Domain Lines as Fractional Vortices

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Introduction

Domain walls natural objects in supersymmetric theories

There are the following alternatives (without taking in account the trivial translational modulus):

a) A **single** domain wall interpolating two vacua.

b) A **moduli space** of solution. The modulus can be promoted to a massless field on the wall world-volume (for example $\mathcal{N} = 2$ SQED)

c) A **discrete number** $\neq 1$ of solutions (expected in $\mathcal{N} = 1$ Super-Yang-Mills)
Domain Lines interpolating between discrete Domain Wall solutions: an explicit solitonic example at weak coupling
Fields & Superpotential

$\mathcal{N} = 2$ U(1) gauge theory

Bosonic Fields:

* U(1) gauge vector superfield
* superfield $A$ with zero charge
* $N_f = 2$ squark superfields $Q_B, \tilde{Q}_B$ from hypermultiplets

$$W = \frac{1}{\sqrt{2}} AQ_B \tilde{Q}_B + m_B Q_B \tilde{Q}_B + \frac{\beta}{\sqrt{2}} (Q_1 \tilde{Q}_2 - \tilde{Q}_1 Q_2) - \frac{1}{2\sqrt{2}} \xi A,$$

$m_1 = -m_2 = m$

$\beta \neq 0$ we are breaking the extended supersymmetry, and also the flavor group

$$U(1) \times U(1) \rightarrow U(1)$$
Two Vacua

1) \[ a = -\sqrt{2m^2 - \beta^2}, \quad q_1 = \tilde{q}_1^\dagger = \Omega, \quad q_2 = -\tilde{q}_2^\dagger = \omega. \]

2) \[ a = \sqrt{2m^2 - \beta^2}, \quad q_1 = -\tilde{q}_1^\dagger = \omega, \quad q_2 = \tilde{q}_2^\dagger = \Omega. \]

where:

\[
\Omega = \frac{1}{2} \sqrt{\xi + \frac{m\xi}{\sqrt{m^2 - \beta^2/2}}}, \quad \omega = \frac{1}{2} \sqrt{-\xi + \frac{m\xi}{\sqrt{m^2 - \beta^2/2}}}
\]

\[ \beta < \sqrt{2m} \text{ in order to avoid run-away vacua} \]
Sigma Model description

\( \xi >> m, \beta \) we can integrate out \( a \)

Constraints:

\[ |q_1|^2 + |q_2|^2 = |\tilde{q}_1|^2 + |\tilde{q}_2|^2 , \quad \tilde{q}_A q_A = \frac{\xi}{2} \]

Eguchi-Hanson manifold (four real dimensions)
Coordinates: \((r, \theta, \varphi, \psi)\)

Potential induced by \( m, \beta \)

\[
V = \frac{m^2 (4r^2 + \xi^2 \sin^2 \theta)}{\sqrt{4r^2 + \xi^2}} + 2\sqrt{2} m \beta r (\cos \theta \cos \varphi \cos \psi - \sin \varphi \sin \psi) + \\
\beta^2 (4r^2 + 3(\xi/2)^2 + 2\xi^2 \cos(2\theta) \sin^2 \varphi + \xi^2 \cos(2\varphi)) \]
\[
+ \frac{2\sqrt{2} m \beta r (\cos \theta \cos \varphi \cos \psi - \sin \varphi \sin \psi)}{\sqrt{4r^2 + \xi^2}} \]
Wall at $\beta = 0$

Both the vacua and the wall are living in the $S^2$ subspace of Eguchi-Hanson manifold at $\beta = 0$. Vacua: $r = 0$, $\theta = 0, \pi$

Moduli space of BPS domain walls parameterized by an $U(1)$ phase $e^{i\sigma}$

$$\phi(z) = \sigma, \quad r(z) = 0$$

$$\theta(z) = 2 \tan^{-1}(\exp(2mz))$$

Modulus promoted to a field on the wall world-volume:

$$S = \int d^3x \left( \frac{\xi}{4m} (\partial_n \sigma)^2 \right)$$
Wall at $\beta \neq 0$

