STABILITY OF THE FRACTIONAL STEP Θ-SCHEME FOR THE NONSTATIONARY NAVIER-STOKES EQUATIONS

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Abstract. We present stability and convergence analysis of the fractional step Θ-scheme for the Navier–Stokes equations coupled with the of the finite element approximations. The solutions of the underlying Stokes and Burgers problem are described in detail. The asymptotic solution obtained by the numerical stabilization is compared with the analytical solution of the Lid-driven cavity problem.

Introduction

We present stability analysis of the nonstationary Navier-Stokes equations fully discretized by the finite element method and fractional step Θ-scheme.

The main advantage of this approach, developed by R.Glowinski in [G II], consists in the separation of the two underlying problems: solution of the nonlinear parabolic equation and the imposed incompressibility constrain. The method consists in the subsequent solution of the Generalized Stokes problem (linear parabolic equation with the incompressibility condition), of the solution of the Regularized Burgers problem (nonlinear parabolic equation), and, of the correction of the incompressibility condition by solving again the Generalized Stokes problem.

For the sake of completeness we have included a detailed description of the numerical solution of these two underlying problems. The central idea behind the use of the conjugate gradient (CG) solution of the Stokes and Burgers problems is to reduce the update procedure to the solution of the Poisson equation. The implementation can be highly optimized since we deal with just one type of the linear system. On the other hand the use of the CG method to solve the Regularized Burgers problem limits the range of the characteristic scales of the numerical flow.

The paper contains also example comparing the numerical assymptotic solution with the analytical solution of the simple Lid-driven cavity flow. The cavity computations suggest that the method works up to $Re = 3000$ (with the Taylor-Hood velocity/pressure approximation). We observed production of a considerable amount

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of oscillations in the kinetic energy prohibiting the convergence of the CG method for the Regularized Burgers problem above this range.

The paper is organized as follows:

- In section 1 we recall the known theoretical results about the Navier-Stokes equations.
- In Section 2 we describe the Fractional step Θ-time stepping scheme. We also describe the solution of the corresponding Generalized Stokes and Regularized Burgers problems.
- In Section 3 we recall the results about the conforming and nonconforming finite element approximations of the Navier–Stokes equations .
- Section 4 contains the proof of the stability of the fractional step Θ-scheme and the convergence of the finite element approximations.
- In Section 5 we recall the analytically solvable Lid-driven cavity problem to discuss some of the properties of the method.

The derivation of the stability conditions leading to the proof of the Theorem is rather long. The reason is that we cannot use all of the properties of the nonlinear term because of the form of the scheme.

1. Navier–Stokes Equations

The flow of a viscous incompressible fluid is frequently described by the following initial–boundary value problem:

$$\text{(NS)} \qquad \begin{aligned} \vec{u}_t - \nu \triangle \vec{u} + (\vec{u}.\nabla)\vec{u} + \nabla p = \vec{f}, &\quad \text{in } \Omega \times (0,T), \\ \nabla.\vec{u} = 0, &\quad \text{in } \Omega \times (0,T), \\ \vec{u} = \vec{g}, &\quad \text{on } \partial\Omega, \\ u(0) = \vec{u}_0, &\quad \text{in } \Omega, \end{aligned}$$

where $\vec{u} \equiv \vec{u}(x,t), p \equiv p(x,t)$ are the unknown velocity and pressure, respectively, and Ω is a bounded region in $\mathbb{R}^n, n = 2,3$, with a sufficiently regular boundary $\partial\Omega$. Given data of the initial–boundary value problem are the viscosity, ν, the volume force, $\vec{f} \equiv \vec{f}(x,t)$, the boundary velocity, $\vec{g} = \vec{g}(x,t)$, and the initial velocity, $\vec{u}_0 \equiv \vec{u}_0(x)$. The density of the fluid is normalized. We assume that $\int_\Omega \vec{g}.\vec{n}\, dx = 0$, $\vec{n}$ being the unit outward normal on $\partial\Omega$.

