THE STRUCTURE OF THE EIGENVECTORS OF SPARSE MATRICES

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Abstract

In this work we discuss the following question: Given the zero-nonzero pattern of a matrix A with complex entries, what can be said about the zero-nonzero pattern of its eigenvectors? To be more general, what are the possible sparsity patterns for the basis of the maximal invariant subspaces of A associated with each of its eigenvalues? Or even, what is the sparsity pattern of the similarity transformation M such that $M^{-1}AM$ is in Jordan canonical form, i.e, A's Jordan basis. Let struct(A) be the usual directed graph associated with the zero-nonzero pattern of A. The main result of this paper is that there exists a matrix B such that if λ is an eigenvalue of A with algebraic multiplicity m then there are m columns of B that form a basis for the maximal invariant subspace of A associated with λ and such that struct(B) is a subgraph of the graph obtained by adding all the edges of the form (i,i) to the transitive closure of struct(A), which we call rstruct(A). We show that if the defective eigenvalues of A have geometric multiplicity one then the matrix B above can be chosen in such a way that there exists a permutation matrix P for which BP is a Jordan basis of A. We present examples and theorems showing that our results are sharp. Similar results hold for the real Jordan canonical form.

key words sparse matrix, eigenvectors, transitive closure, Jordan canonical form.

1. Introduction

A natural question raised when computing the eigenvectors of a sparse matrix is how much memory and what kind of data structure we need to hold them. In some applications we have interest in the maximal invariant subspaces of A associated with each eigenvalue λ and it would be nice to find bases for these subspaces that are as sparse as possible.

In [G] John Gilbert shows that if $A \in \mathbb{C}^{n\times n}$ has simple eigenvalues and nonzero diagonal entries then there exists an eigenbasis B of A such that

$$\text{struct}(B) \subset \text{transitive closure of } \text{struct}(A), \tag{1.1}$$

where **struct**(A) is the **structure** of the $n \times n$ matrix A, i.e, the directed graph with nodes $1, \ldots, n$ and with an arc from i to j if and only if $a_{i,j} \neq 0$ and the transitive closure of a graph G is the graph that has and arc from i to j if and only if there is a path from i to j in G. An **eigenbasis** is a matrix that has linearly independent eigenvalues of A as its columns. John Gilbert has also shown that for almost all matrices A with simple eigenvalues and nonzero diagonals, any eigenvector matrix is such that

$$\text{struct}(BP) = \text{transitive closure of} \text{struct}(A)$$

for some permutation P of the columns of B. In other words, the inclusion (1.1) is sharp for this class of matrices.

The purpose of this work is to generalize Gilbert's results to matrices with repeated eigenvalues and to weaken the requirement of the diagonal entries being nonzero. It turns out that with a little more effort we can analyze the question of how much memory and what kind of data structure we need to hold a **Jordan basis** for A, where by a Jordan basis we mean (the columns of) a similarity transformation M such that $M^{-1}AM$ is in Jordan canonical form. We can also analyze the structure of the **real Jordan basis** R of A, i.e., the real matrix R such that $R^{-1}AR$ is in real Jordan canonical form.

Our main conclusion is that the correct structure for the eigenvectors is the graph obtained by adding the diagonal arcs to the transitive closure of the structure of the matrix. We call this graph the **reflexive transitive closure** of the structure of A and denote it by rstruct(A). The use of this structure allows us to prove inclusion results of the type (1.1) without restrictions on the sparsity pattern and with milder conditions on the multiplicity of eigenvalues. We prove that our results are sharp for sparsity pattern that allow nonsingular matrices, a generalization of the matrices with nonzero diagonals.

In order to deal with the case of multiple eigenvalues and nondiagonalizable matrices we need the following generalizations of the concept of eigenspace and eigenbase. If λ is an eigenvalue of $A \in \mathbb{C}^{n\times n}$ we call the subspace

$$S_{\lambda,A} = \{v \in \mathbb{C}^n \text{ such that there exists } k \text{ for which } (A-\lambda)^k v = 0\}$$

the **maximal invariant subspace of A associated with the eigenvalue λ.**

We say that $B \in \mathbb{C}^{n\times n}$ is a **generalized eigenbasis** of A if we can find a base for all the $S_{\lambda,A}$ among the columns of B.

In the case when $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{C} - \mathbb{R}$ is an eigenvalue of A we will be interested in the subspace

$$R_{\lambda,A} = \{v \in \mathbb{R}^n \text{ such that there exists } k \text{ for which } [(A-\bar{\lambda})(A-\lambda)]^k v = 0\}$$

or the **maximal invariant subspace of A associated with the eigenvalues λ and $\bar{\lambda}$.** We say that $B \in \mathbb{R}^{n\times n}$ is a **real generalized eigenbasis** of A if we can find bases for all $S_{\lambda,A}$ and $R_{\lambda,A}$ among the columns of B. Any (real) Jordan basis is an example of a (real) generalized eigenbasis.

Using the terminology above, we can state the main result in this paper as follows

Theorem 1. *For every matrix $A \in \mathbb{C}^{n\times n}$ there exists a generalized eigenbasis B with struct(B) $\subset$ rstruct(A).*

The analogous Theorem for real matrices is

Theorem 1'. *For every matrix $A \in \mathbb{R}^{n\times n}$ there exist a real generalized eigenbasis B with struct(B) $\subset$ rstruct(A).*

Let us denote by B_j the jth column of the matrix B. The matrix B in Theorem 1 will be a Jordan basis if the columns associated with each eigenvalue λ are consecutive, $B_j, \dots, B_{j+m-1}$, say, and $(A-\lambda)B_i = B_{i-1}$ if $j < i < j+m$. The next Theorem shows that there exists B satisfying this extra requirement if the defective eigenvalues of A have geometric multiplicity one. Later in this section we show that this may not be true if A has a defective eigenvalue with geometric multiplicity bigger than one.

Theorem 2. *If the defective eigenvalues of $A \in \mathbb{C}^{n\times n}$ have geometric multiplicity one then A has a Jordan basis M for which there exists a permutation matrix P such that struct(MP) $\subset$ rstruct(A).*

An analogous result for real matrices is given by the following Theorem.

Theorem 2'. *If the defective eigenvalues of $A \in \mathbb{R}^{n\times n}$ have geometric multiplicity one then A has a real Jordan basis M for which there exists a permutation matrix P such that struct(MP) $\subset$ rstruct(A).*

Proofs of Theorems 1 and 2 and their real versions are given in section 3. The simplest example of what can go wrong if the hypothesis of Theorems 2 or 2' does not hold is the family of matrices

$$F(a,b) = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $a, b \neq 0$. The graph struct($F(a,b)$) has incidence matrix

$$E = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and rstruct($F(a,b)$) has incidence matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrices $F(a,b)$ have zero as a defective triple eigenvalue with geometric multiplicity two and eigenvectors of the form $v(x,y)=(x,by,-ay)$, for x or y nonzero. Therefore, any matrix M containing the eigenvectors of $F(a,b)$ among its columns has a column of the form $(x,by,-ay)$, $y\neq 0$. This implies that there is no permutation matrix P such that $\mathrm{struct}(MP)\subset \mathrm{rstruct}(F(a,b))$.

