On some regularity problems in the theory of Navier Stokes equation

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Abstract

We present some results obtained jointly with Professor Vladimír Šverák, in the study of some problems in the regularity theory of Navier Stokes equations, and some Liouville theorems for time-dependent Stokes system in domains jointly with Professor Vladimír Šverák and Gregory Seregin. In the first part of the thesis, we prove that the regularity of weak solution (called Leray solution) depends only locally on the regularity properties of the initial data, at least for a short time. This observation is then used to prove existence of scale-invariant solutions to the Navier Stokes equation with -1-homogeneous initial data without smallness condition. The main point of the result is that it seems to be out of reach of perturbation methods, and it provides valuable insights into the possible non-uniqueness of Leray-Hopf solutions, which is a long standing open problem in this area. In the second part of the thesis, we give a simple proof of the existence of initial data with minimal L^3-norm for potential Navier-Stokes singularities, recently established in "Gallagher, I., Koch, G.S., Planchon, F., *A profile decomposition approach to the $L^\infty_t(L^3_x)$ Navier-Stokes regularity criterion*, Math. Ann. (published online July 2012)" with techniques based on profile decomposition. Our proof is more elementary, and is based on suitable splittings of initial data and energy methods. The main difficulty in the L^3 case is the lack of compactness of the imbedding $L^3_{\mathrm{loc}} \hookrightarrow L^2_{\mathrm{loc}}$. In the third part of the thesis, we characterize bounded ancient solutions to the time-dependent Stokes system with zero boundary value in various domains, including the half-space.

Contents

Chapter 1

Introduction

In the first part of the thesis, we consider the classical Cauchy problem for the incompressible Navier-Stokes equation in $R^3 \times (0,\infty)$

$$\left.\begin{array}{rcl} u_t + u\nabla u + \nabla p - \Delta u & = & 0 \\ \operatorname{div} u & = & 0 \end{array}\right\} \qquad \text{in } R^3 \times (0,\infty)\,, \tag{1.0.1}$$

$$u|_{t=0} = u_0 \qquad \text{in } R^3\,. \tag{1.0.2}$$

We recall that the problem is invariant under the scaling

$$\begin{array}{rcccl} u(x,t) & \to & u_\lambda(x,t) & = & \lambda u(\lambda x, \lambda^2 t)\,, \\ p(x,t) & \to & p_\lambda(x,t) & = & \lambda^2 p(\lambda x, \lambda^2 t)\,, \\ u_0(x) & \to & u_{0\lambda}(x) & = & \lambda u_0(\lambda x)\,, \end{array} \tag{1.0.3}$$

where $\lambda > 0$. We say that a solution u is *scale-invariant* if $u_\lambda = u$ and $p_\lambda = p$ for each $\lambda > 0$. Similarly, we say that an initial condition u_0 is scale-invariant, if $u_{0\lambda} = u_0$ for each $\lambda > 0$. This is of course the same as requiring that u_0 be $(-1)-$ homogeneous.

One of our goals in this thesis is to give a proof of the following result.

Theorem 1.0.1 *Assume u_0 is scale-invariant and locally Hölder continuous in $R^3\setminus\{0\}$ with* div $u_0 = 0$ *in R^3. Then the Cauchy problem (1.0.1), (1.0.2) has at least one scale-invariant solution u which is smooth in $R^3 \times (0,\infty)$ and locally Hölder continuous in $R^3 \times [0,\infty) \setminus \{(0,0)\}$.*

Previously this result has been known only under suitable smallness conditions on u_0, see for example [5, 22]. For small u_0 one can also prove uniqueness (in suitable function classes). It is quite conceivable that uniqueness may fail for large data. We will comment on this point in more detail below.

An important tool is local-in-space regularity estimates for weak solutions to the Navier Stokes equation near the initial time $t = 0$. It is known that if $u_0 \in L^q$ for $q \geq 3$, then the initial value problem (1.0.1), (1.0.2) has a unique local-in-time "mild solution" defined on some time interval $(0, T)$, which is smooth in $R^3 \times (0, T)$ and has in many respects the same regularity as the solution of the heat equation in $R^3 \times [0, T)$ for times close to $t = 0$, see for example [7, 15]. A natural question is under which condition this result can be localized in space: if u_0 is a quite general initial condition for which a generalized suitable weak solution u in the sense of [27] is defined and $u_0|_{B_r} \in L^q(B_r)$ for some $q > 3$, say, can we conclude that u is regular in $B_{\frac{r}{2}} \times [0, t_1)$ for some time $t_1 > 0$? We prove that this is indeed the case under quite general assumptions, which include u_0 which is in $L^2_{\rm loc}$ and $\int_{B_{x,r}} |u_0|^2 \, dx \to 0$ for $x \to \infty$. Due to non-local effects of the pressure the solution u in $B_{\frac{r}{2}} \times [0, t_1)$ may not have the same amount of regularity as the solution of the heat equation in this situation, but the non-local effects are limited to the influence of the "harmonic part of the pressure" in a suitable pressure decomposition. We can formulate this type of results somewhat loosely in the following statement.

(S) *Modulo the usual (and quite mild) non-local influences of the pressure, local regularity of the initial data propagates for at least a short time.*

We refer the reader to Section 2, Chapter 2 for precise statements. Statement (S), in addition to being of independent interest, is one of the main ingredients of our proof of Theorem 1.0.1.

Results in the direction of (S) can be found already in the classical paper [3]. More recently, related questions about vorticity propagation have been studied in [41]. Our main result concerning (S), Theorem 2.2.1, takes a somewhat different angle on (S).

Our proof of (S) (see also Theorem 2.2.1) is based on a combination of techniques from [17, 24, 26, 27, 32]. Heuristically, the main point is that one can obtain a sufficient control of the energy flux into "good regions" from the rest of the space, see Section 3. Once we know that only small amount of energy can move into the "good region" one can use (a slight modification of) partial regularity schemes in [24, 26] to prove regularity.

To prove Theorem 1.1, we seek the solution $u(x,t)$ in the form

$$u(x,t) = \frac{1}{\sqrt{t}}\, U\left(\frac{x}{\sqrt{t}}\right). \tag{1.0.4}$$

The Navier-Stokes equation for u gives

$$-\Delta U - \frac{1}{2}U - \frac{1}{2}x\nabla U + U\nabla U + \nabla P = 0, \qquad \operatorname{div} U = 0\,, \tag{1.0.5}$$

in R^3. For a scale-invariant u_0 the problem of finding a scale-invariant solution of the Cauchy problem (1.0.1), (1.0.2) is equivalent to the problem of finding a solution of (1.0.5) with the asymptotics

$$|U(x) - u_0(x)| = o\left(\frac{1}{|x|}\right), \qquad x \to \infty\,. \tag{1.0.6}$$

The problem (1.0.5), (1.0.6) is reminiscent of the classical Leray's problem of finding steady-state solution of the Navier-Stokes equation in a bounded domain, with given boundary conditions for U. We will show that one can solve this problem using the Leray-Schauder degree theory, just as in the case of the bounded domain. The non-trivial part is to find the right functional-analytic setup and establish the necessary a-priori estimates. The main difficulty is to find good estimates near ∞. This difficulty will be overcome by applying statement (S) above. Heuristically it is clear that when u is given by (1.0.4), then estimates of u near $t = 0$ are closely related to estimates of U near ∞. In Section 3 of Chapter 2 we will make this more precise.

As in the case of the bounded domains, the Leray-Schauder approach gives existence of the solutions, but not uniqueness. In the case of bounded domains one does not generically expect uniqueness for large data, and this non-uniqueness is in fact expected to be quite typical in the context of the steady Navier-Stokes, once the data is large. Could this also be the case for the problem (1.0.5), (1.0.6)? This would lead to non-uniqueness for the Cauchy problem (1.0.1), (1.0.2) with scale-invariant u_0, and by a suitable truncation of u_0 at large $|x|$ possibly also to non-uniqueness for the Leray-Hopf solutions of the Cauchy problem for $u_0 \in L^2$. We plan to address these issues in future work.

In the second part of the thesis we consider the existence of minimal blow-up initial data in $L^3(R^3)$ for Navier-Stokes equations (NSE). It is well known that if divergence free u_0 belongs to one of a number of 'critical' spaces with respect to the natural scaling

$$\begin{aligned} u(x,t) &\longrightarrow \lambda u(\lambda x, \lambda^2 t) \quad \text{for} \quad \lambda > 0, \\ u_0(x) &\longrightarrow \lambda u_0(\lambda x) \quad \text{for} \quad \lambda > 0, \end{aligned}$$

such as $L^3(R^3)$ and $\dot{H}^{1/2}(R^3)$, NSE has a unique local 'mild solution' (see [20, 22] and references therein). It is not clear whether such mild solutions exist for all time or whether singularities might develop in finite time. In [31] it was shown that there exists a minimal blowup initial data in $\dot{H}^{1/2}(R^3)$ assuming that some initial data in $\dot{H}^{1/2}$ would produce finite time singularity. A natural question is if the result of [31] can be extended to the $L^3(R^3)$ setting. The main tools of [31] are the stability of singularities and compactness for a sequence of suitable weak solutions uniformly bounded in energy norm, certain estimates of the so called 'Leray solutions' together with a uniqueness theorem of Leray solutions when we have a 'good solution'. One of the crucial points in the proof of uniqueness is the compactness of the embedding $\dot{H}^{1/2}_{loc}(R^3) \hookrightarrow L^2_{loc}(R^3)$. In the case of $L^3(R^3)$ we lose this compactness. Thus the question is whether one can avoid using the compactness. This was done in [11], by using the technique of profile decompositions. Here we present another way to overcome the difficulty. Moreover, we also recover the compactness of the set of 'minimal blow up initial data' in $L^3(R^3)$ modulo translations and scalings (see Section 3 of Chapter 3 below), while in [11] compactness was proved in a weaker Besov space. Our main tool is the following simple observation:

Lemma 1.0.1 *Let u be a Leray solution with divergence free initial data $u_0 \in L^3(R^3)$. Then there exists a nonnegative function $h(t)$ depending only on $\|u_0\|_{L^3(R^3)}$, such that $\lim_{t\to 0+} h(t) = 0$ and*

$$\|u(\cdot,t) - e^{\Delta t}u_0\|_{L^2(B_1(x_0))} \leq h(t) \quad \text{for any} \quad x_0 \in R^3, \quad \text{a.e.} \quad 0 \leq t < 1. \tag{1.0.7}$$

The proof is based on a suitable splitting of the solutions. We remark that results in similar spirit (with more detailed estimates) have been obtained in [6] for strong solutions. The important point is $h(t)$ in the lemma only depends on the L^3-norm of initial data, which gives a certain uniformity of strong continuity in $L^2_{loc}(R^3)$ at time 0 for a sequence of solutions with initial data uniformly bounded in $L^3(R^3)$. This lemma is, in fact, already sufficient to extend the arguments in [31] to the L^3 case. We present in some detail proofs of certain estimates and uniqueness results for Leray solutions which were proved in Lemarié-Rieusset [27] and used in [31], since in the situation considered here the proofs significantly simplify. We will often refer to [31], as the general ideas are similar and we provide more detailed proof of some points which in [31] were only sketched, and vice versa.

In the third part of the thesis, we prove some Liouville theorems for time-dependent Stokes system in domains. More precisely we show that any solution $u(x,t) \in L^\infty(\Omega \times (-\infty, 0))$ to

$$\left.\begin{aligned} \partial_t u - \Delta u + \nabla p &= 0 \\ \text{div } \ u &= 0 \end{aligned}\right\} \qquad \text{in } \Omega \times (-\infty, 0), \text{ and} \tag{1.0.8}$$

$$u|_{\partial\Omega} = 0 \tag{1.0.9}$$

satisfies

$u(x,t) =$

$$\begin{cases} 0 & \text{if } \Omega \subset R^n \text{ is a bounded domain and } n \geq 2, \\ u(t) & \text{if } \Omega = R^n \text{ and } n \geq 2, \\ u(t, x_n) \text{ with } u_n = 0 & \text{if } \Omega \text{ is half space } x_n > 0 \text{ and } n \geq 2, \\ a(t) + O(\frac{1}{|x|^{n-2}}) \text{ as } |x| \to \infty & \text{if } \Omega \subset R^n \text{ is an exterior domain with } n \geq 3. \end{cases} \tag{1.0.10}$$

Throughout the thesis we assume that the domains Ω have smooth boundary. One has to be somewhat careful with the definition of the boundary condition $u|_{\partial\Omega} = 0$ since a-priori we only assume u to be bounded, with no further regularity assumptions. The usual definition is the following:

Definition 1.0.1 *We call $u(x,t) \in L^\infty(\Omega \times (0,\infty))$ a very weak ancient solution to equations (1.0.8)(1.0.9) if*

$$\int_{-\infty}^0 \int_\Omega u(x,t)(\partial_t \phi + \Delta \phi)(x,t) dx dt = 0 \tag{1.0.11}$$

for any $\phi \in C_c^\infty(\overline{\Omega} \times (-\infty, 0))$ with div $\phi = 0$, $\phi|_{\partial\Omega \times (-\infty,0)} = 0$; and

$$\int_{-\infty}^0 \int_\Omega u(x,t) \nabla \psi(x,t) dx dt = 0 \tag{1.0.12}$$

for any $\psi \in C_c^\infty(\overline{\Omega} \times (-\infty, 0))$.

Solutions defined in this way are often called very weak solutions in the literature and we also use this terminology. For smooth solutions the definition coincides with the usual one, as one can easily check by integration by parts. Our main result is as follows:

Theorem 1.0.2 *Let* u *be a bounded very weak ancient solution to equation (1.0.8), and equation (1.0.9), in the sense that* u *satisfies equations (1.0.11), (1.0.12). Then* u *is given by (1.0.10).*

Remarks: The results are essentially sharp. This is obvious in the cases when Ω is R^n or a bounded domain. In the case of a half space, one can take

$$u = (u_1(t, x_n), \dots, u_{n-1}(t, x_n), 0),$$

where u_i verifies: $\partial_t u_i - \partial_n^2 u_i = f_i(t), \quad u_i|_{x_n=0} = 0$ for $1 \le i \le n-1$, where $f_i \in C_c^\infty(-\infty, 0)$. Then u is a solution to equations (1.0.8),(1.0.9). This is the example given in [34]. In exterior domain, the decay rate we obtain is as good as that of the fundamental solution of steady Stokes equations.

Our work is motivated by boundary regularity for the Navier-Stokes equations. The main interest is in the case of the half-space, the other case are included for completeness. The connection between regularity and Liouville-type theorems is of course classical. In the context of the Navier-Stokes equations it is discussed for example in [23, 33]. In a recent note [34] a bounded shear flow for unsteady Stokes equations is constructed which is not fully regular although the boundary value is zero. This example simplifies earlier constructions of [19]. The lack of boundary regularity in the time-dependent case is in contrast with the case of steady Stokes equations, see e.g. [18]. In the time-dependent Stokes equations and Navier-Stokes equations, one usually treats pressure as an auxiliary variable, determined by u. Such treatment is valid as long as we have some decay of u at spatial infinity. On the other hand, it has been known that in unbounded domains, if we do not assume decay of u, the pressure may act as an external force 'driving' the fluid motion, as in the case of [34]. In such situations, we lose boundary regularity even with the vanishing boundary conditions. In this context, our result could be understood as showing that the solutions in [34] are in some sense the only obstacle to full boundary regularity (in suitable solution classes).

Notation: We will use standard notations in the literature. For instance, $B_R(x_0)$ denotes a ball centered at x_0 with radius R in R^3, $B_R := B_R(0)$; for $z_0 = (x_0, t_0)$, $Q(R, z_0) := B_R(x_0) \times (t_0 - R^2, t_0]$, $Q_R := Q(R, (0,0))$; $\overline{A}$ denotes the closure of A, $A \Subset \mathcal{O}$ means the closure of A is a compact subset of $\mathcal{O}$, $\partial_i := \frac{\partial}{\partial x_i}$; for any f in $\mathcal{O}$, $⨍_{\mathcal{O}} f := \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} f$. We also use the following standard notations: for vectors a and v, $a \otimes v$ is a matrix with $(a \otimes v)_{ij} = a_i v_j$; for two matices a, b, $(a : b) = a_{ij} b_{ij}$ where we assume the usual Einstein summation convention; for a matrix valued function $f = (f_{ij})$, div f is a vector with $(\text{div } f)_i = (\sum_j \partial_j f_{ij})$; $(u)_{Q(R,z_0)} := ⨍_{Q(R,z_0)} u dz$, $(u)_r := (u)_{Q_r}$; $(p)_{B_R(x_0)}(t) := ⨍_{B_{R(x_0)}} p(x,t) dx$, $(p)_r(t) := (p)_{B_r}(t)$; $Y(u, p, Q(R, z_0)) := (⨍_{Q(R,z_0)} |u - (u)_{Q(R,z_0)}|^3 dz)^{1/3} + R(⨍_{Q(R,z_0)} |p - (p)_{B_R(x_0)}(t)|^{3/2} dz)^{2/3}$; $Y(u, p, Q_R) := Y(u, p, Q(R, (0,0)))$. We use C to denote an absolute and often large positive number, c a positive small absolute number, ϵ the positive small numbers, $C(\alpha, \beta, \dots)$ when the number depends on the parameters α, β, ... $C^{\alpha}_{\text{par}}(\mathcal{O})$ denotes the Hölder space with respect to the parabolic distance when $\mathcal{O}$ is a space time domain. We adopt the convention that nonessential constants can change from line to line. We use u_0 as a divergence free initial data throughout the thesis, unless defined otherwise.

Chapter 2

Local-in-space estimate near initial time for Leray solution and forward self similar solutions

In this chapter we study the local-in-space estimates near initial time for weak solutions to Navier Stokes equation, and show the existence of forward self similar solution with -1 homogeneous initial data. Our presentation is organized as follows. In section 1, we prove an 'ϵ-regularity' criteria for a generalized Navier-Stokes system; in section 2, we study the local in space near initial time smoothness of Leray solutions; in section 3 we study the asymptotics of forward self similar solutions to Stokes and Navier-Stokes equations; in section 4 we prove the existence of forward self similar solutions for large -1 homogeneous initial data.

2.1 ϵ-regularity criteria

Our goal in this section is to prove an ϵ-regularity criteria similar to that of Caffarelli-Kohn-Nirenberg for a generalized Navier-Stokes equation. Our setting is as follows: Let $\mathcal{O}$ be an open subset of $R^3_x \times R_t$, $a \in L^m_{loc}(\mathcal{O})$ with $m > 5$ (not necessarily an integer), div $a = 0$. We call a pair of functions (u, p) suitable weak solution to

$$\left.\begin{aligned} \partial_t u - \Delta u + a\cdot\nabla u + \text{div }\ (a\otimes u) + u\cdot\nabla u + \nabla p &= 0 \\ \text{div }\ u &= 0 \end{aligned}\right\} \quad (x,t)\in\mathcal{O}, \qquad (2.1.1)$$

if $u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(\mathcal{O}')$ for any open subset $\mathcal{O}' \subseteq \overline{\mathcal{O}'} \Subset \mathcal{O}$, $p \in L^{3/2}_{loc}(\mathcal{O})$, such that (u,p) satisfies equations (2.1.1) in the sense of distributions in $\mathcal{O}$, and

$$\partial_t \frac{|u|^2}{2} - \Delta \frac{|u|^2}{2} + |\nabla u|^2 + \text{div }\ \left(\frac{|u|^2}{2}(u+a)\right) + u \text{ div }\ (a\otimes u) + \text{div }\ (up) \leq 0, \quad (2.1.2)$$

in the sense of distributions. Recall that a distribution v in $\mathcal{O}$ is called nonnegative if $(v,\phi)\geq 0$ for any $\phi\in C_c^\infty(\mathcal{O})$ with $\phi \geq 0$; u div $(a\otimes u)$ is a distribution with

$$(u \text{ div }\ (a\otimes u), \phi) = -\int_{\mathcal{O}} a_i u_j \partial_j u_i \phi(x,t)dxdt - \int_{\mathcal{O}} a_i u_j u_i \partial_j \phi(x,t)dxdt.$$

The terms in 2.1.2 make sense due to the regularity assumptions and $u\in L^{10/3}_{loc}(\mathcal{O})$ by known multiplicative inequalities.
The main theorem in this section can be stated as the following:

Theorem 2.1.1 (ϵ-regularity criterion)
Let (u,p) be a suitable weak solution to equations (2.1.1) in Q_1 with $a\in L^m(Q_1)$, $m>5$, div $a=0$. *Then there exists $\epsilon_0=\epsilon_0(m)>0$ with the following property: if*

$$\left(\fint_{Q_1} |u|^3\ dxdt\right)^{1/3} + \left(\fint_{Q_1} |p|^{3/2}\ dxdt\right)^{2/3} + \left(\fint_{Q_1} |a|^m\ dxdt\right)^{1/m} \leq \epsilon_0, \qquad (2.1.3)$$

then u is Hölder continuous in $Q_{1/2}$ with exponent $\alpha=\alpha(m)>0$ and

$$\|u\|_{C^\alpha_{\text{par}}(Q_{1/2})} \leq C(m,\epsilon_0). \qquad (2.1.4)$$

Remarks: The proof of this theorem follows the general line of presentation in [8,24,26]. There are some additional complications due to the new terms $a\cdot\nabla u + \text{div }\ (a\otimes u)$ as we shall see below.

Before going into the proof of the theorem, we need the following two lemmas to be used below.