Functional setting. The following subsets of the space $[H^1(\Omega)]^n$ will be frequently used

$$H_g(\Omega) = \{\vec{v} \in [H^1(\Omega)]^n \mid \vec{v}|_{\partial\Omega} = \vec{g}\},$$
$$V_g(\Omega) = \{\vec{v} \in [H^1(\Omega)]^n \mid \nabla.\vec{v} = 0, \vec{v}|_{\partial\Omega} = \vec{g}\}.$$

We also use the spaces $H_0(\Omega)$ and $V_0(\Omega)$, which are defined as above, provided the functions have zero trace on the boundary $\partial\Omega$. Sometimes we refer to these spaces as either H or V. Also, $[H_0^1(\Omega)]^n$ is endowed with the product structure and consequently the spaces H, V are equipped with a scalar product and the Hilbert norm given by $((\vec{u},\vec{v})) \equiv \sum_{i=1}^n \int_\Omega \nabla u_i.\nabla v_i\, dx$ and by $\|\vec{u}\| = \sqrt{((\vec{u},\vec{u}))}$, respectively. All the considered spaces are Hilbert spaces for this norm. Consequently, the $[L^2(\Omega)]^n$ scalar product and the corresponding norm are given by $(\vec{u},\vec{u}) = \sum_{i=1}^n \int_\Omega u_i v_i \, dx$ and by $|\vec{u}| = \sqrt{(\vec{u},\vec{u})}$. For the usual Bochner spaces of the time dependent functions with values in some Banach space X we use the notation:

$$L^p(0,T;X) = \{u \mid u : (0,T) \to X, \text{measurable}; \int_0^T \|u(\tau)\|_X^p \, d\tau < +\infty\},$$

with the standard modification for $p = \infty$. Moreover, we denote

$$b(\vec{u},\vec{v},\vec{w}) \equiv ((\vec{u}.\nabla)\vec{v},\vec{w}) .$$

The form $b(.,.,.)$ is defined and trilinear continuous on $[H^1(\Omega)]^n \times [H^1(\Omega)]^n \times [H^1(\Omega)]^n$.

In this notation we define the weak solution of the problem (NS) as follows:

Definition. *Given $\vec{u}_0 \in V_0$ and $\vec{f} \in L^2(0,+\infty;[L^2(\Omega)]^n)$ the weak solution of the initial boundary value the problem (NS), defined on some interval $[0,T]$, $T > 0$, is a function $\vec{u}$ such that*

$$\text{(WNS)} \qquad \begin{gathered} \vec{u} \in L^\infty(0,T;V_g), \\ (\vec{u}_t,\vec{\varphi}) + \nu((\vec{u},\vec{\varphi})) + b(\vec{u},\vec{u},\vec{\varphi}) = (\vec{f},\vec{\varphi}), \quad \forall \vec{\varphi} \in V_0, \\ \vec{u}(0) = \vec{u}_0. \end{gathered}$$

We note that there exists unique weak solution to the problem (WNS) on some time interval $[0,T]$. Moreover, it is known that there exists unique strong solution $u \in L^\infty(0,T;V_g) \cap L^2(0,T;V_0 \cap [H^2(\Omega)]^n)$, $u_t \in L^2(0,T;[L^2(\Omega)]^n)$, for all $T > 0$, in two space dimensions, which is uniformly bounded in the H^1 norm, i.e. $\|\vec{u}(t)\| \leq C < +\infty$, where $C \equiv C(\nu,\vec{g},\vec{f},\Omega)$. In addition, the corresponding pressure p satisfying (NS) is in $L^2(0,T;H^1(\Omega))$. Under somewhat stronger assumptions $\vec{u}_0 \in [H^2(\Omega)]^n$ and $\vec{f}_t \in L^2(0,+\infty;[L^2(\Omega)]^n)$ we have $\vec{u} \in L^\infty(0,T;[H^1(\Omega)]^n)$, $u_t \in L^\infty(0,T;[L^2(\Omega)]^n) \cap L^2(0,T;[H^1(\Omega)]^n)$ and $p \in L^\infty(0,T;H^1(\Omega))$.
In the case of the three space dimensions the strong solution is known to exist and to be unique on some interval $[0,T_1]$, where $T_1 = C/(1+\|\vec{u}_0\|^2)^2$. The weak solution, which exists for all $T > 0$, coincides with the strong solution at least on the interval $[0,T_1]$ but its uniqueness is not proved. These results can be found e.g. in Heywood[H], Ladyzhenskaya[L], Temam[T I].

