# Computability of Preference, Utility, and Demand

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**Abstract:** This paper studies consumer theory from the bounded rationality approach proposed in Richter and Wong (1996a), with a "uniformity principle" constraining the magnitudes (prices, quantities, etc.) and the operations (to perceive, evaluate, choose, communicate, etc.) that agents can use. In particular, we operate in a computability framework, where commodity quantities, prices, consumer preferences, utility functions, and demand functions are computable by finite algorithms (Richter and Wong (1996a)).

We obtain a computable utility representation theorem. We prove an existence theorem for computable maximizers of quasiconcave computable utility functions (preferences), and prove the computability of the demand functions generated by such functions (preferences). We also provide a revealed preference characterization of computable rationality for the finite case. Beyond consumer theory, the results have applications in general equilibrium theory (Richter and Wong (1996a)).

**JEL Classification:** D11, C63

**Keywords:** Bounded rationality, Consumer theory, Convex Optimization, Revealed Preference Analysis, Recursive Analysis, Utility Representation

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## 1. INTRODUCTION

One approach to bounded rationality limits the operations that agents can perform to a class of "simple" decision rules. As noted in Richter and Wong (1996b) it is natural to limit magnitudes, as well as operations, to simple classes. In this paper, we examine agents whose operations and magnitudes are both limited by the agents' abilities in performing computations: we impose a computability bound on their rationality. The results are of interest not only for consumer theory, but have applications in the study of general equilibrium in a boundedly rational context (Richter and Wong (1996a)).

In the classical approach, agents can perform arbitrarily complicated calculations on arbitrarily complicated magnitudes. Such extreme calculating abilities have been recognized as unrealistic in many situations (cf. Simon (1959, 1978)). In Richter and Wong (1996a), we also pointed out the unrealistic nature of perceiving and communicating arbitrary magnitudes. To incorporate "realistic" limitations on calculation and communication, we first ask what operations, or calculations, human beings can perform. A widely accepted view is that routine human calculating abilities are limited to those of the ideal computing machines known as Turing machines. (Cf. Turing (1950)).

Even within the bounds of Turing computability, there are interesting subclasses (polynomial-time Turing machines, finite automata, etc.; cf. Davis (1983)). In this paper, however, we examine the consequences of assuming our agents' calculating abilities coincide with those of Turing machines. (This is known as Church's Thesis (cf. Church (1935).)

We will apply this Turing computability notion to magnitudes as well as to operations. In particular, our agents can only use "computable" numbers (as prices, commodity quantities, utility parameters, etc.) — i.e. those reals whose decimal (hence digital, etc.) representations can be computed by Turing machines. One realistic consequence is this: Since each Turing machine can be represented by a single integer (Gödel number), each computable real can be represented by an integer code. Thus agents can communicate prices and quantities without needing an uncountable vocabulary.

Similarly, binary preference rankings of our agents are determined by Turing machines. Indeed, all functions we consider will also be "computable."

Beyond bounded rationality, results in the present context can have useful implications for applied economics — in particular, for computational economics. For example, if an economist seeks an algorithm to solve a constrained maximization problem, he needs to know whether such an algorithm exists. By our definition of computability, this is equivalent to existence of a computable maximizing vector (whose coordinates are computable reals). Several of our theorems assert such existence.

As mentioned in Richter and Wong (1996a), this computability approach inherits some mathematical difficulties in proving theorems. For example, the indifference relations of preferences are in general non-computable (cf. Richter and Wong (1996a), Wong (1994)); so we cannot follow traditional constructions of utility representation (e.g. Debreu (1954)), which would require computability for indifference relations. Also, our commodity space, the $l$-dimensional space of computable real numbers, is not topologically complete (cf. Rice (1954); cf. Footnote 12 below); and our consumers' budget sets are not compact. Thus we cannot follow standard proofs for maximizer existence theorems. Indeed, the Heine-Borel Theorem fails for computable reals (cf. Zalsavskii (1962)); there are well-known counterexamples (e.g. Kreisel (1958)) to the existence of computable maximizers for computable functions. (Brouwer's Fixed-Point Theorem also fails for computable reals, cf. Orevkov (1964), Baigger (1985), Richter and Wong (1996a)).

Nevertheless, we prove the following results. They demonstrate that many of the tools that have been proved fruitful for classical economic analysis are also available for the present version of computable economics.

1) (Computable Utility Representation) Every computable preference has a computable utility representation; the converse is also true.

2) (Existence of Computable Demand) Every quasiconcave computable utility function admits a computable maximizer in every computable budget set.

3) (Computability of Demand Function) Every demand function that is generated by a quasiconcave computable utility function is computable.

4) (Characterization of Computable Rationality) Given any finite set $X$ of computable price-income observations, a computable and exhaustive "demand" function $h$ on $X$ is generated by some computable preference (or utility) if and only if $h$ satisfies the Strong Axiom of Revealed Preference.

We also prove that, for a denumerable set $X$, the Strong Axiom and computability properties do not suffice to characterize computable rationality. This contrasts sharply with the classical view (Richter (1966)) where the Strong Axiom characterizes rational choice behavior.

Beyond economics applications, our theorems also have implications in recursive analysis. For example, (1) suggests that the computability of preferences and their numerical (utility) representations are essentially the same — at least in some interesting spaces. Also, traditional approaches to ensuring existence of computable maximizers rely on certain technical conditions such as the "isolatedness" of maximizers and the effective uniform continuity of the given functions (cf. Grzegorczyk (1955), Ko (1991, p. 75). Our result (2) provides an alternative way — through quasiconcavity. This may be a more natural assumption in applications; it is widely used, for example, in optimization theory.

## 2. STATEMENT OF RESULTS

Our notation and terminology will be those of Richter and Wong (1996a). We take as given the formal notions of algorithms: recursive functions and partial recursive functions (cf. Kleene (1952); for economic motivations, see Richter and Wong (1996a)). In Appendix I we will give a summary of the recursive analysis notions (a computable number, an integer code of a computable real, etc.) used here.

Roughly speaking, a real vector (or number) $x$ is computable if there is an algorithm, a Turing machine $M$, or a partial recursive function $\phi$ that generates a sequence of rational approximations of $x$ (see Appendix I). An integer $n$ is a code of $x$ if $n$ is a Gödel number of $M$ or $\phi$ (see Appendix I).

**Notation.**

1. $\mathbb{N} = \{0, 1, 2, \cdots\}$ is the set of natural numbers.
2. "$\phi(x)\downarrow$" means "$x \in \text{dom}(\phi)$," i.e. the function $\phi$ is defined at $x$.
3. $\mathbb{R}_c$ is the set of computable real numbers.
4. $\mathbb{R}_c^l$ is the $l$-dimensional space of $\mathbb{R}_c$.
5. "$x \geqq y$" means "$x_i \geqq y_i$ for all $i$," for all $x, y \in \mathbb{R}_c^l$,
   "$x \gg y$" means "$x_i > y_i$ for all $i$" for all $x, y \in \mathbb{R}_c^l$.
6. $\|\cdot\|$ is the Euclidean norm, i.e. $\|x\| = (\sum_{i=1}^l x_i)^{1/2}$ for all $x \in \mathbb{R}_c^l$.
7. $\mathbb{R}_{c+}^l = \{x \in \mathbb{R}_c^l : x_i \geqq 0\}$.
8. $\mathbb{R}_{c++}^l = \{x \in \mathbb{R}_c^l : x \gg 0\}$.
9. $B_c(p, w) = \{x \in \mathbb{R}_{c+}^l : p \cdot x \leqq w\}$ for all $(p, w) \in \mathbb{R}_{c++}^l \times \mathbb{R}_{c++}$.
10. $[x, y]_c = \{\lambda x + (1-\lambda)y : \lambda \in \mathbb{R}_c \ \&\ 0 \leqq \lambda \leqq 1\}$ for all $x, y \in \mathbb{R}_c^l$
    $(x, y)_c = \{\lambda x + (1-\lambda)y : \lambda \in \mathbb{R}_c \ \&\ 0 < \lambda < 1\}$ for all $x, y \in \mathbb{R}_c^l$.
11. "$A(n, x)$" means "$n$ is a code of $x$," where $n \in \mathbb{N}$ and $x \in \mathbb{R}_c^l$.

**Computable Preference and Computable Utility.** In traditional economic analysis, there are two standard approaches to modeling rational agents: using utility functions or using preference relations. It is well-known that these two approaches are in general not equivalent, but will be equivalent under certain continuity assumptions. In Theorem 1 below, we show that the equivalence also holds under computability restrictions.

Let the consumption space be $\mathbb{R}^l_{c+}$. Consider a *strict preference* $\succ$ on $\mathbb{R}^l_{c+}$, i.e. $\succ$ is an asymmetric and negatively transitive binary relation. We recall from Richter and Wong (1996a, Definition 1) that $\succ$ is *computable* if there is an algorithm, or a Turing machine that determines, for all $x, y \in \mathbb{R}^l_{c+}$, whether $x \succ y$, by using codes of $x$ and $y$. More precisely, $\succ$ is determined by some partial recursive function $\phi(\,\cdot\,,\,\cdot\,)$, i.e. for all $x, y \in \mathbb{R}^l_{c+}$ and all $n, m \in \mathbb{N}$:

$$\text{if } A(n,x) \;\&\; A(m,y), \text{ then } x \succ y \;\Leftrightarrow\; \phi(n,m) = 1, \tag{1}$$

(cf. the notion of a listable set in Moschovakis (1964b)).

Similarly, for $X = \mathbb{R}^l_{c+}$ or any subset of $\mathbb{R}^l_c$, a function $f : X \to \mathbb{R}^m_c$ is *computable* (cf. Moschovakis (1964b)) if there is an algorithm, or a Turing machine that transforms the codes of $x \in X$ into the codes of $f(x)$. More precisely, $f$ is determined by some partial recursive function $\psi(\,\cdot\,)$, i.e. for all $x \in X$ and all $n \in \mathbb{N}$:

$$A(n,x) \qquad \text{implies} \qquad \psi(n)\downarrow \;\&\; A(\psi(n), f(x)) \tag{2}$$

The following result shows the equivalence between computable preferences and computable utility functions. This provides useful tools in computable economics, as standard utility representation theorems do in classical economics.

**Theorem 1 (Computable Preference and Computable Utility Function).**

a) *Every computable function* $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ *represents some computable strict preference* $\succ$ *on* $\mathbb{R}^l_{c+}$, *i.e.*

$$u(x) > u(y) \;\Leftrightarrow\; x \succ y \qquad \text{for every } x, y \in \mathbb{R}^l_{c+}. \tag{3}$$

b) *Conversely, every computable strict preference* $\succ$ *on* $\mathbb{R}^l_{c+}$ *is represented by some computable function* $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$.

*Proof:* See Section 3 below.

**Remark 1.**

1) Notice that $\mathbb{R}^l_{c+}$ is a countable set; so if we drop the computable conditions on $\succ$ and $u$ in Theorem 1, then it becomes an immediate consequence of Cantor's method (cf. Debreu (1954)). However, the indifference relation of a computable preference $\succ$ is generally non-computable (e.g. when $\succ$ is monotone (see Wong (1994)); so Theorem 1(b) is by no means obvious from Cantor's method.

2) Although Theorem 1(b) imposes no explicit continuity assumption on $\succ$, the computability of $\succ$ does imply that $\succ$ is (pointwise) *continuous*, i.e.

> for every $x, y \in \mathbb{R}^l_{c+}$, if $x \succ y$, then there is a positive $\epsilon \in \mathbb{R}_c$ such that $x' \succ y$ and $x \succ y'$ for all $x', y' \in \mathbb{R}^l_{c+}$ with $||x - x'|| < \epsilon$ and $||y - y'|| < \epsilon$. (4)

(See Richter and Wong (1996a, Fact 2); cf. Moschovakis (1964b)).[(1)] [(2)] However, since $\mathbb{R}^l_{c+}$ is not locally compact ($\mathbb{R}^l_{c+}$ is not even topologically complete), (pointwise) continuity does not imply "local uniform continuity." Hence our computable $\succ$ has a weaker continuity property than that of the (classical) continuous preferences defined on $\mathbb{R}^l_{+}$.

**Computable Demand.** We now study the demand of a consumer whose preference on $\mathbb{R}^l_{c+}$ is computable. We will give sufficient conditions for the existence of a computable demand bundle and the computability of a demand function. These results ensure that we can make use of the important notion of demand in computable economics analysis as we do in the classical economics analysis.

By Theorem 1 we can focus on the maximization of computable utility functions on *computable budget sets* $B_c(p, w) = \{x \in \mathbb{R}^l_{c+} : p \cdot x \leqq w\}$, where $(p, w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$.

As one might expect from the fact that the underlying space $\mathbb{R}^l_{c+}$ and the budget sets $B_c(p, w)$ are topologically incomplete, proving existence of maximizers is a nontrivial task. Indeed, well-known examples (cf. Kreisel (1958)) in recursive analysis show that a computable utility function $u$ does not necessarily have a computable maximizer on a computable budget. In fact, this negative result still holds even when $u$ satisfies a condition called effective uniform continuity (EUC). However, the existence of a maximizer does hold if $u$ is *c-quasiconcave* (i.e. $\lambda u(x)+$

(1) Similarly, a computable utility function is continuous; see below.