Lemma 2.1.1 *Let f be a nonnegative nondecreasing bounded function defined on $[0,1]$ with the following property:*
for any $3/4 \leq s < t < 1$ and some positive constants $0 < \theta < 1$, $M > 0$, $\beta > 0$, we have

$$f(s) \leq \theta f(t) + \frac{M}{(t-s)^{\beta}}. \tag{2.1.5}$$

Then,

$$\sup_{s \in [0,3/4]} f(s) \leq C(\theta, \beta, M), \tag{2.1.6}$$

for some postive constant depending only on θ, β, M.

Remarks: The lemma is well-known, one can find a proof for example in [9].

Our next lemma is an estimate of a generalized Stokes system.

Lemma 2.1.2 *Let $a \in L^m(Q_1)$, with $\mathrm{div}\ a = 0$ and $(\int_{Q_1} |a|^m dxdt)^{1/m} \leq M$, for some positive $M > 0$, $m > 5$, let $\lambda \in R^n$, $|\lambda| \leq M$, $f = (f_{ij}) \in L^m(Q_1)$ with $(\int_{Q_1} |f|^m dxdt)^{1/m} \leq M$. Let $u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_1)$ and $p \in L^{3/2}(Q_1)$ with*

$$\left(\int_{Q_1} |u|^3 \ dxdt\right)^{1/3} + \left(\int_{Q_1} |p|^{3/2} \ dxdt\right)^{2/3} \leq M, \tag{2.1.7}$$

Assume (u,p) satisfies

$$\left.\begin{aligned} \partial_t u - \Delta u + a \cdot \nabla u + \lambda \cdot \nabla u + \mathrm{div}\ \ (a \otimes u) + \nabla p &= \mathrm{div}\ \ f \\ \mathrm{div}\ \ u &= 0 \end{aligned}\right\} \quad \text{in} \quad Q_1, \tag{2.1.8}$$

in the sense of distributions. Then u is Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m) > 0$ and

$$\|u\|_{C^{\alpha}_{\mathrm{par}}(Q_{1/2})} \leq C(m, M). \tag{2.1.9}$$

Proof: The proof is based on a standard application of elliptic estimates. We sketch some of the details below. We first show that if $u \in L^q(Q_R)$, $q \geq 3$, then $u \in L^{\tilde{q}}(Q_{R-\delta})$, for some $\tilde{q} > q$. Thus we obtain an improvement in the regularity of u, and after finitely many such improvements we can conclude $u \in L^q$ for a sufficiently large q to use elliptic estimates and conclude that u is in a Hölder space. Here we can assume $R > 3/4$ and δ is a small positive number. Let us rewrite the equations of (u, p) as

$$\left.\begin{aligned} \partial_t u - \Delta u + \nabla p &= \text{div} \ \ (f - a \otimes u - u \otimes a - u \otimes \lambda) \\ \text{div} \ \ u &= 0 \end{aligned}\right\} \quad \text{in} \ \ Q_R.$$

By Hölder inequality, we see $h := f - a \otimes u - u \otimes a - u \otimes \lambda \in L^{\frac{mq}{m+q}}(Q_R)$. Taking divergence in the first equation, we obtain

$$\Delta p = \text{div} \ \ \text{div} \ \ (f - a \otimes u - u \otimes a - u \otimes \lambda) \quad \text{in} \ \ Q_R.$$

Set

$$p_1 = \Delta^{-1} \text{div} \ \ \text{div} \ \ ((f - a \otimes u - u \otimes a - u \otimes \lambda)\chi_{B_R}),$$

and write $p = p_1 + p_2$. Then $\Delta p_2 = 0$ in Q_R. Recall that R is in $[3/4, 1]$. By elliptic estimates, we get

$$\|p_1\|_{L^{\frac{mq}{m+q}}(Q_R)} \leq C\|h\|_{L^{\frac{mq}{m+q}}(Q_R)}.$$

Since $\frac{mq}{m+q} > 3/2$, we see p_2 verifies estimate

$$\|p_2\|_{L_t^{3/2} C_x^2(Q_{R-\delta/2})} \leq C(\delta, M),$$

with δ being a small positive number. Then,

$$\partial_t u - \Delta u = -\nabla p_1 - \nabla p_2 + \text{div} \ \ h \quad \text{in} \ \ Q_{R-\delta/2},$$

where p_1, p_2 and h satisfy above estimates. For a smooth cutoff function η with $\eta \equiv 1$ in $Q_{R-3\delta/4}$ and $\eta \equiv 0$ outside $Q_{R-\delta/2}$, set

$$\begin{aligned} v_1(\cdot, t) &= \int_{-\infty}^{t} e^{\Delta(t-s)}[-\nabla(p_1\eta)(\cdot, s) + \text{div} \ \ (h\eta)(\cdot, s)]ds, \\ v_2(\cdot, t) &= -\int_{-\infty}^{t} e^{\Delta(t-s)}\nabla(p_2\eta)(\cdot, s)ds. \end{aligned}$$

Write $u = v_1 + v_2 + v_3$. By estimates of heat equation, we see $\|v_2\|_{L^\infty(Q_{R-\delta})} \leq C(\delta, M)$. As for v_1, by Young's inequality and the properties of heat kernel, we see $v_1 \in L^r(Q_{R-\delta})$ for any $r > 0$ such that

$$\frac{1}{r} > \frac{1}{q} + \frac{1}{m} - \frac{1}{5}.$$

Since v_3 satisfies heat equation in $Q_{R-3\delta/4}$, we see v_3 is smooth in $Q_{R-\delta}$. Thus in summary, we get,

$u \in L^q(Q_R)$ implies $u \in L^r(Q_{R-\delta})$ with $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}(\frac{1}{5} - \frac{1}{m})$.

Since $m > 5$, after applying this boostraping argument for finitely many times, we can conclude $u \in L^{r_0}(Q_{5/8})$ with r_0 sufficiently large such that

$$|a||u| \in L^{\frac{m+5}{2}}(Q_{5/8}). \tag{2.1.10}$$

Then we can go back to the decompositions $v_1,\ \ v_2,\ \ $ and v_3, it is not difficult to verify that all of them are Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m)$. If we keep track of the constants in the above process, it's clear we also have the estimates claimed in the lemma. Alternatively, one can use the closed graph theorem with appropriate function spaces to obtain the estimates, we omit the details here. The lemma is proved.

Now we can return to the proof of Theorem 2.1.1. We first prove the following 'oscillation lemma', which roughly speaking asserts that if u is of 'small oscillation' in Q_1, then the oscillation is even smaller in Q_θ for $\theta < 1$. Let us recall from the notation list that $Y(u,p,Q(R,z_0)) := (⨍_{Q(R,z_0)} |u - (u)_{Q(R,z_0)}|^3 dz)^{1/3} + R(⨍_{Q(R,z_0)} |p - (p)_{B_R(x_0)}(t)|^{3/2} dz)^{2/3}$, and $Y(u,p,Q_R) := Y(u,p,Q(R,(0,0)))$.

Lemma 2.1.3 (Oscillation lemma)
Let (u,p) be a suitable weak solution to equations (2.1.1) in Q_1 with $a \in L^m(Q_1)$, $m > 5$, div $\ a = 0$*, $\|a\|_{L^m(Q_1)} \le c$, $|(u)_1| \le M$, for some small absolute number $c > 0$, and some positive number M. Then for any $\theta \in (0,1/3)$, there exists an $\epsilon = \epsilon(\theta, M, m) > 0$, $C_1(M,m) > 0$, and $\alpha = \alpha(m) > 0$ such that if*

$$Y(u,p,Q_1) + |(u)_1| \left(⨍_{Q_1} |a|^m \ dxdt \right)^{1/m} < \epsilon,$$

then

$$Y(u,p,Q_\theta) \le C_1(M,m)\theta^\alpha \left(Y(u,p,Q_1) + |(u)_1| \left(⨍_{Q_1} |a|^m \ dxdt \right)^{1/m} \right).$$

Proof: Suppose the lemma is false. Then there exists (u_i,p_i) and a_i with the following properties:

$$\begin{aligned}
&|(u_i)_1| \le M, \quad \|a_i\|_{L^m(Q_1)} \le c, \quad \text{div} \ \ a_i = 0,\\
&Y(u_i,p_i,Q_1) + |(u_i)_1| \left(\fint_{Q_1} |a_i|^m \ dxdt \right)^{1/m} = \epsilon_i \to 0+ \quad \text{as} \ \ i \to +\infty,\\
&Y(u_i,p_i,Q_\theta) > C_1(M,m)\theta^\alpha \epsilon_i,
\end{aligned}$$

and (u_i,p_i) satisfies equations (2.1.1) and inequality (2.1.2) with a replaced by a_i.
Set

$$\begin{aligned}
v_i &= \frac{u_i - (u_i)_1}{\epsilon_i},\\
q_i &= \frac{p_i - (p_i)_1(t)}{\epsilon_i},\\
f_i &= \frac{a_i \otimes (u_i)_1}{\epsilon_i}.
\end{aligned}$$

Then we have $(v_i)_1 = 0$, div div $f_i = 0$,

$$\begin{aligned}
&\left(\fint_{Q_1} |v_i|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |q_i|^{3/2} dxdt\right)^{2/3} + \left(\fint_{Q_1} |f_i|^m dxdt\right)^{1/m} \le 1, \text{and},\\
&\left(\fint_{Q_\theta} |v_i - (v_i)_\theta|^3 dxdt\right)^{1/3} + \theta \left(\fint_{Q_\theta} |q_i - (q_i)_\theta(t)|^{3/2} dxdt\right)^{2/3} > C_1(M,m)\theta^\alpha.
\end{aligned}$$

Moreover, (v_i,q_i) satisfies:

$$\left.\begin{aligned}
\partial_t v_i - \Delta v_i + (\epsilon_i v_i + a_i + (u_i)_1)\cdot \nabla v_i + \text{div} \ (a_i \otimes v_i + f_i) + \nabla q_i &= 0\\
\text{div} \ v_i &= 0
\end{aligned}\right\} \quad (2.1.11)$$

in the sense of distributions in Q_1 and

$$\begin{aligned}
&\partial_t \tfrac{|v_i|^2}{2} - \Delta \tfrac{|v_i|^2}{2} + |\nabla v_i|^2 + \text{div} \ \left(\tfrac{|v_i|^2}{2} (\epsilon_i v_i + (u_i)_1 + a_i) \right)\\
&+ v_i \ \text{div} \ \ (f_i + a_i \otimes v_i) + \text{div} \ \ v_i q_i \le 0,
\end{aligned} \qquad (2.1.12)$$

in the sense of distributions in Q_1. Here again the terms make sense due to our regularity assumptions and the interpretation of v_i div $(a_i \otimes v_i + f_i)$ as the one below inequalities (2.1.2).

Since $v_i \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q_1)$ and v_i satisfies equations (2.1.11), we can change the value of v_i on a set of measure zero such that $t \to v_i(\cdot, t)$ is continuous from $(-1, 0)$ to $L^2_w(B_1(0))$, the weak L^2 space.

From inequality (2.1.12) we obtain,

$$\begin{aligned}
&\int_{B_1(0)} \frac{|v_i|^2}{2}(x,t)\phi(x,t)dx + \int_{-1}^{t}\int_{B_1(0)} |\nabla v_i|^2 \phi(x,s)dxds \\
&\leq \int_{-1}^{t}\int_{B_1(0)} \frac{|v_i|^2}{2}(\partial_t \phi + \Delta\phi)dxds + \int_{-1}^{t}\int_{B_1(0)} \frac{|v_i|^2}{2}[(u_i)_1 + a_i + \epsilon_i v_i]\nabla\phi dxds \\
&+ \int_{-1}^{t}\int_{B_1(0)} [(f_i + a_i \otimes v_i) : (\nabla v_i \phi + v_i \otimes \nabla\phi)]dxds + \int_{-1}^{t}\int_{B_1(0)} q_i v_i \nabla\phi dxds,
\end{aligned}$$

for any $\phi \geq 0$ with $\phi \in C_c^\infty(B_1(0) \times (-1, t])$.

Let us define

$$E_i(r) = \text{ess}\sup_{-r^2 < t \leq 0} \int_{B_r} \frac{|v_i|^2}{2}(x,t)dx + \int_{-r^2}^{0}\int_{B_r} |\nabla v_i|^2(x,s)dxds, \tag{2.1.13}$$

for $0 < r < 1$. By known multiplicative inequalities we have

$$\|v_i\|^2_{L^{10/3}(Q_r)} \leq C E_i(r). \tag{2.1.14}$$

Then for any $1/2 < r_1 < r_2 \leq 1$, if we choose nonnegative test function ϕ with support in Q_{r_2} appropriately, we obtain the following estimates, with the help of the above local energy estimates, Hölder inequality, and the estimates on v_i, q_i, a_i:

$$\begin{aligned}
E(r_1) \quad &\leq \quad \frac{C}{(r_2 - r_1)^2} + \frac{C}{r_2 - r_1}\int_{Q_{r_2}} \frac{|v_i|^2}{2}\left(M + |a_i| + \epsilon_i |v_i|\right) dxds \\
&+ \quad \int_{Q_{r_2}} |f_i||\nabla v_i| + |a_i||v_i||\nabla v_i| dxdt + \frac{C}{r_2 - r_1}\int_{Q_{r_2}} |f_i||v_i| + |a_i||v_i|^2 + |q_i||v_i| dxdt \\
&\leq \quad \frac{C(M)}{(r_2 - r_1)^2} + \left(\int_{Q_{r_2}} |f_i|^2 dxdt\right)^{1/2} E(r_2)^{1/2} + \|a_i\|_{L^5(Q_{r_2})}\|v_i\|_{L^{10/3}(Q_{r_2})}\|\nabla v\|_{L^2(Q_{r_2})} \\
&\leq \quad \frac{C(M)}{(r_2 - r_1)^2} + C E(r_2)^{1/2} + C\|a\|_{L^m(Q_1)} E(r_2) \\
&\leq \quad \frac{C(M)}{(r_2 - r_1)^2} + (C\|a_i\|_{L^m(Q_1)} + 1/2) E(r_2).
\end{aligned}$$

Note that we have $\|a_i\|_{L^m(Q_1)} \leq c$ with c small. So if we choose c such that $Cc < 1/2$, then we can apply Lemma 2.1.1 and conclude that $E(3/4) \leq C(M, m)$. That is, v_i are uniformly bounded in $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{3/4})$. Thus by known embedding theorems and

the fact that v_i satisfies equations (2.1.11) (which provide crucial regularity in t), we can choose a subsequence of v_i (which we still denote as v_i), such that for some $\lambda \in R^3$, $v \in L^3(Q_{3/4})$, $q \in L^{3/2}(Q_{3/4})$ and $a, f \in L^m(Q_{3/4})$ with div $a = 0$, we have
$v_i(\cdot,t) \rightharpoonup v(\cdot,t)$ weakly in $L^2(B_{3/4})$ for every $t \in (-(\frac{3}{4})^2, 0)$,
$v_i \to v$ strongly in $L^3(Q_{3/4})$,
$q_i \rightharpoonup q$ weakly in $L^{3/2}(Q_{3/4})$,
$(u_i)_1 \to \lambda$, $a_i \rightharpoonup a$ weakly in $L^m(Q_{3/4})$,
$f_i \rightharpoonup f$ in $L^m(Q_{3/4})$.
Moreover, we have

$$\left(\int_{Q_{3/4}} |v|^3 dxdt\right)^{1/3} + \left(\int_{Q_{3/4}} |q|^{3/2} dxdt\right)^{2/3} + \left(\int_{Q_{3/4}} (|f| + |a|)^m dxdt\right)^{1/m} \le C,$$

and $|\lambda| \le M$.
From equations (2.1.11) for (v_i, q_i), we see

$$\left.\begin{aligned} \partial_t v - \Delta v + \lambda \cdot \nabla v + \text{div } (a \otimes v + v \otimes a + f) + \nabla q &= 0 \\ \text{div } v &= 0 \end{aligned}\right\} \quad \text{in } Q_{3/4}. \quad (2.1.15)$$

By Lemma 2.1.2 on generalized Stokes system, we see for some $\alpha = \alpha(m) > 0$, v is Hölder continuous in $Q_{1/2}$ with exponent α, with respect to parabolic distance. More precisely,

$$|v(x_1,t_1) - v(x_2,t_2)| \le C(M,m)\left(|x_1 - x_2| + |t_1 - t_2|^{1/2}\right)^{\alpha}.$$

Since $v_i \to v$ strongly in $L^3(Q_{3/4})$, we see

$$\left(\fint_{Q_\theta} |v_i - (v_i)_\theta|^3 dxdt\right)^{1/3} \le C(M,m)\theta^{\alpha},$$

for i sufficiently large.
Note that from equations (2.1.11) we have

$$-\Delta q_i = \text{div div } \left(\epsilon_i v_i \otimes v_i + v_i \otimes a_i + a_i \otimes v_i\right).$$

Let $q_i = q_i^1 + q_i^2$, where

$$q_i^1 = (-\Delta)^{-1}\text{div div } \left((\epsilon_i v_i \otimes v_i + v_i \otimes a_i + a_i \otimes v_i)\chi_{B_{3/4}}\right),$$

with $\chi_{B_{3/4}}$ being the characteristic function on $B_{3/4}$. Since v_i strongly converges to v in $L^3(Q_{4/3})$, we have $q_i^1 - \tilde{q}_i$ strongly converges to 0 in $L^{3/2}(Q_{4/3})$, where

$$\tilde{q}_i = (-\Delta)^{-1}\text{div div} \quad \Big((v \otimes a_i + a_i \otimes v)\chi_{B_{3/4}}\Big).$$

Since v is bounded, we obtain by estimates of Riesz operators $\tilde{q}_i \in L^m(Q_{1/2})$. Thus

$$\begin{aligned}
&\theta \left(\fint_{Q_\theta} |\tilde{q}_i|^{3/2} \, dxdt \right)^{3/2} \\
&\leq \theta \left(\fint_{Q_\theta} |\tilde{q}_i|^m \, dxdt \right)^{1/m} \leq C(M,m)\theta^{1-5/m}.
\end{aligned}$$

Therefore, for i sufficiently large, we have

$$\theta \left(\fint_{Q_\theta} |q_i^1|^{3/2} \, dxdt \right)^{3/2} \leq C(M,m)\theta^{1-5/m}.$$

By definition, $\Delta q_i^2 = 0$ in $Q_{3/4}$ and $\left(\fint_{Q_{3/4}} |q_i^2|^{3/2} \, dxdt \right)^{2/3} \leq C$. Thus by elliptic estimates, we obtain,

$$\begin{aligned}
\theta \left(\fint_{Q_\theta} |q_i^2 - (q_i^2)_\theta(t)|^{3/2} dxdt \right)^{2/3} &\leq C\theta \left(\theta^{3/2} \fint_{-\theta^2}^{0} \|\nabla q_i^2(\cdot,t)\|^{3/2}_{L^\infty(B_{5/12})} dt \right)^{2/3} \\
&\leq C\theta \left(\theta^{-1/2} \int_{-\theta^2}^{0} \int_{B_{1/2}(0)} |q|^{3/2} dxdt \right)^{2/3} \\
&\leq C\theta^{2/3}.
\end{aligned}$$

Therefore, summarizing the above, we see

$$\theta \left(\fint_{Q_\theta} |q_i - (q_i)_\theta(t)|^{3/2} dxdt \right)^{2/3} \leq C(M,m)\theta^{\min\{2/3, 1-5/m\}},$$

for i sufficiently large. This, together with the estimates on v_i, shows

$$Y(v_i, q_i, Q_\theta) \leq C(M,m)\theta^\alpha,$$

for i sufficiently large, if we choose $\alpha(m)$ sufficiently small. This contradicts $Y(v_i, q_i, Q_\theta) \geq C_1(M,m)\theta^\alpha$ if we choose $C_1(M,m) > 2C(M,m)$. Thus the lemma is proved.

The above lemma admits the following iterations.