Now that we know what can go wrong if the hypothesis of Theorem 2 fails, it is natural to ask how tight the inclusion of these theorems is when their hypothesis hold. We present an example showing how Theorem 2 can overestimate the structure of the Jordan basis. Let us take the family of matrices

$$F(a,b,c,d)=\begin{pmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{pmatrix},$$

with $a,b,c,d\neq 0$. In this case $\mathrm{rstruct}(F(a,b,c,d))$ is a clique. However, if $ab+cd\neq 0$ then $F(a,b,c,d)$ has simple eigenvalues, $0,\pm\sqrt{ab+cd}$, with eigenvectors

$$M=\begin{pmatrix} a & a & c \\ \sqrt{ab+cd} & -\sqrt{ab+cd} & 0 \\ d & d & -b \end{pmatrix}.$$

If $ab+cd=0$, then 0 is a defective eigenvalue of $F(a,b,c,d)$ and Theorem 2 does not apply. Therefore, in all the cases to which Theorem 2 applies it overestimates the Jordan basis by one nonzero entry, M_{23} in M the above.

The main reason for the overestimate above is the fact that $\mathrm{struct}(F(a,b,c,d))$ is a **singular structure**, i.e, any matrix with this structure is singular. Theorem 2 will overestimate the structure of the Jordan Basis for all (nonzero) matrices with a singular structure. In order to get a sharp estimate in this case we need to deflate the eigenvalue zero before applying Theorem 2.

The situation for a nonsingular structures S is quite different. In this case we have Theorem 3, which states, roughly, that Theorem 2 is sharp for "almost all" matrices with structure S. In order to formalize the "almost all" in the last sentence we need to make some definitions. Notice that the set of complex matrices with $\mathrm{struct}(A)\subset S$, which we call $\mathbb{C}(S)$, can be identified with $\mathbb{C}^m$, where m is the number of edges in S. By enumerating the edges (i,j) of S we can associate to each matrix $A\in\mathbb{C}(S)$ the vector $V(A)\in\mathbb{C}^m$ given by $v_k=A_{i,j}$, where (i,j) is the kth edge of S. The set $\mathbb{R}(S)$ and the map $V:\mathbb{R}(S)\to\mathbb{R}^m$ are defined analogously. We can then state the following Theorem.

Theorem 3 *If S is a nonsingular structure, with n nodes and m edges, then there exists a polynomial $Q:\mathbb{C}^m\to\mathbb{C}$, $Q\neq 0$, such that if $A\in\mathbb{C}(S)$ and $Q(V(A))\neq 0$ then any generalized eigenbasis B of A is such that struct$(BP)=$ rstruct(S) for some permutation matrix P.*

As a corollary of Theorem 3 we have that for almost all matrices A with structure S we have that any Jordan basis M for A is such that $\mathrm{struct}(MP)=\mathrm{rstruct}(A)$ for some permutation matrix P.

The generalization of Theorem 3' to the real case is not as straightforward as before, basically because the subspaces associated with complex eigenvalues give us the freedom to introduce two extra zeros in its basis. For example, if B is a real generalized eigenbasis for A given by Theorem 1' and B_1 and B_2 are the basis for $R_{\lambda,A}$ then, if we look at the proof of Theorem 1', we see that if we replace B_1 and B_2 by any (nonsingular) linear combinations of B_1 and B_2 the new matrix B will also be a generalized eigenbasis for A with struct$(B) \subset$ rstruct(A). We can then choose these linear combinations to zero out two entries of B. The next Theorem shows that for almost all matrices this is as far as we can go in reducing the structure of B.

Theorem 3' *If S is a nonsingular structure, with n nodes and m edges then there exists a polynomial $Q : \mathbb{R}^m \to \mathbb{R}$, $Q \neq 0$, such that if $A \in \mathbb{R}^{n\times n}$, struct$(A) = S$ and $Q(V(A)) \neq 0$ then A has simple eigenvalues and there exists an eigenbasis B of A that satisfies the following:*

a) struct(B) = rstruct(A).

b) If $\lambda \in \mathbb{C} - \mathbb{R}$ is an eigenvalue of A, $\{B_i, B_j\}$ is a basis of $R_{\lambda,A}$ and $e_{ri} = e_{si} = 1$ then the vectors $(b_{ri}, b_{si})^T$ and $(b_{rj}, b_{sj})^T$ are linearly independent.

Theorem 3' implies that if $Q(V(A)) \neq 0$ then we can introduce at most two extra zeros per complex eigenvalue in B' by combining its columns associated with complex eigenvalues.

In the next sections we prove the Theorems above. The basic idea behind the proof is quite simple, however we found it hard to formalize then in a simpler way then what is presented in the next sections. The basic idea is to look at matrices of the form

$$\begin{pmatrix} A_{11} & \dots & A_{1k} \\ 0 & \ddots & \vdots \\ 0 & 0 & A_{kk} \end{pmatrix},$$

where k is the number of strongly connected components of struct(A). These matrices have (generalized) eigenvectors of the form

$$V = \begin{pmatrix} V_1 \\ \vdots \\ V_j \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where V_j is an eigenvector of A_{jj} and the other V_i are obtained by back substitution, or by solving the equations

$$(A_{ii} - \lambda)V_i = V_i' - \sum_{l=i+1}^{j} A_{il}V_l, \tag{1.2}$$

where $V' = 0$ for eigenvectors or V' is found inductively for generalized eigenvalues. In principle there are two things that can go wrong when trying to solve (1.2): $A_{ii} - \lambda$ may be singular or the solution may have the wrong sparsity pattern. The proofs of Theorems 1 and 1' basically show that having $A_{ii} - \lambda$ singular does not really matter because we can "fix" the right hand side in (1.2). As for sparsity, the basic idea is that if V_i is forced to be zero then all the products $A_{il}V_l$ will also necessarly be zero and everything works fine. Theorems 2 and 2' show that if we cannot solve (1.2) for the columns of a Jordan basis associated with λ then λ must be defective and have geometric multiplicity at least two. Finally, the proofs of Theorems 3 and 3' are just a question of finding the correct polynomials, using resultants, and showing that there exist matrices with the given sparsity pattern for which these polynomials do not vanish.

Some remarks about notation. Along the paper the number of columns of a matrix A is denoted by cols(A) and the number of rows is rows(A). The (i, j) entry of A is a_{ij}. A capital A_{ij} denotes a submatrix of A and A_j is the $j$tj column of A.

We have discussed some of the results in this work with Alex Pothen, Charles Johnson and Richard Brualdi. The proofs of Theorems 3 and 3' were influenced by discussions with Hans Weinberger. We would like to thank Nick Higham for painting the first draft of this paper in red. He didn't have the opportunity to look at the final version though and we are solely responsible for the contents of it.