Plot of the potential at some sections at constant $r$ and at $\theta = \pi/3, \pi/2, 2/3\pi$. There are always two symmetric minima, one at $\phi = \psi = \pi/2$ and the other at $\phi = \psi = 3\pi/2$. BPS wall equations can be solved in correspondence of these two values.

$$q_1 = \pm i\Omega \cos(\eta/2) + \omega \sin(\eta/2), \quad q_2 = \pm i\omega \cos(\eta/2) + \Omega \sin(\eta/2)$$

$$\tilde{q}_1 = \mp i\Omega \cos(\eta/2) - \omega \sin(\eta/2), \quad \tilde{q}_2 = \pm i\omega \cos(\eta/2) + \Omega \sin(\eta/2)$$

$$a = -\sqrt{2m^2 - \beta^2} \left( \cos \eta \pm i \frac{\beta}{\sqrt{2m}} \sin \eta \right)$$
Metastable walls

The BPS wall solutions at $\beta \neq 0$ correspond to $\sigma = \pi/2, 3\pi/2$.

Ansatz for a “meta-solution“ at generic $\sigma$:

\[ q_1 = (e^{i\sigma}\Omega \cos(\eta/2) + \omega \sin(\eta/2))/A, \]
\[ q_2 = (e^{i\sigma}\omega \cos(\eta/2) + \Omega \sin(\eta/2))/A, \]
\[ \tilde{q}_1 = (e^{-i\sigma}\Omega \cos(\eta/2) - \omega \sin(\eta/2))A, \]
\[ \tilde{q}_2 = (-e^{-i\sigma}\omega \cos(\eta/2) + \Omega \sin(\eta/2))A. \]

where $\eta(z)$ is a profile function which is calculated by minimization of the action and the factor $A$ is introduced in order to maintain the sigma model constraints.
At the first non trivial order in $\beta$, the following potential is generated:

$$V = \frac{\beta^2 \xi}{2m} \cos(2\sigma) + O(\beta^3)$$
There are two different kinks interpolating the two world-volume vacua

\[ \mathcal{H} = \int d^3x \left( \frac{\xi}{4m} (\partial_n \sigma)^2 + \frac{\beta^2 \xi}{2m} \cos^2 \sigma \right) = \]

\[ = \int d^3x \left\{ \frac{\xi}{2m} \left( \frac{1}{\sqrt{2}} (\partial_x \sigma) \mp \beta \cos \sigma \right)^2 \mp \frac{\xi \beta}{\sqrt{2}m} \partial (\sin \sigma) \right\} \].
The fate of a bulk vortex

Bulk: \( T = 2\pi \xi \)

Domain line: \( T = \sqrt{2\beta \xi / m} \)

The Domain Line carries a magnetic flux which is \(1/2\) of the flux carried by a bulk vortex.
Sine Gordon vs. Chern-Simon

World-volume scalar field $\sigma$:

$$S = \int d^3 x \left( \frac{\xi}{4m} (\partial_n \sigma)^2 \right).$$

Duality: $F_{nm}^{(2+1)} = \frac{e_{2+1}^{2+1}}{2\pi} \epsilon_{nmk} \partial^k \sigma$,

$$S = \int d^3 x \left( -\frac{1}{4e^2} (F_{mn}^{2+1})^2 \right).$$

Chern-Simon term (proposed for domain walls of $\mathcal{N} = 1$ Super-Yang-Mills):

$$S_{CS} = \frac{1}{2\pi} \epsilon_{nmk} A_n \partial_m A_k.$$

Different from the Sine-Gordon potential in our weakly coupled example!
We can introduce external magnetic monopoles embedding the $U(1)$ theory in a $SU(2)$ one.

It is possible to build a static configuration where a monopole is a junction of two domain lines (similar to what happen in the bulk for the non-abelian string).
Conclusion

* A Domain-Line soliton has been built in a weak coupled theory

* The effective world-volume description involves a Sine-Gordon theory with two vacua

* The domain line carries a magnetic flux which is 1/2 of a bulk Abrikosov-Nielsen-Olesen vortex