Lemma 2.1.4 (Iteration of the oscillation lemma)
Let (u,p), M, $\epsilon(\theta,M,m)$, $C_1(M,m)$, $\alpha(m)$, c and a be as in the above lemma, with $|(u)_{Q_1}| \le M/2$. Let $\beta = \alpha/2$. Choose $\theta \in (0,1/3)$ such that $C_1(M,m)\theta^{\alpha-\beta} < 1$, and $\theta < c_1$ with $c_1 = c_1(M,m)$ being some small number. Then there exists $\epsilon_*(\theta,M,m)$ sufficiently small, such that if

$$Y(u,p,Q_1) + M\left(\fint_{Q_1} |a|^m dxdt\right)^{1/m} < \epsilon_*, \tag{2.1.16}$$

then for any $k = 1,2,\dots,$ we have

$$|(u)_{Q_{\theta^{k-1}}}| \le M, \tag{2.1.17}$$

$$Y(u,p,Q_{\theta^{k-1}}) + |(u)_{\theta^{k-1}}|\left(\fint_{Q_{\theta^{k-1}}} |a|^m dxdt\right)^{1/m}\theta^{k-1} < \epsilon_* \le \epsilon(\theta,M,m), \tag{2.1.18}$$

$$Y(u,p,Q_{\theta^k}) \le \theta^\beta\left(Y(u,p,Q_{\theta^{k-1}}) + |(u)_{\theta^{k-1}}|\left(\fint_{Q_{\theta^{k-1}}} |a|^m dxdt\right)^{1/m}\theta^{k-1}\right) \tag{2.1.19}$$

Proof: We prove the lemma by induction.
For $k=1$, the conclusion follows from Lemma 2.1.3, if we choose ϵ_* such that $\epsilon_* < \epsilon(\theta,M,p)$. Suppose the conclusion is true for $k \le k_0$, $k_0 \ge 1$, we show it remains true for $k = k_0+1$.
By induction

$$\begin{aligned}
&|(u)_{Q_{\theta^{k-1}}}| \le M,\\
&Y(u,p,Q_{\theta^{k-1}}) + |(u)_{\theta^{k-1}}|\left(\fint_{Q_{\theta^{k-1}}} |a|^m dxdt\right)^{1/m}\theta^{k-1} < \epsilon_*,\\
&Y(u,p,Q_{\theta^k}) \le \theta^\beta\left(Y(u,p,Q_{\theta^{k-1}}) + |(u)_{\theta^{k-1}}|\left(\fint_{Q_{\theta^{k-1}}} |a|^m dxdt\right)^{1/m}\theta^{k-1}\right) \le \theta^\beta\epsilon_*,
\end{aligned}$$

for all $k \le k_0$. Thus,

$$\begin{aligned}
Y(u,p,Q_{\theta^k}) &\le \theta^\beta\left(Y(u,p,Q_{\theta^{k-1}}) + \theta^{k-1}M\left(\fint_{Q_{\theta^{k-1}}} |a|^m dxdt\right)^{1/m}\right)\\
&\le \theta^\beta\left(Y(u,p,Q_{\theta^{k-1}}) + \theta^{(k-1)(1-5/m)}M\left(\fint_{Q_1} |a|^m dxdt\right)^{1/m}\right)\\
&\le \theta^\beta Y(u,p,Q_{\theta^{k-1}}) + \theta^{k\beta_1}\epsilon_*
\end{aligned}$$

for all $k \leq k_0$, with $\beta_1 = \min\{\beta, 1 - 5/m\}$. Simple calculations with a repeated use of the above inequalities show:

$$Y(u, p, Q_{\theta^k}) \leq \theta^{k\beta} Y(u, p, Q_1) + k\theta^{k\beta_1}\epsilon_*, \quad \forall k \leq k_0.$$

Thus,

$$\begin{aligned}
|(u)_{Q_{\theta^{k_0}}}| &\leq \sum_{k=1}^{k_0} |(u)_{Q_{\theta^k}} - (u)_{Q_{\theta^{k-1}}}| + |(u)_{Q_1}| \\
&\leq \sum_{k=1}^{k_0} \left(\fint_{Q_{\theta^k}} |u - (u)_{Q_{\theta^{k-1}}}|^3 dxdt \right)^{1/3} + |(u)_{Q_1}| \\
&\leq \theta^{-5/3} \sum_{k=1}^{k_0} Y(u, p, Q_{\theta^{k-1}}) + |(u)_{Q_1}| \\
&\leq \theta^{-5/3} \sum_{k=1}^{k_0} \left(\theta^{(k-1)\beta}\epsilon_* + \epsilon_*(k-1)\theta^{(k-1)\beta_1} \right) + M/2 \\
&\leq \theta^{-5/3}(1 - \theta^\beta)^{-1}\epsilon_* + \theta^{-5/3}\epsilon_* C(\beta_1, \theta) + M/2.
\end{aligned}$$

If we choose $\epsilon_* = \epsilon_*(\theta, M, m)$ to be sufficiently small, we see

$$|(u)_{Q_{\theta^{k_0}}}| \leq M.$$

Moreover,

$$\begin{aligned}
&Y(u, p, Q_{\theta^{k_0}}) + \theta^{k_0}|(u)_{\theta^{k_0}}| \left(\fint_{Q_{\theta^{k_0}}} |a|^m dxdt \right)^{1/m} \\
&\leq \theta^\beta \epsilon_* + \theta^{(1-5/m)k_0}\epsilon_* < \epsilon_*,
\end{aligned}$$

if we choose $\theta < c(M, m)$ to be sufficiently small. Set

$$\begin{aligned}
u(x,t) &= \frac{1}{\theta^{k_0}} v(\frac{x}{\theta^{k_0}}, \frac{t}{\theta^{2k_0}}), \\
p(x,t) &= \frac{1}{\theta^{2k_0}} q(\frac{x}{\theta^{k_0}}, \frac{t}{\theta^{2k_0}}), \quad \text{and} \\
a(x,t) &= \frac{1}{\theta^{k_0}} b(\frac{x}{\theta^{k_0}}, \frac{t}{\theta^{2k_0}}).
\end{aligned}$$

One can verify that (v,q) is a suitable weak solution to equations (2.1.1) with a replaced by b in Q_1. Moreover,

$$
\begin{aligned}
& Y(v,q,Q_1) + |(v)_{Q_1}| \left(\fint_{Q_1} |b|^m dxdt \right)^{1/m} \\
&= \theta^{k_0} \left(Y(u,p,Q_{\theta^{k_0}}) + \theta^{k_0} |(u)_{Q_{\theta^{k_0}}}| \left(\fint_{Q_{\theta^{k_0}}} |a|^m dxdt \right)^{1/m} \right) < \epsilon_*, \\
& \left(\int_{Q_1} |b|^m dxdt \right)^{1/m} \leq \theta^{k_0 - \frac{5k_0}{m}} \left(\int_{Q_1} |a|^m dxdt \right)^{1/m} < c.
\end{aligned}
$$

Thus, by Lemma 2.1.3, we obtain,

$$
Y(v,q,Q_\theta) \leq \theta^\beta \left(Y(v,q,Q_1) + |(v)_{Q_1}| \left(\fint_{Q_1} |b|^m dxdt \right)^{1/m} \right), \tag{2.1.20}
$$

that is,

$$
Y(u,p,Q_{\theta^{k_0+1}}) \leq \theta^\beta \left(Y(u,p,Q_{\theta^{k_0}}) + |(u)_{\theta^{k_0}}| \left(\fint_{Q_{\theta^{k_0}}} |a|^m dxdt \right)^{1/m} \theta^{k_0} \right). \tag{2.1.21}
$$

The lemma is then proved.

By translation and dilation, we obtain the following corollary.

Corollary 2.1.1 *Let (u,p) be a suitable weak solution to equations (2.1.1) in $Q(R,z_0)$, with $a \in L^m(Q(R,z_0))$, $\mathrm{div}\ \ a = 0$, $|(u)_{Q(R,z_0)}|R < M/2$, θ, β are as in the above. Then there exists $\epsilon_* = \epsilon_*(\theta, M, m)$ such that*

$$
RY(u,p,Q(R,z_0)) + RM \left(\fint_{Q(R,z_0)} |a|^m dxdt \right)^{1/m} < \epsilon_*
$$

implies, for $k \geq 1$:

$$
\begin{aligned}
& R|(u)_{Q(\theta^{k-1}R,z_0)}| \leq M, \quad \text{and} \\
& Y(u,p,Q(\theta^k R, z_0)) \\
& \leq \theta^\beta \left(Y\left(u,p,Q(\theta^{k-1}R,z_0)\right) + R\theta^{k-1} |(u)_{Q_{\theta^{k-1}R}}| \left(\fint_{Q(\theta^{k-1}R,z_0)} |a|^m dxdt \right)^{1/m} \right).
\end{aligned}
$$

Proof of Theorem 2.1.1:

It is clear if we choose ϵ_0 sufficiently small, we can apply Corollary 2.1.1 in $Q(1/2, z_0)$ for any $z_0 \in Q_{1/2}$. Note that $|(u)_{Q_{\theta^k R}}|$ is bounded and $m > 5$. Thus we can conclude

$$Y(u, p, Q(z_0, Q_{\theta^k})) \leq C(\theta, M, m)\theta^{k\alpha},$$

for some $\alpha = \alpha(m)$, where we can choose $M < 1$, $\theta = \theta(M, m) = \theta(m)$. (There is a slight abuse of notation, in particular, this α is smaller than those appearing in the oscillation lemma). In particular,

$$\left(\fint_{Q(\theta^k, z_0)} |u - (u)_{Q(\theta^k, z_0)}|^3 dxdt\right)^{1/3} \leq C(\theta, M, m)\theta^{k\alpha},$$

for all $z_0 \in Q_{1/2}$ and $k \geq 1$. By Campanato's lemma, we conclude u is Hölder continuous in $Q_{1/2}$. The theorem is proved.

In applications, it is cumbersome to have the "smallness condition" on a. We can remove this condition and get the following theorem.

Theorem 2.1.2 (Improved ϵ-regularity criteria)
Let (u, p) be a suitable weak solution to equations (2.1.1) in Q_1, with $a \in L^m(Q_1)$, div $a = 0$, *$\|a\|_{L^m(Q_1)} \leq M$, for some $M > 0$ and $m > 5$. Then there exists $\epsilon_1 = \epsilon_1(m, M) > 0$ with the following properties: if*

$$\left(\fint_{Q_1} |u|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |p|^{3/2} dxdt\right)^{2/3} \leq \epsilon_1,$$

then u is Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m) > 0$ and

$$\|u\|_{C^{\alpha}_{\text{par}}(Q_{1/2})} \leq C(m, \epsilon_1, M) = C(m, M). \tag{2.1.22}$$

Proof: Choose $0 < R_0 < 1/2$, a small positive number to be determined below. For any $z_0 = (x_0, t_0) \in Q_{1/2}$, we would like to apply a scaled version of Theorem 2.1.1 for (u, p) in $Q(R_0, z_0)$. Set

$$\begin{aligned} u(x,t) &= \frac{1}{R_0} v\left(\frac{x - x_0}{R_0}, \frac{t - t_0}{R_0^2}\right), \\ p(x,t) &= \frac{1}{R_0^2} q\left(\frac{x - x_0}{R_0}, \frac{t - t_0}{R_0^2}\right), \\ a(x,t) &= \frac{1}{R_0} b\left(\frac{x - x_0}{R_0}, \frac{t - t_0}{R_0^2}\right). \end{aligned}$$

We see that (v, q) is a suitable weak solution to equations (2.1.1) with a replaced by b in Q_1. Moreover,

$$\|b\|_{L^m(Q_1)} \leq R_0^{1-5/m} \|a\|_{L^m(Q(R_0,z_0))} \leq C R_0^{1-5/m} M,$$

and

$$\begin{aligned}
&\left(\fint_{Q_1} |v|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |q|^{3/2} dxdt\right)^{2/3} \\
&= R_0 \left(\fint_{Q(R_0,z_0)} |u|^3 dxdt\right)^{1/3} + \left(\fint_{Q(R_0,z_0)} |p|^{3/2} dxdt\right)^{2/3} R_0^2 \\
&\leq C(R_0 R_0^{-5/3} + R_0^2 R_0^{-10/3})\epsilon_1 \leq C R_0^{-4/3} \epsilon_1.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(\fint_{Q_1} |v|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |q|^{3/2} dxdt\right)^{2/3} + \left(\fint_{Q_1} |b|^m dxdt\right)^{1/m} \\
&\leq R_0^{1-5/m} M + C R_0^{-4/3} \epsilon_1.
\end{aligned}$$

Thus, if we choose R_0 such that $R_0^{1-5/m} M < \epsilon_0/2$, fix R_0, $R_0 = R_0(M, m)$ and choose ϵ_1 such that $C R_0^{-4/3} \epsilon_1 < \frac{\epsilon_0}{2}$. Then we can apply Theorem 2.1.1 to (v, q) and conclude v is Hölder continuous in $Q_{1/2}$. Scale back and collect all constants, the theorem is then proved.

2.2 Local in space near initial time smoothness of Leray solutions

In this section, we use the 'ϵ-regularity' theorem proved in the last section to study the local in space near initial time smoothness of the so called Leray solutions. Our setting is as follows.

Let $u_0 \in L^2_{loc}(R^3)$ with div $u_0 = 0$ and $\sup_{x_0 \in R^3} \int_{B_1(x_0)} |u_0|^2 dx < \infty$. We recall the definition of Leray solutions in [27], see also [17].

Definition 2.2.1 *(Leray solution) A vector field $u \in L^2_{loc}(R^3 \times [0, \infty))$ is called a Leray solution to Navier-Stokes equations with initial data u_0 if it satisfies:*

i) $\text{ess}\sup_{0\le t<R^2}\sup_{x_0\in R^3}\int_{B_R(x_0)}\frac{|u|^2}{2}(x,t)dx+\sup_{x_0\in R^3}\int_0^{R^2}\int_{B_R(x_0)}|\nabla u|^2dxdt<\infty$, *and*

$$\lim_{|x_0|\to\infty}\int_0^{R^2}\int_{B_R(x_0)}|u|^2(x,t)dxdt=0,$$

for any $R<\infty$.

ii) for some distribution p *in* $R^3\times(0,\infty)$, (u,p) *verifies Navier Stokes equations*

$$\left.\begin{aligned}\partial_t u-\Delta u+u\cdot\nabla u+\nabla p&=0\\ div\ \ u&=0\end{aligned}\right\}\quad in\ R^3\times(0,\infty), \tag{2.2.1}$$

in the sense of distributions and for any compact set $K\subseteq R^3$, $\lim_{t\to 0+}\|u(\cdot,t)-u_0\|_{L^2(K)}=0$.

iii) u is suitable in the sense of Caffarelli-Kohn-Nirenberg, more precisely, the following local energy inequality holds:

$$\int_0^\infty\int_{R^3}|\nabla u|^2\phi(x,t)dxdt\le\int_0^\infty\int_{R^3}\frac{|u|^2}{2}(\partial_t\phi+\Delta\phi)+\frac{|u|^2}{2}u\cdot\nabla\phi+pu\cdot\nabla\phi dxdt \tag{2.2.2}$$

for any smooth $\phi\ge 0$ *with* $\text{supp}\ \ \phi\Subset R^3\times(0,\infty)$. *The set of all Leray solutions starting from* u_0 *will be denoted as* $\mathcal{N}(u_0)$.

Remarks: For general existence result of Leray solutions, see [4, 21, 27]. In the case the initial data is in $L^2(R^3)$, the notion of Leray-Hopf weak solutions is often used (see [24] for example). The difference is that Leray-Hopf weak solutions belong to $L_t^\infty L_x^2\cap L_t^2\dot{H}_x^1(R^3\times[0,\infty))$. It is clear that our definition includes such solutions. Note that we impose a decay condition on u in i). This condition allows us to calculate p in the following way: $\forall(x,t)\in B_r(x_0)\times(0,t_*)\subseteq R^3\times(0,\infty)$, take a smooth cutoff function ϕ with $\phi|_{B_{2r}(x_0)}=1$, then there exists a function $p(t)$ depending only on x_0,r,t,ϕ (we suppress the dependence on x_0,r,ϕ in our notation) such that for $(x,t)\in B_r(x_0)\times(0,t_*)$

$$p(x,t)=-\Delta^{-1}\text{div div}(u\otimes u\phi)-\int_{R^3}(k(x-y)-k(x_0-y))\,u\otimes u(y,t)\,(1-\phi(y))\,dy+p(t) \tag{2.2.3}$$

where $k(x)$ is the kernel of Δ^{-1}div div.

The right hand side is well defined since u satisfies the estimates in i) and

$$|k(x-y)-k(x_0-y)|=O(\frac{1}{|x_0-y|^4})\ \text{ as }\ |y|\to\infty. \tag{2.2.4}$$

The situation is similar to extending the domain of singular integrals to bounded functions, see for example [27] and [39].

For Leray solution $u \in \mathcal{N}(u_0)$, we have the following a priori estimates, first proved in [27], see also a simpler proof in [17]. These estimates have played an important role in [17, 31], see also [32].

Lemma 2.2.1 (A priori estimate for Leray solutions)
Let $\alpha = \sup_{x_0 \in R^3} \int_{B_R(x_0)} \frac{|u_0|^2}{2}(x)dx < \infty$ *for some* $R > 0$ *and let* u *be a Leray solution with initial data* u_0*. Then there exists some small absolute number* $c > 0$ *such that for* λ *satisfying* $0 < \lambda \leq c \min\{\alpha^{-2}R^2, 1\}$*, we have*

$$\operatorname{ess}\sup_{0 \leq t \leq \lambda R^2} \sup_{x_0 \in R^3} \int_{B_R(x_0)} \frac{|u|^2}{2}(x,t)dx + \sup_{x_0 \in R^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} |\nabla u|^2(x,t)dxdt \leq C\alpha. \tag{2.2.5}$$

Remarks: Note that from the formula (4.4.3) and the a priori estimate of u, we get the following estimate for p which will be useful:

$$\sup_{x_0 \in R^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} |p - p(t)|^{3/2} dxdt \leq C\alpha^{3/2} R^{1/2}. \tag{2.2.6}$$

In the above estimate on p, more precisely, $p(t) = p_{x_0,R}(t)$. That is, we need to choose some appropriate constants $p_{x_0,R}(t)$ to satisfy the inequality. The reason is that the decay assumption on u is too weak to imply decay of p, and thus the pressure may have a large mean value for large $|x_0|$ which must be subtracted to satisfy the above estimate. The point here is that such constants depending on x_0, R, t exist. This remark is effective throughout the paper.

Now we can prove our our first important result.

Theorem 2.2.1 *Let* $u_0 \in L^2_{loc}(R^3)$ *with* $\sup_{x_0 \in R^3} \int_{B_1(x_0)} |u_0|^2(x)dx \leq \alpha < \infty$*. Suppose* u_0 *is in* $L^m(B_2(0))$ *with* $\|u_0\|_{L^m(B_2(0))} \leq M < \infty$ *and* $m > 3$*. Let us decompose*[1] $u_0 =$

[1] *Such decomposition is well-known. One can for example first localize* u_0 *using a smooth cutoff function, and then use Bogovskii's lemma to deal with the divergence-free condition. See for example [1, 14].*

$u_0^1 + u_0^2$ with div $u_0^1 = 0$, *$u_0^1|_{B_{4/3}} = u_0$,* supp $u_0^1 \Subset B_2(0)$ *and $\|u_0^1\|_{L^m(R^3)} \le C(M,m)$. Let a be the locally in time defined mild solution to Navier-Stokes equations with initial data u_0^1. Then there exists a positive $T = T(\alpha, m, M) > 0$, such that any Leray solution $u \in \mathcal{N}(u_0)$ satisfies:*
$u - a \in C^\gamma_{\text{par}}(\overline{B_{1/2}} \times [0,T])$, and $\|u - a\|_{C^\gamma_{\text{par}}(\overline{B_{1/2}} \times [0,T])} \le C(M,m,\alpha)$, for some $\gamma = \gamma(m) \in (0,1)$.