2. Proofs of theorems 1, 1', 2 and 2'

In this section we present proofs of theorems 1,1', 2 and 2'. The proofs for the real Jordan form and the real generalized eigenbasis are adaptations of the proofs for the complex case. The theorems follow from the lemmas below.

Lemma 1 *Let $A \in \mathbb{C}^{n \times n}$ and let E be the incidence matrix of rstruct(A). If $v \in \mathbb{C}^n$ is such that there exists j for which $v_i = 0$ if $e_{ij} = 0$ then for any $\lambda \in \mathbb{C}$ we have that $\tilde{v} = (A - \lambda)v$ also has the property that $e_{ij} = 0 \Rightarrow \tilde{v}_i = 0$.*

Proof of lemma 1: If $e_{ij} = 0$ then $v_i = 0$ and $\tilde{v}_i = \sum_{l \neq i} a_{il}v_l$. Since rstruct$(A)$ is strongly connected and $e_{ij} = 0$, for every $l \neq i$ we have that either $e_{il} = 0$ or $e_{lj} = 0$, otherwise $(i, l), (l, j)$ would be a path connecting i and j. This implies that either $a_{il} = 0$ or $v_l = 0$, what shows that $a_{il}v_l = 0$ for all $l \neq i$. Therefore $\tilde{v}_i = 0$ and the proof of lemma 1 is finished •

As a corollary of the lemma 1 we have that if $e_{ij} = 0$ then the ith components of $(A - \lambda)^k v$ and $[(A - \bar{\lambda})(A - \lambda)]^k v$ are zero for all k.

Lemma 2 *For any $A \in \mathbb{C}^{n \times n}$ and $k \in I\!N$ the dimension of the kernel of A^k is less than or equal to k times the dimension of the kernel of A.*

Proof of lemma 2: Left to the reader •

We state now the main lemmas of this section. In fact, these lemmas are more general then theorems 1 and 1'. They are also the main step in the proofs of theorem 2 and 2'.

We prove the theorems after the proof of these lemmas. From now to the end of this section the reading becomes quite technical.

Lemma 3 *Given $A \in \mathbb{C}^{n\times n}$ there exists $B \in \mathbb{C}^{n\times n}$ with struct(B) $\subset$ rstruct(A) and such that if λ is an eigenvalue of A with algebraic multiplicity m then there are columns B_{j_i} of B, $1 \leq i \leq m$, such that*

$$Span\{B_{j_i},\ 1 \leq i \leq m\} = S_{\lambda,A} \tag{2.3}$$

and

$$(A-\lambda)B_{j_i} \in Span\{B_{j_k} \text{ such that } 1 \leq k < i \text{ and } e_{j_k,j_i} \neq 0\}, \tag{2.4}$$

where E is the incidence matrix of rstruct(A).