(2) Not all computable binary relations are continuous. Continuity is a special result for our asymmetric and negatively transitive relation. Cf. Richter and Wong (1996a, Remark 13(4)).

$(1-\lambda)u(y) \geqq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}^l_{c+}$ and all $\lambda \in [0,1]_c = \{\lambda \in \mathbb{R}_c : 0 \leqq \lambda \leqq 1\}$) and satisfies EUC (see Wong (1996)). In Theorem 2, we go further, and show that we can drop EUC, retaining only the economically natural quasiconcavity condition.

**Theorem 2 (Existence of a Computable Demand Bundle).** *Let $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ be computable and c-quasiconcave. Then for every $(p, w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$, the function $u$ admits a computable maximizer on $B_c(p, w)$.*

*Proof.* See Section 4.

**Remark 2.**

1) By Theorem 1, Theorem 2 can be rephrased in terms of computable preferences; see Richter and Wong (1996a, Theorem 2).

2) Theorem 2 shows the possibility of using algorithms in maximizing quasiconcave objective functions. That is because by our definition, computability of a maximizer $x$ is equivalent to existence of an algorithm that calculates digital approximations of $x$ to any desired degree of accuracy within a known time.

3) As in Remark 1(2), a computable function $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ is (pointwise) *continuous*, i.e.

$$\text{for every } x \in \mathbb{R}^l_{c+} \text{ and every positive } \epsilon \in \mathbb{R}_c, \text{ there is a positive } \delta \in \mathbb{R}_c \text{ such that } |u(x) - u(y)| < \delta \text{ for all } y \in \mathbb{R}^l_{c+} \text{ with } \|x - y\| < \delta \tag{5}$$

(see Moschovakis (1964b, Theorem 3 and Corollary 4.1); cf. Richter and Wong (1996a, Fact 2)). Hence, although we have dropped EUC as an hypothesis from Theorem 2, we still get some continuity as a conclusion.

Theorem 2 asserts only the existence of computable demand bundles, but not a computable demand function. In Theorem 3, without making any new assumptions, we show the computability of any demand function $h : \mathbb{R}^l_{c++} \times \mathbb{R}_{c++} \to \mathbb{R}^l_{c+}$ that is *generated* by some computable c-quasiconcave utility function $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$, i.e.

$$h(p, w) = \operatorname{argmax}_{x \in B_c(p,w)} u(x) \tag{6}$$

for all $(p, w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$.

**Theorem 3 (Computability of Demand Function).** *Let $h : \mathbb{R}^l_{c++} \times \mathbb{R}_{c++} \to \mathbb{R}^l_{c+}$ be a demand function generated by some computable and c-quasiconcave function $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$. Then $h$ is computable.*

*Proof:* See Section 4 below.

**Remark 3.**

1) Since the function $h : I\!R^l_{c++} \times I\!R_{c++} \to I\!R^l_{c+}$ is computable, it is also continuous (Moschovakis (1964b, Theorem 3); cf. Richter and Wong (1996a, Fact 2)).

2) By Theorem 1, Theorem 3 can be rephrased in terms of computable preferences; see Richter and Wong (1996a, Theorem 3).

3) We establish Theorem 3 by using Proposition 1(2), whose proof uses an algorithm that finds $h(p, w)$ for any given $(p, w) \in I\!R^l_{c++} \times I\!R_{c++}$. (Cf. Bridges' algorithmic proof (1992) for existence of a demand function in the context of constructive mathematics (cf. Bishop (1967))).

4) Let $u : I\!R^l_{c+} \to I\!R_c$ be *strictly c-quasiconcave*, i.e. $u(\lambda x + (1 - \lambda)y) > \min\{u(x), u(y)\}$ for all $\lambda \in (0, 1)_c$ and all pairs of distinct $x, y \in I\!R^l_{c+}$.[3] Clearly strict c-quasiconcavity implies c-quasiconcavity (but not vice versa). Suppose $u$ is also computable. Then by Theorem 2 and the definition of strict c-quasiconcavity, it is clear that $u$ admits a unique computable maximizer on every computable budget set, so $u$ generates a demand function $h : I\!R^l_{c++} \times I\!R_{c++} \to I\!R^l_{c+}$. By Theorem 3 this $h$ is indeed computable.

5) Let $u : I\!R^l_{c+} \to I\!R_c$ be *strictly monotone*, i.e. $u(x) > u(y)$ for all $x, y \in I\!R^l_{c+}$ with $x \geqq y$ and $x \neq y$.[4] If $h : I\!R^l_{c++} \times I\!R_{c++} \to I\!R^l_c$ is the demand function generated by $u$, then it is clear that $h$ is *exhaustive*, i.e. $p \cdot h(p) = w$ for all $(p, w) \in I\!R^l_{c++} \times I\!R_{c++}$.

6) In contrast to classical models, computable demand functions and computable excess demand functions (cf. Remark 9(2,II) in Appendix II) need not satisfy the standard "boundary conditions," even when they are generated by strictly c-quasiconcave and strictly monotone utility functions. (Cf. Footnotes 3 and 4; see Remark 9 in Appendix II.)

**Revealed Preference Analysis.** Consider a computable consumer whose computable utility function $u : I\!R^l_{c+} \to I\!R_c$ is strictly c-quasiconcave and strictly monotone. By Remarks 3(4) and 3(5), the consumer has a computable and exhaustive demand function on $I\!R^l_{c++} \times I\!R_{c++}$; so for any $X \subseteqq I\!R^l_{c++} \times I\!R_{c++}$, the restriction

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(3) **The strict c-quasiconcavity of $u$ on $I\!R^l_{c+}$ does not imply strict quasiconcavity of the extension of $u$ to all of $I\!R^l_+$. This is true even when $u$ has a unique continuous extension to $I\!R^l_+$ (see Appendix II).**

(4) **As with strict c-quasiconcavity (cf. Footnote 3) the strict monotonicity property fails for continuous extension from $I\!R^l_{c+}$ to $I\!R^l_+$ (cf. Appendix II).**

$h$ of his demand is a computable function from $X$ to $\mathbb{R}^l_{c+}$ and is exhaustive on $X$ (i.e. $p \cdot h(p) = w$ for all $(p, w) \in X$). Moreover, $h$ clearly satisfies Houthakker's Strong Axiom of Revealed Preference (cf. Richter (1966)): the binary relation $S$ on $\mathbb{R}^l_{c+}$ is acyclic (i.e. $xSyS \cdots zSx$ is impossible), where $S$ is defined by "$xSy$ if $x \neq y$ & $x = h(p, w)$ & $y \in B_c(p, w)$ for some $(p, w) \in X$."

Thus the computability, exhaustiveness, and Strong Axiom properties are necessary for a demand function $h : X \to \mathbb{R}^l_{c+}$ to be the demand of such "classical-rational" computable consumer. The following result, which is a computable version of Matzkin and Richter (1991, Theorem 1), shows the converse for the finite case. Notice that the utility function $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ that we obtain in Theorem 4 is *strictly c-concave* (rather than just c-quasiconcave), i.e. $u(\lambda x + (1-\lambda)y) > \lambda u(x) + (1-\lambda)u(y)$ for all $\lambda \in (0,1)_c$ and all $x, y \in \mathbb{R}^l_{c+}$ with $x \neq y$.

**Theorem 4 (Characterization of Computable Classical-Rationality).** *Let $X$ be a finite set in $\mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$. If $h : X \to \mathbb{R}^l_{c+}$ is computable,*(5) *exhaustive, and satisfies the Strong Axiom of Revealed Preference, then there is a computable, strictly monotone, and strictly c-concave function $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ that generates $h$ on $X$, i.e. (6) holds on $X$.*

*Proof:* See Section 5 below.

**Remark 4.** When $X$ is not finite, the Strong Axiom by itself does not necessarily imply existence of a utility generator. To see this, consider the function $h$ defined in Hurwicz and Richter (1971, p. 65). The function $h$ is exhaustive, and satisfies the Strong Axiom. Also it is easily verified that $h$ is computable on $\mathbb{R}^2_{c++} \times \mathbb{R}_{c++}$. As in Hurwicz and Richter (p. 66), for the computable point $\bar{x} = (1, 0)$ we can find a computable point $y$ in $\mathbb{R}^2_{c+}$ that is revealed worsen than $\bar{x}$ and find a (not-necessarily-computable) sequence of computable points $a_0, a_1, \cdots$ in $\mathbb{R}^2_{c+}$ convergent to $\bar{x}$ and such that each $a_k$ is revealed worsen than $y$. Suppose $h$ is rationalized by a (not-necessarily-computable) utility function $u$. Then $u$ must satisfy $u(\bar{x}) > u(y) > u(a_k)$, so $u$ cannot be continuous at $\bar{x}$, hence $u$ cannot be computable (cf. (5)).

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(5) Notice that when $X \subsetneqq \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$ is finite, then every function $h : X \to \mathbb{R}^l_{c+}$ is computable, since $h$ is the restriction of some (computable) function $g = (g_1, g_2)$ whose components $g_i$ are polynomials with computable coefficients.

## 3. PROOF OF THEOREM 1

**Proof of Theorem 1(a).** Let $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ be computable. Then it is clear that (3) defines a strict preference relation $\succ$ on $\mathbb{R}^l_{c+}$.

To show the computability for $\succ$, we pick a partial recursive function $\phi(\,\cdot\,)$ that determines $u$. We also pick a partial recursive function $\psi(\,\cdot\,,\,\cdot\,)$ that determines the computable relation $>$ (cf. Richter and Wong (1996a, Footnote 14)). We now show that $\succ$ is determined by the partial recursive function $\varphi(n, n') = \psi(\phi(n), \phi(n'))$. For any $x, x' \in \mathbb{R}^l_{c+}$ and any $n, n' \in \mathbb{N}$ with $A(n, x)$ and $A(n', x')$, we have: $A(\phi(n), u(x))$ and $A(\phi(n'), u(x'))$; so $u(x) > u(x') \;\Leftrightarrow\; \psi(\phi(n), \phi(n')) = 1$, and hence $x \succ x' \;\Leftrightarrow\; \varphi(n, n') = 1$. Q.E.D.

In our proof of Theorem 1(b), we will derive a computable utility representation $u$ for an arbitrary computable preference $\succ$. We will begin with a common approach (cf. Debreu (1954)) to construction of utility representations: we start by defining a function on a countable $\succ$-dense subset, and then take limits. But to obtain the computability of $u$, we need to ensure the computability of the construction process. Therefore our construction cannot employ the weakly preferred or indifference relations of $\succ$, because they are generally noncomputable. So we modify the common approach and use only the strictly preferred relation.

In order to make use of the computability of $\succ$ in a "deterministic" (recursive) way, we will use the following observation: Let $\succ$ be a computable strict preference on $\mathbb{R}^l_{c+}$. Then $\succ$ is determined by some partial recursive function $\phi$ as given in (1). By the Normal Form Theorem (cited in Fact 1(1), page 26) we can pick a recursive function[6] $\Psi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n, n' \in \mathbb{N}$: $\phi(n, n') = 1 \;\Leftrightarrow\; (\exists m \in \mathbb{N})[\Psi(m, n, n') = 0]$; hence by (1), for all $x, x' \in \mathbb{R}^l_{c+}$ and all $n, n' \in \mathbb{N}$:

$$\text{if } A(n, x) \;\&\; A(n', x'), \text{ then: } x \succ x' \;\Leftrightarrow\; (\exists m \in \mathbb{N})[\Psi(m, n, n') = 0]. \tag{7}$$

(Intuitively, $\Psi(m, n, n')$ can be interperted as "the algorithm $\Psi$ takes $m$ steps to determine $x \succ x'$.")

Also, we will take as given the notion of a recursive sequence of rational vectors (cf. Appendix I).

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(6) Recall that by definition a recursive function is *total*, thus $\Psi(m, n, n') \downarrow$ for all $m, n, n' \in \mathbb{N}$, i.e. $\mathrm{dom}(\Psi) = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

**Proof of Theorem 1(b).** Let $\succ$ be a computable strict preference relation on $\mathbb{R}^l_{c+}$. We need to construct a computable function that represents $\succ$.

If $\succ$ is trivial, i.e. $x \sim y$ for all $x, y \in \mathbb{R}^l_{c+}$, then $\succ$ can be represented by any constant computable function, say $u(x) = 0$.