Remark: We can certainly choose $T(M,m) > 0$ such that a is defined on $R^3 \times [0, T(M,m)]$. The point of the theorem is that regularity of solution to Navier-Stokes equations depends locally on initial data, as least when Hölder continuity is concerned.
Proof: By assumption a solves the Cauchy problem for Navier-Stokes equations with initial data u_0^1 in $R^3 \times [0, T_1]$, where $T_1 = T_1(M,m)$, namely:

$$\left.\begin{aligned} \partial_t a - \Delta a + a \cdot \nabla a + \nabla \tilde{p} &= 0 \\ \text{div } a &= 0 \end{aligned}\right\} \qquad \text{in } R^3 \times (0, T_1), \tag{2.2.7}$$

$$\text{and} \quad a(\cdot, 0) = u_0^1. \tag{2.2.8}$$

It is well-known how to construct the so called mild solution to Navier-Stokes equations, see for example [20, 22, 28]. In our case, it is even simpler, since $u_0^1 \in L^m$ with $m > 3$ is subcritical with respect to the natural scaling of the equation. We can follow the arguments in the Appendix of [8], and obtain $a \in L^{\frac{5m}{3}}(R^3 \times (0, T_1)))$ with $\|a\|_{L^{\frac{5m}{3}}(R^3 \times (0,T_1))} \le CM$. Note that $\frac{5m}{3} > 5$ since $m > 3$. Moreover, by the estimates on a and by treating the nonlinear term as pertubation, we can recover a local energy estimate for a:

$$\text{ess} \sup_{0<t<T_1} \int_{B_1(x_0)} \frac{|a|^2}{2}(x,t)dx + \int_0^{T_1} \int_{B_1(x_0)} |\nabla a|^2(x,t)dxdt \le C(M,m),$$

for any $x_0 \in R^3$. Write $u = a + v$, we can verify that v satisfies:

$$\left.\begin{aligned} \partial_t v - \Delta v + v \cdot \nabla v + a \cdot \nabla v + \text{ div } (a \otimes v) + \nabla q &= 0 \\ \text{div } v &= 0 \end{aligned}\right\}, \tag{2.2.9}$$

in the sense of distributions in $R^3 \times (0, T_1)$, here $q = p - \tilde{p}$ with p being the associated pressure for u; and the local energy inequality

$$\partial_t \frac{|v|^2}{2} - \Delta \frac{|v|^2}{2} + |\nabla v|^2 + \text{div } (\frac{|v|^2}{2}(v + a)) + v \text{ div } (a \otimes v) + \text{div } (vq) \le 0,$$

in the sense of distributions in $R^3 \times (0,T_1)$;

$$\lim_{t\to 0+} \|v(\cdot,t)-u_0^2\|_{L^2(B_1(x_0))}=0, \quad \text{for any } \ x_0\in R^3.$$

Note also that $u_0^2|_{B_{4/3}}\equiv 0$. Since (u,p) satisfies the a priori estimates in Lemma 2.2.1 (and the remarks below it), $(a,\tilde{p})$ is regular, we obtain the following estimates for (v,q) in $B_2(0)\times[0,T_2)$, $T_2=T_2(\alpha,M,m)$:

$$\begin{aligned}&\operatorname*{ess\ sup}_{0<t<T_2}\frac{1}{2}\int_{B_2(0)}|v|^2(x,t)dx+\int_0^{T_2}\int_{B_2(0)}|\nabla v|^2(x,s)dxds\\&+\left(\int_0^{T_2}\int_{B_2(0)}|q|^{3/2}dxds\right)^{2/3}\le C(\alpha,m,M).\end{aligned}$$

From the local energy inequality for v, and $\lim_{t\to 0+}\|v(\cdot,t)\|_{L^2(B_{4/3}(0))}=0$, we obtain

$$\begin{aligned}&\frac{1}{2}\int_{B_{4/3}}|v|^2(x,t)\phi(x)dx+\int_0^t\int_{B_{4/3}}|\nabla v|^2(x,t)\phi(x)dxds\\&\le\int_0^t\int_{B_{4/3}}\frac{|v|^2}{2}\Delta\phi dxds+\int_0^t\int_{B_{4/3}}\frac{|v|^2}{2}(v+a)\nabla\phi dxds\\&+\int_0^t\int_{B_{4/3}}[a\otimes v:(\nabla v\phi+v\otimes\nabla\phi)]+qv\cdot\nabla\phi dxds,\end{aligned}$$

where $\phi\in C_c^\infty(B_{4/3})$, $\phi|_{B_1}\equiv 1$, $\phi\ge 0$.
By multiplicative inequalities, we know

$$\left(\int_0^{T_2}\int_{B_2(0)}|v|^{10/3}dxdt\right)^{3/10}\le C(\alpha,m,M).$$

Thus from the above, we see by Schwartz inequality:

$$\frac{1}{2}\int_{B_1(0)}|v|^2(x,t)dx+\int_0^t\int_{B_1(0)}|\nabla v|^2(x,s)dxds\le C(\alpha,m,M)t^{\min\{1/30,\frac{m-3}{5m}\}},$$

for $t<T_2$. From

$$\Delta q=-\text{div div }(v\otimes v+a\otimes v+v\otimes a),$$

we can see $q\in L^{5/3}_{loc}$. Thus

$$\left(\int_0^t\int_{B_1(0)}|q|^{3/2}dxds\right)^{2/3}\le C(\alpha,m,M)t^{1/15}.$$

The importance of these estimates lies in the fact that they provide crucial "quantitative" information on the decay in time as $t \to 0+$. Now for t_0 fixed, whose precise value is to be determined later, extend v, q to $B_1(0) \times (-1 + t_0, t_0]$ by setting $v = 0$, $q = 0$ for $(x, t) \in B_1 \times (-1 + t_0, 0]$. Extend a to $B_1(0) \times (-1 + t_0, t_0]$ by setting $a(t, x) = 0$ for $t < 0$. The extended function (v, q) is a suitable weak solution to the generalized Navier-Stokes equations (2.1.1) with the extended a in $B_1(0) \times [-1 + t_0, t_0]$. Note here that

$$\lim_{t \to 0+} \|v(\cdot, t)\|_{L^2(B_1(0))} = 0$$

plays a crucial role: it guarantees that $\partial_t v$ and $\partial_t \frac{|v|^2}{2}$ will not cause any problem across $t = 0$. Then clearly if we choose $t_0 = t_0(\alpha, m, M)$ sufficiently small, we can apply Theorem 2.1.2 and conclude v is Hölder continuous in $B_{1/2} \times [0, t_0]$, with $\|v\|_{C^\gamma_{\text{par}}(B_{1/2} \times [0, t_0])} \leq C(\alpha, m, M)$, for some $\gamma = \gamma(m)$. The theorem is proved.

For applications below, we state the following simple (and certainly well-known) lemma for heat equation without proof.

Lemma 2.2.2 *We have the following estimates:*
1. If $u_0 \in C^\beta(R^3)$ for some $\beta \in (0, 1)$, then $e^{\Delta t} u_0(x) \in C^\beta_{\text{par}}(R^3 \times [0, 1])$, with

$$\|e^{\Delta t} u_0(x)\|_{C^\beta_{\text{par}}(R^3 \times [0,1])} \leq C \|u_0\|_{C^\beta(R^3)}. \tag{2.2.10}$$

2. If $f \in L^\infty(R^3 \times [0, 1])$, then $\int_0^t \nabla e^{\Delta(t-s)} f(\cdot, s) ds \in C^\beta_{\text{par}}(R^3 \times [0, 1])$ for any $\beta \in (0, 1)$, and

$$\left\| \int_0^t \nabla e^{\Delta(t-s)} f(\cdot, s) ds \right\|_{C^\beta_{\text{par}}(R^3 \times [0,1])} \leq C(\beta) \|f\|_{L^\infty(R^3 \times [0,1])}. \tag{2.2.11}$$

The above theorem implies the following result.

Theorem 2.2.2 (Local Hölder regularity of Leray solutions)
Let $u_0 \in L^2_{loc}(R^3)$ with $\sup_{x_0 \in R^3} \int_{B_1(x_0)} |u|^2(x) dx \leq \alpha < \infty$. Suppose u_0 is in $C^\gamma(B_2(0))$ with $\|u_0\|_{C^\gamma(B_2(0))} \leq M < \infty$. Then there exists a positive $T = T(\alpha, \gamma, M) > 0$, such that any Leray solution $u \in \mathcal{N}(u_0)$ satisfies:

$$u \in C^\gamma_{\text{par}}(\overline{B_{1/4}} \times [0, T]), \quad \text{and} \quad \|u\|_{C^\gamma_{\text{par}}(\overline{B_{1/4}} \times [0,T])} \leq C(M, \alpha, \gamma). \tag{2.2.12}$$

Proof: Let us decompose $u_0 = u_0^1 + u_0^2$ with div $u_0^1 = 0$, $u_0^1|_{B_{4/3}(0)} = u_0$, supp $u_0^1 \Subset B_2(0)$ and $\|u_0^1\|_{C^\gamma(R^3)} \le CM$. Let a be the mild solution to Navier-Stokes equations with initial data u_0^1 in $R^3 \times (0, T(M))$. Then Theorem 2.2.1 implies that $u - a$ is Hölder continuous with some exponent $\beta \in (0, \gamma)$ in $B_{1/2} \times [0, T]$ with some $T = T(\alpha, \gamma, M) \in (0, T(M))$. Since the initial data u_0^1 for a is in C^γ, it is not difficult to show that $a \in C^\gamma_{\text{par}}(R^3 \times (0, T))$. Thus u is Hölder continuous with exponent β in $B_{1/2} \times [0, T(M)]$. By using a routine boostraping argument, one can improve the exponent to γ. Since this argument will be used one more time below, we sketch some of the details here for the reader's convenience. Note that u is Hölder continuous in $\overline{B_{1/2}} \times [0, T]$, thus from the representation formula (4.4.3) for p and estimates for Riesz transform, we know p is bounded in $\overline{B_{7/16}} \times [0, T]$ modulo some function $p(t)$. Now rewrite the equation for u as

$$\partial_t u - \Delta u = -\text{div } (u \otimes u) - \nabla p. \tag{2.2.13}$$

Choose a smooth cutoff function η with $\eta \equiv 1$ on $\overline{B_{3/8}}$ and $\eta \equiv 0$ outside $B_{7/16}$. Write

$$\begin{aligned} u_1(\cdot, t) &= \int_0^t e^{\Delta(t-s)}[-\text{div } (u \otimes u\eta) - \nabla(p\eta)](\cdot, s) ds, \\ u_2(\cdot, t) &= e^{\Delta t}(u_0 \eta). \end{aligned}$$

Let $u = u_1 + u_2 + u_3$. By Lemma 2.2.2 we see u_1 and u_2 are Hölder continuous with exponent β. Note that u_3 satisfies

$$\partial_t u_3 - \Delta u_3 = 0 \quad \text{in} \quad B_{3/8} \times [0, T],$$

and $u_3(\cdot, 0)|_{B_{3/8}} = 0$. Thus u_3 is smooth in $\overline{B_{1/4}} \times [0, T]$. In summary u is Hölder continuous in $\overline{B_{1/4}} \times [0, T]$ with exponent β. Then the theorem is proved.

2.3 Estimates of forward self similar solutions to Navier-Stokes and Stokes equations

In this section, we start to study forward self similar solutions to Navier-Stokes equations and a related nonhomogeneous Stokes system. Our setting is as follows.

Let u be a Leray solution with initial data u_0. Suppose $\lambda u_0(\lambda x) = u_0(x)$, $\lambda u(\lambda x, \lambda^2 t) = u(x,t)$ for any $\lambda > 0$. We also assume $u_0|_{\partial B_1(0)} \in C^\infty(\partial B_1(0))$. Then it is easy to see

$$|\nabla^\alpha u_0(x)| \le \frac{C(\alpha, u_0)}{|x|^{1+|\alpha|}}, \quad \forall \quad |\alpha| \ge 0.$$

Our first main result in this section is the following theorem.

Theorem 2.3.1 (A-priori estimate for forward self similar solutions)
Let divergence free initial data u_0 be scale-invariant, $u \in \mathcal{N}(u_0)$ be scale-invariant. Then $U(\cdot) := u(\cdot, 1)$, the solution profile at time $t = 1$, belongs to $C^\infty(R^3)$ and

$$|\partial^\alpha \left(U(x) - e^\Delta u_0(x)\right)| \le \frac{C(\alpha, u_0)}{(1+|x|)^{3+|\alpha|}}, \quad \forall \quad |\alpha| \ge 0. \tag{2.3.1}$$

Remarks: Here and below, constants $C(u_0, \dots), T(u_0, \dots)\dots$ only depend on the magnitude of u_0 and its finitely many derivatives on the unit sphere. Similar estimates with more precise asymptotics have been proved in [2] when the initial data is small in appropriate senses.

Proof: Apply Lemma 2.2.1 with $R = 1$, we see (set $M := \|u_0\|_{C(\partial B_1)}$)

$$\sup_{0<t<T_1} \frac{1}{2}\int_{B_1(0)} |u(x,t)|^2 dx + \int_0^{T_1}\int_{B_1(0)} |\nabla u(x,t)|^2 dxdt \le C(M), \quad T_1 = T_1(M). \tag{2.3.2}$$

For fixed $t_* < T_1$, with t_* to be determined later, since $u(x,t) = \frac{1}{\sqrt{t}} u(\frac{x}{\sqrt{t}}, 1) = \frac{1}{\sqrt{t}} U(\frac{x}{\sqrt{t}})$, we have

$$\begin{aligned} C(M) &\ge 1/2 \int_{B_1(0)} |u(x,t_*)|^2 dx + \int_{t_*/2}^{t_*}\int_{B_1(0)} |\nabla u(x,t)|^2 dxdt \\ &\ge \frac{\sqrt{t_*}}{2}\int_{B_{\frac{1}{\sqrt{t_*}}}(0)} |u(x,1)|^2 dx + \frac{\sqrt{t_*}}{8}\int_{B_{\frac{1}{\sqrt{t_*}}}(0)} |\nabla u(x,1)|^2 dx. \qquad (2.3.3) \\ &\ge \frac{\sqrt{t_*}}{2}\int_{B_{\frac{1}{\sqrt{t_*}}}(0)} |U(x)|^2 dx + \frac{\sqrt{t_*}}{8}\int_{B_{\frac{1}{\sqrt{t_*}}}(0)} |\nabla U(x)|^2 dx. \qquad (2.3.4) \end{aligned}$$

On the other hand, for $\forall x_0$, $|x_0| = 8$, since $u_0 \in C^\infty(B_4(x_0))$, we can apply Theorem 2.2.1 and some simple boostraping arguments to show the following:
there exists $T_2 = T_2(M) > 0$ such that $\forall$ α,

$$\|\partial_t \partial_x^\alpha u\|_{L^\infty(B_{1/8}(x_0)\times[0,T_2])} \le C(\alpha, u_0),$$

this is true for any $u \in \mathcal{N}(u_0)$.
Since $\forall\ \lambda > 0$, $\lambda u(\lambda x, \lambda^2 t)$ is also a Leray solution with initial data u_0, we obtain

$$|\lambda^{|\alpha|+1}\partial^\alpha u(\lambda x_0, \lambda^2 t) - \partial^\alpha u_0(x_0)| \le C(\alpha, u_0)t,$$

for any $\lambda > 0$, $|\alpha| \ge 0$, $t \le T_2(u_0)$.
Take $\lambda = \frac{1}{\sqrt{t}}$, we obtain $|(\frac{1}{\sqrt{t}})^{|\alpha|+1}\partial^\alpha u(\frac{x_0}{\sqrt{t}}, 1) - \partial^\alpha u(x_0)| \le C(\alpha, u_0)t$.
Setting $y = \frac{x_0}{\sqrt{t}}$, and using the homogeneity of $\partial^\alpha u_0$, we get

$$|\partial^\alpha U(y) - \partial^\alpha u_0(y)| \le \frac{C(\alpha, u_0)}{|y|^{|\alpha|+3}}, \quad \forall \ \ |y| > \frac{8}{\sqrt{T_2}}. \tag{2.3.5}$$

Now choose t_* sufficiently small, $t_* = t_*(M)$, we see from inequality (2.3.3):

$$\int_{B_{\frac{16}{\sqrt{T_2}}}} (|U(y)|^2 + |\nabla U(y)|^2) dy \le C(M).$$

Since $u(x,t)$ satisfies Navier-Stokes equations, it is easy to verify U satisfies

$$\left.\begin{aligned} -\Delta U - \tfrac{x}{2}\cdot\nabla U - \tfrac{U}{2} + U\cdot\nabla U + \nabla P &= 0 \\ \text{div} \ \ U &= 0 \end{aligned}\right\} \quad \text{in } R^3. \tag{2.3.6}$$

Thus elliptic estimates give

$$\|U(\cdot)\|_{C^k(B_{\frac{9}{\sqrt{T_2}}})} \le C(k, M).$$

These estimates, combined with the properties of heat equation, finish the proof.

For later use, let us study a nonhomogeneous Stokes system with singular forcing. Our result is the following lemma.

Lemma 2.3.1 (Decay for the linear singularly forced Stokes system)
Let $f \in C(R^3)$, suppose $v \in L_t^\infty L_x^\gamma(R^3 \times (0,T))$ for any $T < \infty$, and some $\gamma > 1$, suppose v satisfies

$$\left.\begin{aligned} \partial_t v - \Delta v + \nabla p &= t^{-3/2} f(\tfrac{x}{\sqrt{t}}) \\ \text{div} \ \ v &= 0 \end{aligned}\right\} \quad \text{in } R^3 \times (0,\infty), \tag{2.3.7}$$

for some distribution p, and $\lim_{t\to 0+}\|v(\cdot,t)\|_{L^\gamma(R^3)}=0$. *Then*
i) if $\tilde v$ also satisfies the above conditions, then $v=\tilde v$.
ii) if f satisfies $M:=\sup_{x\in R^3}(1+|x|)^3|f(x)|<\infty$, *then*

$$v(\cdot,t)=\int_0^t e^{\Delta(t-s)}P\frac{1}{s^{3/2}}f(\frac{\cdot}{\sqrt{s}})ds, \tag{2.3.8}$$

where P is the Helmholtz projection operator. Let $V(x)=v(x,1)$, then $\|V\|_{C^{1,\alpha}(B_R)}\le C(\alpha,R)M$ for $\alpha\in(0,1)$ and

$$\sup_{x\in R^3}\big((1+|x|)^2|V(x)|+(1+|x|)^3|\nabla V(x)|\big)\le CM. \tag{2.3.9}$$

iii) if f satisfies $M:=\sup_{x\in R^3}(1+|x|)^4|f(x)|<\infty$, then v is given by formula (2.3.8). Let $V(x)=v(x,1)$, then $\|V\|_{C^{1,\alpha}(B_R)}\le C(\alpha,R)M$ for $\alpha\in(0,1)$ and

$$\sup_{x\in R^3}\big((1+|x|)^3|V(x)|+(1+|x|)^4|\nabla V(x)|\big)\le CM. \tag{2.3.10}$$

Proof: The uniqueness is easy. We only need to show that if $f=0$ and for some $\gamma_1,\ \gamma_2>1$,

$$v\in L_t^\infty(L_x^{\gamma_1}+L_x^{\gamma_2})(R^3\times(0,T))\ \text{ for any }\ T>0\ \text{ and,}$$
$$\lim_{t\to 0+}\|v(t,\cdot)\|_{(L_x^{\gamma_1}+L_x^{\gamma_2})(R^3)}=0,$$

then $v=0$. Set $\omega=\text{curl }\ v$, then $\partial_t\omega-\Delta\omega=0$ in $R^3\times(0,\infty)$. Since

$$\lim_{t\to 0+}\|v(\cdot,t)\|_{(L^{\gamma_1}+L^{\gamma_2})(R^3)}=0,$$

we can extend ω to $R^3\times R$ by setting $\omega=0$ for $v<0$, and the extended function, which we still denote as ω, satisfies $\partial_t\omega-\Delta\omega=0$ in $R^3\times R$. Here again there is no problem showing that the equation is satisfied across $t=0$ since ω decays to 0 as $t\to 0+$. One can for example first mollify ω in x and the mollified function is smooth in both x and t. For the mollified function the claim is clear, then we can pass to the limit to show our claim. Since we have bounds for ω in some negative Sobolev space and $\omega=0$ for $t<0$, we conclude $\omega\equiv 0$. Thus $\Delta v=0$ in $R^3\times(0,\infty)$. Therefore $v\equiv 0$.
Let us now prove part ii) and part iii). By the uniqueness result, we only need to prove

the claimed estimates. Denote the kernel of Pe^{Δ} by $k(x)$, then $k(\cdot) \in L^{1+\epsilon}(R^3)$ for any $\epsilon > 0$. By Young's inequality it is easy to get

$$\begin{aligned}
&\| \int_0^t e^{\Delta(t-s)} P s^{-3/2} f(\frac{\cdot}{\sqrt{s}}) ds \|_{L_x^{\frac{1+\epsilon}{1-\epsilon}}} \\
&\le C(\epsilon) \int_0^t \|(t-s)^{-3/2} k(\frac{\cdot}{\sqrt{t-s}})\|_{L_x^{1+\epsilon}} \|s^{-3/2} f(\frac{\cdot}{\sqrt{s}})\|_{L_x^{1+\epsilon}} ds \\
&\le C(\epsilon) M t^{1-\frac{3\epsilon}{1+\epsilon}}.
\end{aligned}$$

Thus,

$$v(\cdot, t) = \int_0^t e^{\Delta(t-s)} P \frac{1}{s^{3/2}} f(\frac{\cdot}{\sqrt{s}}) ds.$$

Now let us prove the decay estimates of V. The proof is a direct consequence of the following inequality (which can be proved by simple calculations) with $\alpha,\ \beta = 3,\ 4$ and $R := |x| > 8$:

$$\int_0^1 \int_{R^3} \frac{1}{(|x-y| + \sqrt{1-t})^\alpha} \frac{1}{(|y| + \sqrt{t})^\beta} dydt \le \begin{cases} R^{-3} \log R & \text{if } \ \alpha = \beta = 3, \\ R^{-\alpha-\beta+4} & \text{otherwise.} \end{cases} \tag{2.3.11}$$

For part ii), we have

$$\begin{aligned}
|V(x)| &\le \int_0^1 \int_{R^3} \frac{1}{(|x-y| + \sqrt{1-t})^3} \frac{1}{(|y| + \sqrt{t})^3} dydt \le |x|^{-3} \log |x|, \\
|\nabla V(x)| &\le \int_0^1 \int_{R^3} \frac{1}{(|x-y| + \sqrt{1-t})^4} \frac{1}{(|y| + \sqrt{t})^3} dydt \le |x|^{-3},
\end{aligned}$$

for $|x| > 8$. For part iii), we have

$$\begin{aligned}
|V(x)| &\le \int_0^1 \int_{R^3} \frac{1}{(|x-y| + \sqrt{1-t})^3} \frac{1}{(|y| + \sqrt{t})^4} dydt \le |x|^{-3}, \\
|\nabla V(x)| &\le \int_0^1 \int_{R^3} \frac{1}{(|x-y| + \sqrt{1-t})^4} \frac{1}{(|y| + \sqrt{t})^4} dydt \le |x|^{-4},
\end{aligned}$$

for $|x| > 8$. Thus the decay estimates are proved. Since V also satisfies an elliptic equation:

$$\left.\begin{aligned} -\Delta V - \tfrac{x}{2} \cdot \nabla V - \tfrac{V}{2} + \nabla P &= f \\ \text{div } \ V &= 0 \end{aligned}\right\} \quad \text{in } R^3, \tag{2.3.12}$$

the estimates in $B_C(0)$ is simple.

2.4 Existence of forward self similar solution for large initial data

In this section we shall prove the existence of scale-invariant solutions without assuming the initial data is small. Let us briefly recall the general strategy of the proof already outlined in the introduction before going into technical details. For any scale-invariant initial data u_0, we introduce a parameter $\mu \in [0,1]$. When μ is small, the existence and uniqueness of a scale-invariant solution u_μ with initial data μu_0 is known. Then we increase μ. Theorem 2.3.1 provides us with crucial a priori estimates sufficient for defining Leray-Schauder degree (we will introduce the precise definitions of the relevant operators and function spaces below) for all $\mu \in [0,1]$, which is continuous in μ and is equal to 1 when μ is small. Since the degree is integer-valued, we can conclude that the degree is identically 1 for all $\mu \in [0,1]$, which by the degree theory implies existence of scale-invariant solutions for initial data μu_0 with $\mu \in [0,1]$. Thus there exists a scale-invariant solution with initial data u_0 (taking $\mu = 1$). Let us emphasize that the asymptotics for large x obtained in Theorem 2.3.1 is important, since it ensures that certain operator defined below is compact (note that we work in the whole space).