To avoid the difficulties of the discretization of the divergence-free space V it is convenient to work with the following formulation (in the form of the saddle point problem) which is equivalent to (WNS) for sufficiently regular functions:

Find $\{\vec{u},p\} \in L^\infty(0,T;H_g) \times L^2(\Omega)$ such that

$$\text{(SNS)} \qquad \begin{aligned} (\vec{u}_t,\vec{\varphi}) + \nu((\vec{u},\vec{\varphi})) + b(\vec{u},\vec{u},\vec{\varphi}) - (p,\nabla.\vec{\varphi}) &= (\vec{f},\vec{\varphi}), \quad \forall \vec{\varphi} \in H_0, \\ (\nabla.\vec{u},q) &= 0, \qquad\quad \forall q \in L^2(\Omega), \\ \vec{u}(0) &= \vec{u}_0. \end{aligned}$$

The pressure p is unique in $L^2(\Omega)/\mathbb{R}$ in both definitions (WNS) and (SNS) .

2. Fractional step method and Navier-Stokes equations

Fractional step (or splitting up) methods can be applied to the Navier–Stokes equations in many different ways. We will use the version advocated in R.Rannacher [R], where some more details and comparisons with other time stepping schemes (Euler, Crank–Nicolson, etc.) can be found.

In order to treat the main difficulties of the problem, the incompressibility and the nonlinearity, separately, we define Generalized Stokes problem and Regularized Burgers problem by splitting up the sum $\nu((\vec{u},\vec{\varphi}))+b(\vec{u},\vec{u},\vec{\varphi})$ into two parts, $\alpha\nu((\vec{u},\vec{\varphi}))$ and $(1-\alpha)\nu((\vec{u},\vec{\varphi}))+b(\vec{u},\vec{u},\vec{\varphi})$, considered on the space V and the space H, respectively.

Denoting $\vec{u}^n \equiv \vec{u}(x,t_n)$, $\vec{f}^n \equiv \vec{f}(x,t_n)$, simple version of the fractional step method applied to the Navier–Stokes equations reads:

Generalized Stokes problem. Find $\{\vec{u}^{n+\frac{1}{2}},p^{n+\frac{1}{2}}\}\in\{H_g\times L^2(\Omega)\}$ such that

$$\begin{gathered}\frac{1}{\triangle t}(\vec{u}^{n+\frac{1}{2}},\vec{\varphi})+\alpha\nu((\vec{u}^{n+\frac{1}{2}},\vec{\varphi}))-(p^{n+\frac{1}{2}},\nabla.\vec{\varphi})=\\ \frac{1}{\triangle t}(\vec{u}^{n},\vec{\varphi})-(1-\alpha)\nu((\vec{u}^{n},\vec{\varphi}))-b(\vec{u}^n,\vec{u}^n,\vec{\varphi})+(\vec{f}^n,\vec{\varphi}),\quad\forall\vec{\varphi}\in H_0,\\ (\nabla.\vec{u}^{n+\frac{1}{2}},q)=0,\quad\forall q\in L^2(\Omega).\end{gathered}\tag{FRNS1}$$