Now we assume $\succ$ is non-trivial. We can pick a recursive sequence $\{v_k\}$ of rational vectors $\mathbb{R}^l_{c+}$ as given in Lemma 1 below. Then it is easy (e.g. using a simple computable analogue of Cantor's method) to algorithmically map the set $\{v_k : k \in \mathbb{N}\}$ onto the set of rational numbers in (0,1) in an order-preserving fashion (cf. Moschovakis (1964a, pp. 58–59)); more precisely, we take as given a recursive sequence $\{r_k\}$ such that:

$$\begin{array}{l} 1)\ \{r_k\} \text{ enumerates all rational numbers in } (0,1); \\ 2)\ r_k > r_{k'} \ \Leftrightarrow\ v_k \succ v_{k'} \qquad \text{for all } k, k' \in \mathbb{N}. \end{array} \tag{8}$$

For each $x \in \mathbb{R}^l_{c+}$, we define:

$$u(x) = \begin{cases} 1 & \text{if } x \text{ is a } \succ\text{-maximal element, i.e. } \nexists y_{y \in \mathbb{R}^l_{c+}}\, y \succ x; \\ \inf\{r_k : v_k \succ x\} & \text{if } x \text{ is not a } \succ\text{-maximal element.} \end{cases} \tag{9}$$

Notice that by Lemma 1(1), if $x$ is not a maximal element, then $\{v_k : v_k \succ x\}$ is non-empty, so $u(x)$ is well-defined; thus we have $u : \mathbb{R}^l_{c+} \to \{\alpha \in \mathbb{R} : 0 \leqq \alpha \leqq 1\}$. We now show that $u$ represents $\succ$. Consider any $x, y \in \mathbb{R}^l_{c+}$. Suppose $x \succ y$. Then from Lemma 1(1) for some $n, n'$ we have $x \succ v_n \succ v_{n'} \succ y$, so by (8(2)) and (9) we have $u(x) \geqq r_n > r_{n'} \geqq u(y)$. Conversely, suppose $u(x) > u(y)$. Then by (8(1)) for some $n, n'$ we have $u(x) > r_n > r_{n'} > u(y)$, so by (9) and (8(2)) we have $x \not\prec v_n \succ v_{n'} \succ y$, hence $x \succ y$, where "$x \not\prec v_n$" means "$\neg v_n \succ x$." Thus $u$ represents $\succ$.

To prove the computability of $u(\cdot)$, we will construct a partial recursive function $\gamma(\cdot, \cdot)$ that selects from the sequence $\{r_{k'} : k' \in \mathbb{N}\}$ given in (8) the rational approximations $r_{\gamma(n,k)}$ of the utilities $u(x)$ of bundles $x \in \mathbb{R}^l_{c+}$ within small errors $2^{-k}$. Formally, for all $x \in \mathbb{R}^l_{c+}$ and all $n \in \mathbb{N}$, we will show

$$A(n, x) \ \Rightarrow\ \forall k_{k \in \mathbb{N}}[\gamma(n,k)\downarrow\ \&\ |r_{\gamma(n,k)} - u(x)| \leqq 2^{-k}]. \tag{10}$$

Since $\{r_k\}_{k \in \mathbb{N}}$ is a recursive sequence of rational vectors, it follows that $u$ is a computable function from $\mathbb{R}^l_{c+}$ into $[0,1]_c$ (cf. Fact 3, page 29). The rest of this proof is devoted to constructing such $\gamma$.

We can pick a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ the generates codes of $v_k$, i.e.

$$A(\varphi(k), v_k) \text{ for all } k \in \mathbb{N} \tag{11}$$

(cf. Remark 7(1,3), page 26). Since $\succ$ is computable, the Normal Form Theorem (cf. Fact 1(1), page 26) guarantees that we can pick a recursive function $\Psi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as given in (7) above.

Now we will find sequences $\{a_{nt}\}_{t\in\mathbb{N}}$ of upper approximations of the utilities $u(x)$. To do this, for all $n, t \in \mathbb{N}$, we define

$$C_{nt} = \{k \leqq t : (\exists m \leqq t)[\Psi(m, \varphi(k), n) = 0]\}. \tag{12}$$

(Thus if $A(n, x)$, then $C_{nt}$ can be considered as the set of all $k \leqq t$ such that $\Psi$ can determine $v_k \succ x$ within $t$ steps.) For all $n, t \in \mathbb{N}$, We define:

$$a_{nt} = \begin{cases} 1 & \text{if } C_{nt} = \emptyset, \\ \min\{r_k : k \in C_{nt}\} & \text{if } C_{nt} \neq \emptyset. \end{cases} \tag{13}$$

Thus for all $n \in \mathbb{N}$, the numbers $a_{nt}$ are non-increasing in $t$; and if $A(n, x)$, then by (9) $a_{nt} \searrow u(x)$ as $t \to \infty$.

We need to compute the rate of convergence for sequences $\{a_{nt}\}_{t\in\mathbb{N}}$. To do this, we will find sequences $\{b_{nt}\}_{t\in\mathbb{N}}$ of lower approximations of the utilities $u(x)$. First, paralleling (12) for all $n, t \in \mathbb{N}$, we define:

$$D_{nt} = \{k \leqq t : (\exists m \leqq t)[\Psi(m, n, \varphi(k)) = 0]\}. \tag{14}$$

(Thus if $A(n, x)$, then $D_{nt}$ can be considered as the set of all $k \leqq t$ such that $\Psi$ can determine $x \succ v_k$ within $t$ steps.) For all $n, t \in \mathbb{N}$, we define:

$$b_{nt} = \begin{cases} 0 & \text{if } D_{nt} = \emptyset, \\ \max\{r_k : k \in D_{nt}\} & \text{if } D_{nt} \neq \emptyset. \end{cases} \tag{15}$$

Thus for all $n \in \mathbb{N}$, the numbers $b_{nt}$ are non-decreasing in $t$.

In order to transform the upper approximations $a_{nt}$ and lower approximations $b_{nt}$ into effective approximations $r_{\gamma(n,k)}$, it will be useful to ensure that for all $x \in \mathbb{R}^l_{c+}$ and all $n \in \mathbb{N}$, if $A(n, x)$, then:

$$\begin{aligned} &1)\ b_{nt} \leqq u(x) \leqq a_{nt} \text{ for all } t \in \mathbb{N}, \\ &2)\ b_{nt} \to u(x) \text{ and } a_{nt} \to u(x) \text{ as } t \to \infty. \end{aligned} \tag{16}$$

Condition (16(1)) is easily checked from (7), (9), (13) and (15). We have also noticed that $a_{nt} \searrow u(x)$ as $t \to \mathbb{N}$. It remains to show that $b_{nt} \to u(x)$ as $t \to \infty$. Suppose

$u(x) = 0$. Then (15) and (16(1)) implies that $b_{nt} = 0$ for all $t$, so $b_{nt} \to u(x)$ as $t \to \infty$. Suppose $u(x) > 0$. Then for every $\epsilon > 0$, by (8(1)) there are some $\bar{k}, k'$ such that $u(x) - \epsilon < r_{\bar{k}} < r_{k'} < u(x)$, so $x \not\prec v_{k'} \succ v_{\bar{k}}$ by (9) and (8(2)), hence $x \succ v_{\bar{k}}$. Therefore, by (7) and (14) we have $\bar{k} \in D_{nt}$ for all large $t$, and so by (15) we have $b_{nt} \geqq r_{\bar{k}} > u(x) - \epsilon$ for all large $t$. Hence $b_{nt} \to u(x)$ as $t \to \infty$. Thus (16(2)) also holds.

Now, to define the desired $\gamma(\cdot, \cdot)$ we first define a partial recursive function $\iota(\cdot, \cdot)$ by

$$\iota(n,k) = \min\{t \in \mathbb{N} : |a_{nt} - b_{nt}| \leqq 2^{-(k+1)}\}. \tag{17}$$

So for all $x \in \mathbb{R}^l_{c+}$ and all $n \in \mathbb{N}$, if $A(n,x)$, then by (16(2)) and (17) we have:

$$\iota(n,k)\downarrow \text{ and } |u(x) - a_{n,\iota(n,k)}| \leqq 2^{-(k+1)} \qquad \text{for all } k \in \mathbb{N}. \tag{18}$$

Then we define the desired partial recursive function $\gamma(\cdot, \cdot)$ by

$$\gamma(n,k) = \min\{\tilde{k} \in \mathbb{N} : \iota(n,k)\downarrow \ \& \ (|a_{n,\iota(n,k)} - r_{\tilde{k}}| \leqq 2^{-(k+1)})\}. \tag{19}$$

Thus for all $x \in \mathbb{R}^l_{c+}$ and all $n \in \mathbb{N}$, if $A(n,x)$, then for all $k \in \mathbb{N}$, by (18), (19) and (8(1)) we have $\gamma(n,k)\downarrow$ and $|u(x) - r_{\gamma(n,k)}| \leqq |u(x) - a_{n,\iota(n,k)}| + |a_{n,\iota(n,k)} - r_{\gamma(n,k)}| \leqq 2^{-k}$. Hence $\gamma$ satisfies (10); so $u(\cdot)$ is a computable function from $\mathbb{R}^l_{c+}$ into $[0,1]_c$.
Q.E.D.

**Lemma 1.** *Let $\succ$ be a computable strict preference on $\mathbb{R}^l_{c+}$. Assume $\succ$ is nontrivial, i.e. $x \succ y$ for some $x, y \in \mathbb{R}^l_{c+}$. Then there exists a recursive sequence $\{v_k\}_{k \in \mathbb{N}}$ of rational vectors $v_k \in \mathbb{R}^l_{c+}$ satisfying:*

*1) (separability) for all $x, y \in \mathbb{R}^l_{c+}$, if $x \succ y$, then $x \succ v_k \succ y$ for some $k$;*

*2) (nonconstancy) $v_k \not\sim v'_k$ if $k \neq k'$;*[7]

*3) (unboundedness) for all $v_k$, there are some $x, y \in \mathbb{R}^l_{c+}$ with $x \succ v_k \succ y$.*

*Proof of Lemma 1.* In Step 1 below, we will find a recursive sequence $\{(a_k, b_k)\}_{k \in \mathbb{N}}$ such that:

$$\{(a_k, b_k)\}_{k \in \mathbb{N}} \text{ enumerates all pairs of rational vectors } a, b \in \mathbb{R}^l_{c+} \text{ with } a \succ b. \tag{20}$$

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[7] "$v_k \not\sim v_{k'}$" means "$(v_k \succ v_{k'})$ or $(v_{k'} \succ v_k)$."

Then in Step 2 below, we will find a recursive sequence $\{v_k\}_{k\in\mathbb{N}}$ of rational vectors such that:

$$\begin{aligned} &\text{i) } v_0 \not\succ v_k, \cdots, v_{k-1} \not\succ v_k \qquad \text{for all } k > 1, \\ &\text{ii) } a_k \succ v_k \succ b_k \qquad \text{for all } k. \end{aligned} \tag{21}$$

Clearly, properties (21(i,ii)) ensures that $\{v_k\}$ satisfies properties (2,3) in Lemma 1. To see property (1) in Lemma 1, by (20) and (21(ii)) it clearly suffices to show that for all $x, y \in \mathbb{R}^l_{c+}$:

$$\text{if } x, y \in \mathbb{R}^l_{c+} \text{ with } x \succ y, \text{ then } x \succ v \succ y \text{ for some rational vector } v \in \mathbb{R}^l_{c+} \tag{22}$$

To see (22), let $x, y \in \mathbb{R}^l_{c+}$ with $x \succ y$. Since $\succ$ is computable, we can apply Fact 2, page 27, which ensures that $x \succ w \succ y$ for some $w \in \mathbb{R}^l_{c+}$, so (22) follows from continuity of $\succ$.

The rest of the proof is devoted to the construction of such $\{(a_k, b_k)\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$ as claimed in (20) and (21).

*Step 1) Constructing* $\{(a_k, b_k)\}_{k\in\mathbb{N}}$. We can pick a recursive sequence $\{(\tilde{a}_k, \tilde{b}_k)\}_{k\in\mathbb{N}}$ that enumerates all pairs of rational vectors in $\mathbb{R}^l_{c+}$. We can also pick two recursive functions $\psi_1$ and $\psi_2$ generating codes of $\tilde{a}_k$ and $\tilde{b}_k$, i.e.

$$A(\psi_1(k), \tilde{a}_k) \text{ and } A(\psi_2(k), \tilde{b}_k) \text{ for all } k \in \mathbb{N} \tag{23}$$

(cf. Remark 7(1,3) in Appendix I). We pick a partial recursive $\phi$ that determines $\succ$, so by (1) for all $k \in \mathbb{N}$: $\tilde{a}_k \succ \tilde{b}_k$ if and only if $k \in E$, where $E$ is the (recursively enumerable) set $\{k \in \mathbb{N} : \phi(\psi_1(k), \psi_2(k)) = 1\}$ in $\mathbb{N}$. Since $\succ$ is non-trivial, it follows easily from (22) that $E$ is non-empty. Therefore, there exists[(8)] a recursive function $\zeta$ such that $\zeta(\mathbb{N}) = E$; so the recursive sequence $\{(a_k, b_k)\}_{k\in\mathbb{N}}$ of rational vectors

$$(a_k, b_k) = (\tilde{a}_{\zeta(k)}, \tilde{b}_{\zeta(k)}) \tag{24}$$

enumerates all $(\tilde{a}_k, \tilde{b}_k)$ with $\tilde{a}_k \succ \tilde{b}_k$, i.e. $\{(a_k, b_k)\}$ satisfies (20).