Theorem 2.4.1 *Let $u_0 \in C^\infty(R^3\backslash\{0\})$ satisfy $\lambda u_0(\lambda x) = u_0(x)$ for all $\lambda > 0$, $\mathrm{div}\ \ u_0 = 0$. Then there exists $u \in C^\infty(R^3 \times (0,\infty))$, with $\lambda u(\lambda x, \lambda^2 t) = u(x,t)$ for all $\lambda > 0$, and $u \in \mathcal{N}(u_0)$, that is, u satisfies*

$$\left.\begin{aligned} \partial_t u - \Delta u + u\cdot\nabla u + \nabla p &= 0 \\ \mathrm{div}\ \ u &= 0 \end{aligned}\right\} \quad \textit{in } R^3 \times (0,\infty) \textit{ for some } p. \tag{2.4.1}$$

Moreover, let $U(x) = u(x,1)$, then

$$|\partial^\alpha \left(U(x) - e^{\Delta} u_0(x)\right)| \le \frac{C(\alpha, u_0)}{(1+|x|)^{3+|\alpha|}}.$$

Proof: By Theorem 2.3.1, it suffices to show there exists $u \in \mathcal{N}(u_0)$ with the scaling

$$\lambda u(\lambda x, \lambda^2 t) = u(x,t) \quad \text{for all } \lambda > 0. \tag{2.4.2}$$

Denote

$$X = \{V \in C^1(R^3) : \mathrm{div}\ \ V = 0, \quad \sup_{x\in R^3}\left((1+|x|)^2|V(x)| + (1+|x|)^3|\nabla V(x)|\right) < \infty\}. \tag{2.4.3}$$

For any $V \in X$, we define a natural norm

$$\|V\|_X = \sup_{x \in R^3} \left((1+|x|)^2|V(x)| + (1+|x|)^3|\nabla V(x)|\right). \tag{2.4.4}$$

Set $U_0 = e^{\Delta}u_0$. Introduce a parameter $\mu \in [0,1]$, set $U_{0\mu} = \mu U_0$. We will follow Leray's method to prove the existence of $u_\mu \in \mathcal{N}(\mu u_0)$ with $\lambda u_\mu(\lambda x, \lambda^2 t) = u_\mu(x,t)$ for all $\lambda > 0$ and $\mu \in [0,1]$. Due to the scaling invariance of $u_\mu(x,t)$, we are essentially seeking the profile function $U_\mu(x) = u_\mu(x,1)$. For ease of notation, we will suppress the dependence of u and U on μ below. $U(x)$ satisfies

$$\left.\begin{aligned} -\Delta U + U\cdot\nabla U - \tfrac{U}{2} - \tfrac{x}{2}\cdot\nabla U + \nabla P &= 0 \\ \text{div } U &= 0 \end{aligned}\right\} \quad \text{in } R^3, \tag{2.4.5}$$

and the asymptotics $|U(x) - U_{0\mu}(x)| = o(\frac{1}{|x|})$ as $|x| \to \infty$. We will solve U in the following form

$$U = U_{0\mu} + V, \quad \text{where} \quad V \in X. \tag{2.4.6}$$

It is clear $u(x,t) = \frac{1}{\sqrt{t}}U(\frac{x}{\sqrt{t}}) \in \mathcal{N}(\mu u_0)$ if and only if $U(x)$ satisfies the above elliptic system and $U(x) = U_{0\mu} + V$ for some $V \in X$, by Theorem 2.3.1. Thus we have reduced the problem to finding $V \in X$, with

$$\left.\begin{aligned} -\Delta V + V\cdot\nabla V + U_{0\mu}\cdot\nabla V + V\cdot\nabla U_{0\mu} - \tfrac{V}{2} - \tfrac{x}{2}\cdot\nabla V + \nabla P &= -U_{0\mu}\cdot\nabla U_{0\mu} \\ \text{div } V &= 0 \end{aligned}\right\} \tag{2.4.7}$$

in R^3. We rewrite the above as:

$$-\Delta V - \frac{V}{2} - \frac{x}{2}\cdot\nabla V + \nabla P = -V\cdot\nabla V - U_{0\mu}\cdot\nabla V - V\cdot\nabla U_{0\mu} - U_{0\mu}\cdot\nabla U_{0\mu}. \tag{2.4.8}$$

Since $V \in X$, V satisfies the above equation if and only if $v(x,t) := \frac{1}{\sqrt{t}}V(\frac{x}{\sqrt{t}})$ satisfies

$$\left.\begin{aligned} \partial_t v - \Delta v + \nabla p &= t^{-3/2}F(\tfrac{x}{\sqrt{t}}) \\ \text{div } v &= 0 \\ v(\cdot,0) &= 0 \end{aligned}\right\}, \tag{2.4.9}$$

where

$$F = -V\cdot\nabla V - U_{0\mu}\cdot\nabla V - V\cdot\nabla U_{0\mu} - U_{0\mu}\cdot\nabla U_{0\mu} \tag{2.4.10}$$

has the decay properties in Lemma 2.3.1. Thus for such F, equation (2.4.9) is uniquely solvable, we denote the solution profile at time 1 as $\mathcal{G}(F) \in X$. This enables us to consider the following equivalent formulation,

$$\text{find } V \in X \text{ with } \; V = \mathcal{G}(-V \cdot \nabla V - U_{0\mu} \cdot \nabla V - V \cdot \nabla U_{0\mu} - U_{0\mu} \cdot \nabla U_{0\mu}). \qquad (2.4.11)$$

Let $K : X \times [0,1] \to X$ be defined as:

$$\forall \; V \in X, \;\; \mu \in [0,1], \;\; K(V,\mu) := \mathcal{G}(U_{0\mu}\nabla U_{0\mu}) + \mathcal{G}(U_{0\mu}\nabla V + V\nabla U_{0\mu} + V\nabla V). \; (2.4.12)$$

Note $\mathcal{G}(U_{0\mu}\nabla U_{0\mu}) = \mu^2\mathcal{G}(U_0\nabla U_0)$ has a one-dimensional range. The second term by estimates in Lemma 2.3.1 is compact. The compactness is due to the local $C^{1,\alpha}$ estimates and the fast decay at inifinity. Note also that K is quadratic in V in the highest order term. Using this fact one can check that K is differentiable in V and μ (we omit the routine details). Thus we conclude $K \in C^1(X \times [0,1])$ is compact. Therefore we are reduced to solve the following abstract problem:

$$\text{find } V \in X, \text{ such that } V + K(V,\mu) = 0, \text{ where } \mu \in [0,1].$$

At this stage, we are in a position to apply Leray's method, see for example [29]. We need the following conditions to be verified:

1. Solvability for μ small. This is already done, for example in [5, 16], note that it also follows from a simple implicit function theorem in our formulation. In the language of Leray Schauder degree theory, we can verify $d(I + K(\cdot,\mu), B_M(0), 0) = 1$ for μ small and some fixed $M > 0$.
2. A priori estimate for solutions. This is done, in Theorem 2.3.1.
3. Compactness and continuity of K. This follows from the estimates of $\mathcal{G}$.

Thus we can apply Leray's method, and conclude that for each $\mu \in [0,1]$, there exists a solution $V \in X$ to $V + K(V,\mu) = 0$. Take $\mu = 1$, the theorem is proved.

With the existence theorem for smooth (away from 0) -1 homogeneous initial data, we can obtain existence results for not so smooth initial data. We illustrate the method with Hölder continuous (away from 0) initial data, although more general initial data can be considered.

Theorem 2.4.2 *Let $u_0 \in C^\alpha_{\text{loc}}(R^3 \setminus \{0\})$ with $\alpha \in (0,1)$, $\lambda u_0(\lambda x) = u_0(x)$ for all $\lambda > 0$, and* div $u_0 = 0$ *in R^3. Denote $M = \|u_0\|_{C^\alpha(\partial B_1)}$. Then there exists $u \in \mathcal{N}(u_0)$, and u satisfies $u(x,t) = \lambda u(\lambda x, \lambda^2 t)$ for all $\lambda > 0$. Moreover, let $U(x) = u(x,1)$. Then $U \in C^\infty(R^3)$ with*

$$|U(x) - e^\Delta u_0(x)| \leq \frac{C(M)}{(1+|x|)^{1+\alpha}}. \tag{2.4.13}$$

Proof: Let us choose $u_0^\epsilon \in C^\infty(R^3 \setminus \{0\})$ with $\lambda u_0^\epsilon(\lambda x) = u_0^\epsilon(x)$ for all $\lambda > 0$, div $u_0^\epsilon = 0$ in R^3, $\|u_0^\epsilon\|_{C^\alpha(\partial B_1(0))} \leq CM$, and $\|u_0^\epsilon - u_0\|_{C(\partial B_1)} \to 0$ as $\epsilon \to 0+$. We can construct such u_0^ϵ by first mollifying u_0 on the unit sphere and then using the scaling invariance and applying Helmholtz projection operator to form u_0^ϵ. We only note that the scaling invariance is preserved by the Helmholtz projection. By Theorem 2.4.1, we can find $u^\epsilon \in \mathcal{N}(u_0^\epsilon)$ with $\lambda u^\epsilon(\lambda x, \lambda^2 t) = u^\epsilon(x,t)$, for all $\lambda > 0$. Let $U^\epsilon(x) = u^\epsilon(x,1)$, then $u^\epsilon(x,t) = \frac{1}{\sqrt{t}} U^\epsilon(\frac{x}{\sqrt{t}})$. For any $x_0 \in R^3$ with $|x_0| = 8$, since $u_0^\epsilon \in C^\alpha(B_4(x_0))$ with $\|u_0^\epsilon\|_{C^\alpha(B_4(x_0))} \leq C(M)$, by Theorem 2.2.2, there exists $T(M) > 0$, such that $u^\epsilon \in C^\alpha_{\text{par}}(B_{1/2} \times [0, T(M)])$ and $\|u^\epsilon\|_{C^\alpha_{\text{par}}(B_{1/2} \times [0,T(M)])} \leq C(M)$. Thus,

$$|\frac{1}{\sqrt{t}} U^\epsilon(\frac{x_0}{\sqrt{t}}) - u_0^\epsilon(x_0)| \leq C(M) t^{\alpha/2}, \quad \text{for } \ t < T(M). \tag{2.4.14}$$

By the homogeneity of u_0^ϵ, we get

$$|U^\epsilon(\frac{x_0}{\sqrt{t}}) - u_0^\epsilon(\frac{x_0}{\sqrt{t}})| \leq C(M) t^{1/2+\alpha/2}, \quad \text{for } \ t < T(M). \tag{2.4.15}$$

Notice that $|x_0| = 8$ is arbitrary, we get

$$|U^\epsilon(x) - u_0^\epsilon(x)| \leq \frac{C(M)}{|x|^{1+\alpha}} \quad \text{for } \ |x| > C_1(M). \tag{2.4.16}$$

Moreover, by following the same arguments in the proof of Theorem 2.3.1, we can obtain

$$\|U^\epsilon\|_{C^k(B_R(0))} \leq C(k, M, R) \quad \text{for } \ \forall R > 0. \tag{2.4.17}$$

By combining the above estimates and using elementary properties of heat equation, we get

$$|U^\epsilon(x) - e^\Delta u^\epsilon(x)| \leq \frac{C(M)}{(1+|x|)^{1+\alpha}}, \quad \text{for } \ x \in R^3. \tag{2.4.18}$$

Note also that since u^ϵ satisfies the Navier-Stokes equations for $t > 0$, U^ϵ satisfies

$$\left.\begin{aligned} -\Delta U^\epsilon + U^\epsilon \cdot \nabla U^\epsilon - \tfrac{x}{2} \cdot \nabla U^\epsilon - \tfrac{U^\epsilon}{2} + \nabla P^\epsilon &= 0 \\ \text{div } \ U^\epsilon &= 0 \end{aligned}\right\} \text{in } \ R^3.$$

By the estimates on U^ϵ, we can pass to a subsequence $\epsilon_i \to 0+$, such that $U^{\epsilon_i} \to U$ in $C^2(B_R(0))$ for all $R > 0$. Thus U satisfies

$$\left.\begin{aligned} -\Delta U + U\cdot\nabla U - \tfrac{x}{2}\cdot\nabla U - \tfrac{U}{2} + \nabla P &= 0 \\ \text{div }\ U &= 0 \end{aligned}\right\} \text{in } \ R^3, \tag{2.4.19}$$

and

$$|U(x) - e^{\Delta}u_0(x)| \le \frac{C(M)}{(1+|x|)^{1+|\alpha|}} \quad \text{for all} \quad x \in R^3. \tag{2.4.20}$$

Setting $u(x,t) = \frac{1}{\sqrt{t}}U(\frac{x}{\sqrt{t}})$, we can easily verify that u satisfies all the conditions in our theorem.

Chapter 3

Minimal $L^3(R^3)$ initial data for potential Navier Stokes singularities

In this chapter we show the existence of a minimal blow-up initial data in $L^3(R^3)$ provided there exists an initial data in $L^3(R^3)$ which produces singular solution. Our presentation is organized as follows. In section 1, we study the weak solutions called Leray solutions, recall some well known lemmas and obtain localized energy estimates for such solutions; in section 2 we study the blow up behavior of mild solutions and introduce our main technical Lemma 3.2.2; in section 3, we prove our main theorem.

3.1 Leray solutions

In [28] J.Leray showed, among many other important results, the existence of a globally defined weak solution $u(x,t)$ to (2.2.1) with $u_0 \in L^2$ using a priori energy estimates. The regularity and uniqueness of such solutions are open. Later, Calderon [4] generalized Leray's theory of weak solutions to the case $u_0 \in L^p$. In [27], Lemarié-Rieusset constructed global weak solutions with initial data in the space of uniformly locally integrable functions with certain decay at infinity. Here, we recall some results in [27] and present their proofs in some detail for the sake of completeness.

Definition 3.1.1 *(Leray solution) $u \in L^2_{loc}(R^3 \times [0,\infty))$ is called a Leray solution to NSE with initial data u_0 if it satisfies:*

i) $\operatorname{ess\,sup}_{0\le t<R^2} \sup_{x_0\in R^3} \int_{B_R(x_0)} \frac{|u|^2}{2}(x,t)dx + \sup_{x_0\in R^3}\int_0^{R^2}\int_{B_R(x_0)} |\nabla u|^2 dxdt < \infty$*, and*

$$\lim_{|x_0|\to\infty} \int_0^{R^2}\int_{B_R(x_0)} |u|^2(x,t)dxdt = 0, \tag{3.1.1}$$

for any $R<\infty$.

ii) for some distribution p in $R^3\times(0,\infty)$, (u,p) verifies NSE (2.2.1) in the sense of distributions and for any compact set $K\subseteq R^3$, $\lim_{t\to 0+}\|u(\cdot,t)-u_0\|_{L^2(K)}=0$.

iii) u is suitable in the sense of Caffarelli-Kohn-Nirenberg, more precisely, the following local energy inequality holds:

$$\int_0^\infty\int_{R^3} |\nabla u|^2\phi(x,t)dxdt \le \int_0^\infty\int_{R^3} \frac{|u|^2}{2}(\partial_t\phi+\Delta\phi)+\frac{|u|^2}{2}u\cdot\nabla\phi+pu\cdot\nabla\phi dxdt, \tag{3.1.2}$$

for any smooth $\phi\ge 0$ with $\operatorname{supp}\ \phi \Subset R^3\times(0,\infty)$*. The set of all Leray solutions starting from u_0 will be denoted as $\mathcal{N}(u_0)$.*

Remarks: In the case the initial data is in $L^2(R^3)$, the notion of Leray-Hopf weak solution is often used (see [24] for example). The difference is that Leray-Hopf weak solution is in $L^\infty_t L^2_x \cap L^2_t\dot H^1_x(R^3\times[0,\infty))$. The definition above is a modification of the definition found in [27] by adding the decay condition (3.1.1) to ensure uniqueness. An alternative definition, where (3.1.1) is replaced by a condition on the pressure, can be found in [21]. The existence of Leray solutions for very general inital data is proved in [27]. In our situation with initial data u_0 in L^3 we can follow [4, 31] or see section 4 below. We note that condition (3.1.1) allows us to calculate p in the following way: $\forall B_r(x_0)\times(0,t_*)\subseteq R^3\times(0,\infty)$, take a smooth cutoff function ϕ with $\phi|_{B_{2r}(x_0)}=1$, then there exists a function $p(t)$ depending only on x_0,r,t,ϕ (we suppress the dependence on x_0,r,ϕ in our notation) such that for $(x,t)\in B_r(x_0)\times(0,t_*)$

$$p(x,t) = -\Delta^{-1}\mathrm{div}\ \mathrm{div}\,(u\otimes u\phi) - \int_{R^3} (k(x-y)-k(x_0-y))\,u\otimes u(y,t)\,(1-\phi(y))\,dy + p(t), \tag{3.1.3}$$

where $k(x)$ is the kernel of $\Delta^{-1}\mathrm{div}\ \mathrm{div}$.

The right hand side is well defined since u satisfies the estimates in i) and

$$|k(x-y)-k(x_0-y)| = O\left(\frac{1}{|x_0-y|^4}\right) \quad \text{as} \quad |y|\to\infty. \tag{3.1.4}$$

The situation is similar to extending the domain of singular integrals to bounded functions, see for example [27] and [39]. There are many other possibilities in choosing a decay condition (such as by imposing conditions on the pressure used in [**?**], already mentioned above). Condition (3.1.1) works well for our purposes here. It should be noted that some decay of u at spatial infinity is needed if we wish p to be given as in (4.4.3). For instance, as observed by many authors, if we take $u(x,t) = f(t)$, $p(x,t) = -f'(t)\cdot x$, then (u,p) verifies the conditions for the definition of Leray solution except the decay requirement. In this case, p is not given as the above formula though the right hand side is still well defined.

We will use the following version of the local energy estimates due to Lemarié-Rieusset [27]:

Lemma 3.1.1 *(A priori estimate for Leray solution)*
Let $\alpha = \sup_{x_0\in R^3}\int_{B_R(x_0)}\frac{|u_0|^2}{2}(x)dx < \infty$ *for some* $R>0$ *and let* u *be a Leray solution with initial data* u_0. *Then for* λ *satisfying* $0<\lambda\le\epsilon_0\min\{\alpha^{-2}R^2,1\}$ *with some small absolute number* $\epsilon_0>0$, *we have*

$$\operatorname{ess}\sup_{0\le t\le\lambda R^2}\sup_{x_0\in R^3}\int_{B_R(x_0)}\frac{|u|^2}{2}(x,t)dx + \sup_{x_0\in R^3}\int_0^{\lambda R^2}\int_{B_R(x_0)}|\nabla u|^2(x,t)dxdt \le C\alpha. \tag{3.1.5}$$

Proof: Since u is suitable and $u(t,\cdot)$ converges to u_0 locally in L^2 as $t\to 0+$, we obtain by local energy estimate:

$$\begin{aligned}
&\int_{R^3}\frac{|u|^2}{2}(x,t)\phi(x-x_0)dx + \int_0^t\int_{R^3}|\nabla u|^2\phi(x-x_0)dxds\\
&\le \int_{R^3}\frac{|u|^2}{2}(x,0)\phi(x-x_0)dx + \int_0^t\int_{R^3}\frac{|u|^2}{2}\Delta\phi(x-x_0)dxds\\
&\quad + \int_0^t\int_{R^3}\frac{|u|^2}{2}u\cdot\nabla\phi(x-x_0) + pu\cdot\nabla\phi(x-x_0)dxds,
\end{aligned}$$

for a.e. $t>0$, where ϕ is a nonnegative smooth cutoff function with $\phi=1$ in $B_R(0)$, supp $\phi \Subset B_{2R}(0)$ and $|\nabla\phi|\le\frac{C}{R}$. For $\lambda<1$, denote

$$A(\lambda) := \operatorname{ess}\sup_{0\le t\le\lambda R^2}\sup_{x_0\in R^3}\int_{R^3}\frac{|u|^2}{2}(x,t)\phi(x-x_0)dx + \sup_{x_0\in R^3}\int_0^{\lambda R^2}\int_{R^3}|\nabla u|^2(x,t)\phi(x-x_0)dxdt. \tag{3.1.6}$$

Sobolev embedding theorem gives

$$\sup_{x_0\in R^3}\int_0^{\lambda R^2}\int_{B_{2R}(x_0)}|u|^3(x,t)dxdt \le CA(\lambda)^{3/2}R^{1/2}\lambda^{1/4} \quad \text{if } \lambda\le 1,$$

for some absolute number $C>0$. We apply formula (4.4.3) to p in $B_{2R}(x_0)\times(0,\lambda R^2)$:

$$p(x,t) = -\Delta^{-1}\text{div}\;\;\text{div}\,(u\otimes u\psi) - \int_{R^3}(k(x-y)-k(x_0-y))\,(u\otimes u(y,t)(1-\psi(y)))\,dy + p(t),$$

where ψ is a smooth cutoff function with $\psi|_{B_{4R}(x_0)}=1$, $0\le\psi\le 1$, ψ vanishes outside $B_{8R}(x_0)$ and $|\nabla\psi|\le\frac{C}{R}$. Then by elliptic estimates and

$$|k(x-y)-k(x_0-y)| \le \frac{CR}{|x_0-y|^4} \quad \text{for} \quad |x_0-y|\ge 4R, |x-x_0|\le 2R, \tag{3.1.7}$$

we easily obtain

$$\begin{aligned}
&\|p(x,t)-p(t)\|_{L^{3/2}(B_{2R}(x_0)\times(0,\lambda R^2))}\\
&\le C\left(\|u\|^2_{L^3(B_{8R}(x_0)\times(0,\lambda R^2))} + \|R^{-3}A(\lambda)\|_{L^{3/2}(B_{2R}(x_0)\times(0,\lambda R^2))}\right)\\
&\le C\lambda^{1/6}A(\lambda)R^{1/3}, \quad \text{for} \quad \lambda\le 1.
\end{aligned}$$

Thus we obtain from the local energy inequality for $\lambda\le 1$ and a. e. $t\le\lambda R^2$:

$$\int_{R^3}\frac{|u|^2}{2}(x,t)\phi(x-x_0)dx+\int_0^t\int_{R^3}|\nabla u|^2\phi(x-x_0)dxds \le \alpha+C\lambda A(\lambda)+CA(\lambda)^{3/2}\lambda^{1/4}R^{-1/2}.$$

Taking sup over $x_0\in R^3$ and $t\le\lambda R^2$, we get

$$A(\lambda)\le\alpha+CA(\lambda)\lambda+CA(\lambda)^{3/2}\lambda^{1/4}R^{-1/2}. \tag{3.1.8}$$

Note that $A(\lambda)$ is a priori bounded which is critical in our lemma. Also, we note that $A(\lambda)$ is non-decreasing in λ and from (4.3.2) it is not hard to see that $A(\lambda)$ is continuous in λ. (We note that this conclusion does not imply the continuity of the map $t\to\int_{R^3}|u(x,t)|^2\psi(x)dx$, which is unclear, in general.) From the above estimate for $A(\lambda)$, the lemma follows easily by the usual "continuation in λ" argument when ϵ_0 is chosen sufficiently small. Note that from the formula (4.4.3) and the a priori estimate of u, we get the following estimate for p which will be useful:

$$\sup_{x_0\in R^3}\int_0^{\lambda R^2}\int_{B_R(x_0)}|p-p(t)|^{3/2}dxdt\le C\alpha^{3/2}R^{1/2}. \tag{3.1.9}$$

Remarks: In the above estimate on p, more precisely, $p(t) = p_{x_0,R}(t)$. That is, we need to choose some appropriate constants $p_{x_0,R}(t)$ to satisfy the inequality. The point here is that such constants depending on x_0, R, t exist. This remark is effective throughout the paper.