Proof of lemma 3: The proof uses two levels of induction. The first level is an induction on the number of strongly connected components of the matrix and the second is an induction on the columns associated with a given eigenvalue λ. The scheme of the proof is as follow:

```
begin induction on strongly connected components
  argue
  begin induction on columns associated with lambda
    argue
  end induction on columns associated with lambda
end induction on strongly connected components.
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Let us start the induction on the number of strongly connected components of struct(A). If rstruct(A) has only one strongly connected component then rstruct(A) is a clique and we can take B to be any Jordan basis for A. Let us assume that the lemma 3 holds for matrices M for which rstruct(M) have $k-1$ strongly connected components and let us take A such that rstruct(A) has k strongly connected components. Without loss of generality we can assume that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A' \end{pmatrix}, \tag{2.5}$$

where rstruct(A_{11}) is a clique and struct(A') has $k-1$ strongly connected components. This implies that rstruct(A) has incidence matrix

$$E = \begin{pmatrix} E_{11} & E_{12} \\ 0 & E' \end{pmatrix}, \tag{2.6}$$

where E' is the incidence matrix of rstruct(A'). We construct now a matrix

$$B = \begin{pmatrix} K & L \\ 0 & B' \end{pmatrix}, \tag{2.7}$$

satisfying the conditions required by lemma 3. We choose K to be any Jordan basis for A_{11} and take B' to be any matrix obtained by applying lemma 3 to A'. Notice that $b'_{i'j'} = b_{ij}$ for $i' = i - \text{cols}(A_{11})$ and $j' = j - \text{cols}(A_{11})$.

From now on if k is an index of a matrix then k' will denote $k - \text{cols}(A_{11})$.

The proof proceeds by constructing, for every eigenvalue λ of B', columns of L in such a way if $B'_{j'} \in S_{\lambda,A'}$ then

$$B_j = \begin{pmatrix} L_{j'} \\ B'_{j'} \end{pmatrix} \in S_{\lambda,A}.$$

Let m be the algebraic multiplicity of λ as an eigenvalue of A_{11} and m' the algebraic multiplicity of λ as an eigenvalue of A'. (Take $m = 0$ if λ is not an eigenvalue of A_{11}.) In order to simplify the notation, we assume that K_l, $1 \le l \le m$ form a basis for $S_{\lambda,A_{11}}$ and $B'_{j'}$, $1 \le j' \le m'$ are the columns of B' that form a basis for $S_{\lambda,A'}$.

At this point we begin the second induction, to determine the columns $L_{j'}$. Let us assume that we have determined $L_{t'}$ for every $1 \le t' < j'$ in such a way that (2.4) is satisfied and with $L_{t'} = 0$ if $e_{it} = 0$ for some $1 \le i \le \text{rows}(L)$. (Since there is no $1 \le t' < j' = 0$ we can certainly do that for $j' = 0$.)

Since we obtained B' applying lemma 3 to A', equation (2.4) shows that there exist $\alpha_{l'}$ such that

$$(A - \lambda)B'_{j'} = \sum_{l'=1}^{j'-1} \alpha_{l'} B'_{l'},$$

with

$$\alpha_{l'} = 0 \text{ if } e'_{l'j'} = 0. \tag{2.8}$$

Since the K_l form a basis for $S_{\lambda,A_{11}}$ there exist β_s, for $1 \le s \le m$, and $L_{j'}$ such that

$$(A_{11} - \lambda)L_{j'} + A_{22}B'_{j'} = \sum_{s=1}^{m} \beta_s K_s + \sum_{l'=1}^{j'-1} \alpha_{l'} L_{l'}. \tag{2.9}$$

This last equation implies that $(A - \lambda)B_j \in \text{Span}\{B_i \text{ such that } 1 \le i < j\}$ and that B_j is in $S_{\lambda,A}$. In order to complete the induction on the L_j we need to show that if $e_{ij} = 0$ for some $1 \le i \le \text{rows}(A_{11})$ then we can take $L_{j'} = 0$ and $\beta_s = 0$ for all s in equation (2.9).

In fact, suppose that $e_{ij} = 0$. Since rstruct(A) is transitively closed this implies that for any given l either $e_{il} = 0$ or $e_{lj} = 0$. Let l be such that $\text{cols}(A_{11}) < l < j$, or equivalently $1 \le l' < j'$. If $e_{lj} = 0$ then $e'_{l'j'} = 0$ and (2.8) implies that $\alpha_{l'} = 0$. Therefore $\alpha_{l'} L_{l'} = 0$ if $e_{lj} = 0$. If $e_{il} = 0$ then, since struct(A_{11}) is strongly connected, $e_{tl} = 0$ for all $1 \le t \le \text{rows}(A_{11})$ and, since we are assuming that L_l have the correct sparsity pattern for $l < j$, $L_{l'} = 0$ and again $\alpha_{l'} L_{l'} = 0$. This shows that $\sum_{l'=1}^{j'-1} \alpha_{l'} L_{l'} = 0$.

Since $e_{ij} = 0$ we have that either $e_{il} = 0$ or $e'_{l'j'} = 0$ for any l. Therefore either $(A_{22})_{il'} = 0$ or $b'_{l'j'} = 0$. This implies that $(A_{22}B'_{j'})_i = 0$. Moreover, since struct(A_{11}) is strongly connected, $e_{ij} = 0$ for all $1 \le i \le \text{rows}(A_{11})$ what shows that $A_{22}B'_{j'} = 0$.

The last two paragraphs imply that if $e_{ij} = 0$ for some $1 \leq i \leq \text{rows}(A_{11})$ then $L_{j'} = 0$ and $\beta_s = 0$ for $1 \leq s \leq m$ satisfy equation (2.9). This is the end of the inductive construction of the columns of L.

We are done with the proof of lemma 3 •

Lemma 3' *Given $A \in \mathbb{R}^{n\times n}$ there exists $B \in \mathbb{R}^{n\times n}$ with struct(B) $\subset$ rstruct(A) and such that if λ is an eigenvalue of A with algebraic multiplicity m then if $\lambda \in \mathbb{R}$ there are columns B_{j_i} of B, $1 \leq i \leq m$, such that*

$$Span\{B_{j_i},\ 1 \leq i \leq m\} = S_{\lambda,A}$$

and

$$(A-\lambda)B_{j_i} \in Span\{B_{j_k} \text{ such that } 1 \leq k < i \text{ and } e_{j_k,j_i} \neq 0\},$$

where E is the incidence matrix of rstruct(A). If $\lambda \in \mathbb{C} - \mathbb{R}$ then there are columns B_{j_i} of B, $1 \leq i \leq 2m$, such that $j_{i+1} = j_i + 1$,

$$Span\{B_{j_i},\ 1 \leq i \leq m\} = R_{\lambda,A} \tag{2.10}$$

and

$$(A-\bar{\lambda})(A-\lambda)B_j \in Span\{B_{j_k} \text{ such that } 1 \leq k < 2i-1 \text{ and } e_{j_k,j_{2i}} \neq 0\} \tag{2.11}$$

for $j = 2i-1$ and $2i$. Moreover $e_{kj_{2i}} = e_{kj_{(2i-1)}}$ for every k.

The requirement that $e_{kj_{2i}} = e_{kj_{(2i-1)}}$ for every k in lemma 3' is necessary for proving theorem 2'. It basically shows that any basis for the space Span$\{B_{2j-1}, B_{2j}\}$ satisfying the same criterea regarding sparsity.

Proof of lemma 3': This proof is an adaptation of the proof of lemma 3. The proofs are exactly the same up to the induction to construct the columns of L and in the case when λ is real. We proceed now with the case $\lambda \in \mathbb{C} - \mathbb{R}$.

In this case we want to choose the columns $L_{j'}$ in such a way that if $B'_j \in R_{\lambda,A}$ then