Regularized Burgers problem. Find $\vec{u}^{n+1}\in H_g$ such that

$$\begin{gathered}\frac{1}{\triangle t}(\vec{u}^{n+1},\vec{\varphi})+(1-\alpha)\nu((\vec{u}^{n+1},\vec{\varphi}))+b(\vec{u}^{n+1},\vec{u}^{n+1},\vec{\varphi})=\\ \frac{1}{\triangle t}(\vec{u}^{n+\frac{1}{2}},\vec{\varphi})-\alpha\nu((\vec{u}^{n+\frac{1}{2}},\vec{\varphi}))+(p^{n+\frac{1}{2}},\nabla.\vec{\varphi})+(\vec{f}^{n+\frac{1}{2}},\vec{\varphi}),\quad\forall\vec{\varphi}\in H_0.\end{gathered}\tag{FRNS2}$$

2.1 Fractional step Θ-scheme for the Navier-Stokes equations.

Further refinement of the splitting up approach consists in introduction of further correction of the solution with respect to the incompressibility condition.

We divide the time interval $[t_n,t_{n+1}]$ of the length $\triangle t$ into the three subintervals $[t_n,t_{n+\Theta})$, $[t_{n+\Theta},t_{n+(1-\Theta)})$ and $[t_{n+(1-\Theta)},t_{n+1}]$ of the lengths $\Theta\triangle t$, $(1-2\Theta)\triangle t$ and $\Theta\triangle t$, respectively. Using this partition, we obtain $\vec{u}^{n+1}$ from $\vec{u}^n$ by first solving (FRNS1) for $\vec{u}^{n+\Theta}$ then by solving (FRNS2) for $\vec{u}^{n+(1-\Theta)}$ with $\vec{u}^{n+\frac{1}{2}}$ replaced by $\vec{u}^{n+\Theta}$ and finally by solving (FRNS1) for $\vec{u}^{n+1}$ with $\vec{u}^n$ replaced by $\vec{u}^{n+(1-\Theta)}$.

Different implementation of this method can be found in Glowinski [G II].

Let $\Theta_b\equiv 1-2\Theta$ and $\beta\equiv 1-\alpha$. Assuming $\Theta\in(0,\frac{1}{2})$ and $\alpha\in(0,1)$, the scheme in the saddle point formulation reads:

First step. Find $\{\vec{u}^{n+\Theta},p^{n+\Theta}\}\in H_g\times L^2(\Omega)$ such that

$$\begin{gathered}\frac{1}{\Theta\triangle t}(\vec{u}^{n+\Theta},\vec{\varphi})+\alpha\nu((\vec{u}^{n+\Theta},\vec{\varphi}))-(p^{n+\Theta},\nabla.\vec{\varphi})=\\ \frac{1}{\Theta\triangle t}(\vec{u}^{n},\vec{\varphi})-\beta\nu((\vec{u}^{n},\vec{\varphi}))-b(\vec{u}^n,\vec{u}^n,\vec{\varphi})+(\vec{f}^n,\vec{\varphi}),\quad\forall\vec{\varphi}\in H_0,\\ (\nabla.\vec{u}^{n+\Theta},q)=0,\quad\forall q\in L^2(\Omega).\end{gathered}$$

Second step. Find $\vec{u}^{n+(1-\Theta)} \in H_g$ such that

$$\frac{1}{\Theta_b \triangle t}(\vec{u}^{n+(1-\Theta)},\vec{\varphi}) + \beta\nu((\vec{u}^{n+(1-\Theta)},\vec{\varphi})) + b(\vec{u}^{n+(1-\Theta)},\vec{u}^{n+(1-\Theta)},\vec{\varphi}) =$$
$$\frac{1}{\Theta_b \triangle t}(\vec{u}^{n+\Theta},\vec{\varphi}) - \alpha\nu((\vec{u}^{n+\Theta},\vec{\varphi})) + (p^{n+\Theta},\nabla.\vec{\varphi}) + (\vec{f}^{n},\vec{\varphi}), \quad \forall\vec{\varphi} \in H_0.$$