We have the following simple corollary that will be useful below.

Corollary 3.1.1 *Let u be a Leray solution with initial data $u_0 \in L^3(R^3)$. Let p be the associated pressure. Then for $\forall r > 0$,*

$$\int_0^{r^2}\int_{B_r(x_0)} |\nabla u|^2 dxds + \text{ess}\sup_{0\le t\le r^2}\int_{B_r(x_0)} \frac{|u|^2(x,t)}{2}dx$$

$$\le \frac{C\|u_0\|^2_{L^3(R^3)} r}{\sqrt{\epsilon_0 \min\{\|u_0\|^{-4}_{L^3(R^3)}, 1\}}}, \quad \text{and} \tag{3.1.10}$$

$$\int_0^{r^2}\int_{B_r(x_0)} |p - p(t)|^{3/2} dxds \le \frac{C\|u_0\|^3_{L^3(R^3)} r^2}{\epsilon_0 \min\{\|u_0\|^{-4}_{L^3(R^3)}, 1\}}, \tag{3.1.11}$$

for any $x_0 \in R^3$.

Proof: For each $r > 0$, let $R = \frac{r}{\sqrt{\epsilon_0 \min\{\|u_0\|^{-4}_{L^3(R^3)},1\}}} > r$. We shall apply Lemma 3.1.1 with this R. We have:

$$\begin{aligned}\alpha &= \sup_{x_0\in R^3}\int_{B_R(x_0)} |u_0|^2 dx \le \sup_{x_0\in R^3}\left(\int_{B_R(x_0)} |u_0|^3 dx\right)^{2/3} R \\ &\le \|u_0\|^2_{L^3(R^3)} R.\end{aligned}$$

Thus we can choose $\lambda = \epsilon_0 \min\{\|u_0\|^{-4}_{L^3(R^3)}, 1\} \le \epsilon_0 \min\{\alpha^{-2}R^2, 1\}$. Note that by our choice of R and λ, $\lambda R^2 = r^2$, therefore from Lemma 3.1.1, the lemma follows.

We will need the following uniqueness result, see for example [27, 36] for a proof.

Lemma 3.1.2 *Suppose (u,p), (v,q) are two Leray solutions with the same initial data u_0, and $v \in L^5(R^3 \times [0,T))$ for any $T < \infty$. Then $u = v$ in $R^3 \times (0,\infty)$ almost everywhere.*

The following version of ϵ-regularity criteria of Caffarelli-Kohn-Nirenberg will be important for us in the sequel:

Lemma 3.1.3 *Let (u,p) be a suitable weak solution to NSE in $Q_1 := B_1(0) \times (-1,0)$ with $u \in L_t^\infty L_x^2(Q_1) \cap L_t^2 \dot{H}^1(Q_1)$ and $p \in L^{3/2}(Q_1)$, in the sense that (u,p) verifies NSE as distributions and they satisfy local energy inequality. Then there exists an absolute constant $\epsilon_0 > 0$, with the following property:*
if $(\int_{Q_1} |u|^3 dxdt)^{1/3} + (\int_{Q_1} |p|^{3/2} dxdt)^{2/3} \le \epsilon_0$, then $\|\nabla^k u\|_{L^\infty(Q_{1/2})} \le C_k$ for some constants C_k, $k = 0, 1, \ldots$

A sketch of a short proof can be found for example in [26], a detailed one in [24].

We recall the following lemmas proved in [31]:

Lemma 3.1.4 *(compactness) Let (u^k, p^k), $k = 1, 2, \ldots$ be a sequence of suitable weak solutions such that u^k are uniformly bounded in the energy space $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ on compact subsets of open set $\mathcal{O} \subset R^3 \times R$ and p^k are uniformly bounded in $L_t^{3/2} L_x^{3/2}$ on compact subsets of $\mathcal{O}$. Then the sequence u^k is compact in $L_t^3 L_x^3$ on compact subsets of $\mathcal{O}$. Moreover, if $u^k \to u$ in $L_t^3 L_x^3$ on compact subsets of $\mathcal{O}$ and $p^k \rightharpoonup p$ in $L_t^{3/2} L_x^{3/2}$ on compact subsets of $\mathcal{O}$, then (u,p) is again a suitable weak solution.*

Lemma 3.1.5 *(Stability of Singularities) In the situation of lemma 3.1.4, assume that $z^k \in \mathcal{O}$ are singular points of (u^k, p^k), $k = 1, 2, \ldots$, and that $z^k \to z_0 \in \mathcal{O}$. Then z_0 is a singular point of (u,p).*

We refer readers to [31] for the proofs, and here we only recall that a point z_0 is called a singular point of a suitable weak solution u to NSE if u is not bounded in any neighborhood of z_0.

3.2 Mild solutions with initial data in $L^3(R^3)$

In this section we collect some well known results about mild solutions, and introduce some splitting arguments which are useful in the proof of our main result.
One can rewrite NSE as an integral equation:

$$u(\cdot,t) = e^{\Delta t} u_0 - \int_0^t e^{\Delta(t-s)} P \mathrm{div}\ u \otimes u(\cdot,s) ds, \tag{3.2.1}$$

where P is the Helmholtz projection operator. It is well known that (3.2.1) has a global solution if initial data is small in L^3, and for arbitrary initial data in L^3 a unique local in time solution $u \in C([0,T_*), L^3(R^3))$ with a number of additional properties such as $u \in L^5(R^3 \times (0,T))$ for $T < T_*$, here T_* denotes the maximal existence time. See [20,22] and references therein for more details. Take any $v \in \mathcal{N}(u_0)$, by the uniqueness Lemma 3.1.2 and the remark below it, we know $v = u$ on $R^3 \times [0,T_*)$. Thus when we consider properties of solutions only on $R^3 \times [0,T_*)$, there is no confusion to assume u has been properly extended to $R^3 \times [0,\infty)$ as a Leray solution. We will make use of this observation below.

A priori there could be a number of reasons why T_* could be finite, we first show that it can only be due to the formation of a 'singular point':

Lemma 3.2.1 *Let $u \in C([0,T_*), L^3(R^3))$ be the mild solution to NSE with initial data u_0 and T_* is the maximal existence time. Suppose $T_* < \infty$, then there exists $z_0 = (x_0, T_*)$ such that $\forall r > 0$, $\operatorname{ess\,sup}_{Q_r(z_0)} |u| = +\infty$.*

Proof: We write $u_0 = a + b$, with $\|a\|_{L^3(R^3)} \leq \epsilon$, $\|b\|_{L^2(R^3)} < \infty$, where ϵ is a sufficiently small number to be chosen later. There are a number of ways in which one can perform such a decomposition. One can for example take

$$a = P(u_0 I_{|u_0|<\lambda}), \quad b = P(u_0 I_{|u_0|\geq\lambda}), \tag{3.2.2}$$

and take λ sufficiently small. If we choose ϵ small enough, we can apply global existence result for small data for NSE and get a global mild solution v with initial data a. Then $w = u - v$ satisfies

$$\left.\begin{aligned} \partial_t w - \Delta w + v \cdot \nabla w + w \cdot \nabla v + w \cdot \nabla w + \nabla q &= 0 \\ \operatorname{div}\ w &= 0 \end{aligned}\right\} \quad (x,t) \in R^3 \times (0,T_*), \tag{3.2.3}$$

$$w(\cdot,0) = b \qquad \text{in } R^3. \tag{3.2.4}$$

We claim the following estimate:

$$\operatorname*{ess\,sup}_{0\leq t\leq T_*} \|w(\cdot,t)\|^2_{L^2(R^3)} + \int_0^{T_*}\int_{R^3} |\nabla w|^2(x,t)\,dxdt \leq C(\|b\|_{L^2(R^3)}, \|v\|_{L^5(R^3\times(0,T_*))}, T_*). \tag{3.2.5}$$

There are two ways in which we can prove this claim. Since this type of argument will be used several times we provide them both here.

In the first approach, note that regular solution $\tilde{w}$ (by 'regular' we mean $\tilde{w}$ is smooth with sufficient decay) to equations (3.2.3) (3.2.4) satisfies the following, by a simple integration by parts:

$$\begin{aligned}
&\frac{d}{dt}\int_{R^3}\frac{|\tilde{w}|^2}{2}(x,t)dx+\int_{R^3}|\nabla\tilde{w}|^2(x,t)dx\\
&\leq\int_{R^3}(\tilde{w}\cdot\nabla\tilde{w})v(x,t)dx\\
&\leq\|v(\cdot,t)\|_{L^5_x}\|\tilde{w}(\cdot,t)\|_{L^{10/3}_x}\|\nabla\tilde{w}(\cdot,t)\|_{L^2_x}\\
&\leq C\|v(\cdot,t)\|_{L^5_x}\|\tilde{w}(\cdot,t)\|^{2/5}_{L^2_x}\|\nabla\tilde{w}(\cdot,t)\|^{8/5}_{L^2_x}\\
&\leq\frac{1}{2}\int_{R^3}|\nabla\tilde{w}|^2(x,t)dx+C\|v(\cdot,t)\|^5_{L^5}\int_{R^3}|\tilde{w}(\cdot,t)|^2dx.
\end{aligned}$$

In the above, we have used Hölder inequality, interpolation inequality

$$\|\tilde{w}\|_{L^{10/3}_x}\leq\|\tilde{w}\|^{2/5}_{L^2_x}\|\tilde{w}\|^{3/5}_{L^6_x},$$

and Sobolev embedding $\dot{H}^1\hookrightarrow L^6$ in R^3.

Thus we obtain:

$$\frac{d}{dt}\int_{R^3}\frac{|\tilde{w}|^2}{2}(x,t)dx\leq C\|v(\cdot,t)\|^5_{L^5_x}\int_{R^3}|\tilde{w}|^2dx.$$

Since $\int_0^{T_*}\|v(\cdot,t)\|^5_{L^5_x}dt$ is bounded, we get from Gronwall's inequality:

$$\sup_{0\leq t\leq T_*}\|\tilde{w}(\cdot,t)\|^2_{L^2(R^3)}+\int_0^{T_*}\int_{R^3}|\nabla\tilde{w}|^2(x,t)dxdt\leq C(\|b\|_{L^2(R^3)},\|v\|_{L^5(R^3\times(0,T_*))},T_*).\tag{3.2.6}$$

With this a priori estimate at hand, we can then follow Leray's arguments in constructing global weak solutions and obtain a weak solution $\tilde{w}$ to equations (3.2.3) (3.2.4) and $\tilde{w}$ satisfies the energy inequality (3.2.6). We can also require $\tilde{w}$ to satisfy the appropriate local energy inequality (see (3.2.8) below). Since $v+\tilde{w}$ is also a Leray solution with initial data u_0, by uniqueness result of lemma (3.1.2) and the remark below it we must have $w=\tilde{w}$ on $R^3\times[0,T_*)$ and thus w satisfy (3.2.6).

Alternatively one can also derive the inequality directly from the equation as follows.

w clearly satisfies the following integral equation:

$$w(\cdot,t)=e^{\Delta t}w_0-\int_0^t e^{\Delta(t-s)}P\mathrm{div}\ \left(v\otimes w(\cdot,s)+w\otimes v(\cdot,s)+w\otimes w(\cdot,s)\right)ds,\tag{3.2.7}$$

where $w_0 = b \in L^2(R^3)$, $t < T_*$.

Since both $u, v \in C([0,T_*), L^3(R^3))$ we conclude $w \in C([0,T_*), L^3(R^3))$. From this fact and the integral equation (3.2.7) together with $b \in L^2(R^3)$ we obtain $w \in C([0,T_*), L^2(R^3))$ from known estimates of the integral equation. Clearly, w satisfies the following local energy inequality (where q is the associated pressure for w):

$$\begin{aligned}
&\int_{R^3} \frac{|w|^2}{2}(x,t)\phi(x)dx + \int_0^t \int_{R^3} |\nabla w|^2 \phi(x)dx \\
&\leq \int_{R^3} \frac{|w|^2}{2}(x,0)\phi(x)dx + \int_0^t \int_{R^3} \frac{|w|^2}{2}\Delta\phi + qw\cdot\nabla\phi dxds \\
&\quad + \int_0^t \int_{R^3} \frac{|w|^2}{2}(w+v)\cdot\nabla\phi - (w\cdot\nabla v)w\phi dxds,
\end{aligned} \tag{3.2.8}$$

where ϕ is a nonnegative smooth cutoff function. By local theory of mild solutions (see [20]), we know

$$\sup_{0\leq t\leq T_*} \sqrt{t}\|\nabla v(\cdot,t)\|_{L^3_x} < \infty. \tag{3.2.9}$$

Thus

$$\int_0^T \int_{R^3} |w||\nabla v||w|dxds \leq \int_0^T \|\nabla v(\cdot,t)\|_{L^3_x}\|w(\cdot,t)\|^2_{L^3}dx < \infty, \tag{3.2.10}$$

for any $T < T_*$. This, together with $q \in L^\infty((0,T), L^{3/2}(R^3))$, $w \in C([0,T), L^3 \cap L^2(R^3))$ for any $T < T_*$, implies that we can take the cutoff function in the local energy inequality (3.2.8) to be $\phi(\frac{x}{R})$ with $\phi|_{B_1} \equiv 1$, and send $R \to \infty$. We obtain

$$\begin{aligned}
&\int_{R^3} \frac{|w|^2}{2}(x,t)dx + \int_0^t \int_{R^3} |\nabla w|^2 dxds \\
&\leq \int_{R^3} \frac{|w|^2}{2}(x,0)dx - \int_0^t \int_{R^3} (w\cdot\nabla v)wdxds \\
&= \int_{R^3} \frac{|w|^2}{2}(x,0)dx + \int_0^t \int_{R^3} (w\cdot\nabla w)vdxds.
\end{aligned}$$

The last identity holds since the first inequality implies

$$\int_0^t \int_{R^3} |\nabla w|^2 dxds < \infty,$$

which is the key. Then we can proceed as in the first approach to finish the proof of the claim.

From Corollary 3.1.1, we know

$$\sup_{x_0\in R^3, 0\le t\le T_*}\int_{B_1(x_0)}|v|^2(x,t)dx+\sup_{x_0\in R^3}\int_0^{T_*}\int_{B_1(x_0)}|\nabla v|^2+|\tilde p-\tilde p(t)|^{3/2}dxdt\le C(\epsilon)C(T_*), \tag{3.2.11}$$

where $\tilde p$ is the associated pressure for v. Observe from equation (3.2.3) and estimate (3.2.5) that $q\in L^{3/2}(R^3\times(0,T_*))$ and $w\in L^3(R^3\times(0,T_*))$. Since $C(\epsilon)\to 0$ as $\epsilon\to 0$, we can choose ϵ sufficiently small and find $R>0$ sufficiently large, such that for $|x_0|>R$, we can apply the ϵ-regularity criteria to $u=v+w$ in $Q_{\min(\sqrt{T_*/2},1)}(x_0,T_*)$. Thus we are able to conclude the following:

there exists a compact set $K\subseteq R^3$ such that u is bounded in $(R^3\backslash K)\times[T_*/2,T_*]$.

Now suppose the lemma is not true. Then u is bounded in a neighborhood of any point in $R^3\times[T_*/2,T_*]$. Thus by the compactness of K, we see u is bounded in $R^3\times[T_*/2,T_*]$. Since v is bounded in $R^3\times[T_*/2,T_*]$, we conclude w is also bounded in $R^3\times[T_*/2,T_*]$. This means $u(\cdot,t)$ is bounded in L^3 as well as L^∞, as t approaches T_*. Then local existence theory for NSE tells us we can continue u beyond T_*, a contradiction.

Remark 3.2.1 *The splitting argument in the above proof has played a key role in [4], and its variations were subsequently used in [12, 27]. Such splitting is very useful in obtaining estimates for u even when we approach the blow up time. Let $u\in C([0,1),L^3(R^3))$ be a mild solution with initial data u_0. Still take the above decomposition $u_0=a+b$ with a,b defined as above. Denote $\alpha=\|u_0\|_{L^3}$. We immediately see*

$$\|a\|_{L^6(R^3)}\le C\lambda^{1/2}\alpha^{1/2},\quad \|b\|_{L^2(R^3)}\le C\lambda^{-1/2}\alpha^{3/2}. \tag{3.2.12}$$

If we choose $\lambda=\lambda(\alpha)$ so small such that $\|a\|_{L^6(R^3)}$ is smaller than some absolute number, fix such $\lambda=\lambda(\alpha)$, then we can conclude from local existence theory of NSE the existence of a mild solution v to NSE with initial data a in $R^3\times[0,2)$ with

$$\text{ess sup}_{\;0\le t\le 3/2}\|v(\cdot,t)\|_{L^6}\le C. \tag{3.2.13}$$

Then w as defined above satisfy (3.2.3) with initial data b. Now we can use the usual energy estimate as in the above lemma to bound

$$\sup_{0\le t\le 1}\|w(\cdot,t)\|^2_{L^2(R^3)}+\int_0^1\int_{R^3}|\nabla w|^2dxdt\le C(\alpha). \tag{3.2.14}$$

Thus, $u = v + w$ *satisfies*

$$\sup_{0\le t\le 1} \|u(\cdot,t)\|_{L^2+L^6} \le C(\alpha). \tag{3.2.15}$$

Of course one can do the same thing if the blowup time is T*, though the estimates will also depend on* T *then. The point here is this estimate is uniform as long as* L^3 *norm of* u_0 *stays bounded. This observation will be useful later. And in fact, this estimate is stronger than the one in Lemma 3.1.1 for general Leray solutions, since it implies decay of* u *at spatial infinity while Lemma 3.1.1 does not imply any decay of solutions. The difference is that here we are dealing with an a priori regular solution.*

The following estimate is the main new observation that enables us to work in L^3.