*Step 2) Constructing* $\{v_k\}$. We pick a recursive sequence $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ that enumerates all rational vectors in $\mathbb{R}^l_{c+}$. First, by (22) we can pick $n_0$ with $a_0 \succ \tilde{v}_{n_0} \succ b_0$, and

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[(8)] Every non-empty recursively enumerable set $A$ is the image of of $\mathbb{N}$ under a recursive function (cf. Kleene (1952), page 306).

set $\xi(0) = n_0$. Given any $\xi(0), \xi(1), \cdots, \xi(k)$, by (22) there is some (indeed many) $n$ satisfying:

$$\tilde{v}_{\xi(0)} \not\sim \tilde{v}_n, \cdots, \tilde{v}_{\xi(k)} \not\sim \tilde{v}_n \text{ and } a_{k+1} \succ \tilde{v}_n \succ b_{k+1}. \tag{25}$$

So we can pick such $n$ and set $\xi(k+1) = n$. Therefore, the the rational vectors $v_k = \tilde{v}_{\xi(k)}$ form a sequence $\{v_k\}_{k \in I\!N}$ satisfying (21).

To obtain the recursiveness of $\{v_k\}$, we need to ensure that the function $\xi(\cdot)$ is recursive. We can obtain this by using a recursive function $\Psi$ given as in (7), codes $\psi_1(\zeta(k))$ of $a_k$, codes $\psi_2(\zeta(k))$ of $b_k$ (see (23)), and codes of $\tilde{v}_n$. For example, we pick a recursive function $\varphi$ that generates codes of $\tilde{v}_n$ i.e.

$$A(\varphi(n), \tilde{v}_n) \qquad \text{for all } n \in I\!N. \tag{26}$$

Then we set $\xi(0) = n_0$ as above. We continue as follows. At any $k$-th stage, we are given $\zeta(0), \cdots, \zeta(k)$. Notice that for all $n \in I\!N$, by (7), (22),(23), and (26) $n$ satisfies (25) if and only if there exists some $m \in I\!N$ such that

$$P_k^{\not\sim}(m, \varphi(n), \varphi(\xi(0)), \cdots, \varphi(k)) \;\&\; P_k(m, \varphi(n)), \tag{27}$$

where these recursive predicates are defined by

$$\begin{aligned} P_k^{\not\sim}&(m, \varphi(\xi(0)), \cdots, \varphi(\xi(k)), \varphi(n)) \\ &= \wedge_{i=1}^{k}[(\exists m' \leqq m)(\Psi(m', \varphi(\xi(i)), \varphi(n)) = 0) \\ &\qquad \vee (\exists m' \leqq m)(\Psi(m', \varphi(n), \varphi(\xi(i))) = 0)] \\ P_k(m, \varphi(n)) &= [(\exists m' \leqq m)(\Psi(m', \psi_1(\zeta(k+1)), \varphi(n)) = 0) \\ &\qquad \& \; (\exists m' \leqq m)(\Psi(m', \varphi(n), \psi_2(\zeta(k+1))) = 0)], \end{aligned} \tag{28}$$

Then we can define $\xi(k+1) = \min\{n \leqq \bar{M} : (\exists m \leqq \bar{M})[(m,n) \text{ satisfies } (27)]\}$, where $\bar{M} = \min\{M \in I\!N : (\exists n, m \leqq M)[(m,n) \text{ satisfies } (27)]\}$. It is clear that $\xi(\cdot)$ is recursive and for all $k \in I\!N$, (25) holds with $n = \xi(k+1)$, as we desire. Q.E.D.

## 4. PROOF OF THEOREMS 2 AND 3

We take as given the notions of a recursive sequence of rational vectors and a computable sequence of computable vectors (cf. Appendix I).

To prove Theorems 2 and 3, we will make use of Proposition 1 below, which asserts, under the (CC) assumption below, the existence of computable maximizers, and asserts existence of an algorithm for finding unique computable maximizers.

**Comment 1.** To motivate the (CC) assumption, consider the maximization of a given computable function $u$. To find a computable maximizer, a standard approach is this: 1) First find $\alpha_k$-best elements $x_k$, i.e. $\sup u - u(x_k) \leqq \alpha_k$, where $\alpha_k \searrow 0$. 2) Then ensure that the $x_k$ converge to some $x$. To obtain the computability of $x$, by recursive completeness (cf. Remark 7(2), page 26) it suffices to ensure the computability for both the $x_k$ sequence and its rate of convergence. If $u$ is computable, it is easy to obtain the computability of such $x_k$ sequence. If $u$ also satisfies an effective uniform continuity property (EUC), then it is easy to approximate effectively the sets $\mathcal{P}_{\alpha_k}$ of $\alpha_k$-best elements (cf. Pour-El and Richards (1989, p. 40-41) and Ko (1991, p. 73 (proof of Theorem 3.1)), and hence to approximate effectively their radii. Notice that $x_{k'} \in \mathcal{P}_{\alpha_{k'}} \subseteqq \mathcal{P}_{\alpha_k}$ for all $k, k'$ with $k' \geqq k$. Therefore, if we further know that the radii of $\mathcal{P}_{\alpha_k} \searrow 0$,[9] then (cf. Pour-El and Richard (1989, p. 20, Corollary 2a)) we can compute the rate of convergence of the radii of $\mathcal{P}_{\alpha_k}$, and also the rate of convergence of the vectors $x_k$ (cf. Grzegorczyk (1955)).

**Remark 5.** In the following Proposition 1, we replace the standard EUC assumption by quasiconcavity, but we retain the standard convergence condition for the $\mathcal{P}_{\alpha_k}$, formulated as follows (cf. Remark 6 below):

(Convergence Condition for $u, (p,w)$) for all computable sequences $\{\alpha_k\}$ of computable reals, if $\mathcal{P}_{\alpha_k} \neq \emptyset$ for all $k$ and if $\alpha_k$ converges non-decreasingly to $\sup_{y \in B_c(p,w)} u(y)$, then $\mathrm{rad}(\mathcal{P}_{\alpha_k}) \to 0$, (CC)

where $\mathcal{P}_{\alpha_k} = \{y \in B_c(p,w) : u(y) \geqq \alpha_k\}$, and $\mathrm{rad}(\mathcal{P}_{\alpha_k}) = \sup\{\|x-y\| : x, y \in \mathcal{P}_{\alpha_k}\}$.

In fact, in our proof from quasiconcavity we find a covering property ((CP) below) for $u$ that allows us to approximate effectively the sets $\mathcal{P}_{\alpha_k}$ in a manner very similar to the case where $u$ satisfies EUC. Then (CC) permits us, as usual, to find a computable maximizer.

**Proposition 1.** *Let $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ is computable and c-quasiconcave.*

*1) Then for every $(p,w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$ satisfying (CC), there exists a computable maximizer $\bar{x}$ of $u$ on $B_c(p,w)$;*

*2) Moreover, if (CC) holds for every $(p,w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$, then (6) defines a computable function $h : \mathbb{R}^l_{c++} \times \mathbb{R}_{c++} \to \mathbb{R}^l_{c+}$.*

Before proving Proposition 1, we will apply Proposition 1(1) to prove Theorem 2, and apply Proposition 1(2) to prove Theorem 3.

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[9] Cf. the assumption of unique maximizer in Grzegorczyk (1955, Theorem 4) and the assumption of isolated maximizers in Pour-El and Richards (1989, p. 41, Remark) and Ko (1991, p. 75, Corollary 3.2(b)).

**Proof of Theorem 2.** (As in Wong (1996, proof of Theorem 1.)) Let $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ be computable and c-quasiconcave. We will prove Theorem 2 by induction on $l$.

Let $l = 1$. Let $(p, w) \in \mathbb{R}_{c++} \times \mathbb{R}_{c++}$; so $B_c(p, w)$ is the computable interval $[0, w/p]_c = \{x \in \mathbb{R}_c : 0 \leqq x \leqq w/p\}$. There are two cases:

Case 1) Suppose (CC) holds. Then Proposition 1(1) shows the existence of a computable maximizer of $u$ on $B_c(p, w)$.

Case 2) Suppose (CC) fails. Then there is a (computable) sequence of numbers $\alpha_n \in \mathbb{R}_c$ that converges non-decreasingly to $\sup_{y \in B_c(p,w)} u(y)$ but $\text{rad}(\mathcal{P}_{\alpha_n}) \not\to 0$. Then we can pick a (not-necessarily-computable) subsequence $\{\alpha_{n_k}\}$, and pick two (not-necessarily-computable) sequences $a_k, b_k \in \mathcal{P}_{\alpha_{n_k}}$ such that $\sup_{k \in \mathbb{N}} a_k < \inf_{k \in \mathbb{N}} b_k$. We now show that there are many computable maximizers. For example, we can pick any rational (hence computable) $\bar{x}$ with $\sup_{k \in N} a_k < \bar{x} < \inf_{k \in \mathbb{N}} b_k$; so for all $k$ we have $a_k < \bar{x} < b_k$, hence $\bar{x} \in \mathcal{P}_{\alpha_{n_k}}$ by c-quasiconcavity. Therefore, we have $u(\bar{x}) \geqq \alpha_{n_k} \to \sup_{y \in B_c(p,w)}$, and so $u(\bar{x}) = \sup_{y \in B_c(p,w)}$. Thus $\bar{x}$ is a computable maximizer of $u$ on $B_c(p, w)$. Hence Theorem 2 holds for $l = 1$.

Now, we let $l > 1$, and let Theorem 2 hold for $l - 1$. Consider any $(p, w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$. Again there are two cases:

Case I) Suppose (CC) holds. Then Proposition 1(1) ensures the existence of a computable maximizer of $u$ on $B_c(p, w)$.

Case II) Suppose (CC) fails. Then there is a (computable) sequence of numbers $\alpha_n \in \mathbb{R}_c$ that converges non-decreasingly to $\sup_{y \in B_c(p,w)} u(y)$ but $\text{rad}(\mathcal{P}_{\alpha_n}) \not\to 0$. We can pick a (not-necessarily-computable) sequence $\{\alpha_{n_k}\}$ from $\{\alpha_n\}$, and pick (not-necessarily-computable) sequences $a_k, b_k \in \mathcal{P}_{n_k}$ such that for some coordinate $i = 1, \cdots, l$, one has: $\sup_{k \in \mathbb{N}} (a_k)_i < \inf_{k \in \mathbb{N}} (b_k)_i$. We now show that there are many computable maximizers. For example, we can pick a rational (hence computable) number $r$ between with $\sup_{k \in \mathbb{N}} (a_k)_i < r < \inf_{k \in \mathbb{N}} (b_k)_i$. Then we can consider the set $B^r_c = \{x \in B_c(p, w) : x_i = r\}$ as a computable budget set in $\mathbb{R}^{l-1}_{c+}$; the restriction $u|_{H^r_c} : H^r_c = \{x \in \mathbb{R}^l_{c+} : x_i = r\} \to \mathbb{R}_c$ is c-quasiconcave and computable. By the induction hypothesis there exists an $\bar{x} \in B^r_c \subseteqq B_c(p, w)$ that maximizes $u$ on $B^r_c(p, w)$. Notice that for all $k \in \mathbb{N}$, the computable vector

$$y_k = \frac{r - (a_k)_i}{(b_k)_i - (a_k)_i} b_k + (1 - \frac{r - (a_k)_i}{(b_k)_i - (a_k)_i}) a_k \tag{29}$$

belongs to $B^r_c$; and the c-quasiconcavity of $u$ implies that $u(y_k) \geqq \min\{u(a_k), u(b_k)\} \geqq \alpha_{n_k}$. Therefore, we have $u(\bar{x}) \geqq u(y_k) \geqq \alpha_{n_k} \to \sup_{y \in B_c(p,w)} u(y)$, and so $u(\bar{x}) = \sup_{y \in B_c(p,w)} u(y)$. Thus $\bar{x}$ is a computable maximizer of $u$ on $B_c(p, w)$. Hence

Theorem 2 holds for $l$. Q.E.D.

**Remark 6.** Let $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ be c-quasiconcave and computable, and $(p, w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$. Then $(p, w)$ satisfies (CC) if and only if there is a unique computable maximizer $\bar{x}$ of $u$ on $B_c(p, w)$. To see the "if," we suppose by contradiction that $(p, w)$ fails to satisfy (CC). Then by Case 2 and Case II in the proof of Theorem 2 above there are many computable maximizers; contradicting the uniqueness of $\bar{x}$. To see the "only if", suppose $(p, w)$ satisfies (CC), then Proposition 1(1) ensures the existence of a computable maximizer. Suppose by contradiction there are distinct maximizers $a, b \in B_c(p, w)$, then for the sequence $\alpha_k = \max_{x \in B_c(p,w)} u(x)$, we have $\text{rad}(\mathcal{P}_{\alpha_k}) \geqq \|a - b\| > 0$, so (CC) fails to hold.

**Proof of Theorem 3.** Let $h : \mathbb{R}^l_{c++} \times \mathbb{R}_{c++} \to \mathbb{R}^l_{c+}$ be a demand function generated by some computable and c-quasiconcave function $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$. Then by definition for each $(p, w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$, the vector $h(p, w)$ is the unique maximizer on $B_c(p, w)$, so $(p, w)$ satisfies (CC) (see Remark 6). Then by Proposition 1(2) our given function $h$, defined by (6), is computable. Q.E.D.

**Proof of Proposition 1.** We use ideas of the standard approach mentioned in Comment 1 above, with the modifications mentioned in Remark 5.