Third step. Find $\{\vec{u}^{n+1}, p^{n+1}\} \in H_g \times L^2(\Omega)$ such that

$$\frac{1}{\Theta\triangle t}(\vec{u}^{n+1},\vec{\varphi}) + \alpha\nu((\vec{u}^{n+1},\vec{\varphi})) - (p^{n+1},\nabla.\vec{\varphi}) = \frac{1}{\Theta\triangle t}(\vec{u}^{n+(1-\Theta)},\vec{\varphi})$$
$$-\beta\nu((\vec{u}^{n+(1-\Theta)},\vec{\varphi})) - b(\vec{u}^{n+(1-\Theta)},\vec{u}^{n+(1-\Theta)},\vec{\varphi}) + (\vec{f}^{n+1},\vec{\varphi}), \quad \forall\vec{\varphi} \in H_0,$$
$$(\nabla.\vec{u}^{n+1}, q) = 0, \quad \forall q \in L^2(\Omega).$$

On each time level we have to solve two saddle point problems and one nonlinear Dirichlet problem.

2.2 Solution of the Stokes problem.

The first and the third step can be solved by the modified version of the conjugate gradient algorithm (c.f. R.Glowinski [G I], Chp. VII, §5.8.7.3.4). This method appears to be quite robust and efficient.

Consider $W \subset L^2(\Omega)$ defined by $W = \{q \in L^2(\Omega) \mid \int_\Omega q(x)\,dx = 0\}$. The solution method is based on the following Proposition, proved in Crouzeix [C].

Proposition. *Let operator $\mathcal{A} : L^2(\Omega) \to L^2(\Omega)$ be defined by*

$$\mathcal{A}q = \nabla.\vec{v},$$

where, for $q \in L^2(\Omega)$, the function $\vec{v} \in H_0$ is the solution of the equation

$$\beta\vec{v} - \nu \triangle \vec{v} = -\nabla q, \quad \text{in } \Omega. \tag{2.2.1}$$

For $\beta > 0$ and $\nu > 0$ the operator $\mathcal{A}$ is W–elliptic, self-adjoint and automorphic from W to W. □

We apply the Proposition in the following way. Let $\{\vec{u}, p\} \in H_g \times L^2(\Omega)$ be the weak solution of the following Stokes problem

$$\begin{aligned} \alpha\vec{u} - \nu \triangle \vec{u} + \nabla p &= \vec{f}, \quad \text{in } \Omega,\\ \nabla.\vec{u} &= 0, \quad \text{in } \Omega,\\ \vec{u} &= \vec{g}, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.2.2}$$

and let $\vec{u}_0 \in H_g$ be the weak solution of the Dirichlet problem

$$\begin{aligned} \beta\vec{u}_0 - \nu \triangle \vec{u}_0 &= \vec{f}, \quad \text{in } \Omega,\\ \vec{u}_0 &= \vec{g}, \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2.3}$$

Existence of the strong solutions (i.e. solutions in $H^2(\Omega) \times H^1(\Omega)$ and $H^2(\Omega)$, respectively) in both cases justifies the subtraction of (2.2.2) and (2.2.3) and we obtain

$$\alpha(\vec{u} - \vec{u}_0) - \nu \triangle (\vec{u} - \vec{u}_0) = -\nabla p, \quad \text{in } \Omega,$$
$$\vec{u} - \vec{u}_0 \in H_0.$$

Applying the above Proposition and the incompressibility condition, it follows that the pressure is given by the unique solution in $L^2(\Omega)/\mathbb{R}$ of the equation

$$\text{(SP)} \qquad \mathcal{A}p = -\nabla.\vec{u}_0.$$

2.3 Conjugate gradient solution of the problem (SP). For arbitrary $p^0 \in L^2(\Omega)$ and $\vec{u}^0$ being the solution of (2.2.3) we set $z^0 \equiv \nabla.\vec{u}^0$ and $w^0 \equiv z^0$. Then, for $n \geq 0$, we compute $p^{n+1}, z^{n+1}, w^{n+1}$ from p^n, z^n, w^n as follows.

