k\backslash G_\mathbb{A}/K_\mathbb{A})$ and is the distribution on the right-hand side of the differential equation $(\Delta - \lambda_w)u = \theta$. Proposition 8.1.1 below puts θ in a Sobolev space, so the operator Δ puts a solution u to the differential equation also in some Sobolev space.

The definition of Eisenstein-Sobolev spaces (6.14) with norms (6.13), and meromorphic continuation of Eisenstein series $E_{s,F}$ imply that we can spectrally decompose the triple-product distribution as in 6.3

$$\theta = \sum_{F \text{ on } G} \langle \theta, F\rangle \cdot F + \frac{\langle \theta, 1\rangle \cdot 1}{\langle \theta, 1\rangle} + \frac{1}{4\pi i}\iint_{\rho + i\mathfrak{a}^*} \langle \theta, E_s^{\min}\rangle \cdot E_s^{\min} ds + \textit{[other degenerate]}$$
$$+ \sum_{\substack{F \text{ cfm} \\ \text{on } M^{\text{Sieg}}}} \left(\sum_{\Upsilon_F \sim F} \langle \theta, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} \langle \theta, E_{s,F}^{\text{Sieg}}\rangle \cdot E_{s,F}^{\text{Sieg}} ds \right)$$
$$+ \sum_{\substack{F \text{ cfm} \\ \text{on } M^{\text{Kling}}}} \left(\sum_{\Upsilon_F \sim F} \langle \theta, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} \langle \theta, E_{s,F}^{\text{Kling}}\rangle \cdot E_{s,F}^{\text{Kling}} ds \right) \tag{8.3}$$

where F in the first sum are cuspforms in $L^2(G_k \backslash G_{\mathbb{A}}/K_{\mathbb{A}})$ and where the spectral coefficients $\langle \theta, -\rangle$ are defined by 8.2.

The spectral coefficients of the cuspform terms $\langle \theta, F\rangle$ are in $L^2(G_k \backslash G_{\mathbb{A}})$ because in this case (8.2) is the integral of a rapid-decay function against a rapid-decay function. Similarly, $\Upsilon_F \to (\langle \theta, \Upsilon_F\rangle)$ is in $L^2(G_k \backslash G_{\mathbb{A}})$ because $\Upsilon_F \in L^2(G_k \backslash G_{\mathbb{A}})$.

In the continuous spectrum, the Eisenstein series $E_{s,F}^{\text{Kling}} \in C^\infty_{\text{Mod}}(G_k \backslash G_{\mathbb{A}})$ are integrated against fixed cuspforms on the restriction $f_1 \otimes f_2 \in C^\infty_{\text{Rap}}(H_k \backslash H_{\mathbb{A}})$. Lemma 3.1.2 shows that $E_{s,F} \in C^\infty_{\text{Mod}}(G_k \backslash G_{\mathbb{A}})$ implies $E_{s,F}|_{H_k \backslash H_{\mathbb{A}}} \in C^\infty_{\text{Mod}}(H_k \backslash H_{\mathbb{A}})$, so Section 3.3 applies to H.

We restrict our investigation to a small subspace of the spectrum of θ. The idea is to put the restriction of θ in a Sobolev space inside $H^{-1}(G_k \backslash G_{\mathbb{A}})$ so the solutions to the differential equation $(\Delta - \lambda_w)u = \theta$ are in $H^1(G_k \backslash G_{\mathbb{A}})$, as required to be eigenfunctions of Friedrichs extension. Fix a cuspform F on the Levi component of the Klingen parabolic M^{Kling}. Consider the subspace spanned by the Speh form term and corresponding continuous spectrum

$$\begin{aligned} \theta^{\text{Kling}} &= \sum_{\Upsilon_F \sim F} \langle \theta, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} \langle \theta, E_{s,F}^{\text{Kling}}\rangle \cdot E_{s,F}^{\text{Kling}} ds \\ &= \sum_{\Upsilon_F \sim F} \langle \theta, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} \mathcal{E}\theta \cdot E_{s,F}^{\text{Kling}} ds \end{aligned} \tag{8.4}$$

To satisfy the hypotheses of Theorem 1.2.1 we need

8.1.1 Proposition 1) θ is not in $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$.
2) $\theta \in \mathfrak{E}_F^{-r}$ for some $r > 0$.