It will be useful to approximate utility values of a dense subset of $\mathbb{R}^l_{c+}$. Therefore, we fix a recursive sequence $\{v_n\}_{n \in \mathbb{N}}$ that enumerates all rational vectors $v \gg 0$ in $\mathbb{R}^l_c$. For such $\{v_n\}_{n \in \mathbb{N}}$, we have a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ with $A(\varphi(n), v_n)$ for all $n \in \mathbb{N}$ (see Remark 7(1,3) in Appendix I). We can also pick a partial recursive function $\phi(\cdot)$ that determines $u$, so $\phi(\varphi(n)) \downarrow$ & $A(\varphi(n), u(v_n))$ for all $n \in \mathbb{N}$ (see (2)); therefore, $\{u(v_n)\}_{n \in \mathbb{N}}$ is a computable sequence (see Remark 7(3) in Appendix I), hence by definition (see Appendix I) there is a recursive double sequence $\{r_{nk}\}_{k \in \mathbb{N}}$ of rational numbers such that:

$$|r_{nk} - u(v_n)| \leqq 2^{-k} \qquad \text{for all } n, k \in \mathbb{N}. \tag{30}$$

So this will allow us to approximate the values of $u$ on the $v_n$, as closely as desired.

**Proof of Proposition 1, Part 1.** Let $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ be any computable and c-quasiconcave function. Consider any $(p, w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$. Then by definition we can pick a recursive sequence $\{(q_k, a_k)\}_{k \in \mathbb{N}}$ of rational vectors in $\mathbb{R}^l_c \times \mathbb{R}_c$ such that:

$$\|(p, w) - (q_k, a_k)\| \leqq 2^{-k} \qquad \text{for all } k \in \mathbb{N}. \tag{31}$$

Suppose $(p, w)$ satisfies (CC). Then applying the algorithm given below, from

$\{(q_k, a_k)\}_{k\in\mathbb{N}}$ we can find a recursive function $\psi : \mathbb{N} \to \mathbb{N}$ such that:

$$\begin{aligned}&1)\ v_{\psi(n)} \in B_c(p,w) \qquad \text{for all } n \in \mathbb{N};\\ &2)\ \|v_{\psi(n)} - v_{\psi(n')}\| \leqq 2^{-n} \qquad \text{for all } n, n' \in \mathbb{N} \text{ with } n' \geqq n;\\ &3)\ v_{\psi(n)} \to \sup_{x\in B_c(p,w)} u(x) \text{ as } n \to \infty.\end{aligned} \tag{32}$$

By recursive completeness (cf. Remark 7(2) in Appendix I) it follows immediately from (32(1,2)) that $v_{\psi(n)} \to \bar{x}$ for some $\bar{x} \in B_c(p,w)$. Notice that $u$ is continuous (since computable), so it follows from (32(3)) that the computable vector $\bar{x}$ is a maximizer of $u$ on $B_c(p,w)$.

We will now give the algorithm, which consists of sub-algorithms I through V.

*I) Enumerating a dense set in $B_c(p,w)$.* We will find a recursive function $\iota : \mathbb{N} \to \mathbb{N}$ such that:

$$\{v_{\iota(k)}\}_{k\in\mathbb{N}} \text{ enumerates all rational vectors in } \mathrm{Int}(B_c(p,w)), \tag{33}$$

where $\mathrm{Int}(B_c(p,w)) = \{x \in \mathbb{R}^l_{c+} : (x \gg 0) \ \&\ (p \cdot x < w)\}$.

By (31) for all $n$ we have: $p \cdot v_n < w$ if and only if there exists some $k \in \mathbb{N}$ satisfying:

$$(q_k \cdot v_n) < a_k - 2^{-k}(1 + \sum_{i=1}^{l}(v_n)_i). \tag{34}$$

For all $t \in \mathbb{N}$ define:

$$C_t = \{n \leqq t : (\exists k \leqq t)[(n,k) \text{ satisfies } (34)]\}. \tag{35}$$

Then we can define:

$$\iota(t) = \min\{n \in C_{\tilde{\gamma}(t)}\}, \tag{36}$$

where $\tilde{\gamma}(0)$ is the least $t' \in \mathbb{N}$ with $C_{t'} \neq \emptyset$, and for all $t > 1$, we define $\tilde{\gamma}(t)$ to be the least $t' \in \mathbb{N}$ such that:

$$(\exists n' \in C_{t'})[\wedge_{i=0}^{t-1} n' \neq \iota(i)]. \tag{37}$$

Then it is easily checked that $\iota(\cdot)$ is a recursive function, and that the recursive sequence $\{v_{\iota(n)}\}_{n\in\mathbb{N}}$ enumerates all $v_n$ with $p \cdot v_n < w$, hence (33) holds.

*II) Approximating $\{u(v_{\iota(n)})\}_{n\in\mathbb{N}}$ and $\sup_{x\in B_c(p,w)} u(x)$.* We pick a recursive sequences $\{r_{nk}\}_{n,k\in\mathbb{N}}$ as given in (30) above, and then we define a non-decreasing recursive sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of rational vectors by

$$\alpha_n = \max\{r_{\iota(n'),n} : n' \leqq n\} - 4 \cdot 2^{-n}. \tag{38}$$

By (33) the set $\{v_{\iota(n)} : n \in \mathbb{N}\}$ is dense in $B_c(p, w)$, so by continuity of $u(\cdot)$ we have:

$$\alpha_n \to \sup_{y \in B_c(p,w)} u(y) \qquad \text{as } n \to \infty. \tag{39}$$

*III) Approximating $B_c(p, w)$ with a convexity covering condition.* We will construct a recursive function $n \mapsto N_n$ from $\mathbb{N}$ into $\mathbb{N}$ such that every $n \in \mathbb{N}$ satisfies the following property with $N = N_n$:

(Convexity Covering Property) for all $x \in B_c(p, w)$, there are some $m, t_0, t_1, \cdots, t_l, K, n' \leqq N$ such that:

1) $\|x - v_{\iota(n')}\| \leqq 2^{-N}$ and $v_{\iota(n')} \in \cup\{[x, y]_c : y \in U_c(v_{\iota(m)}, 2^{-K})\}$, (CP)

2) $U_c(v_{\iota(m)}, 2^{-K}) \subseteqq \mathrm{co}_c\{v_{\iota(t_0)}, \cdots, v_{\iota(t_l)}\}$,

3) $r_{\iota(t_0),n}, \cdots, r_{\iota(t_l),n} \geqq \alpha_n + 2^{-n}$,

where $U_c(y, \epsilon)$ denotes the computable closed ball $\{x \in \mathbb{R}^l_c : \|x - y\| \leqq \epsilon\}$ for all $y \in \mathbb{R}^l_c$ and all computable $\epsilon > 0$, and $\mathrm{co}_c\{v_{\iota(t_0)}, \cdots, v_{\iota(t_l)}\}$ is the computable convex hull $\{\sum_{i=0}^{l} \lambda_i v_{\iota(t_i)} : \lambda_0, \cdots, \lambda_l \in [0, 1]_c, \sum_{i=0}^{l} \lambda_i = 1\}$. (As we will see in Stage IV below, this (CP) permits us to approximate the sets $\mathcal{P}_{\alpha_n}$ of "almost best" vectors in a manner very similar to the case where $u$ is effectively uniformly continuous on $B_c(p, w)$ (cf. Pour-El and Richards (1989, p. 40-41), and Ko (1991, p. 73))).

In order to obtain (CP(2,3)), it will be useful to show that for all $n \in \mathbb{N}$, if $K \in \mathbb{N}$ is sufficiently large, then

$$\text{there exist some } m, t_0, \cdots, t_l \leqq K \text{ satisfying (CP(2,3)).} \tag{40}$$

To see (40), consider any $n \in \mathbb{N}$. We pick an $m$ with $r_{\iota(m),n} = \max_{n' \leqq n} r_{\iota(n'),n}$, so $u(v_{\iota(m)}) > \alpha_n + 2 \cdot 2^{-n}$ (see (38) and (30)). Recall from (33) that $v_{\iota(m)} \in \mathrm{Int}(B_c(p, w))$; so there is a small positive $\epsilon > 0$ such that $U_c(v_{\iota(m)}, \epsilon) \subseteqq \mathrm{Int}(B_c(p, w))$. By working with a smaller $\epsilon$, by (33) we can assume that for all $K$ with $2^{-K} < \epsilon$, there exist some $t_0, \cdots, t_l$ satisfying (CP(2)). Since $u$ is continuous, by working with a still smaller $\epsilon$ we can assume that $u(x) > \alpha_n + 2 \cdot 2^{-n}$ for all $x \in U_c(v_{\iota(m)}, \epsilon)$, so by (30) $t_0, \cdots, t_l$ also satisfies (CP(3)) (even with strict inequalities). Hence (40) holds for all $K \geqq m, t_0, \cdots, t_l$ with $2^{-K} < \epsilon$.

Therefore, from (40) we can define a recursive function $n \mapsto K_n$ by

$$K_n = \min\{K \in \mathbb{N} : K \text{ satisfies (40) }\}. \tag{41}$$

In order to obtain (CP(3)), it will be useful to find finite approximations $\{v_{\iota(n')} : n' \leqq \mathbb{N}\}$ of $B_c(p, w)$ with small errors $2^{-d_N}$, as in (45) below. First, recall that

$p \gg 0$; therefore by (31) for all large $N \in I\!N$ we have:

$$q_N - 2^{-N}e \gg 0, \tag{42}$$

where $e = (1, 1, \cdots, 1)$. Then $B_c(p, w) \subseteqq B_c(q_N - 2^{-k}e, a_N + 2^{-N})$, so

$$c_N \geqq \sup_{y,y' \in B_c(p,w)} \|y - y'\|, \tag{43}$$

where $c_N$ is the maximum of the distances $\|y - y'\|$ between the vertices $y, y'$ of the simplex $B_c(q_N - 2^{-k}e, a_N + 2^{-N})$, i.e.

$$c_N = \max_{i,j \leqq l} \sqrt{\left(\frac{a_N + 2^{-N}}{(q_N)_i - 2^{-N}}\right)^2 + \left(\frac{a_N + 2^{-N}}{(q_N)_j - 2^{-N}}\right)^2}, \tag{44}$$

which is also equal to $\max\{\|y - y'\| : \ y, y' \in B_c(q_N - 2^{-N}e, a_N + 2^{-N})\}$.

Consider any $n \in I\!N$. Notice that the set $\{v_{\iota(n')} : n' \in I\!N\}$ is dense in $B_c(p, w)$, and $c_N \to \sup_{y,y' \in B_c(p,w)} \|y - y'\|$ as $N \to \infty$; so for all large $N \in I\!N$, in addition to (42) we have:

$$\cup_{n' \leqq N} U_c(v_{\iota(n')}, 2^{-d_N}) \supseteqq B_c(q_N - 2^{-N}e, a_N + 2^{-N}), \tag{45}$$

where $d_N$ is the least $d \in I\!N$ such that:

$$2^{-d} \leqq \min\{2^{-(n+1)}, \frac{2^{-(n+1)}}{\max\{1, c_N\}} 2^{-K_n}\}. \tag{46}$$

Therefore, from (45) and (42) we can now define the recursive function $n \mapsto N_n$ by

$$N_n = \min\{N : N \geqq K_n \text{ and satisfies (42) \& (45)}\}. \tag{47}$$

We now show that the function $n \mapsto N_n$ satisfies (CP). Consider any $n \in I\!N$ and any $x \in B_c(p, w)$. By (40) there exist some $m, t_0, t_1, \ldots, t_l \leqq K_n \leqq N_n$ satisfying (CP(2,3)). It remains to find a $v_{n'}$ as given in (CP(1)). To do this, we first define $\lambda' = 2^{-(n+1)}/\max\{1, c_{N_n}\}$ and define $x' = \lambda' v_{\iota(m)} + (1 - \lambda')x$, so $\|x' - x\| = \lambda' \|x - v_{\iota(m)}\|$, hence by (43) we have:

$$\|x' - x\| \leqq \lambda' c_{N_n} \leqq 2^{-(n+1)}. \tag{48}$$

Also, by (46) we have $U_c(x', 2^{-d_{N_n}}) \subseteqq U_c(x', \lambda' 2^{-K_n}) = \{\lambda' y + (1 - \lambda')x : y \in U_c(v_{\iota(m)}, 2^{-K_n})\}$, so

$$U_c(x', 2^{-d_{N_n}}) \subseteqq \cup\{[x, y]_c : y \in U_c(v_{\iota(m)}, 2^{-K_n})\}. \tag{49}$$

Since $x' \in B_c(p,w)$, by (45) (with $N = N_n$) there is some $n' \leqq N_n$ with $x' \in U_c(v_{\iota(n')}, 2^{-d_{N_n}})$ and so

$$v_{\iota(n')} \in U_c(x', 2^{-d_{N_n}}), \tag{50}$$

hence by (49) $v_{\iota(n')} \in \cup\{[x,y]_c : y \in U_c(v_{\iota(m)}, 2^{-K_n})\}$. Also, from (50) and (46) we have $\|x' - v_{\iota(n')}\| \leqq 2^{-(n+1)}$, so $\|x - v_{\iota(n')}\| \leqq \|x - x'\| + \|x' - v_{\iota(n')}\| \leqq 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}$. Hence $v_{\iota(n')}$ satisfies (CP(1)) with $v_{\iota(m)}$.