First step (Descent).

$$\rho_n = \frac{(z^n, w^n)}{(\mathcal{A}w^n, w^n)}, \quad p^{n+1} = p^n - \rho_n w^n.$$

Second step (New descent direction).

$$\begin{aligned} z^{n+1} &= z^n - \rho_n \mathbb{A} w^n, \\ \gamma_n &= \frac{|z^{n+1}|^2}{|z^n|^2}, \\ w^{n+1} &= z^{n+1} + \gamma_n w^n. \end{aligned}$$

Since the operator $\mathcal{A}$ is defined through the solution of (2.2.1), we use the following implementation of the algorithm (2.3).

Initialization. Let $p^0 \in H^1(\Omega)$ be arbitrary. Find $\vec{u}^0 \in H_g$ such that

$$\alpha(\vec{u}^0, \vec{\varphi}) + \nu((\vec{u}^0, \vec{\varphi})) = (\vec{f}, \vec{\varphi}) - (\nabla p^0, \vec{\varphi}), \quad \forall \vec{\varphi} \in H_0.$$

Find $z^0 \in H^1(\Omega)$ such that

$$\begin{aligned} (z^0, \phi) &= (\nabla.\vec{u}^0, \phi), \quad \forall \phi \in H^1(\Omega), \\ w^0 &\equiv z^0. \end{aligned}$$

Then, for $n \geq 0$, we compute $p^{n+1}, z^{n+1}, w^{n+1}$ from p^n, z^n, w^n as follows.

First step (Descent). Find $\vec{\chi} \in H_0$ such that

$$\alpha(\vec{\chi}^n, \vec{\varphi}) + \nu((\vec{\chi}^n, \vec{\varphi})) = -(\nabla w^n, \vec{\varphi}), \quad \forall \vec{\varphi} \in H_0.$$

Find $\vartheta \in H^1(\Omega)$ such that

$$\begin{aligned} (\vartheta^n, \phi) &= (\nabla.\vec{\chi}^n, \phi), \quad \forall \phi \in H^1(\Omega), \\ \rho_n &= \frac{(z^n, w^n)}{(\vartheta^n, w^n)}, \quad \text{and} \\ p^{n+1} &= p^n - \rho_n w^n. \end{aligned}$$

Second step (New descent direction). Find $z^{n+1} \in H^1(\Omega)$ such that

$$(z^{n+1}, \phi) = (p^{n+1}, \phi), \quad \forall \phi \in H^1(\Omega),$$
$$\gamma_n = \frac{|z^{n+1}|^2}{|z^n|^2},$$
$$w^{n+1} = z^{n+1} + \gamma_n w^n.$$

Stop condition: $\rho_n \|w^n\| \leq \varepsilon \ll 1$.

Remark 1. The above version differs through the introduction of $\vartheta \in H^1(\Omega)$ from the version described in Glowinski [G I]. □

Remark 2. Corresponding velocity is computed by Initialization step with p^0 replaced by p^{n+1}. The Initialization step for the first and the third step of the Θ-scheme reads:

Find $\vec{u}^0 \in H_g$ such that

$$\frac{1}{\Theta \triangle t}(\vec{u}^0, \vec{\varphi}) + \alpha\nu((\vec{u}^0, \vec{\varphi})) = \frac{1}{\Theta \triangle t}(\vec{u}^m, \vec{\varphi}) - \beta\nu((\vec{u}^m, \vec{\varphi}))$$
$$-b(\vec{u}^m, \vec{u}^m, \vec{\varphi}) + (\vec{f}^m, \vec{\varphi}) + (p^m, \nabla . \vec{\varphi}), \quad \forall \vec{\varphi} \in H_0, \quad m = n, \text{ or } n + \Theta.$$

For $m = n + \Theta, \vec{f}^m = \vec{f}^{n+1}$. We use the pressure from the preceding time level as the starting value instead of an arbitrary choice of p^0. □

2.4 Solution of the Regularized Burgers problem. We use the conjugate gradient method to minimize the cost functional J arising from the least square formulation.