Proof. 1) Let $s = \sigma + it$ with s in the critical strip, $|\sigma| \ll 1$ and $|t| > 1$. Then for weight k

$$\int_*^\infty \Bigg| L(s-2, f_1 \times f_2 \times F) \\ \times (-1)2^{6-4s}\pi^{3-s}\zeta(2s)^{-1}\zeta(4s-2)^{-1}\frac{\Gamma(s-1)^3\Gamma(s-2)}{\Gamma(2s-2)\Gamma(s)}\Bigg|^2 dt = \infty$$

The important terms are the gamma factors. Consequences of the Stirling Approximation give

$$\Gamma(\sigma + it) \sim e^{-\pi|t|/2}|t|^{\sigma-\frac{1}{2}}$$

Then,

$$\frac{\Gamma(s-1)^3\Gamma(s-2)}{\Gamma(2s-2)\Gamma(s)} \sim \frac{(e^{-\pi|t|/2}|t|^{-1})^3(e^{-\pi|t|/2}|t|^{-2})}{(e^{-\pi|t|}|t|^{-\frac{3}{2}})(e^{-\pi|t|/2}|t|^0)} \sim \frac{e^{\pi|t|}}{|t|^{\frac{7}{2}}}$$

Since $e^{\pi|t|}$ dominates $|t|^{\frac{7}{2}}$ as $t \to +\infty$, $\langle\theta, E_{s,F}^{\text{Kling}}\rangle^2 = \infty$ for $|\sigma| \ll 1$.
2) Proposition 8.1.3 below and 1) imply that $\Theta \subset H^{-r}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}/K)$ for some $r \in \mathbb{R}_{\geq 0}$. The isometric isomorphism $\mathcal{E}_{s,F} : H^r(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}) \to \mathfrak{E}_F^r$ (6.11) implies $\theta \in \mathfrak{E}_F^{-r}$. □

8.1.2 Subconvexity Assumption We need a strong subconvexity assumption on $L(s, f_1 \otimes f_2 \otimes F)$ to obtain a distribution θ in $H^{-1}(G_k\backslash G_{\mathbb{A}})$ to satisfy (1.4), so the Friedrichs extension behaves as required.

8.1.3 Proposition Characterized as a distribution, the triple-product L-function

$$L(1/2 + it, f_1 \times f_2 \times F) \quad \text{(with cuspforms } f_1, f_2, F \text{ on } \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2)$$

as a period of Eisenstein series (ref. 8.1) is in $H^{-3}(G_k\backslash G_{\mathbb{A}})$ and is not in $H^{-1}(G_k\backslash G_{\mathbb{A}})$.

Proof. (8.1) is of degree $n = 8$ (cf., [Garrett 1987]). The functional equation and Phragmén-Lindelöf imply the convexity bound

$$L(1/2 + it, f_1 \times f_2 \times F) \ll_\varepsilon |t|^{n/4+\varepsilon} \tag{8.5}$$

which gives a second-moment bound

$$\int_*^T \left| L(\frac{1}{2} + it, f_1 \times f_2 \times F) \right|^2 \ll_\varepsilon T^{16 \cdot \frac{1}{4} + \varepsilon + 1} \tag{8.6}$$

Find the order of $L(1/2+it, f_1 \times f_2 \times F)$ so $\theta \in H^{-1}(G_k\backslash G_{\mathbb{A}})$by expanding θ spectrally (for fixed cuspform $f_1 \otimes f_2$) with cuspidal data F on the Levi component,

$$\theta = \sum_{\Upsilon_F \sim F} \langle \theta, \Upsilon_F \rangle \cdot \Upsilon_F + \frac{1}{4\pi i} \int_{(1/2)} \langle \theta, E_{s,F} \rangle \cdot E_{s,F} ds$$