*IV) Giving sharp upper bounds for rad($\mathcal{P}_{\alpha_n}$).* We will find a recursive sequence $\{s_n\}$ of rational numbers such that:

$$\text{rad}(\mathcal{P}_{\alpha_n}) \leqq \sqrt{s_n} + 2\cdot 2^{-n} \leqq \text{rad}(\mathcal{P}_{\alpha_n - 2\cdot 2^{-n}}) + 2\cdot 2^{-n} \text{ for all } n \in I\!N. \tag{51}$$

For each $n \in I\!N$, we define:

$$D_n = \{n' \in I\!N : (n' \leqq N_n) \ \& \ (r_{\iota(n'),n} \geqq \alpha_n - 2^{-n})\}, \tag{52}$$

and define

$$s_n = (\max\{\|v_{\iota(n')} - v_{\iota(n'')}\| : n', n'' \in D_n\})^2. \tag{53}$$

We now show (51). Consider any $n$. By (30) and (52), $n' \in D_n$ implies $u(v_{n'}) \geqq \alpha_n - 2\cdot 2^{-n}$; so $\{v_{\iota(n')} : n' \in D_n\} \subseteqq \mathcal{P}_{\alpha_n - 2\cdot 2^{-n}}$. Hence the second inequality of (51) follows.

To show the first inequality in (51), it clearly suffices to consider any $x \in \mathcal{P}_{\alpha_n}$ (i.e. $x \in B_c(p,w)$ with $u(x) \geqq \alpha_n$) and show that $\|x - v_{\iota(n')}\| \leqq 2^{-n}$ for some $n' \in D_n$. First, (CP) ensures that there exists some $n' \in I\!N_n$ such that $\|x - v_{\iota(n')}\| \leqq 2^{-n}$ and with the property that there exist some $t_0, \cdots, t_l \in I\!N$ satisfying (CP(3)) and there exists a $y \in \text{co}_c(\{v_{t_0}, \cdots, v_{t_l}\})$ with $v_{\iota(n')} \in [x,y]_c$. Notice that (CP(3)) and (30) imply that $u(v_{\iota(t_i)}) \geqq \alpha_n$ for all $t_i$; so by c-quasiconcavity of $u(\cdot)$ we have $u(y) \geqq \alpha_n$ and also $u(v_{\iota(n')}) \geqq \alpha_n$. Then (30) ensures that $r_{n',n} \geqq \alpha_n - 2^{-n}$, so $n' \in D_n$. Therefore, the first inequality in (51) follows.

*V) Approximating a computable maximizer effectively.* We will give (in (57) below) a recursive function $\psi$ satisfying (32).

By (CC), we have $\text{rad}(\mathcal{P}_{\alpha_n - 2\cdot 2^{-n}}) \to 0$ as $n \to \infty$, so by (51) we can define a recursive function $\gamma : I\!N \to I\!N$ by:

$$\begin{aligned}\gamma(0) &= \min\{n' \in I\!N : \sqrt{s_{n'}} + 2\cdot 2^{-n'} \leqq 2^{-0}\} \\ \gamma(n) &= \min\{n' \in I\!N : [n' > \gamma(n-1)] \ \& \ [\sqrt{s_{n'}} + 2\cdot 2^{-n'} \leqq 2^{-n}]\} \text{ for all } n > 0.\end{aligned} \tag{54}$$

Thus $\gamma(n)$ is increasing in $n$, and

$$\sqrt{s_{\gamma(n)}} + 2 \cdot 2^{-\gamma(n)} \leqq 2^{-n} \text{ for all } n \in \mathbb{N}. \tag{55}$$

By (38) we can define the recursive function $n \mapsto M_n$ by

$$M_n = \min\{m \leqq \gamma(n) : r_{\iota(m),\gamma(n)} \geqq \alpha_{\gamma(n)} + 2^{-\gamma(n)}\}. \tag{56}$$

Now we define the desired recursive function $\psi : \mathbb{N} \to \mathbb{N}$ by

$$\psi(n) = \iota(M_n). \tag{57}$$

To complete our proof, we now show that the function $\psi$ satisfies (32). First, recall from (33) that $\{v_{\iota(k)}\}_{k\in\mathbb{N}} \subseteqq B_c(p,w)$, so the sequence $\{v_{\psi(n)}\}_{n\in\mathbb{N}}$ immediately satisfies (32(1)).

To see (32(3)), notice that for all $n \in \mathbb{N}$ by (56) we have: $r_{\iota(M_n),\gamma(n)} \geqq \alpha_{\gamma(n)} + 2^{-\gamma(n)}$; so $u(v_{\psi(n)}) = u(v_{\iota(M_n)}) \geqq \alpha_{\gamma(n)}$ by (30), hence

$$v_{\psi(n)} \in \mathcal{P}_{\alpha_{\gamma(n)}}. \tag{58}$$

Recall that $\gamma(n)$ is increasing in $n$; so by (39) we have $\alpha_{\gamma(n)} \to \sup_{x\in B_c(p,w)} u(x)$, hence (32(3)) follows from (58).

To see (32(2)), notice that $\alpha_n$ are non-decreasing in $n$, so for all $n, n' \in \mathbb{N}$, if $n' \geqq n$, then by (58) we have $v_{\psi(n')} \subseteqq \mathcal{P}_{\alpha_{\gamma(n')}} \subseteqq \mathcal{P}_{\alpha_{\gamma(n)}}$, so we have $\|v_{\psi(n')} - v_{\psi(n)}\| \leqq \mathrm{rad}(\mathcal{P}_{\alpha_{\gamma(n)}}) \leqq \sqrt{s_{\gamma(n)}} + 2 \cdot 2^{-\gamma(n)}$ by (51), hence $\|v_{\psi(n')} - v_{\psi(n)}\| \leqq 2^{-n}$ by (55). This shows (32(2)). Q.E.D.

**Proof of Proposition 1, Part 2.** Let $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ be computable and c-quasiconcave.

In the following, we will sketch a modification of the algorithm given in the proof of Proposition 1(1), which will then give a partial recursive function $\Psi(\cdot,\cdot)$ similar to the function $\psi$ constructed in (57).

First, we pick a recursive sequence $\{(\tilde{q}_k, \tilde{a}_k)\}_{k\in\mathbb{N}}$ that enumerates all rational vectors in $\mathbb{R}^l_c \times \mathbb{R}_c$; and pick a partial recursive function $\tau(\cdot,\cdot)$ such that for all $(p,w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$ and all $T \in \mathbb{N}$:

$$A(T,(p,w)) \Rightarrow \forall k \in \mathbb{N}[\tau(T,k)\downarrow \;\&\; \|(\tilde{q}_{\tau(T,k)}, \tilde{a}_{\tau(T,k)}) - (p,w)\| \leqq 2^{-k}] \tag{59}$$

(cf. Remark 7(4), page 26). Then by using an argument similar to (35)-(37), from $\tau$ it is easy to find a partial recursive function $\tilde{\iota}(\cdot,\cdot)$ such that for all $(p,w) \in$

$\mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$ and all $T \in \mathbb{N}$:

$$\text{if } A(T,(p,w)), \text{ then } \tilde{\imath}(T,k)\downarrow \text{ for all } k \in \mathbb{N} \text{ and } \{v_{\tilde{\imath}(T,k)}\}_{k\in\mathbb{N}} \text{ satisfies (33) with } v_{\imath(k)} = v_{\tilde{\imath}(T,k)}. \tag{60}$$

For all $n, T \in \mathbb{N}$, we can define (as in (38))

$$\alpha_{T,n} = \max\{r_{\tilde{\imath}(T,n'),n} : n' \leqq n\} - 4 \cdot 2^{-n}; \tag{61}$$

so for all $T \in \mathbb{N}$, if $A(T,(p,w))$ for some $(p,w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$, then $\alpha_{T,n}\downarrow$ for all $n \in \mathbb{N}$. Along the lines of (39)-(47) and (52)-(57), from the functions $\tilde{\imath}$ and values $\alpha_{T,n}$ one can easily find a partial recursive function $\Psi(\,\cdot\,,\,\cdot\,)$ such that for all $(p,w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$ and $T \in \mathbb{N}$:

$$\begin{aligned}&\text{if } A(T,(p,w)) \text{ and } (p,w) \text{ satisfies (CC), then (as in (32))}\\ &1)\ \Psi(T,n)\downarrow \text{ and } v_{\Psi(T,n)} \in B_c(p,w) \text{ for all } n \in \mathbb{N};\\ &2)\ \|v_{\Psi(T,n)} - v_{\Psi(T,n')}\| \leqq 2^{-n} \text{ for all } n,n' \in \mathbb{N} \text{ with } n' \geqq n;\\ &3)\ u(v_{\Psi(T,n)}) \to \sup_{x\in B_c(p,w)} u(x) \text{ as } n \to \infty.\end{aligned} \tag{62}$$

Recall that each $(p,w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$ satisfies (CC). So by Remark 6 the equation (6) defines a function $h : \mathbb{R}^l_{c++} \times \mathbb{R}_{c++} \to \mathbb{R}^l_{c+}$. Notice that for all $T \in \mathbb{N}$ and all $(p,w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$ with $A(T,(p,w))$, the function $\Psi(T,\,\cdot\,)$ is recursive. So by recursive completeness of $\mathbb{R}^l_{c+}$ and (62), it follows that for all $T \in \mathbb{N}$ and all $(p,w) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$:

$$A(T,(p,w)) \ \Rightarrow\ \forall n_{n\in\mathbb{N}}[\Psi(T,n)\downarrow \ \&\ \|h(p,w) - v_{\Psi(T,n)}\| \leqq 2^{-n}]. \tag{63}$$

It follows that the function $h$ is computable (cf. Fact 3, page 29). Q.E.D.

## 5. PROOF OF THEOREM 4

**Proof of Theorem 4.** We will prove Theorem 4 by carrying over Theorem 1 of Matzkin and Richter (1991) to our computability context. Let $X = \{(p^1,w^1),\cdots,(p^k,w^k)\}$, and let $x^i = h(p^i,w^i)$, where $h : X \to \mathbb{R}^l_{c+}$ is exhaustive and satisfies the Strong Axiom ($h$ is then computable, by Footnote 5). Lemma 1 in Matzkin and Richter (1991) ensures that there exist real numbers $\mu^1,\lambda^1,\cdots,\mu^k,\lambda^k$ satisfying their finite system of linear inequalities (3.3(a-d)) with the (computable) parameters $p^1,x^1,\cdots,p^k,x^k$. Since $\mathbb{R}_c$ is a real closed (ordered) field, Tarski's algorithm (1951) ensures that these $\mu^i, \lambda^i$ can be chosen to be computable reals.[10] For such $\mu^i$ and $\lambda^i$, Matzkin and Richter's proof of their Lemma 2 shows that for any $T > 0$ and any small $\epsilon > 0$, the utility function $U : \mathbb{R}^l_+ \to \mathbb{R}$ defined by their equation (4.18) is strictly monotone and strictly concave on $\mathbb{R}^l_+$, and such that each $h(p^i, w^i)$ uniquely maximizes $U$ on $B(p^i, w^i) = \{x \in \mathbb{R}^l_+ : p^i \cdot x \leqq w^i\}$. We can pick a computable $T > 0$ and a small computable $\epsilon > 0$. Then it is clear that the restriction $U|_{\mathbb{R}^l_{c+}}$ is a computable, strictly monotone and strictly c-concave function from $\mathbb{R}^l_{c+}$ into $\mathbb{R}_c$, and satisfies (6) on $X$. Q.E.D.

## APPENDIX I

We will review some recursive analysis notions, assuming the notions of recursive and partial recursive functions are understood.

We begin by stating two basic theorems in recursion theory in the following Fact 1; standard proofs can be found in Kleene (1952).

**Fact 1.**

*1) (Kleene's Normal Form Theorem) There is a recursive function $U : \mathbb{N} \to \mathbb{N}$ and recursive functions $R^k : \mathbb{N}^{k+2} \to \mathbb{N}$, where $k = 1, 2, \cdots$, such that for every partial recursive function $\phi(x_1, \cdots, x_k)$, there exists at least one $n \in \mathbb{N}$ (called a Gödel number of $\phi$) such that:*

$$\phi(x_1, \cdots, x_k) = \Phi^k(n, x_1, \cdots, x_k) \tag{64}$$

*for all $x_1, \cdots, x_k \in \mathbb{N}$, where*

$$\Phi^k(n, x_1, \cdots, x_k) = U(\min\{t \in \mathbb{N} : R^k(n, x_1, \cdots, x_k, t) = 0\}). \tag{65}$$

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[10] More specifically, the existence of $\mu^i, \lambda^i$ satisfying Matzkin and Richter's (3.3(a-d)) can be stated in the first order predicate language of ordered fields with parameters $p^i_j, x^i_j$ from the real closed ordered field $\mathbb{R}_c$. It is a well-known consequence of Tarski's Theorem on the elimination of quantifiers for real closed ordered fields, that: *) any sentence with parameters from a real closed ordered field $\mathcal{A}$ that is true in any real closed ordered field containing those parameters is also true in $\mathcal{A}$. In our case, the coefficients are from $\mathbb{R}^l_c$, so solvability in the reals implies solvability in $\mathbb{R}^l_c$.