Lemma 3.2.2 *Let* u *be a Leray solution with divergence free initial data* $u_0 \in L^3(R^3)$*. Then there exists a nonnegative function* $h(t)$ *depending only on* $\|u_0\|_{L^3(R^3)}$*, such that* $\lim_{t\to 0+} h(t) = 0$ *and*

$$\|u(\cdot,t) - e^{\Delta t}u_0\|_{L^2(B_1(x_0))} \le h(t), \tag{3.2.16}$$

for any $x_0 \in R^3$*, a.e.* $0 \le t < 1$*.*

Proof: We use a splitting argument which is slightly different from the usual ones. We refer the reader to the paper [**?**] for another example of a splitting argument based on a comparison with the linear equation. Denote $\alpha := \|u_0\|_{L^3(R^3)}$. For any $\epsilon > 0$, we split $u_0 = a + b$, with

$$a = P(u_0 I_{|u_0|<M}), \quad b = P(u_0 I_{|u_0|\ge M}), \tag{3.2.17}$$

where M is some large number to be chosen later. Clearly,

$$\|a\|_{L^6(R^3)} \le C\sqrt{M}\alpha^{1/2},$$
$$\|b\|_{L^2(R^3)} \le \frac{C}{\sqrt{M}}\|u_0\|^{3/2}_{L^3(R^3)} = \frac{C}{\sqrt{M}}\alpha^{3/2}.$$

Choose M such that

$$\frac{C}{\sqrt{M}}\alpha^{3/2} < \frac{\epsilon}{100}, \tag{3.2.18}$$

and we fix this M from now on, thus $M = M(\epsilon,\alpha)$. From the local existence theory of NSE, we can find $T = T(\epsilon,\alpha) > 0$, and a mild solution $v \in C([0,T), L^6(R^3))$ for NSE with initial data a, enjoying a number of other properties. Among them in particular, we have

$$\sup_{0\le t<T}\|v(\cdot,t)\|_{L^6(R^3)} \le C(\epsilon,\alpha). \tag{3.2.19}$$

$w = u - v$ satisfies:

$$\left.\begin{aligned} \partial_t w - \Delta w + v\cdot\nabla w + w\cdot\nabla v + w\cdot\nabla w + \nabla q &= 0\\ \text{div } \ w &= 0\\ w(\cdot,0) &= b \end{aligned}\right\} \ (x,t)\in R^3\times(0,T), \tag{3.2.20}$$

$$\tag{3.2.21}$$

By estimates of u from Corollary 3.1.1 and estimates on v, local energy estimates for w (3.2.8) and parabolic regularity, one can conclude (we omit the routine calculations): there exists $T_1(\alpha,\epsilon)>0$ such that

$$\sup_{x_0\in R^3, t\le T_1(\alpha,\epsilon)}\|v(\cdot,t)-e^{\Delta t}a\|_{L^2(B_1(x_0))} \le \frac{\epsilon}{10},$$

$$\sup_{x_0\in R^3, t\le T_1(\alpha,\epsilon)}\|w(\cdot,t)\|_{L^2(B_1(x_0))} \le \frac{\epsilon}{2}.$$

Thus, from $u = v + w$, we obtain

$$\|u(\cdot,t)-e^{\Delta t}u_0\|_{L^2(B_1(x_0))} < \epsilon, \tag{3.2.22}$$

for any $x_0\in R^3$, a.e. $t\le T_1(\alpha,\epsilon)$. From this, the lemma follows easily.

3.3 The main theorem

For any divergence free $u_0\in L^3(R^3)$, denote $T_{\max}(u_0)$ as the maximal time of existence for the mild solution for NSE starting from u_0. Define $\rho_{\max} = \sup\{\rho : T_{\max}(u_0) = \infty$ for every divergence free $u_0\in L^3(R^3)$ with $\|u_0\|_{L^3(R^3)} < \rho\}$. Also define $\mathcal{M} := \{u_0\in L^3(R^3) : T_{\max}(u_0)<\infty, \|u_0\|_{L^3(R^3)} = \rho_{\max}\}$.

Theorem 3.3.1 *Suppose $\rho_{\max}<\infty$. Then $\mathcal{M}$ is nonempty, and moreover, $\mathcal{M}$ is compact with respect to L^3-norm modulo translations and scalings. That is, for any sequence u_0^k in $\mathcal{M}$, there exist x_k, λ^k such that $\lambda^k u_0^k(\lambda^k(x-x_k))$ has a convergent subsequence in $L^3(R^3)$.*

Proof: By the definition of $\rho_{\max}$ and the assumption that $\rho_{\max} < \infty$, there exists a sequence of divergence free initial data u_0^k such that $T_{\max}(u_0^k) < \infty$ (thus $\|u_0^k\|_{L^3(R^3)} \geq \rho_{\max}$) and $\|u_0\|_{L^3(R^3)} \to \rho_{\max}$. By Lemma (3.2.1) we know there are singular points for mild solutions u^k corresponding to u_0^k. By translations and scalings

$$u_0^k \to \lambda^k u_0^k(\lambda^k(x - x_k)),$$

for some λ^k, x_k, we can assume the first singularity is at time 1 and is $(x,t) = (0,1)$. We still denote the sequence as u_k (u_0^k correspondingly) after translations and scalings. By Lemma (3.2.2), we have

$$\sup_{x_0 \in R^3} \|u^k(\cdot,t) - e^{\Delta t} u_0^k\|_{L^2(B_1(x_0))} \leq h(t) \quad \text{for } t < 1, \tag{3.3.1}$$

for some nonnegative function $h(t)$ with $\lim_{t \to 0+} h(t) = 0$. Note that Corollary 3.1.1 and Remark 3.2.1 imply uniform boundedness of u^k in local energy norm and

$$\sup_{0 \leq t \leq 1} \|u^k(\cdot,t)\|_{L^2 + L^6} \leq C(\rho_{\max}). \tag{3.3.2}$$

By compactness as in Lemma 3.1.4, and weak continuity in t we can find a subsequence of u^k (which we still denote as u^k) and a suitable weak solution u to NSE, such that:
$u^k \to u$ in $L^3(B_1(x_0) \times (0,1))$, for all $x_0 \in R^3$,
$u^k(\cdot,t) \rightharpoonup u(\cdot,t)$ in $L^2(B_1(x_0))$ for every $t \in [0,1]$ and $x_0 \in R^3$, and
$u_0^k \rightharpoonup u_0$ in $L^3(R^3)$.
Moreover, by stability of singularity in Lemma 3.1.5 $(x,t) = (0,1)$ is a singular point of u. From estimate (3.3.1) and the weak convergence of $u^k(\cdot,t)$, u_0^k, we get

$$\sup_{x_0 \in R^3} \|u(\cdot,t) - e^{\Delta t} u_0\|_{L^2(B_1(x_0))} \leq h(t). \tag{3.3.3}$$

Since $h(t) \to 0$ as $t \to 0$, we see

$$u(\cdot,t) \to u_0 \quad \text{in} \quad L^2(B_1(x_0)) \quad \text{for any} \quad x_0.$$

Furthermore,

$$\sup_{0 \leq t \leq 1} \|u(\cdot,t)\|_{L^2 + L^6} \leq C(\rho_{\max}), \tag{3.3.4}$$

by the weak convergence of u^k and the estimate (3.3.2). Thus u satisfies the decay condition at spatial infinity required in the definition of Leray solutions. Summarzing above, we see u is a Leray solution with initial data u_0. By uniqueness result of Lemma 3.1.2 and the remark below it, we see that the mild solution starting with u_0 must have a singularity in $R^3 \times [0,1]$. Since

$$\|u_0\|_{L^3(R^3)} \leq \liminf \|u_0^k\|_{L^3(R^3)} \leq \rho_{\max}, \tag{3.3.5}$$

by the definition of $\rho_{\max}$, we must have $\|u_0\|_{L^3(R^3)} = \rho_{\max}$. Thus we have $u_0^k \rightharpoonup u_0$ in L^3 and $\|u_0^k\|_{L^3} \to \|u_0\|_{L^3}$. Since $L^3(R^3)$ is uniformly convex as a Banach space, this also implies u_0^k converges strongly to u_0 in $L^3(R^3)$. The theorem is proved.

From results above the following corollary can be proved following the same arguments as in [31]:

Corollary 3.3.1 *Assume that every solution of the Cauchy problem 2.2.1 with* $u_0 \in L^3(R^3)$ *is regular, i.e.* $T_{\max} = +\infty$ *for each* $u_0 \in L^3(R^3)$*. Then for* $l = 0, 1, 2, \ldots$ *there exist functions* $F_l : [0,\infty) \to [0,\infty)$ *such that*

$$t^{(l+1)/2} \sup_x |\nabla^l u(x,t)| \leq F_l(\|u_0\|_{L^3}) \quad \textit{for all } t > 0. \tag{3.3.6}$$

Chapter 4

Liouville theorems for time-dependent Stokes system

In this chapter we prove some Liouville theorems for time-dependent Stokes system in domains. Our presentation is organized as follows. In section 1 we introduce some technical lemmas to be used below. Section 2 deals with the simple cases when Ω is a bounded domain or the whole space. Section 3 and 4 deal with the more subtle cases when Ω is a half space or an exterior domain. For the exterior domains, we use a standard extension argument together with some estimates of linear Stokes system. For the half space, which is the most interesting case, we use Fourier transform. There is also a proof based on duality arguments, which requires some additional point-wise estimates of solutions to linear Stokes system in half space. The estimates may be of independent interest, but the calculations are somewhat lengthy. This alternative proof will appear elsewhere.

4.1 Some technical lemmas

In the sequel we will make use standard mollifications. For completeness we include the following standard lemma:

Lemma 4.1.1 *Let Ω be as above, $u \in L^\infty_{x,t}(\Omega \times (-\infty, 0))$. Take a standard smooth* *cutoff function $\eta(t)$ with supp $\eta \Subset (0,1)$ and $\int \eta = 1$. For each $\epsilon > 0$, we define u^ϵ as a distribution in $\Omega \times (-\infty, 0)$ in the following way,*

$$(u^\epsilon, \phi) = \int_{-\infty}^{0} \int_{\Omega} u(x,t) \int_{-\infty}^{0} \frac{1}{\epsilon} \eta(\frac{s-t}{\epsilon}) \phi(x,s) ds dx dt \tag{4.1.1}$$

for any smooth ϕ with supp $\phi \Subset \Omega \times (-\infty, 0)$.
Then u^ϵ is a bounded function with bounded distributional derivatives $\partial_t^k u^\epsilon$, $k = 0, 1, 2, \cdots$. Moreover, we have the following estimates:

$$\|\partial_t^k u^\epsilon\|_{L^\infty(\Omega \times (-\infty,0))} \leq C\epsilon^{-k} \|u\|_{L^\infty(\Omega \times (-\infty,0))}. \tag{4.1.2}$$

Proof and Remarks: The proof follows immeditely from well-known properties of convolution. We note that due to our special choice of the support of η, the mollified function u^ϵ is still defined in $\Omega \times (-\infty, 0)$. It is clear from definition that u^ϵ converge weakly* to u in $L^\infty(\Omega \times (-\infty, 0))$. It is also clear that, after possibly changing the value of u^ϵ on a set of measure zero, the map $t \to u^\epsilon(\cdot, t)$ is continuous from $(-\infty, 0)$ to $L^p(K)$ for any $K \Subset \Omega$, $1 < p < \infty$.

Let u be a bounded distributional solution to the linear Stokes equations (1.0.8) in Q_1 with some distribution p. It is well known that we have regularity of u in x, for almost every t in $Q_{1/2}$. We can not, however, expect to have any regularity in t for u, or any reasonable estimate on p in general, assuming only that u is bounded in Q_1. This point is usually illustrated with the example where $u(t,x) = f(t)$, $p = -f'(t) \cdot x$. Here $f(t)$ is bounded, but $f'(t)$ can be arbitrarily large. On the other hand, if we assume some estimate on $\partial_t u$, then we can improve estimates on p. The following lemma summarizes the above discussion.

Lemma 4.1.2 *Let u be a bounded distributional solution to linear unsteady Stokes equations in Q_1. Let $\|u\|_{L^\infty_{x,t}(Q_1)} \leq 1$. Then for any multi-index α with $|\alpha| \geq 0$, $\|\partial_x^\alpha u\|_{L^\infty_{x,t}(Q_{1/2})} \leq C(n, \alpha)$. If in addition $\|\partial_t u\|_{L^\infty_{x,t}(Q_1)} \leq M$, then there exists a pressure field $p(x,t)$ such that (1.0.8) is satisfied and $\|\partial_x^\alpha p\|_{L^\infty_{x,t}(Q_{1/2})} \leq C(\alpha, M, n)$ for any multi-index α with $|\alpha| \geq 0$.*

Proof: For the first part of the lemma, note that the vorticity $\omega_{ij} := \partial_i u_j - \partial_j u_i$, $1 \leq i, j \leq n$, satisfies heat equation $\partial_t \omega_{ij} - \Delta \omega_{ij} = 0$ in Q_1. Thus ω_{ij} is smooth with all derivatives bounded by constants depending only on n in $Q_{3/4}$. From the divergence free condition, we get $\Delta u_i = -\sum_{j=1}^n \partial_j \omega_{ij}$. Then the first part of the lemma follows from interior estimate of Laplace equations. For the second part, note that $\|\nabla p\|_{L^\infty_{x,t}(Q_{3/4})} \leq C(n, M)$ from the assumption on $\partial_t u$ and first part of the lemma. Since we also have $\Delta p = 0$, the estimate follows.

Remark: The pressure is only determined up to an arbitrary function of t. (If we change p to $p + c(t)$, equation (1.0.8) is not affected.) In estimates below we will usually assume a suitable choice of $c(t)$.

We shall need the following extension result (which is interesting in its own right) below.

Lemma 4.1.3 *(Extension of divergence-free vector field) For any smooth compactly supported vector field $g = (g_1, \cdots, g_{n-1}, 0)$ in R^{n-1}, there exists a smooth divergence free vector field $\phi = (\phi_1, \cdots, \phi_n)$ with* supp $\phi \Subset \overline{R^n_+}$ *such that $\phi|_{x_n=0} = 0$ and $\frac{\partial \phi}{\partial x_n}|_{x_n=0} = g$.*

Proof: We seek ϕ in the form of $\phi_i = \sum_{j=1}^n \partial_j w_{ij}$, with some $w_{ij} \in C_c^\infty(\overline{R^n_+})$ and $w_{ij} = -w_{ji}$ for $1 \leq i, j \leq n$. Note that under such conditions on w_{ij}, div $\phi = 0$ is automatically satisfied. To satisfy boundary conditions for ϕ, we need:

$$\sum_{j=1}^n \frac{\partial w_{ij}}{\partial x_j}|_{x_n=0} = 0 \quad \text{for } 1 \leq i \leq n, \tag{4.1.3}$$

and

$$\sum_{j=1}^n \frac{\partial^2 w_{ij}}{\partial x_n \partial x_j}|_{x_n=0} = g_i \quad \text{for } 1 \leq i \leq n,\ g_n\text{=0}. \tag{4.1.4}$$

It is easy to verify that the n-th equation in (4.1.4) is also automatically satisfied once the rest of the equations in the above are satisfied. To satisfy equations (4.1.3)(4.1.4), we first require $w_{ij}|_{x_n=0} = 0$, for all $1 \leq i, j \leq n$. Then equations (4.1.3) become $\frac{\partial w_{in}}{\partial x_n}|_{x_n=0} = 0$ for $1 \leq i \leq n$. We further require that $w_{ij} = 0$ if $1 \leq i, j \leq n-1$, then equations (4.1.4) reduce to $\frac{\partial^2 w_{in}}{\partial^2 x_n}|_{x_n=0} = g_i$ for $1 \leq i \leq n-1$. Summarizing the above analysis, it is sufficient to find $w_{in} \in C_c^\infty(\overline{R_+^n})$ for $1 \le i \le n-1$, $w_{in} = -w_{ni}$ satisfying

$$\left.\begin{array}{ll} w_{in}|_{x_n=0} & = 0 \\ \frac{\partial w_{in}}{\partial x_n}|_{x_n=0} & = 0 \\ \frac{\partial^2 w_{in}}{\partial^2 x_n}|_{x_n=0} & = g_i \end{array}\right\} \quad \text{for } 1 \le i \le n-1. \tag{4.1.5}$$

It is clear that we can always find such w_{in}. Thus ϕ satisfying conditions in the lemma exists.

We collect some facts about the operator $|\nabla|$ which will be used in our proofs.

For $f \in \mathcal{S}(R^n)$, we define $|\nabla| f(x) = (|\xi|\hat{f}(\xi))^\vee(x)$ where we have used Fourier transform

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-ix\cdot\xi} f(x)dx$$

and inverse Fourier transform

$$\check{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

One can write $|\nabla| f = \sum_{j=1}^n -\frac{\partial_j}{|\nabla|}\partial_j f = \sum_{j=1}^n R_j \partial_j f$, where R_j denotes the Riesz transform. Clearly $|\nabla|$ can be considered as a continuous operator from $\mathcal{S}(R^n) \to L^1(R^n)$. By duality, we can extend $|\nabla|$ to act on $L^\infty(R^n)$ according to the usual fromula $\langle |\nabla| f, \phi\rangle = \langle f, |\nabla|\phi\rangle$ for any $\phi \in \mathcal{S}(R^n)$.

We recall the following obvious continuity result.

Lemma 4.1.4 *Let $|\nabla|$: $L^\infty(R^n) \longmapsto \mathcal{S}'(R^n)$ be defined as above. If $u_m \in L^\infty$, with u_m converges weakly* to u in L^∞ (viewed as the dual of $L^1(R^n)$), then $|\nabla| u_m$ converges to $|\nabla| u$ in $\mathcal{S}'(R^n)$*

Proof: This follows directly from the definitions.

Recall the definition of Hölder norm in R^n:

$$\|u\|_{C^{m,\alpha}(R^n)} := \sum_{i=0}^m \sup_{|\beta|\le m} \sup_{x\in R^n} |\partial^\beta u(x)| + \sup_{|\beta|=m} \sup_{x,y\in R^n,\ \ x\ne y} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|}{|x-y|^\alpha}$$

for any $m \ge 0$, $0 < \alpha < 1$. The Hölder space $C^{m,\alpha}(R^n)$ is consisted of all u with $\|u\|_{C^{m,\alpha}} < \infty$. We will use the following estimate:

Lemma 4.1.5 $|\nabla| : C^{m+1,\alpha}(R^n) \mapsto C^{m,\alpha}(R^n)$ *is bounded, for* $m \geq 1, \quad 0 < \alpha < 1.$

Proof: This follows from the representation $|\nabla| = \sum_j R_j \partial_j$ and the Schauder estimates for the Riesz transform.

We will denote by $|\nabla'|$ the analogue of $|\nabla|$ acting only on the variables $x_1, \ldots, x_{n-1}$. For Schwartz function f, $|\nabla'| f(x', x_n) = (2\pi|\xi'|\hat{f}(\xi', x_n))^{\vee}(x')$, where the Fourier transform and inverse Fourier transform are both with respect to the first $n-1$ variables. From definition, it is clear if $f(x', x_n, t) \in L^\infty(R^{n-1} \times (x_1, x_2) \times (t_1, t_2))$, then $|\nabla'| f \in \mathcal{D}'(R^{n-1} \times (x_1, x_2) \times (t_1, t_2))$. Moreover, if $\partial^l_{x'} \partial^k_n \partial^m_t f \in L^\infty$, then $|\nabla'| \partial^l_{x'} \partial^k_n \partial^m_t f = \partial^l_{x'} \partial^k_n \partial^m_t |\nabla'| f$ in $\mathcal{D}'(R^{n-1} \times (x_1, x_2) \times (t_1, t_2))$.

For bounded harmonic functions in the upper half spaces, we have the following result (see also [42], for example).

Lemma 4.1.6 *Let* f *be a bounded harmonic function in the upper half space* R^n_+*, we have* $(\partial_n f + |\nabla'| f)(x) = 0$ *in the sense of distributions in* R^n_+.

Remarks: By classical regularity for harmonic functions and lemma 4.1.5, we see both $\partial_n f$ and $|\nabla'| f$ are smooth functions in the interior of R^n_+.

Proof: Let $P(x, y)$ be the Poisson kernel. By classical representation results there exists a $g \in L^\infty(R^{n-1})$, such that $f(x) = \int_{R^{n-1}} P(x, y) g(y)\, dy$. By approximation and continuity properties of $|\nabla'|$ we can assume without loss of generality that g is smooth and compactly supported. Applying Fourier transform in the $x_1, \ldots, x_{n-1}$ variables, we have $\hat{f}(\xi', x_n) = \hat{g}(\xi') e^{-|\xi'| x_n}$ and the result follows.

4.2 The cases $\Omega = R^n$ or a bounded domain

In this section, we first deal with the (easy) cases when $\Omega = R^n$ or Ω is a bounded domain. Recall that our goal is to show that bounded very weak ancient solutions to (1.0.8)(1.0.9) are given by (1.0.10).

1. $\Omega = R^n$.

In this case, it is not difficult to see that equations (1.0.11)(1.0.12) are equivalent to

$$\left.\begin{aligned} \partial_t u - \Delta u + \nabla q &= 0 \\ \text{div} \ \ u &= 0 \end{aligned}\right\} \quad \text{in } R^n \times (-\infty, 0)$$

in the sense of distributions for some $q \in \mathcal{D}'(R^n \times (-\infty, 0))$.

For $1 \leq i, j \leq n$, let $\omega_{ij} = \partial_j u_i - \partial_i u_j$, then clearly $\partial_t \omega_{ij} - \Delta \omega_{ij} = 0$ in $R^n \times (-\infty, 0)$. Since ω_{ij} are bounded in some negative Sobolev space, we immediatly get ω_{ij} are bounded functions from parabolic regularity. Thus ω_{ij} are so called bounded ancient solution to heat equation, and consequently ω_{ij} =constants c_{ij}. Since u is divergence free, we get $\Delta u_i = -\sum_{j=1}^n \partial_j \omega_{ij} = 0$ in $R^n \times (-\infty, 0)$. Therefore $u(t, x) = f(t)$ for some bounded measurable f a.e t. This completes the proof when $\Omega = R^n$.

2. Ω is a bounded domain.