Since each of the finite many Υ_F (attached to F) are in $L^2(G_k\backslash G_{\mathbb{A}}/K)$, θ is in $H^{-\ell}(G_k\backslash G_{\mathbb{A}}/K)$ for $\ell \geq 0$ if

$$|\theta|^2_{H^{-\ell}} = \int_{(1/2)} (1 + |\lambda_{E_{s,F}}|)^{-\ell} \left| \langle \theta, E_{s,F} \rangle \right|^2 ds < \infty$$

Substituting

$$\langle \theta, E_{s,F} \rangle = L(\frac{1}{2} + it, f_1 \times f_2 \times F) \times \text{simpler terms}$$

we need,

$$\int_0^\infty (1 + |\lambda_{E_{1/2+it,f}}|)^{-\ell} \left| L(\frac{1}{2} + it, f_1 \times f_2 \times F) \times \zeta \right|^2 dt < \infty$$

This is equivalent to

$$\int_0^\infty \frac{1}{(1+t^2)^\ell} \left| L(\frac{1}{2} + it, f_1 \times f_2 \times F) \right|^2 dt < \infty$$

Hence, it suffices to have

$$\int_*^T \left| L(\frac{1}{2} + it, f_1 \times f_2 \times F) \right|^2 \ll_\varepsilon T^{2\ell - \varepsilon} \tag{8.7}$$

Comparison with the second moment bound (8.6) gives the result. □

Since analytical propeties of L-functions follow from their Eisenstein series, which are moderate-growth functions with respect to iT, [Garrett 2015] uses a second moment bound, a residue calculation and Cauchy's theorem, to show for general L-functions

$$L(s_0)^2 = \int_0^\infty \frac{e^{(s-s_0)^2}}{s - s_0} \cdot L(1/2 + it)^2 \, dt$$

for $s = 1/2 + it$ and $s_0 = \sigma + it_0$, where $\sigma > 1/2$. Further estimates in [Garrett 2015] translated to the relevant L-function show that (8.7) implies a strong pointwise bound

$$L(1/2 + iT, f_1 \times f_2 \times F) \ll_\varepsilon (1 + T)^{1+\varepsilon} \quad (\text{for all } \varepsilon > 0)$$

This is clearly much stronger than the convexity bound (8.5) and currently unattainable.

Chapter 9

Solving Inhomogeneous Equations Using Spectral Methods

Meromorphic continuation of Eisenstein series $E_{s,f}$ allows us to solve differential equations using spectral methods. Since our objects of investigation are periods of Eisenstein series producing L-functions, and their zeros, we projects solutions to the orthogonal complement to the space of cuspforms. This is the subspace of the continuous spectrum, pseudo-Eisenstein series realized as integrals of Eisenstein series, with a small discrete part, Speh forms, which are residues of Eisenstein series.

9.1 Solutions

9.1.1 Proposition

Let u_w be a solution of

$$(S_\Theta - \lambda_w)u_w = \theta$$

where S_Θ is the invariant Laplacian T_Θ (Definition 7.1.1) restricted to the Hilbert space $\mathfrak{E}^1_F$ (Definition 6.2.1). Then

$$u = \sum_{\Upsilon_F \sim F} \frac{\langle \theta, \Upsilon_F \rangle}{\lambda_F - \lambda_w} \cdot \Upsilon_F + \frac{1}{4\pi i} \int_{(1/2)} \frac{\mathcal{E}_{s,F}(\theta)}{\lambda_s - \lambda_w} \cdot E_{s,F}\, ds \tag{9.1}$$