Though it is easy to apply in our proof, the full strength of Tarski's theorem is not required for our conclusion. As A. Robinson showed (1963), the theory of real closed ordered fields is model complete, a weaker property (Chang and Keisler (1990), p. 202): **) any sentence with parameters from a real closed ordered field $\mathcal{A}$ that is true in some real closed ordered extension of $\mathcal{A}$ is true in $\mathcal{A}$ itself. Again, this implies solvability in $\mathbb{R}^l_c$ for our application.

In fact, since the Matzkin-Richter system of equalities and inequalities is linear, solvability of the system implies solvability in any ordered subfield in which the parameters lie. (Cf. McFadden and Richter (1990), p. 181.)

2) *(Kleene's S-m-n Theorem) There are recursive functions* $S^n_m(y, z_1, \cdots, z_n)$ *such that:*

$$\Phi^{n+m}(y, z_1, \cdots, z_n, x_1, \cdots, x_m) = \Phi^m(S^n_m(y, z_1, \cdots, z_n), x_1, \cdots, x_m) \quad (66)$$

*for all* $y, z_1, \cdots, z_n, x_1, \cdots, x_m \in \mathbb{N}$ *and all* $n, m \in \mathbb{N}$.

We summarize some basic recursive analysis notions (cf. Pour-El and Richards (1989), and Richter and Wong (1996a)).

A sequence $\{v_k\}_{k\in\mathbb{N}}$ of $l$-dimensional vectors of rational numbers is *recursive* if there are recursive functions $\phi^1_1, \phi^1_2, \phi^1_3, \cdots, \phi^l_1, \phi^l_2, \phi^l_3 : \mathbb{N} \to \mathbb{N}$ such that $\phi^i_3(k) \neq 0$ for all $k \in \mathbb{N}$ and all $i = 1, \cdots, l$, and

$$v_k = ((-1)^{\phi^1_1(k)} \frac{\phi^1_2(k)}{\phi^1_3(k)}, \cdots, (-1)^{\phi^l_1(k)} \frac{\phi^l_2(k)}{\phi^l_3(k)}) \text{ for all } k \in \mathbb{N}. \quad (67)$$

Similarly, a double sequence $\{v_{nk}\}_{k,n\in\mathbb{N}}$ of $l$-dimensional rational vectors is *recursive* if there are recursive functions $\phi^1_1, \phi^1_2, \phi^1_3, \cdots, \phi^q_1, \phi^q_2, \phi^q_3 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\phi^i_3(n, k) \neq 0$ for all $n, k \in \mathbb{N}$ and all $i = 1, \cdots, l$, and

$$v_{nk} = ((-1)^{\phi^1_1(n,k)} \frac{\phi^1_2(n, k)}{\phi^1_3(n, k)}, \cdots, (-1)^{\phi^l_1(n,k)} \frac{\phi^l_2(n, k)}{\phi^l_3(n, k)}) \text{ for all } n, k \in \mathbb{N}. \quad (68)$$

An $l$-dimensional real vector $x$ is *computable* if $x$ is the *effective limit* of some recursive sequence $\{v_k\}$,of rational vectors i.e. $\|v_k - x\| \leqq 2^{-k}$ for all $k \in \mathbb{N}$. Similarly, a sequence $\{x_n\}_{n\in\mathbb{N}}$ of (computable) real vectors is *computable* if it is the *effective limit* of some recursive double sequence of rational vectors $v_{nk}$ i.e. $\|x_n - v_{nk}\| \leqq 2^{-k}$ for all $n, k \in \mathbb{N}$.

To define (integer) codes of computable reals, we first fix, for every $k = 1, 2, \cdots$, a bijective recursive function $\Gamma^l : N^{3l} \to \mathbb{N}$. We say an integer $n \in \mathbb{N}$ is a *code* of a computable vector $x \in \mathbb{R}^l_c$, and write $A(n, x)$ if $n = \Gamma^l(n^1_1, n^1_2, n^1_3, \cdots, n^l_1, n^l_2, n^l_3)$ for some Gödel numbers $n^1_1, n^1_2, n^1_3, \cdots, n^l_1, n^l_2, n^l_3$ of some recursive functions $\phi^1_1, \phi^1_2, \phi^1_3, \cdots, \phi^l_1, \phi^l_2, \phi^l_3$ such that (67) defines a sequence $\{v_k\}_{k\in\mathbb{N}}$ of rational vectors whose effective limit is $x$.

**Remark 7.**

1) It is clear that if a sequence of rational vectors in $\mathbb{R}^l_c$ is recursive, then it is computable; the converse is not generally true (cf. Pour-El and Richards (1989, p. 24).

2) The metric space $(\mathbb{R}^l_c, \|\cdot\|)$ is *recursively complete,* i.e. if $\{x_k\}$ is a computable sequence of vectors $x_k \in \mathbb{R}^l_c$ with $\|x_k - x_{k'}\| \leqq 2^{-k}$ for all $k, k' \in \mathbb{N}$ with $k' \geqq k$,

then $x_k \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^l_c$ (cf. Rice (1954)).

3) By means of Fact 1, it is easy to verify from the definition of $A(\cdot, \cdot)$ that a sequence of vectors $x_k \in \mathbb{R}^l_c$ is computable if and only if there is a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ with $A(\varphi(k), x_k)$ for all $k \in \mathbb{N}$.

4) Let $\{w_k\}$ be a recursive sequence that enumerates all rational vectors in $\mathbb{R}^l_{c+}$. By means of Fact 1 and the definition of $A(\cdot, \cdot)$, it is clear that there exists a partial recursive function $\tau(\cdot, \cdot)$ such that for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}^l_c$:

$$A(n, x) \;\Rightarrow\; \forall k_{k \in \mathbb{N}} [\tau(n, k) \downarrow \;\&\; \|w_{\tau(n,k)} - x\| \leqq 2^{-k}]. \tag{69}$$

We now give two facts. The first one, which asserts a connectedness property for a computable strict preference, has been used in our proof of Lemma 1.

**Fact 2.** *Let $\succ$ be a computable strict preference on $\mathbb{R}^l_{c+}$. Then $C_y \cap W_x \neq \emptyset$ for all $x, y \in \mathbb{R}^l_{c+}$ with $x \succ y$, where*

$$\begin{aligned} &C_y \text{ is the (strictly-preferred) set } \{z \in \mathbb{R}^l_{c+} : z \succ y\}, \\ &W_x \text{ is the (strictly-worsen) set } \{z \in \mathbb{R}^l_{c+} : x \succ z\}. \end{aligned} \tag{70}$$

*Proof.* Notice that $\succ$ is computable, so by substituting a code $n$ of $x$ into a partial recursive $\phi$ as given in (1), we obtain a partial recursive function $\phi_1(\cdot) = \phi(n, \cdot)$ that determines $W_x$, i.e. for all $z \in \mathbb{R}^l_{c+}$ and all $m \in \mathbb{N}$:

$$\text{if } A(m, z), \text{ then: } z \in W_x \;\Leftrightarrow\; \phi_1(m) = 1. \tag{71}$$

Similarly, we can find a partial recursive function $\phi_2$ determining $C_y$. Thus $W_x$ and $C_y$ are *listable* sets in $\mathbb{R}^l_{c+}$ in the sense of Moschovakis (1964b, p. 217).

Now suppose by contradiction that $W_x \cap C_y = \emptyset$. Then it will suffice to find two computable sequences of elements $a_k, b_k \in [x, y]_c$ such that $a_0 = x$ and $b_0 = y$ and for all $k \in \mathbb{N}$

$$1)\; a_k \in C_y \text{ and } b_k \in W_x, \qquad 2)\; \|a_{k+1} - b_{k+1}\| = \|a_k - b_k\|/2 \tag{72}$$

For such sequences $\{a_k\}$ and $\{b_k\}$, we have $\|a_k - b_k\| \leqq 2^{-k}\|x - y\|$, so by recursive completeness (cf. Remark 7(2) above) we have $a_k, b_k \to z$ for some $z \in [x, y]_c$. Notice that $W_x \cup C_y = \mathbb{R}^l_c$; so either $z \in W_x$ or $z \in C_y$. Let $z \in C_y$. Since $C_y$ and $W_x$ is a pair of disjoint listable set, Moschovakis (1964b, Corollary 4.1 and Lemma 3) yields an $\epsilon > 0$ with $b \notin W_x$ for all $b \in \mathbb{R}^l_{c+}$ with $\|b - z\| < \epsilon$; so we have $b_k \notin W_x$

for all sufficiently large $k$, contradicting (72(1)) above. Similarly, $z \in W_x$ implies that $a_k \notin C_y$ for all sufficiently large $k$, contradicting (72(1)) again.

To find such $\{a_k\}$ and $\{b_k\}$, we will use a method similar to a recursive analysis proof of an intermediate value theorem (cf. Pour-El and Richards (1989, p. 41, case 2)). First, we fix a recursive sequence $\{r_n\}$ that enumerates all rational numbers in $[0,1]_c$. Then the sequence of vectors $r_n x + (1 - r_n)y$ is computable, so (cf. Remark 7(3)) there is a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that:

$$A(\varphi(n), r_n x + (1 - r_n)y) \qquad \text{for all } n \in \mathbb{N}. \tag{73}$$

We will now define two recursive functions $\psi_1, \psi_2 : \mathbb{N} \to \mathbb{N}$ so that the vectors

$$a_k = r_{\psi_1(k)}x + (1 - r_{\psi_1(k)})y, \qquad b_k = r_{\psi_2(k)}x + (1 - r_{\psi_2(k)})y \tag{74}$$

satisfy (72). First, we set

$$\psi_1(0) = \min\{n : r_n = 1\}, \qquad \psi_2(0) = \min\{n : r_n = 0\}. \tag{75}$$

We continue as follows. At the $k$-th stage, we have defined $\psi_1(k)$ and $\psi_2(k)$ so that the vectors $a_k, b_k$ defined by (74) satisfy (72(1)). Then we will define $\psi_1(k+1)$ and $\psi_2(k+1)$ so that:

$$\begin{aligned} &a_{k+1} = (a_k + b_k)/2 \text{ and } b_{k+1} = b_k \quad \text{if } (a_k + b_k)/2 \in C_y, \\ &a_{k+1} = a_k \text{ and } b_{k+1} = (a_k + b_k)/2 \quad \text{if } (a_k + b_k)/2 \in W_x. \end{aligned} \tag{76}$$

To do this, we set

$$N_{k+1} = \min\{n : r_n = (r_{\psi_1(k)} + r_{\psi_2(k)})/2\}, \tag{77}$$

so

$$r_{N_{k+1}}x + (1 - r_{N_{k+1}})y = (a_k + b_k)/2. \tag{78}$$

By the hypothesis that $C_y \cap W_x = \emptyset$, exactly one of the following will hold:

$$1)\ r_{N_{k+1}}x + (1 - r_{N_{k+1}})y \in C_y, \qquad 2)\ r_{N_{k+1}}x + (1 - r_{N_{k+1}})y \in W_x. \tag{79}$$

Since the functions $\phi_2$ and $\phi_1$ determine the listable sets $C_y$ and $W_x$ respectively (cf. (71)), it follows from (73) that the conditions (79(1)) and (79(2)) are equivalent to the following (80(1)) and (80(2)) respectively:

$$1)\ \phi_2(\varphi(N_{k+1})) = 1, \qquad 2)\ \phi_1(\varphi(N_{k+1})) = 1. \tag{80}$$

Therefore we can set:

$$\psi_1(k+1) = N_{k+1} \text{ and } \psi_2(k+1) = \psi_2(k) \quad \text{if } \phi_2(\varphi(N_{k+1})) = 1$$
$$\psi_1(k+1) = \psi_1(k) \text{ and } \psi_2(k+1) = N_{k+1} \quad \text{if } \phi_1(\varphi(N_{k+1})) = 1. \tag{81}$$

Then from (79) and (78), the vectors $a_{k+1}$ and $b_{k+1}$ defined by (74) for $k+1$ satisfy (76). Hence the sequences $\{a_k\}$ and $\{b_k\}$ defined by (74) satisfy (72). Finally, it is clear that the functions $\psi_1$ and $\psi_2$ are recursive, so the sequences $\{a_k\}$ and $\{b_k\}$ are computable. Q.E.D.

The following fact, which is drawn from Richter and Wong (1996a, Fact 3), is useful for verifying computability of a given function. It has been applied in our proofs for Theorem 1(2) and Proposition 1(2).

**Fact 3 (Richter and Wong (1996a)).** *Let $X \subseteqq \mathbb{R}_c^l$ and $f : X \to \mathbb{R}^m$. Assume there is a recursive sequence of rational vectors $w_k \in \mathbb{R}_c^m$ and a partial recursive function $\gamma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and all $x \in X$:*

$$A(n,x) \;\Rightarrow\; \forall k_{k\in\mathbb{N}}[\gamma(n,k)\downarrow \;\&\; \|w_{\gamma(n,k)} - f(x)\| \leqq 2^{-k}]. \tag{82}$$

*Then $f$ is a computable function from $X$ into $\mathbb{R}_c^m$.*

## APPENDIX II

As noted in Footnotes 3 and 4, the strict c-quasiconcavity and strict monotonicty properties for computable functions $u$ on $\mathbb{R}_{c+}^l$ are weak in the following sense: If we extend a computable utility function $u : \mathbb{R}_{c+}^l \to \mathbb{R}_c$ that is strictly c-quasiconcave and strictly monotone to all of $\mathbb{R}_+^l$, neither of these properties need be preserved, even when there is a unique continuous extension. We demonstrate that fact here.