$$B_j = \begin{pmatrix} L_{j'} \\ B'_{j'} \end{pmatrix} \in R_{\lambda,A}.$$

Let m be the algebraic multiplicity of λ as an eigenvalue of A_{11} and m' the algebraic multiplicity of λ as an eigenvalue of A'. (Take $m = 0$ if λ is not an eigenvalue of A_{11}.) In order to simplify the notation, we assume that K_l, $1 \leq l \leq 2m$ form a basis for $R_{\lambda,A_{11}}$ and $B'_{j'}$, $1 \leq j' \leq 2m'$ are the columns of B' that form a basis for $R_{\lambda,A'}$.

We now use induction to determine the columns $L_{j'}$ one pair at a time. Let us assume that we have determined $L_{2t'}$ and $L_{2t'-1}$ for $1 \leq t' < j'$ in such a way that (2.11) is satisfied and with $L_{2t'} = L_{2t'-1} = 0$ if $e_{i(2t)} = 0$ for some $1 \leq i \leq \text{rows}(L)$ and such that $e_{i(2t)} = e_{i(2t-1)}$ for all $1 \leq i \leq \text{rows}(A_{11})$.

Notice that since struct(identity) $\subset$ rstruct(A), we have that $e_{(2j)(2j)} = 1$. This implies that $e'_{(2j)'(2j)'} = 1$. Therefore $e'_{(2j)'(2j'-1)} = 1$ and since rstruct(A) is strongly connected we have that if $e_{i(2j)} = 1$ then $e_{i(2j-1)} = 1$. Analogously, if $e'_{(2j'-1)(2j)'} = 1$ implies that if $e_{i(2j-1)} = 1$ then $e_{i(2j)} = 1$. Therefore $e_{i(2j)} = e_{i(2j-1)}$ for all i.

Since we obtained B' applying lemma 3' to A', equation (2.11) shows that there exist $\alpha_{rl'}$, for $r = 0$ or 1, such that

$$(A - \bar{\lambda})(A - \lambda)B'_{2l'-r} = \sum_{l'=1}^{2(j'-1)} \alpha_{rl'} B'_{l'},$$

with

$$\alpha_{rl'} = 0 \ \text{ if } \ e'_{l'(2j')} = 0. \tag{2.12}$$

Since the K_l form a basis for $S_{\lambda,A_{11}}$ there exist β_{rs}, for $1 \leq s \leq 2m$, and $L_{rj'}$ such that

$$(A_{11} - \bar{\lambda})(A_{11} - \lambda)L_{2j'-r} + A_{22}B'_{2j'-r} = \sum_{l=1}^{2m} \beta_{rs}K_s + \sum_{i'=1}^{2(j'-1)} \alpha_{rl'}L_{i'}. \tag{2.13}$$

We apply the same argument as in the last proof, replacing $A - \lambda$ by $(A - \bar{\lambda})(A - \lambda)$, to conclude that if $e_{i(2j'-r)} = 0$ then $L_{2j'-r} = 0$ and $\beta_{rs} = 0$ for $1 \leq s \leq 2m$ satisfy (2.13).

This completes the proof of lemma 3' •

Proof of theorem 1: Theorem 1 follows directly from lemma 3 •

Proof of theorem 1': Theorem 1' follows directly from lemma 3' •

Proof of theorem 2: Let B' be a matrix obtained by applying lemma 3 to A. The idea of the proof is to modify B' in order to turn it into a Jordan basis B for A. We construct B by modifying the columns of B' associated with each eigenvalue λ of A. Let m be the algebraic multiplicity of λ and B_{j_i}, $1 \leq i \leq m$, be the columns of B, given by lemma 3, that form a basis for $S_{\lambda,A}$. In order to simplify the notation, we assume that $j_i = i$.

If λ is nondefective, we just take $B_i = B'_i$ for all $1 \leq i \leq m$. In this case it is clear that these columns have the correct sparsity and form the correct basis for $S_{\lambda,A}$.

The case where λ is defective is more difficult. In this case λ has geometric multiplicity one and just copying the columns of B' may not work since B must satisfy the extra condition

$$(A - \lambda)B_j = B_{j-1}, \tag{2.14}$$

for $1 < j \leq m$. In the rest of the proof we show that we can modify the columns of B' in order to achieve (2.14) while preserving the appropriate sparsity pattern.

Let us define

$$S_j = \text{Span}\{B'_i \ \text{ such that } \ i \leq j\}.$$

We leave to the reader the proof that, because of B' satifies (2.3) and (2.4), there exist $B_j \in S_j$, for $1 \leq j \leq m$, such that $B_1 = B'_1$ is an λ-eigenvector of A and such that $(A - \lambda)B_j = B_{j-1}$ if $1 < j \leq m$. Therefore, in order to complete the proof it suffices to show that for any such B we have that $b_{ij} = 0$ if $e_{ij} = 0$. Since $B_j \in S_j$ we only need to show that if $e_{ij} = 0$ then $b'_{it} = 0$ for all $1 \leq t \leq j$. This is a consequence of the following statement: If there exist i, j and $l < j$ such that $e_{ij} = 0$ and $b'_{il} \neq 0$ then the geometric multiplicity of λ as an eigenvalue of A is bigger than one.

In order to prove the last statement, let f be the first l for which $b'_{il} \neq 0$. Since $e_{ij} = 0 \Rightarrow b'_{ij} = 0$ we have that $f < j$. Let us define $v = (A-\lambda)^{j-f} B'_j$. The lemma 1 implies that $v_i = 0$. Equation (2.4) implies that $(A-\lambda)S_t \subset S_{t-1}$, thus $(A-\lambda)^{j-f} S_j \subset S_f$. Therefore $v \in S_f$, or

$$v = \sum_{t=1}^{f} \beta_t B'_t.$$

Looking at the ith coordinate of this last equation we get that $0 = v_i = \beta_f b'_{if}$, because f is the first k for which $b'_{ik} \neq 0$. This implies that $\beta_f = 0$ and as a consequence $v = (A-\lambda)^{j-f} B'_j \in S_{f-1}$. Since $S_j = S_{j-1} \oplus \mathrm{Span}\{B'_j\}$ and

$$(A-\lambda)^{j-f} S_{j-1} \subset S_{f-1},$$

we conclude that

$$(A-\lambda)^{j-f} S_j \subset S_{f-1}.$$

Since

$$(A-\lambda)^{f-1} S_{f-1} = \{0\},$$

this implies that $(A-\lambda)^{j-1} S_j = \{0\}$ and the kernel of $(A-\lambda)^{j-1}$ has dimension at least j. We then use lemma 2 to conclude that the kernel of $A-\lambda$ has dimension bigger than 1. This shows that the geometric multiplicity of λ as an eigenvalue of A is bigger than one.

As a conclusion we have that if the geometric multiplicity of λ as an eigenvalue of A is less or equal to one then we can combine the columns of B in order to obtain a Jordan basis with the appropriate sparsity pattern. The proof of theorem 2 is then complete •

Proof of theorem 2': The proof of theorem 2' is an adaptation of the proof of theorem 2. Taking this last proof into account, we only need to show that we can modify the columns of B' obtained from lemma 3' that correspond to a pair of defective complex eigenvalues, λ and $\bar{\lambda}$, with algebraic multiplicity m and geometric multiplicity one.

As before, let us assume to simplify the notation that the columns B'_j, $1 \leq j \leq 2m$, form a basis for $R_{\lambda,A}$.

Let us call

$$A_\lambda = (A-\bar{\lambda})(A-\lambda).$$

Since $A_\lambda \mathrm{Span}\{B'_1, B'_2\} = \{0\}$, we only need modify the columns $3, \ldots, 2m$ of B' in order to obtain the first $2m$ columns of a matrix B for which

$$A_\lambda \mathrm{Span}\{B_{2j-1}, B_{2j}\} = \mathrm{Span}\{B_{2j-3}, B_{2j-2}\} \tag{2.15}$$