Proof. By Proposition 5.1.3 Eisenstein series with strong-sense cuspidal data are eigenfunctions of differential operators S_Θ descended from Casimir. By construction they are eigenfunctions of the Friedrichs extension $\widetilde{S_\Theta}$, as well. By spectral synthesis, and projecting to the subspace generated by pseudo-Eisenstein series with fixed cuspidal data F

$$\begin{aligned}(S_\Theta - \lambda_w)\left(\sum_{\Upsilon_F \sim F} \langle u_w, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} \langle u_w, E_{s,F}\rangle \cdot E_{s,F}\, ds\right) \\ = \sum_{\Upsilon_F \sim F} \langle \theta, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} \mathcal{E}_{s,F}(\theta) \cdot E_{s,F}\, ds\end{aligned}$$

Looking at the left hand side,

$$(S_\Theta - \lambda_w)\left(\sum_{\Upsilon_F \sim F} \langle u_w, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} \langle u_w, E_{s,F}\rangle \cdot E_{s,F}\, ds\right)$$

By Gelfand-Pettis the operator moves inside the integral, so this is

$$\sum_{\Upsilon_F \sim F} (S_\Theta - \lambda_w)\langle u_w, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} (S_\Theta - \lambda_w)\langle u_w, E_{s,F}\rangle \cdot E_{s,F}\, ds$$

Since Υ_F and $E_{s,F}$ are eigenfunctions of S_Θ, this is

$$\sum_{\Upsilon_F \sim F} (\lambda_F - \lambda_w)\langle u_w, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} (\lambda_s - \lambda_w)\langle u_w, E_{s,F}\rangle \cdot E_{s,F}\, ds$$

Thus, in a suitable Sobolev space

$$\begin{aligned}\sum_{\Upsilon_F \sim F} (\lambda_F - \lambda_w)\langle u_w, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} (\lambda_s - \lambda_w)\langle u_w, E_{s,F}\rangle \cdot E_{s,F}\, ds \\ = \sum_{\Upsilon_F \sim F} \langle \theta, \Upsilon_F\rangle \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} \mathcal{E}_{s,F}(\theta) \cdot E_{s,F}\, ds\end{aligned}$$

By uniqueness of such expressions in Sobolev spaces, we can solve by division. Subsituting for u_w,

$$u_w = \sum_{\Upsilon_F \sim F} \frac{\langle \theta, \Upsilon_F\rangle}{\lambda_F - \lambda_w} \cdot \Upsilon_F + \frac{1}{4\pi i}\int_{(1/2)} \frac{\mathcal{E}_{s,F}(\theta)}{\lambda_s - \lambda_w} \cdot E_{s,F}\, ds \tag{9.2}$$

□

9.2 Main result: The eigenvalues λ_s, ***if any***, are among the zeros of $\theta E_{s,F}$

9.2.1 Theorem *Let θ be the distribution defined in Section 8.1 (ref. 8.2) and S_Θ the operator in Section 7.2 (ref. Definition 7.2.1) restricted to $\mathfrak{E}_c^\infty$ and extended by taking the completion with respect to the norm* (6.13)*. Assuming the subconvexity requirement* (8.7)*, the eigenvalues λ_w of the Friedrichs extension $\widetilde{S_\Theta}$,* ***if any****, only occur for w among the zeros of $\mathcal{E}_{s,F}\theta$.*

Proof. Proposition 8.1.1 imples $\theta \in \mathfrak{E}_F^{-r}$ for some $r > 0$. From Theorem 1.2.1 any solution u_w of

$$(S_\Theta - \lambda_w)u_w = \theta$$

satisfies $(\widetilde{S_\Theta} - \lambda_w)u_w = 0$. By construction of $\widetilde{S_\Theta}$, $\theta u_w = 0$, and $u_w \neq 0$ implies u_w is not a solution of $(S_\Theta - \lambda_w)u_w = 0$ for $\lambda_w \neq 0$. Taking the Eisenstein-Sobolev spectral expansion, as in (6.11), we have

$$\begin{aligned} \mathcal{E}_{s,F}((S_\Theta - \lambda_w)u_w) &= \mathcal{E}_{s,F}(\theta) \\ (\lambda_s - \lambda_w)\mathcal{E}_{s,F}(u_w) &= \mathcal{E}_{s,F}(\theta) \quad \text{(by Gelfand-Pettis)} \end{aligned} \tag{9.3}$$