In this case our goal is to show that bounded very weak ancient solutions u to (1.0.8)(1.0.9) are identically 0. We use a duality argument as follows. For any $f \in C_c^\infty(\Omega \times (0, +\infty))$, let $\tilde{\phi}$ solve

$$\left.\begin{aligned} \partial_t \tilde{\phi} - \Delta \tilde{\phi} + \nabla q &= f \\ \text{div} \ \ \tilde{\phi} &= 0 \end{aligned}\right\} \qquad \text{in } \Omega \times (0, \infty),$$
$$\tilde{\phi}(\cdot, t)|_{\partial\Omega} = 0.$$

The existence, uniqueness and regularity of such solutions are well known, one can see e.g [13]. Moreover, we have $\lim_{t\to\infty} \|\tilde{\phi}(\cdot, t)\|_{L^2(\Omega)} = 0$ (the decay is actually exponential). Take a standard smooth cutoff function $\eta(t)$ with $\eta(t) = 0$ for $t > 2$. For any $R > 0$, let $\phi_R(x, t) = \eta(-\frac{t}{R})\tilde{\phi}(x, -t)$ for $t \in (-\infty, 0)$. Then from equations (1.0.11)(1.0.12) we obtain

$$
\begin{aligned}
0 &= \int_{-\infty}^{0}\int_{\Omega} u(x,t)(\partial_t\phi_R + \Delta\phi_R)dxdt \\
&= \int_{-\infty}^{0}\int_{\Omega} u(x,t)(-\partial_t\tilde{\phi} + \Delta\tilde{\phi})(x,-t)\eta(-\frac{t}{R})dxdt \\
&-\frac{1}{R}\int_{-\infty}^{0}\int_{\Omega} u(x,t)\eta'(-\frac{t}{R})\tilde{\phi}(x,-t)dxdt \\
&= -\int_{-\infty}^{0}\int_{\Omega} u(x,t)f(x,-t)\eta(-\frac{t}{R})dxdt - \frac{1}{R}\int_{-\infty}^{0}\int_{\Omega} u(x,t)\eta'(-\frac{t}{R})\tilde{\phi}(x,-t)dxdt \\
&+\int_{-\infty}^{0}\int_{\Omega} u(x,t)\eta(\frac{t}{R})\nabla q(x)dxdt
\end{aligned}
$$

Using the fact that f is compactly supported in t, q is smooth in x, u is bounded and $\lim_{t\to\infty}\|\tilde{\phi}(\cdot,t)\|_{L^1(\Omega)} = 0$ (since Ω is bounded), we can send $R\to\infty$ and obtain

$$
\int_{-\infty}^{0}\int_{\Omega} u(x,t)f(x,-t)dxdt = 0.
$$

Since f is arbitrary, we must have $u \equiv 0$.

4.3 The case $\Omega = R^n_+$

Now let us deal with the more subtle case when Ω is a half space. In fact one can still use the idea of duality as in the case of bounded domains. In this case, however, one has to study the decay property of solution to the linear Stokes equations quite carefully. One also has to appropriately localize $\tilde{\phi}$ (assuming notations from the last section) since in (1.0.11) the test function ϕ is required to be of compact support. The authors have obtained a proof using such a method, which will appear elsewhere.

Here we take a different approach based on the Fourier transform in which the calculations are simpler.

Let u be as above, take a smooth mollifier $\eta(x',t)$ with *supp* $\eta \Subset B_1(0)\times(0,1) \subseteq R^{n-1}_{x'}\times R_t$ and $\int\eta = 1$. We define the mollified vector field u^ϵ similar as before, again by duality: for any smooth ϕ with *supp* $\phi \Subset R^n_+\times(-\infty,0)$,

$$
(u^\epsilon,\phi) := \int_{-\infty}^{0}\int_{R^n_+} u(x,t)\int_{R^{n-1}}\int_{-\infty}^{0}\epsilon^{-n}\eta(\frac{y'-x'}{\epsilon},\frac{s-t}{\epsilon})\phi(y',x_n,s)dy'dsdxdt.
$$

Again similar as before, one can show u^ϵ is bounded with bounded distributional derivatives $|\partial_t^k \partial_{x'}^\alpha u^\epsilon| \le C(k,\alpha,n)\epsilon^{-k-|\alpha|}\|u\|_{L^\infty_{x,t}}$. We have the following result:

Lemma 4.3.1 *Let u be a bounded very weak ancient solution to (1.0.8)(1.0.9) in $R^n_+ \times (-\infty,0)$, u^ϵ is defined as above. Then u^ϵ is smooth with all derivatives bounded in $\overline{R^n_+} \times (-\infty,0)$. Moreover, u^ϵ still satisfies equations (1.0.11),(1.0.12) and $u^\epsilon(\cdot,t)|_{x_n=0} = 0$.*

Proof: From equations (1.0.11)(1.0.12) and definition of u^ϵ we see

$$\int_{-\infty}^0 \int_\Omega u^\epsilon(x,t)(\partial_t\phi + \Delta\phi)(x,t)dxdt = 0 \tag{4.3.1}$$

for any $\phi \in C_c^\infty(\overline{\Omega} \times (-\infty,0))$ with div $\phi = 0$, $\phi|_{\partial\Omega\times(-\infty,0)} = 0$; and

$$\int_{-\infty}^0 \int_\Omega u^\epsilon(x,t)\nabla\psi(x,t)dxdt = 0 \tag{4.3.2}$$

for any $\psi \in C_c^\infty(\overline{\Omega} \times (-\infty,0))$. These clearly imply

$$\left.\begin{aligned} \partial_t u^\epsilon - \Delta u^\epsilon + \nabla\cdot q &= 0 \\ \text{div } u^\epsilon &= 0 \end{aligned}\right\} \quad \text{in } \mathcal{D}'(R^n_+ \times (-\infty,0)). \tag{4.3.3}$$

We first show u^ϵ is smooth up to boundary $\{x_n = 0\}$. From the differentiablity property of u^ϵ in x', t, we see q is well defined for each $t \in (\infty, 0)$ modulo some $c(t)$. Moreover, from $\Delta q = 0$ and elliptic estimates, we know q is smooth in x away from boundary $\{x_n = 0\}$. Now let us rewrite the n-th equation of (4.3.3) as

$$0 = \partial_t u_n^\epsilon - \Delta u_n^\epsilon + \partial_n q = \frac{\partial}{\partial x_n}(q - \partial_n u_n^\epsilon) - \Delta_{x'} u_n^\epsilon + \partial_t u_n^\epsilon.$$

Note that $\partial_n u_n^\epsilon = -\sum_{i=1}^{n-1} \partial_i u_i^\epsilon$ is bounded up to $x_n = 0$. We see $q - \partial_n u_n^\epsilon$ is bounded up to boundary, thus q is bounded up to boundary $\{x_n = 0\}$. The same argument also shows $\nabla_{x'}^\alpha q$ is bounded up to boundary. Use $\Delta q = 0$ we obtain that q is smooth in spatial variables up to $x_n = 0$. Then from $\partial_n^2 u^\epsilon = \partial_t u^\epsilon - \Delta_{x'} u^\epsilon + \nabla q$ we get $u \in C^2$. By differentiating the equations in x_n and applying similar arguments we obtain smoothness of u^ϵ. Next we show $u^\epsilon|_{x_n=0} = 0$. Since u^ϵ is smooth in $\overline{R^n_+}$, we can use equations (1.0.11)(1.0.12) and integration by parts in equations (4.3.1)(4.3.2) to obtain:

$$\int_{-\infty}^0 \int_{R^{n-1}} u_n^\epsilon \psi dxdt = 0, \tag{4.3.4}$$

$$\int_{-\infty}^0 \int_{R^{n-1}} u^\epsilon \phi dxdt = 0. \tag{4.3.5}$$

Clearly ψ can be arbitrary smooth compactly supported function, thus $u_n^\epsilon|_{x_n=0} \equiv 0$. By lemma 4.1.3, $\phi|_{x_n=0}$ can be any smooth compactly supported vector field with zero n-th component, thus $u_i^\epsilon|_{x_n=0} \equiv 0$ for $1 \le i \le n-1$. Therefore, $u^\epsilon|_{x_n} \equiv 0$.

Now we can prove our main theorem in this section.

Theorem 4.3.1 *Let u be a bounded very weak ancient solution to equations (1.0.8)(1.0.9) in $R_+^n \times (-\infty, 0)$, then we must have $u(x,t) = u(x_n, t)$ and $u_n \equiv 0$.*

Proof: By the above results, it is clear we only need to prove our theorem in the case $u(t,x)$ is smooth up to boundary, with all derivatives bounded and $u|_{x_n=0} = 0$. Then we see $\frac{\partial p}{\partial x_i}$ is bounded, for $1 \le i \le n$. Since $\Delta \frac{\partial p}{\partial x_i} = 0$, from lemma 4.1.6 we get $(\partial_n + |\nabla'|)\frac{\partial p}{\partial x_i} = 0$. Applying operator $(\partial_n + |\nabla'|)$ to the n-th equation of (1.0.8), noting the commutativity of various Fourier multipliers (since u is smooth), we infer that $(\partial_n + |\nabla'|)u_n$ satisfies the heat equation in $R^n \times (-\infty, 0)$ with $(\partial_n + |\nabla'|)u_n$ bounded and $(\partial_n + |\nabla'|)u_n|_{x_n=0} = 0$ (since $\partial_n u_n = -\sum_{i=1}^{n-1} \partial_i u_i$ and $u|_{x_n=0} = 0$). Thus by Liouville's theorem for heat equation in a half space, we get $(\partial_n + |\nabla'|)u_n \equiv 0$, and consequently $\Delta u_n = 0$. Since we also have $u_n|_{x_n=0} = 0$, we see $u_n \equiv 0$. Therefore $\frac{\partial p}{\partial x_n} = 0$. Thus $|\nabla'|\frac{\partial p}{\partial x_i} = 0$ for $1 \le i \le n$. Again applying operator $|\nabla'|$ to the first $n-1$ equations of (1.0.8), we get $|\nabla'|u'$ satisfies heat equation in $R^n \times (-\infty, 0)$ with $(|\nabla'|u')|_{x_n=0} = 0$ and $|\nabla'|u'$ bounded. Using Liouville's theorem for heat equation in a half space again, we obtain $|\nabla'|u' \equiv 0$. Thus $u'(t,x) = u'(t,x_n)$. Summarizing the above, we obtain $u(t,x) = u(t,x_n)$, and $u_n(t,x) \equiv 0$.

4.4 The case Ω is an exterior domain

Let u be a bounded very weak ancient solution to (1.0.8)(1.0.9) in an exterior domain Ω (i.e, the complement of Ω is homeomorphic to a ball), we show $u(x,t) = f(t) + O(\frac{1}{|x|^{n-2}})$ with some bounded f and $n \ge 3$, in this section. More precisely we have the following theorem:

Theorem 4.4.1 *Let u be a bounded very weak ancient solution to equations (1.0.8)(1.0.9) in $\Omega \times (-\infty, 0)$, where $\Omega \subset R^n$ $(n \ge 3)$ is an exterior domain with $\Omega^c \subset B_R$ for some $R > 0$. $\|u\|_{L^\infty_{x,t}} \le 1$. Then there exists a function $a(t)$ with $|a(t)| \le 1$ for almost every t*

such that

$$|u(x,t) - a(t)| \le \frac{C(n,R)}{|x|^{n-2}} \quad \textit{for almost every } |x| \ge 4R \textit{ and } t < 0. \tag{4.4.1}$$

For such purpose, we first mollify u in t variable as in lemma 4.1.1, it is clear that u^ϵ thus obtained still satisfies equations (1.0.11)(1.0.12). Our first goal is to show that u^ϵ is smooth in $\overline{\Omega} \times (-\infty, 0)$ and $u^\epsilon|_{\partial\Omega, t<0} = 0$.

Lemma 4.4.1 *Let u and u^ϵ be as above. Then u^ϵ is smooth in $\overline{\Omega} \times (-\infty, 0)$ and $u^\epsilon|_{\partial\Omega, t<0} = 0$.*

Proof: Clearly u^ϵ verifies

$$\left.\begin{aligned} \partial_t u^\epsilon - \Delta u^\epsilon + \nabla q &= 0 \\ \text{div}\ \ u^\epsilon &= 0 \end{aligned}\right\} \quad \text{in } \Omega \times (-\infty, 0). \tag{4.4.2}$$

In equations (1.0.11)(1.0.12), take test functions as $\eta(t)\phi(x)$, $\eta(t)\psi(x)$ respectively for smooth ϕ, ψ with supp ϕ, supp $\psi \Subset \overline{\Omega}$, div $\phi = 0$, $\phi|_{\partial\Omega} = 0$ and $\eta \in C_c^\infty(-\infty, 0)$. We obtain by integration by parts (and definition of u^ϵ):

$$\begin{aligned} \int_{-\infty}^0 \int_\Omega (-\partial_t u^\epsilon \phi + u^\epsilon \Delta\phi)\eta(t) dx dt &= 0, \\ \int_{-\infty}^0 \int_\Omega u^\epsilon \cdot \nabla\psi \eta(t) dx dt &= 0. \end{aligned}$$

Since η is arbitrary, we get for any $t \in (-\infty, 0)$,

$$\int_\Omega -\partial_t u^\epsilon \phi + u^\epsilon \Delta\phi dx dt = 0, \tag{4.4.3}$$

$$\int_\Omega u^\epsilon \cdot \nabla\psi dx dt = 0. \tag{4.4.4}$$

Take $R > 0$ sufficiently large such that $\Omega^c \subset B_R(0)$. For fixed $t < 0$, we can find $v \in C^{1,1/2}(\Omega \cap B_R)$ satisfying

$$\left.\begin{aligned} -\Delta v + \nabla p &= -\partial_t u^\epsilon(\cdot, t) \\ \text{div}\ \ v &= 0 \end{aligned}\right\} \quad \text{in } \Omega \cap B_R, \tag{4.4.5}$$

$$v|_{\partial\Omega} = 0, \quad v|_{\partial B_R} = u^\epsilon(\cdot, t)|_{\partial B_R}. \tag{4.4.6}$$

Note in the interior of Ω, u^ϵ is smooth by lemma 4.1.2 and definition of u^ϵ. The existence of v follows from well-known results of steady Stokes system, we only remark here that the usual no outflow condition required by existence theory is satisfied in our situation and can be easily seen by setting ψ to be 1 in a neighborhood of $\partial\Omega$ in equation (4.4.4). Set $w = u^\epsilon(\cdot, t) - v$, we claim $w \equiv 0$ in $\Omega \cap B_R$. To prove the claim, take any $\phi \in C^\infty(\overline{\Omega \cap B_R})$ with div $\phi = 0$ and $\phi|_{\partial(\Omega\cap B_R)} = 0$, $\psi \in C^\infty(\overline{\Omega \cap B_R})$, we write $\phi = \phi_1 + \phi_2$, $\psi = \psi_1 + \psi_2$ with the following properties: div $\phi_1 =$ div $\phi_2 = 0$, ϕ_1, ϕ_2, ψ_1, ψ_2 are smooth; ϕ_1, ψ_1 equal ϕ and ψ in a neighborhood of $\partial\Omega$, vanishes in a neighborhood of ∂B_R respectively. The existence of such decomposition of ψ is clear. To obtain such this decomposition for ϕ, one can localize ϕ by a standard cutoff function vanishing in a neighborhood of ∂B_R, then use Bogovski's theorem to deal with the divergence free condition, we omit the details here. With these decompositions, equations (4.4.3)(4.4.4), the definitions of v and the fact that u^ϵ is smooth away from $\partial\Omega$, we easily obtain: $\int_\Omega w\Delta\phi dx = 0$ and $\int_\Omega w\nabla\psi dx = 0$. Thus by result in section 3, this implies $w = 0$. Therefore, $u^\epsilon(\cdot, t) \in C^{1,1/2}(\overline{B_R \cap \Omega})$ and $u^\epsilon|_{\partial\Omega} = 0$. A simple boostraping argument gives smoothness of u^ϵ. The lemma is proved.

Proof of main result of this section

Let us first summarize the above results as follows:

u^ϵ is in $C^\infty(\overline{\Omega} \times (-\infty, 0))$ with all derivatives bounded (with bounds depending on ϵ) and, u^ϵ satisfies

$$\left.\begin{aligned} \partial_t u^\epsilon - \Delta u^\epsilon + \nabla q &= 0 \\ \text{div } u^\epsilon &= 0 \end{aligned}\right\} \quad \text{in } \Omega \times (-\infty, 0), \tag{4.4.7}$$

$$u^\epsilon(\cdot, t)|_{\partial\Omega} = 0. \tag{4.4.8}$$

We extend u^ϵ to R^n by setting $u^\epsilon = 0$ in Ω^c. It is not hard to see the extended u^ϵ satisfies

$$\left.\begin{aligned} \partial_t u^\epsilon - \Delta u^\epsilon + \nabla q &= \mu \\ \text{div } u &= 0 \end{aligned}\right\} \quad \text{in } R^n \times (-\infty, 0). \tag{4.4.9}$$

for the measure $\mu = f^\epsilon(x, t)d\sigma$, where $f^\epsilon = \frac{\partial u^\epsilon}{\partial n} - qn$ on $\partial\Omega$ and $d\sigma$ is the surface measure of $\partial\Omega$. We set

$$v^\epsilon(x, t) = \int_{-\infty}^t Pe^{\Delta(t-s)}\mu(\cdot, s)ds = \int_{-\infty}^t \frac{1}{(t-s)^{n/2}} \int_{\partial\Omega} k(\frac{x-y}{\sqrt{t-s}})f^\epsilon(y, s)d\sigma(y)ds \ ,$$

where P is the Helmholtz projection to divergence free vector field and $k(\cdot)$ is the kernel of Pe^{Δ}. Thus $|k(y)| \leq \frac{C(n)}{(1+|y|)^n}$. Simple calculations show v^ϵ verifies the following estimates: $\|v^\epsilon(\cdot,t)\|_{L^1(B_{2R})} \leq C(R,n,\epsilon)$, and $|v^\epsilon(x,t)| \leq \frac{C(n,\epsilon,R)}{|x|^{n-2}}$, $|\nabla v^\epsilon(x,t)| \leq \frac{C(n,\epsilon,R)}{|x|^{n-1}}$, for $|x| \geq 2R$. In the above calculations $n \geq 3$ is important. One can check if $n = 2$ the integral might diverge. Clearly $w^\epsilon := u^\epsilon - v^\epsilon$ satisfies

$$\left.\begin{aligned} \partial_t w^\epsilon - \Delta w^\epsilon + \nabla q &= 0 \\ \mathrm{div}\ \ w^\epsilon &= 0 \end{aligned}\right\} \quad \text{in } R^n \times (-\infty, 0). \tag{4.4.10}$$

Thus $w^\epsilon = a^\epsilon(t)$ and consequently $u^\epsilon = v^\epsilon + a^\epsilon(t)$. At this stage, we would like to pass ϵ to zero. The decay estimate for v^ϵ, however, depends on ϵ (since the bounds of f^ϵ depends on ϵ). Thus we must first remove this dependence. To do this, let us consider vorticity $\omega_{ij}^\epsilon = \partial_i u_j^\epsilon - \partial_j u_i^\epsilon$ for $1 \leq i,j \leq n$. ω_{ij}^ϵ satisfy $\partial_t \omega_{ij}^\epsilon - \Delta \omega_{ij}^\epsilon = 0$ in $(R^n \backslash B_{2R}) \times (-\infty, 0)$. By interior regularity of solution to heat equation, the scaling invariance $\omega_{ij}^\epsilon(x,t) \to \omega_{ij}^\epsilon(Mx, M^2x)$, and the L^∞ bound on u, we easily conclude:

ω_{ij}^ϵ *is smooth in* $(R^n \backslash B_{3R}) \times (-\infty, 0)$ *with* $|\partial_t \omega_{ij}^\epsilon(x,t)| \leq \frac{C(n,R)}{|x|^3}$, $|x| \geq 3R$, *the estimate being independent of* ϵ.

Now fix $\epsilon = \epsilon_0 > 0$. From estimates of u^{ϵ_0} above, we know $|\omega_{ij}^{\epsilon_0}(x,t)| \leq \frac{C(n,\epsilon_0,R)}{|x|^{n-1}}$, $|x| \geq 3R$. By estimates of $\partial_t \omega_{ij}$ and definition of ω_{ij}^ϵ, we see $|\omega_{ij}(x,t) - \omega_{ij}^{\epsilon_0}(x,t)| \leq \frac{C(n,\epsilon_0,R)}{|x|^3}$, $|x| \geq 3R$. Therefore, $|\omega_{ij}(x,t)| \leq C(n,\epsilon_0,R)(\frac{1}{|x|^3} + \frac{1}{|x|^{n-1}})$, $|x| \geq 3R$. In fact, by a boostrap argument (better decay estimate of ω_{ij} improves estimate of $\partial_t \omega_{ij}$ which in turn implies better decay estimate of ω_{ij}), one can upgrade this estimate to $|\omega_{ij}(x,t)| \leq \frac{C(n,\epsilon_0,R)}{|x|^{n-1}}$. Thus $|\omega^\epsilon(x,t)| \leq \frac{C(n,\epsilon_0,R)}{|x|^{n-1}}$, (now independent of ϵ) for $|x| \geq 3R$. For each fixed $t \in (-\infty, 0)$, u_i^ϵ solves

$$-\Delta u_i^\epsilon = -\sum_{j=1}^{n} \partial_j \omega_{ij}^\epsilon \quad \text{in } R^n \backslash B_{3R}. \tag{4.4.11}$$

From this and the boundedness of u, it is not hard to see u_i^ϵ verifies the following bound: $|u^\epsilon(x,t) - a^\epsilon(t)| \leq \frac{C(n,\epsilon_0,R)}{|x|^{n-2}}$ for $|x| \geq 4R$. Passing $\epsilon \to 0$ the conclusion of this section is reached.

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