for $1 < j \leq m$, while preserving the appropriate sparsity pattern.

Analogously to what we have done in the complex case, let us define

$$S_j = \mathrm{Span}\{B'_i \text{ such that } i \leq 2j\}$$

We leave to the reader the proof that, because of B' satifies (2.10) and (2.11), there exist $B_{2j-1}, B_{2j} \in S_j$, for $1 \leq j \leq 2$, such that $A_\lambda \mathrm{Span}\{B_{2j-1}, B_{2j}\} = \mathrm{Span}\{B_{2j-3}, B_{2j-2}\}$, if $1 < j \leq m$.

Analogously to the last proof, we complete this proof by showing that if there is i such that $e_{i(2j-r)} = 0$ and $b'_{ik} \neq 0$ for $r = 0$ or 1 and $k \leq 2j$ then λ has geometric multiplicity bigger than one.

In fact, let $f = 2h - r$ be the first value of k for which $b'_{ik} \neq 0$. Since B' was obtained from the lemma 3' we have that $e_{i(2j)} = e_{i(2j-1)} = 0$ and $b'_{i(2j)} = b'_{i(2j-1)} = 0$, thus $h < j$. As before, we consider the vectors

$$v^r = A_\lambda^{j-h} B'_{2j-r}, \qquad r = 0, 1.$$

We have that $v^r = \sum_{t=1}^{2h} \beta_{tr} B'_t$. and we can use lemma 1 to conclude that $v_i^r = 0$. This implies that

$$\beta_{(2h-1)0} b'_{i(2h-1)} + \beta_{(2h)0} b' i(2h) = 0 \tag{2.16}$$
$$\beta_{(2h-1)1} b'_{i(2h-1)} + \beta_{(2h)1} b' i(2h) = 0 \tag{2.17}$$

This shows that

$$b'_{i(2h-1)} v^1 + b'_{i(2h)} v^0 \in S_{h-1}.$$

Since $b'_{i(2h-r)} \neq 0$, we have that

$$A_\lambda^{j-h} \mathrm{Span}\{B'_{2j-1}, B'_{2j}\} \subset S_{h-1} \oplus \mathrm{Span}\{u\}$$

for some vector u. Since $A_\lambda^h S_{h-1} = \{0\}$ we have that

$$A_\lambda^{j-1} S_j = A_\lambda^{h-1} A_\lambda^{j-h} S_j = \mathrm{Span}\{A_\lambda^{h-1} u\}$$

Notice that A_j has dimension $2j$. Therefore the kernel of A_λ^{j-1} has dimension at least $2j - 1$. Since the dimension of the kernel of A_λ must be an even integer, it follows from lemma 2 that the dimension of the kernel of A_λ is bigger than two.

As a conclusion we have that if the defective eigenvalues of A have geometric multiplicity one then we can combine the columns of B' in order to obtain a Jordan basis for A. This completes the proof of theorem 2' •

3. Proof of theorem 3 and 3'

In this section we prove theorems 3 and 3'. The main tool for the proof of theorems 3 and 3' is the next lemma:

Lemma 4 *Given polynomials $P_i(z, \lambda)$, $1 \leq i \leq n+1$, $\lambda \in \mathbb{C}^n$ and $z \in \mathbb{C}^m$, with integer coefficients, there exists another polynomial $R(z)$, with integer coefficients, such that $R(z) = 0$ if and only if there exists λ for which $P_i(z, \lambda) = 0$ for all i, $1 \leq i \leq n+1$.*

The polynomial R in lemma 4 is called **resultant of the $P_i's$ with respect to λ.** A discussion of lemma 4 can be found in various books of abstract algebra, as the classical Modern Algebra, by Van der Waeden [VW].

Given $A \in \mathbb{C}^{n\times n}$ we can take $n = 1$, $P_1(V(A), \lambda)$ to be the characteristic polynomial of A and $P_2(V(A), \lambda)$ to be the derivative of P_1 with respect to λ in order to obtain the follwing lemma:

Lemma 5 *There exists a polynomial Σ_n on n^2 variables, $\Sigma_n \neq 0$, and with integer coefficients such that if $A \in \mathbb{C}^{n\times n}$ and $\Sigma_n(V(A)) \neq 0$ then A has simple eigenvalues.*

We should be carefull when using lemma 5 for sparse matrices with a given sparisty pattern S because the restriction of Σ_n to $V(\mathbb{C}(S))$ may be identically zero. That is why the hypothesis about the non-singularity of the sparsity pattern is necessary in theorems 3 and 3'. In the case of a non-singular sparsity pattern we have the lemma

Lemma 6 *If S is a non-singular structure with n nodes then Σ_S, the restriction of Σ_n to $V(\mathbb{C}(S))$, is not identically zero.*

Proof of lemma 6 Since S is a nonsingular structure, there exists a nonsingular matrix $M \in \mathbb{C}(S)$ with determinant different from zero. This implies that one term of the determinat of M is nonzero, what shows that there is a permutation $P = (p_1, \ldots, p_n)$ for which

$$\prod_{i=1}^{n} m_{ip_i} \neq 0,$$

what in implies that all the entries (i, p_i) are allowed to be nonzero. We can then break this permutations into c cycles, with the kth cycle having length l_k. Let A be the matrix defined by $b_{ij} = 0$ if $j \neq p_i$ and $b_{ip_i} = k$ if i is in the kth cycle of P. We leave to the reader the verification that the eigenvalues of A are all the numbers of the form jw, where $1 \leq j \leq c$ and w is a l_jth root of the unity. This shows that we have

$$\sum_{j=1}^{c} l_j = n$$

distinct eigenvalues and we are done with the proof of lemma 6 •

Lemma 6 implies that almost all matrices in $\mathbb{C}(S)$ are such that $\Sigma_n(V(A)) \neq 0$. In other words, almost all matrices in $\mathbb{C}(S)$ have distinct eigenvalues.

As a warm up for the rest of this section let us prove the following lemma

Lemma 7 *There exists a polynomial Π_n on n^2 variables, $\Pi_n \neq 0$, and with integer coefficients such that if $A \in \mathbb{C}^{n\times n}$ has an eigenvector v with one component $v_i = 0$ then $\Pi_n(V(A)) = 0$.*

Proof of lemma 7 Let R_i be the resultant of the characteristc polynomial of A and the characteristc polynomial of the matrix $A^{(i)}$ obtained by removing the ith row and column of A. A has an eigenvector v with $v_i = 0$ if and only if the eigenvalue λ is also an eigenvalue of $A^{(i)}$, or if λ is a root of both characteristic polynomials. Therefore, if we define $P_i(V(A)) = R_i(V(A), V(A^{(i)}))$ we have that A has an eigenvector with $v_i = 0$ if and only if $P_i(V(A)) = 0$. We complete the proof of lemma 7 by taking Π_n as the product of the P_i's and noticing that $\Pi_n(V(A)) \neq 0$ for any matrix with a full eigenvector matrix •

Lemma 7 gives a proof of theorem 3 in the case when S has only one strongly connected component. In this case we have, as expected, that the generalized eigenbasis of almost all matrices with this structure are dense.

The case of more than one strongly connected component reduces to the case where the incidence matrix of S has the form

$$E = \begin{pmatrix} E_{11} & E_{12} & \dots & E_{1k} \\ 0 & E_{22} & & E_{2k} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & E_{kk} \end{pmatrix}. \tag{3.18}$$