Since $\widetilde{S_\Theta}$ is self-adjoint, eigenvalue $\lambda_w = w(w-1)$ is real. Also, $\lambda_w \leq 0$, which implies that $\Re(w) = 1/2$. That is, for $w = a + bi$, $\lambda_w = -1/4 - b^2$, which implies $\lambda_w \leq 1/4$. For $\lambda_s < 0$, $\mathcal{E}_{s,F}(\theta)$ does not have pointwise values as a function of s, but can be evaluated locally by the L^2 norm on neighborhoods of $w = 1/2 + it_0$ with $s = 1/2 + it$. For any $\varepsilon > 0$, let

$$0 < \delta < \frac{\varepsilon}{2(2t_0 + 1)} \tag{9.4}$$

Then, for $|t| \leq |t_0 + \delta|$

$$t^2 - t_0^2 \leq 2t_0\delta + \delta^2 \leq \delta(2t_0 + 1) < \varepsilon \tag{9.5}$$

In a 2δ neighborhood of t_0, (9.3) gives

$$\begin{aligned}
\int_{t_0-\delta}^{t_0+\delta} \left|\mathcal{E}_{1/2+it,F}\theta(1/2+it)\right| \, dt &= \int_{t_0-\delta}^{t_0+\delta} \left|\mathcal{E}_{1/2+it,F}u(1/2+it)(\lambda_s-\lambda_w)\right| \, dt \\
&= \int_{t_0-\delta}^{t_0+\delta} \left|\mathcal{E}_{1/2+it,F}u(1/2+it)(t^2-t_0^2)\right| \, dt \\
&\leq \int_{t_0-\delta}^{t_0+\delta} \left|\mathcal{E}_{1/2+it,F}u(1/2+it)\right| \left|(t^2-t_0^2)\right| \, dt \\
&\leq \left(\int_{t_0-\delta}^{t_0+\delta} \left|\mathcal{E}_{1/2+it,F}u(1/2+it)\right|^2 \, dt\right)^{1/2} \left(\int_{t_0-\delta}^{t_0+\delta} \left|(t^2-t_0^2)\right|^2 \, dt\right)^{1/2}
\end{aligned}$$

by Cauchy-Schwarz-Bunyakowsky. Therefore,

$$\begin{aligned}
&\left(\int_{t_0-\delta}^{t_0+\delta} \left|\mathcal{E}_{1/2+it,F}u(1/2+it)\right|^2 \, dt\right)^{1/2} \left(\int_{t_0-\delta}^{t_0+\delta} \left|(t^2-t_0^2)\right|^2 \, dt\right)^{1/2} \\
&\qquad \leq \|\mathcal{E}_{1/2+it,F}u\|_{L^2(\mathbb{R})} \left(\int_{t_0-\delta}^{t_0+\delta} \left|(t^2-t_0^2)\right|^2 \, dt\right)^{1/2} \\
&\qquad \leq \|\mathcal{E}_{1/2+it,F}u\|_{L^2(\mathbb{R})} (2\delta\varepsilon^2)^{1/2} \quad \text{(by 9.5)} \\
&\qquad \leq \|\mathcal{E}_{1/2+it,F}u\|_{L^2(\mathbb{R})} \varepsilon^{3/2} \quad \text{(since } 2t_0+1 \geq 1\text{)}
\end{aligned}$$

Since $\|\mathcal{E}_{1/2+it,F}u\|_{L^2(\mathbb{R})} < \infty$, normalizing implies that for arbitrary ε and $|s-w| < \delta$,

$$|\mathcal{E}_{s,F}\theta(s)| \leq c_w \varepsilon^{3/2}$$

for some constant c_w. That is, $|\mathcal{E}_{s,F}\theta(s)| = O_{w,\varepsilon}(\varepsilon^{3/2})$. This is what we needed to show. □

Bibliography

[Adams] R. Adams and J. Fournier, *Sobolev Spaces 2nd edition* Academic Press, 2003.