It clearly suffices to find a continuous profile of $\{I_\alpha\}_{\alpha\in\mathbb{R}}$ of non-increasing and (weakly) convex indifference curves in $\mathbb{R}_+ \times \mathbb{R}$ with the properties that:

(83)

1) $(0,\alpha) \in I_\alpha$ for all $\alpha \in \mathbb{R}$,
2) there are many non-computable $\alpha \in \mathbb{R}_+$ such that the indifference curves $I_\alpha$ are horizontal,
3) each computable vector $(x,y) \in \mathbb{R}_+ \times \mathbb{R}$ belongs to some strictly convex and strictly decreasing curve $I_\alpha$,
4) for the function $U$ that assigns each $(x,y) \in \mathbb{R}_+ \times \mathbb{R}$ the unique $U(x,y) \in \mathbb{R}$ with $(x,y) \in I_{U(x,y)}$:
   i) $U$ is continuous,
   ii) the restriction $U|_{\mathbb{R}_{c+}\times\mathbb{R}_c}$ is a computable function from $\mathbb{R}_{c+} \times \mathbb{R}_c \to \mathbb{R}_c$.

To find such $\{I_\alpha\}$, we begin with the following one dimensional Fact, which is a well-known "computable counterexample" to the Heine-Borel Theorem.[11] Our version is due to Beeson (1985, pp. 69–70).

**Fact 4 (cf. Beeson (1985)).** *There are two recursive sequences $a_n, b_n$ of rational numbers with $a_n < b_n$ for all $n \in \mathbb{N}$, where the sequence of intervals $J_n = [a_n, b_n]$ satisfies:*

*1) Any two $J_n$ are disjoint or have only one common endpoint;*

*2) For each computable real $x$, there exist $n, \tilde{n}$ with $x \in (a_{\tilde{n}}, b_n)$ and $b_{\tilde{n}} = a_n$;*

*3) $\sum_{n\in\mathbb{N}}(b_n - a_n) \leqq 1$*

**Remark 8.**

1) Facts 4(3,2) ensure that $\mathbb{R}_+ \setminus \cup_{n\in\mathbb{N}} [a_n, b_n]$ is non-empty but contains no computable reals.

2) Letting $x$ in Fact 4(2) to be $a_n$, then it follows that for every $n$ there is a $\tilde{n}$ with $b_{\tilde{n}} = a_n$; also by Fact 4(1) such $\tilde{n}$ is unique for every $n$. Similarly, for every $n$ there is a unique $\bar{n}$ with $a_{\bar{n}} = b_n$.

We pick such sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$. We define $c_n = (a_n + b_n)/2$ for all $n \in \mathbb{N}$. In the following, we will define a profile $\{f_\alpha\}_{\alpha\in\mathbb{R}}$ of functions $f_\alpha : \mathbb{R}_+ \to \mathbb{R}$, and the desired indifference curves $I_\alpha$ will be defined by:

$$I_\alpha = \{(x, f_\alpha(x)) : x \in \mathbb{R}_+\} \tag{84}$$

for all $\alpha \in \mathbb{R}$.

First, we define the functions

$$f_\alpha(\cdot) = \alpha \qquad \text{for } \alpha \notin \cup_{n\in\mathbb{N}}[a_n, b_n]. \tag{85}$$

Then (83(2)) follows (see Remark 8(1)).

Second, for all $n$, we define the functions $f_{b_n}(\cdot)$ and $f_{c_n}(\cdot)$ by

$$\begin{aligned} f_{b_n}(x) &= c_n + (b_n - c_n)\exp\{-x\}, \\ f_{c_n}(x) &= a_n + (c_n - a_n)\exp\{-x\}. \end{aligned} \tag{86}$$

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[11] **This Fact can be used to find computable functions that have no computable maximizers (cf. Kreisel (1958), Zaslavskii (1962), Beeson (1985, p. 73)), and to find computable functions that have no computable fixed-points (cf. Orekov (1964), Baigger (1985), and Richter and Wong (1996a)).**

Therefore, for all $n \in \mathbb{N}$:

$$\begin{aligned} &1) \text{ the functions } f_{b_n}(\cdot) \text{ and } f_{c_n}(\cdot) \text{ are strictly convex and strictly decreasing,} \\ &2) \ a_n < f_{c_n}(\cdot) \leqq c_n < f_{b_n}(\cdot) \leqq b_n. \end{aligned} \tag{87}$$

Recall from Remark 8(2) that for all $n \in \mathbb{N}$, we have $a_n = b_{\tilde{n}}$ for some $\tilde{n}$, so the functions $f_{a_n}(\cdot)$ have also been defined in (86).

Third, to define the functions $f_\alpha(\cdot)$ for the remaining $\alpha$, we define for all $n \in \mathbb{N}$ the functions

$$\begin{aligned} f_\alpha(x) &= \frac{b_n - \alpha}{b_n - c_n} f_{c_n}(x) + \frac{\alpha - c_n}{b_n - c_n} f_{b_n}(x) \qquad \text{for } c_n < \alpha < b_n \\ f_\alpha(x) &= \frac{c_n - \alpha}{c_n - a_n} f_{b_{\tilde{n}}}(x) + \frac{\alpha - a_n}{c_n - a_n} f_{c_n}(x) \qquad \text{for } a_n < \alpha < c_n, \end{aligned} \tag{88}$$

where $\tilde{n}$ is the unique $\tilde{n} \in \mathbb{N}$ with $a_n = b_{\tilde{n}}$ (see Remark 8(2)).

By (87(2)) and (88), we have:

$$\text{for all } \alpha \in \cup_{n \in \mathbb{N}}[a_n, b_n], \text{ the function } f_\alpha(\cdot) \text{ is strictly convex and strictly decreasing.} \tag{89}$$

Through (85), (86) and (88) we have defined a profile $\{f_\alpha\}_{\alpha \in \mathbb{R}}$ of functions $f_\alpha : \mathbb{R}_+ \to \mathbb{R}$. It is clear that the profile $\{I_\alpha\}_{\alpha \in \mathbb{R}}$ of curves defined by (84) satisfies (83(1)). Also, it is easily checked (see (87(2))) that for all $x \in \mathbb{R}_+$, the values $f_\alpha(x)$ are increasing in $\alpha$; so any two distinct curves $I_\alpha$ cannot intersect. It is also easy to see that the mappings $(x, \alpha) \mapsto f_\alpha(x)$ is continuous.

We now show that every $(x, y) \in \mathbb{R}_+ \times \mathbb{R}$ belongs to some $I_\alpha$.

Case 1) suppose $y \notin \cup_{n \in \mathbb{N}}[a_n, b_n]$, then by (85) we have $y = f_y(x)$, so by (84) $(x, y) \in I_y$.

Case 2) Suppose $y \in [a_n, b_n]$ for some $n$. Then there are $\tilde{n}$ and $\bar{n}$ with $b_{\tilde{n}} = a_n$ and $b_n = a_{\bar{n}}$ (see Remark 8(2)), and so (see (87(2))):

$$f_{b_{\tilde{n}}}(x) \leqq a_n \leqq f_{c_n}(x), f_{b_n}(x), y \leqq b_n < f_{c_{\bar{n}}}(x). \tag{90}$$

Suppose $y = f_{c_n}(x)$ or $f_{b_n}(x)$, then by (84) the vector $(x, y)$ belongs to $I_{c_n}$ or $I_{b_n}$. Suppose $y < f_{c_n}(x)$. By simple calculation (or by the Intermediate Value Theorem) it is easy to find a (unique) $\alpha \in (a_n, c_n)$ with

$$y = \frac{c_n - \alpha}{c_n - a_n} f_{b_{\tilde{n}}}(x) + \frac{\alpha - a_n}{c_n - a_n} f_{c_n}(x) \qquad \text{for } a_n < \alpha < c_n, \tag{91}$$

so by (88(2)) we have $y = f_\alpha(x)$, and hence $(x, y) \in I_\alpha$. Similarly, if $f_{c_n}(x) < y < f_{b_n}(x)$, then $(x, y) \in I_\alpha$ for some (unique) $\alpha \in (c_n, b_n)$; if $f_{b_n}(x) < y$, then $(x, y) \in I_\alpha$ for some (unique) $\alpha \in (b_n, c_{\bar{n}})$.

Thus each $(x, y) \in \mathbb{R}_+ \times \mathbb{R}$ belongs to some $I_\alpha$. Also, for every $(x, y) \in \mathbb{R}_{c+} \times \mathbb{R}_c$, by Fact 4(2) we have $y \in \cup_{n\in\mathbb{N}}[a_n, b_n]$, so Case 2 above shows that $y \in I_\alpha$ for some $\alpha \in \cup_{n\in\mathbb{N}}[a_n, b_n]$; hence (83(3)) follows from (89) and (84). Finally, it is easy to verify (83(4)).

**Remark 9.**

1) It is clear from our construction that the non-extendibility result for strict concavity and strict monotonicity holds even when the original function is effectively locally uniformly continuous.

2) From strict convexity, strict monotonicity, and continuity of prefererence, it follows in the classical case that demand functions $f(p, w)$ satisfy a "boundary condition": $\lim_k \|f(p_k, w_k)\| = \infty$ if $\lim_k w_k > 0$ and $\lim_k (p_k)_i = 0$ for some commodity $i$. This provides a useful tool in general equilibrium analysis (cf. Debreu (1982, Section 3)). However, the boundary condition fails in our computable context.

(I) Example (83) above can be applied to find a computable counterexample to the boundary condition. Let $U$ be constructed as above, so $U|_{\mathbb{R}^2_{c+}} : \mathbb{R}^2_{c+} \to \mathbb{R}_c$ is computable, strictly monotone and strictly c-quasiconcave, and let $h : \mathbb{R}^2_{c++} \times \mathbb{R}_{c++} \to \mathbb{R}^2_{c+}$ be the computable demand function (see (6) and Theorem 3) generated by $U|_{\mathbb{R}^2_{c+}}$. By properties (1,2,3) in Fact 4 we can pick a non-computable real $\bar{x} \in \mathbb{R}_+$ and a recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ such that $b_{\gamma(n)}$ converges increasingly to $\bar{x}$;[12] so by Fact 4(2) we also have $a_{\gamma(n)} \to \bar{x}$. Notice that by (86) the derivative

$$df_{b_{\gamma(n)}}(0)/dx = -(b_{\gamma(n)} - c_{\gamma(n)}) \tag{92}$$

for all $n$, so we have $b_{\gamma(n)} - c_{\gamma(n)} = (1/2)(b_{\gamma(n)} - a_{\gamma(n)}) \to 0$ as $n \to \infty$, hence $df_{b_{\gamma(n)}}(0)/dx \to 0$. Then by (84) it is easy to find a computable sequence of vectors $(p_n, w_n) \in \mathbb{R}^2_{c++} \times \mathbb{R}_{c++}$ such that $h(p_n, w_n) = (0, b_{\gamma(n)})$ for all $n$, and $(p_n, w_n) \to (\bar{p}, \bar{w})$ for some positive $\bar{w} \in \mathbb{R}$ and some $\bar{p} \in \mathbb{R}^2_+$ with $\bar{p}_1 = 0$ and $\bar{p}_2 > 0$.

(II) By modifying the two dimensional example (I), we can find three dimensional counterexample to the boundary condition for excess demand functions. More

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[12] For example, we can set $\gamma(0) = 0$, and for all $n \in \mathbb{N}$, we set $\gamma(n+1)$ to be the unique $\tilde{n} \in \mathbb{N}$ with $a_{\tilde{n}} = b_{\gamma(n+1)}$ (see Remark 8(2)). The recursiveness of $\gamma(\cdot)$ and the monotonicity of $\{b_{\gamma(n)}\}_{n\in\mathbb{N}}$ is clear. Also, Fact 4(3) ensures that the $b_{\gamma(n)}$ sequence is bounded, so $b_{\gamma(n)}$ converges to some real $\bar{x} = \sup_{n\in\mathbb{N}} b_{\gamma(n)}$. It is easily checked from Facts 4(1,2) that this $\bar{x}$ cannot be computable.

precisely, it is easy to find a computable consumer $(u,\omega)$ whose utility function $u : I\!R^3_{c+} \to I\!R_c$ is strictly c-quasiconcave, strictly monotone, and computable, and endowment $\omega$ is strictly positive vector in $I\!R^3_{c+}$, and the excess demand function $f : I\!R^3_{c++} \to I\!R^3_c$ defined by $f(p) = \text{argmax}\{u(x) : x \in I\!R^3_{c+} \ \& \ p \cdot x \leqq p \cdot \omega\} - \omega$ violates the boundary condition. In particular, there is a computable sequence of $p_k \in I\!R^3_{c++}$ such that $||p_k|| = 1$ for all $k$, and $(p_k)_1$ converges to 0 as $k \to \infty$, and $(f(p_k))_1 = (f(p_k))_3 = 0$ for all $k$, and $(f(p_k))_2$ converges to some non-computable $\bar{x} \in I\!R_+$ as $k \to \infty$.

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