where the E_{ij} are either blocks of zeros or blocks of ones and k is the number of strongly connected components of S. The general case can be reduced to this case by permuting the nodes of S or, equivalently, permuting the rows and columns of the matrices A in $\mathbb{C}(S)$, and by adding the self-loops to S, since adding these nodes to struct(A) does not change rstruct(A). Therefore, from now on we will be concerned with matrices of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{13} \\ 0 & A_{22} & & A_{23} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & A_{kk} \end{pmatrix}. \tag{3.19}$$

Each block A_{ii} is square and we denote its dimension by n_i. This implies that the block A_{ij} spans the columns from

$$s_i = 1 + \sum_{t=1}^{i-1} n_t$$

to $s_{(i+1)} - 1$ and the rows from s_j to $s_{(j+1)} - 1$. The eigenvectors of A have the form $v = (V_1^T, \dots, V_k^T)^T$, where V_t is a n_t dimensional vector and there exists $1 \leq j \leq k$ such that

- $V_i = 0$ for $i > j$.
- V_j is an eigenvector of A_{jj}.
- For $1 \leq t < j$ the following recurrence relation is satisfied:

$$(A_{tt} - \lambda)V_t = - \sum_{l=t+1}^{j} A_{tl} V_l \tag{3.20}$$

It follows from this last equation that if λ is a simple eigenvalue of A and the first component of V_j, v_{s_j}, is nonzero then

$$v_i = v_{s_j} \frac{P_{ij}(V(A), \lambda)}{\det(A_{jj}^{(1)} - \lambda) \prod_{l=j+1}^{i} \det(A_{ll} - \lambda)}, \tag{3.21}$$

where P_{ij} is a polynomial with integer coefficients in $m+1$ variables and $A_{jj}^{(1)}$ is the matrix obtained by deleting the first row and column of A_{jj}.

We state now the last two lemmas that we will need to prove theorems 3 and 3'. After that we present a proof of theorems and then finish this section by presenting a proof of the lemmas.

Lemma 8 *Let S be a nonsingular structure with n nodes m edges and incidence matrix (3.18). If $e_{il} \neq 0$ for some l, $s_j \leq l < s_{(j+1)}$ then there exists a polynomial $R_{ij} \neq 0$ in m variables and with integer coefficients such that if $A \in \mathbb{C}(S)$, $R_{ij}(V(A)) \neq 0$ and λ is an eigenvalue of A_{jj}, with eigenvector v, then $v_i \neq 0$.*

Proof of lemma 8 Let us start by making sure that if $R_{ij}(V(A)) \neq 0$ then A_{jj} has simple eigenvalues and has a dense eigenvector matrix. We also want to make sure that A_{jj} has no eigenvalue in commom with the other blocks A_{ll}. This can be achieved by taking

$$R_{ij}(V(A)) = R'_{ij}(V(A)) \, \Sigma_{n_j}(V(A_{jj})) \, \Pi_{n_j}(V(A_{jj})) \prod_{i \neq j} \Delta_{lj}(V(A_{ll}), V(A_{jj})) \tag{3.22}$$

where R'_{ij} is yet to be determined and Δ_{lj} is the resultant of the characteristic polynomials of A_{ll} and A_{jj} with respect to the eigenvalue λ. The Σ_n and Π_n come from lemmas 5 and 7.

Let us then deternime R'. If $R_{ij}(V(A)) \neq 0$ and λ is an eigenvalue of A_{jj} we we can use the formula (3.21) to compute its eigenvector. It is clear than that if we take R'_{ij} to be the resultant with respect to λ of P_{ij}, in equation (3.21) and the characteristic polynomial of A_{jj} we have that for such R'_{ij} $R_{ij}(V(A)) \neq 0$ then $v_i \neq 0$.

In order to complete the proof of lemma 8, we only need to construct a matrix $A \in \mathbb{C}(S)$ for which $R_{ij}(V(A)) \neq 0$, or for which all the polynomials in the product (3.22) are nonzero.

Let us then take $A_{jj} = M$ be a the matrix corresponding to a shift i.e., $m_{i(i+1)} = 1$ for $1 \leq i < n_j$, $m_{n_j 1} = 0$ and the other entries of M are zero. Let us also take $a_{il} = 1$, where l is the value for which $e_{il} \neq 0$. Let all the other entries of A be equal to zero. It is clear that such A is in $\mathbb{C}(S)$. The eigenvalues of A_{jj} are the n_jth roots of unit and the eigenvalues of the other diagonal blocks are all equal to zero. One eigenvector matrix of A_{jj} is given by B with $b_{rs} = w_s^{r-1}$ where w_s, $1 \leq s < n_j$ are the n_jth roots of the unity. This is enough to show that all the polynomials in (3.22) are non zero at $V(A)$, except for R'_{ij}. In order to show that $R'_{ij}(V(A)) \neq 0$ we observe that if w_s is an eigenvalue of A_{jj}

with eigenvector v_i then (3.21) implies that

$$v_i = -\frac{w_s^{l-s_j}}{w_s} \neq 0.$$

Therefore $R_{ij}(V(A)) \neq 0$ and the proof of lemma 8 is complete •

Lemma 9 *Let S be a nonsingular structure with n nodes and m edges and incidence matrix* (3.18). *If $e_{rl} = e_{sl} = 1$ for some l, $s_j \leq l < s_{(j+1)}$ then there exists a polynomial $D_{rsj} \neq 0$ in m variables and with integers coefficients such that if $A \in \mathbb{C}(S)$ and $D_{rsj}(V(A)) \neq 0$ and if λ_1 and λ_2 are distinct eigenvalues of A_{jj}, with eigenvectors $v^{(1)}$ and $v^{(2)}$ then*

$$\begin{vmatrix} v_r^{(1)} & v_r^{(2)} \\ v_s^{(1)} & v_s^{(2)} \end{vmatrix} \neq 0.$$

Proof of lemma 9 This proof is similar to the proof of lemma 8. We will take

$$D_{rsj}(V(A)) = D'_{rsj}(V(A))\, \Sigma_{n_j}(V(A_{jj}))\, \Pi_{n_j}(V(A_{jj})) \prod_{i \neq j} \Delta_{lj}(V(A_{ll}), V(A_{jj})) \qquad (3.23)$$

where D'_{rsj} is yet to be determined and the other polynomials are as in (3.22).

We want to find D'_{rsj} such that if $D_{rsj}(V(A)) \neq 0$ then and λ_1 and λ_2 are distinct eigenvalues of A_{jj} with eigenvectors (for A) $v^{(1)}$ and $v^{(2)}$ then

$$\begin{vmatrix} v_r^{(1)} & v_r^{(2)} \\ v_s^{(1)} & v_s^{(2)} \end{vmatrix} \neq 0. \qquad (3.24)$$

We claim that this will be satisfied if we take D'_{rsl} to be the resultant with respect to (λ_1, λ_2) of the polynomials P_1, P_2 and P_3 below:

$$P_1(V(A), \lambda_1, \lambda_2) = C_A(\lambda_1),$$

$$P_2(V(A), \lambda_1, \lambda_2) = \frac{C_A(\lambda_1) - C_A(\lambda_2)}{\lambda_1 - \lambda_2},$$

where C_A is the characteristic polynomial of A. Notice that P_2 is a polynomial since the $\lambda_1 - \lambda_2$ term cancels out. Finally, $P_3(V(A), \lambda_1, \lambda_2)$ is the numerator of the rational function obtained by using (formally) (3.21) to evaluate (3.24).

In fact, if λ_1 and λ_2 are distinct eigenvalues of A_{jj} then $P_1(V(A), \lambda_1, \lambda_2) = 0$ and $P_2(V(A), \lambda_1, \lambda_2) = 0$. Since $D_{rsj}(V(A)) \neq 0$ we can use (3.21) to compute the eigenvectors associated with λ_1 and λ_2 and we must have (3.24), otherwise $D'_{rsj}(V(A)) = 0$. This concludes the proof that if $D_{rsj}(V(A)) = 0$ then (3.24) holds.