[Bombieri-Garrett 2017] E. Bombieri, P. Garrett, *Designed pseudo-Laplacians*, arXiv:2002.07929v1 [math.NT], `https://www-users.cse.umn.edu/~garrett/m/v/Bombieri-Garrett_current_version.pdf`, 2020.

[Colin de Verdière 1981] Y. Colin de Verdière, *Une nouvelle démonstration du prolongement mèromorphe des séries d'Eisenstein*, C. R. Acad. Sci Paris Sér. I Math **293** (1981), no. 7, 361-363.

[Colin de Verdière 1982] Y. Colin de Verdière, *Pseudo-laplaciens, I*, Ann. Inst. Fourier (Grenoble) **32** (1982) no. 3, xiii, 275-286.

[Colin de Verdière 1983] Y. Colin de Verdière, *Pseudo-laplaciens, II*, Ann. Inst. Fourier (Grenoble) **32** (1983) no. 2, 87-113.

[Faddeev 1967] L. D. Faddeev, *On the Friedrichs model in the theory of perturbations of a continuous spectrum*, in *American Mathematical Society Translations. Series 2, Vol. 62: Five papers on functional analysis*, American Mathematical Society, Providence, R.I. 1967 iii+284 pp.

[Fay 1976] J. Fay, *Fourier coefficients of the resolvent for a Fuchsian group* in *Journal für die riene und angewandte Mathematic*, **293/294**, (1977), 143–203.

[Friedrichs 1934] K.O. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren* in Math. Ann., **109**, (1934), 465-487, 685-713.

[Garrett 1984] P. Garrett, *Pullbacks of Eisenstein series; applications*, in *Automorphic Forms of Several Variables*, Taniguchi Symposium, Katata, ed. I. Satake, Y Morita, 1983, Birkhauser, 1984, 114-136.

[Garrett 1987] P. Garrett, *Decomposition of Eisenstein Series: Rankin Triple Products*, in *Annals of Mathematics*, Second Series, Vol. 125, No 2 (Mar., 1987), 209-235.

[Garrett 2005c] P. Garrett, *Simplest Example of Truncation and Maaß-Selberg Relations*, `http://www.math.umn.edu/~garrett/m/v/maass_selberg_trivial.pdf`.

[Garrett 2011] P. Garrett, *Pseudo-cuspforms, pseudo-Laplacians*,
`http://www.math.umn.edu/~garrett/m/v/pseudo-cuspforms.pdf.`

[Garrett 2012] P. Garrett, *Colin de Verdière's meromorphic continuation of Eisenstein series*,
`http://www.math.umn.edu/~garrett/m/v/cdv_eis.pdf.`

[Garrett 2015] P. Garrett, *From moments to poinwise estimates*,
`http://www.math.umn.edu/~garrett/m/v/moments_to_pointwise.pdf.`

[Garrett 2017] P. Garrett, *Modern Analysis of Automorphic Forms by Examples*, Cambridge University Press, 2018.

[Gelbart-Jacquet 1978] S. Gelbart, H. Jacquet, *A relation between automorphic representations of $GL(2)$ and $GL(3)$*, in *Annales scientifiques de l'É.N.S* 4^e *series*, Vol 11, no. 4, 1978 471-542.

[Grubb 1990] G. Grubb, *Distributions and Operators*, Springer, 1990.

[Harris-Kudla 1991] M. Harris, S. Kudla, *The Central Critical Value of a Triple Product L-Function*, in *Annals of Mathematics*, Second Series, Vol. 133, No 3 (May, 1991), 605-672.

[Harris-Kudla 1992] M. Harris, S. Kudla, *Arithmetic automorphic forms for the nonholomorphic discrete series of $GSp(2)$*, in *Duke Mathematical Journal*, Vol. 66, No 1 (April 1992), 59-120.