In order to complete the proof of lemma 9 we need to show that there exists $A \in \mathbb{C}(S)$ for which $D_{rsj} \neq 0$. Let $M \in \mathbb{C}^{n_j \times n_j}$ be a matrix of which the entries are algebraically independent over the rationals, that is, there is no polynomial $P \neq 0$ with integer coefficients, such that $P(V(M)) \neq 0$. This implies in particular that

$$\begin{vmatrix} m_{il} & m_{it} \\ m_{ul} & m_{ut} \end{vmatrix} \neq 0$$

for all i, l, u, t. Take $A_{jj} = M^{-1}DM$ where D is $n_j \times n_j$ matrix with diagonal $d_{ii} = i$.

Let us then determine the other entries of A. Since S has incidence matrix (3.18), we only need to analyze the three cases;

- a) $s_j \leq r < s < s_{(j+1)}$
- b) $r < s_j \leq s < s_{(j+1)}$
- c) $r < s < s_j$.

Let l be such that $e_{rl} = e_{sl} = 1$, $s_j \leq l < s_{(j+1)}$. In the case a) take all the remaining entries of A to be zero. In the case b) take $a_{rl} = 1$ and all take all the remaining entries of A to be zero. Finally, in case c) take $a_{rl} = a_{sl} = 1$ and take all the remaining entries of A to be zero.

Let us show that $D_{rsj}(A) \neq 0$. It is clear that

$$\Sigma_{n_j}(V(A_{jj}))\, \Pi_{n_j}(V(A_{jj})) \prod_{i \neq j} \Delta_{lj}(V(A_{ll}), V(A_{jj})) \neq 0.$$

Therefore, we only need to show that $D'_{rsj} \neq 0$. This is equivalent to showing that there is no $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ for which

$$P_1(V(A), \lambda_1, \lambda_2) = P_2(V(A), \lambda_1, \lambda_2) = P_3(V(A), \lambda_1, \lambda_2) = 0.$$

In fact, if $P_1(V(A), \lambda_1, \lambda_2) = 0$ then λ_1 is an eigenvalue of A_{jj}. If $P_2(V(A), \lambda_1, \lambda_2) = 0$, in principle, we have two possibilities a) $\lambda_2 \neq \lambda_1$ is an eigenvalue of A_{jj} or b) $\lambda_2 = \lambda_1$ and

$$\frac{\partial C_A}{\partial \lambda}(\lambda_1) = 0.$$

However b) cannot hold, since it would imply that λ_1 is a multiple eigenvalue of A_{jj} but the eigenvalues of A_{jj} are simple. As a conclusion we have that if

$$P_1(V(A), \lambda_1, \lambda_2) = P_2(V(A), \lambda_1, \lambda_2) = 0 \tag{3.25}$$

then λ_1 and λ_2 are distinct eigenvalues of A_{jj}.

We can then use (3.21) to compute the corresponding eigenvectors. We leave to the reader the verification that in the three cases above we have that the determinant in (3.24) is a rational combination of the entries of M (the eigenvector matrix of A) with coefficients different from zero. Since the entries of M are algebraicaly independent, this determinant is nonzero. As a conclusion we have that if (3.25) holds then

$$P_3(V(A), \lambda_1, \lambda_2) \neq 0,$$

what implies that $D'_{rsj}(V(A)) \neq 0$ and completes the proof of lemma 9 •

Proof of theorem 3: whithout loss of generality, we assume that A and S satisfy (3.18) and (3.19). Let $Q : \mathbb{C}^m \rightarrow \mathbb{C}$ be given by

$$Q = \Sigma_S \, \Pi_S \, R_S$$

where R_S is the product of all the R_{ij} from lemma 8 for which e_{is_j} is equal to one and

$$\Pi_S(V(A)) = \prod_{i=1}^{k} \Pi_{n_i}(V_{A_{ii}}).$$

Since each one of the polynomials in the products above is nonzero we have that $Q \neq 0$.

Let A be a matrix for which $Q(V(A)) \neq 0$. Since $\Sigma_S(V(A)) \neq 0$, A has simple eigenvalues and since $\Pi_S(V(A)) \neq 0$, the eigenvectors of the blocks A_{jj} have all their components different from zero. Therefore, for each j, we can apply (3.20) to obtain the other components of the eigenvectors of A corresponding to the eigenvalues of A_{jj}. We can then form an eigenbasis B' for A by taking the lth column of B', for $l = s_j + r < s_{j+1}$, to be the eigenvector associated with the rth eigenvalue of A_{jj} (according to some appropriate ordering). The fact that $R_S(V(A)) \neq 0$ implies that if $e_{il} \neq 0$ then $b'_{il} \neq 0$. Since any other generalized eigenbasis B for A can be obtained by scaling and permuting the columns of B', we are done with the proof of theorem 3 •

Proof of theorem 3' whithout loss of generality, we assume that A and S satisfy (3.18) and (3.19). Let $Q : \mathbb{R}^m \rightarrow \mathbb{R}$ be given by

$$Q = \Sigma_S \, \Pi_S \, D_S,$$

where Π_S is as in the proof of theorem 3 and D_S is the product of the D_{rsj} from lemma 9 for which $e_{rl} = e_{sl} = 1$ for some $s_j \leq l < s_{(j+1)}$. Since all the polynomials in the product above are not identically zero we have that $Q \neq 0$.

Let $A \in \mathbb{R}(S)$ be such that $Q(V(A)) \neq 0$. A has simple eigenvalues and if $\lambda \in \mathbb{C} - \mathbb{R}$ is an eigenvalue of A then λ and $\bar{\lambda}$ are eigenvalues of the same diagonal block, A_{jj}, say. We also have that if v is an λ-eigenvector then $v = x + iy$, where $\sqrt{i} = -1$ and x and y are real and $\bar{v} = x - iy$ is the $\bar{\lambda}$-eigenvector of A seem as a complex matrix. From now on let us assume that r and s are such that $e_{rl}e_{sl} = 1$ for some $s_j \leq l < s_{(j+1)}$. Since $D(V(A)) \neq 0$ we have that $D_{rsj} \neq 0$ and therefore

$$\begin{vmatrix} x_r + iy_r & x_r - iy_r \\ x_s + iy_s & x_s - iy_s \end{vmatrix} = -2i \begin{vmatrix} x_r & y_r \\ x_s & y_s \end{vmatrix} \neq 0.$$

Therefore, $(x_r, x_s)^T$ and $(y_r, y_s)^T$ are linearly independent. This also shows that there exist at most one value of r for which $e_{rl} = 1$ and $x_r = 0$. This implies that for $\alpha > 0$ and small enough we have that the vectors $(x_r, x_s)^T + \alpha(y_r, y_s)^T$ and $\alpha(x_r, x_s)^T + (y_r, y_s)^T$ are linearly independent and have all the components different from zero.

We use this last paragraph to construct a real eigenbasis B' for A as required by theorem 3' in the following way: Let C be a complex eigenbasis for A given by theorem

1. Since $Q(V(A)) \neq 0$ we have that A has simple eigenvalues. If the column C_j is an eigenvector associated with a real eigenvalue, take $B'_j = C_j$. If the eigenvector C_j is associated with $\lambda \in \mathbb{C} - \mathbb{R}$, let C_l be the column associated with $\bar{\lambda}$ and take $B'_j = R + \alpha I$ and $B'_l = I + \alpha R$ where R is the real part of C_j, I is its imaginary part and $\alpha > 0$ is small enough so that if $e_{rj}e_{sj} \neq 0$ then b'_{rj} and $b'_{jl} \neq 0$ and

$$\begin{vmatrix} x_r & y_r \\ x_s & y_s \end{vmatrix} \neq 0.$$

This finishes the proof of theorem 3' •

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