[Hejhal 1981] D. Hejhal, *Some observations concerning eigenvalues of the Laplacian and Dirichlet L-series*, in *Recent Progress in Analytic Number Theory*, ed. H. Halberstam and C. Hooley, vol. 2, Academic Press, NY, 1981, 95-110.

[Iwaniec 2002] H. Iwaniec, *Spectral Methods of Automorphic Forms 2nd Edition*, in *Graduate Studies in Mathematics, Vol. 53*, American Mathematical Society, Providence, R.I., (2002), 220 pp.

[Jacquet 1983] H Jacquet, *On the residual spectrun of $GL(n)$* in *Lie Group Representations II* in *Lecture notes in mathematics*, Vol 1041, Springer-Verlag, New York, 1983

[Kim 1995] H. H. Kim, *The residual spectrum of Sp_4* in *Compositio Mathematica*, Vol. 99, no. 2, 1995, 129-151.

[Langlands 1976] R. P. Langlands, *On the functional equations satisfied by Eisenstein series* in *Lecture notes in mathematics*, Vol 544, Springer-Verlag, New York, 1976

[Langlands 1971] R. P. Langlands, *Euler Products*, Yale University Press, 1971.

[Lax-Phillips 1976] P. Lax, R. Phillips, *Scattering theory for automorphic functions Annals of mathematics studies* **87**, Princeton University Press, Princeton, N.J., 1976.

[Mœglin-Waldspurger 1989] C. Mœglin, J.-L. Waldspurger, *Le spectre résiduel $GL(n)$*, in *Annales scientifiques de l'É.N.S* 4^e *series*, Vol 22, no. 4, 1989, 605-674

[Mœglin-Waldspurger 1995] C. Mœglin, J.-L. Waldspurger, *Spectral Decomposition and Eisenstein Series*, Cambridge University Press, 1995.

[Montgomery 1973] H. L. Montgomery, *The pair correlation of zeros of the zeta function* in *Proc. Symp. Pure Math* **24**, AMS, Providence, R.I., 1973, 181-193.

[Neunhöffer 1973] H. Neunhöffer, *Über die analytische Fortsetzung von Poincaréreihen*, Sitzungsberichte Hiedelberg Akad. Wiss (1973), no. 2.

[Niebur 1973] D. Niebur, *A class of nonanalytic automorphic functions*, Nagoya Math J. **52** (1973), 133-145.

[Piatetski-Shapiro-Rallis 1987] S. Gelbart, I. Piatetski-Shapiro, S. Rallis, *Explicit Constructions of Automorphic L-Functions*, Lecture Notes in Mathematics **1254**, Springer-Verlag, 1987.

[Piatetski-Shapiro-Rallis 1987b] I. Piatetski-Shapiro, S. Rallis, *Rankin triple L functions*, Compositio Mathematica, Martinus Nijhoff Publ., Dordrecht, tome 64, no. 1 1987, 31-115.

[Odlyzko] A Odlyzko: Correspondence about the origins of the Hilbert-Polya Conjecture `http://www.dtc.umn.edu/~odlyzko/polya/index.html.`

[Riesz-Sz.-Nagy 1955] F. Riesz, B. Sz.-Nagy, *Functional Analysis* Frederick Ungar Publishing Co., New York, NY, 1955

[Shahidi 1988] , F. Shahidi, *On the Ramanujan conjecture and fineteness of poles of certain L-functions*, *Annals of Mathematics*, **127**, Princeton University Press, Princeton, N.J., 1988, 547-584.

[Speh 1981] B. Speh, *The Unitary Dual of* $GL(3,\mathbb{R})$ *and* $GL(4,\mathbb{R})$, in *Mathematische Annalen* **258**, Springer-Verlag, Berlin, Germany, (1981/82), no. 2, 113–133.

[Titchmarsh 1986] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function* Oxford University Press, 1986 2nd edition.