The second step of the Θ-method consists, in general, in solving the following problem. Find $\vec{u} \in H_g$ such that

$$\alpha_1 \vec{u} - \alpha_2 \triangle \vec{u} + (\vec{u}.\nabla)\vec{u} = \vec{f}, \quad \text{in } \Omega, \quad \alpha_1, \alpha_2 > 0.$$

We reformulate this problem to find the minimum of the quadratic functional J.

Let $\vec{\xi} \equiv \vec{\xi}(\vec{w}) \in H_0$ be the solution of the state equation with $\vec{w} \in H_g$:

$$\alpha_1(\vec{\xi}, \vec{\varphi}) + \alpha_2((\vec{\xi}, \vec{\varphi})) =$$
$$\alpha_1(\vec{w}, \vec{\varphi}) + \alpha_2((\vec{w}, \vec{\varphi})) + b(\vec{w}, \vec{w}, \vec{\varphi}) - (\vec{f}, \vec{\varphi}), \quad \forall \vec{\varphi} \in H_0.$$

We define the functional $J : H_g \to \mathbb{R}$ by

$$J(\vec{w}) = \frac{\alpha_1}{2}|\vec{\xi}|^2 + \frac{\alpha_2}{2}||\vec{\xi}||^2.$$

Now, the problem in the least square formulation reads: Find $\vec{u} \in H_g$ such that

$$\text{(BP)} \qquad J(\vec{u}) \leq J(\vec{w}), \quad \forall \vec{w} \in H_g.$$

2.5 Conjugate gradient minimization.

We solve the problem (BP) using the well-known Polack-Ribier strategy in implementing the conjugate gradient method to minimize the cost functional.

Initialization. Let $\vec{u}^0 \in H_g$ is arbitrary. Find $\vec{g}^0 \in H_0$ such that

$$\alpha_1(\vec{g}^0, \vec{\varphi}) + \alpha_2((\vec{g}^0, \vec{\varphi})) = < J'(\vec{u}^0), \vec{\varphi} >, \quad \forall \vec{\varphi} \in H_0,$$
$$\vec{z}^0 \equiv \vec{g}^0.$$

Here, $< .,. >$ is the duality pairing between H_0 and H_0^*. We have for the right-hand side

$$< J'(\vec{u}^0), \vec{\varphi} >= \alpha_1(\vec{\xi}, \vec{\varphi}) + \alpha_2((\vec{\xi}, \vec{\varphi})) + ((\vec{\varphi}.\nabla)\vec{u}, \vec{\xi}) + ((\vec{u}.\nabla)\vec{\varphi}, \vec{\xi}).$$

Then, for $n \geq 0$, we compute $\vec{u}^{n+1}, \vec{g}^{n+1}, \vec{z}^{n+1}$ from $\vec{u}^n, \vec{g}^n, \vec{z}^n$ as follows.

First step (Line minimization). Find $\lambda^n \in \mathbb{R}$ such that

$$J(\vec{u}^n - \lambda^n \vec{z}^n) \leq J(\vec{u}^n - \lambda \vec{z}^n), \quad \forall \lambda \in \mathbb{R},$$
$$\vec{u}^{n+1} = \vec{u}^n - \lambda^n \vec{z}^n.$$

Second step (New descent direction). Find $\vec{g}^{n+1} \in H_0$ such that

$$\alpha_1(\vec{g}^{n+1}, \vec{\varphi}) + \alpha_2((\vec{g}^{n+1}, \vec{\varphi})) = < J'(\vec{u}^{n+1}), \vec{\varphi} >, \quad \forall \in H_0.$$

Set (Polack–Ribier strategy)

$$\gamma_{n+1} = \frac{\alpha_1(\vec{g}^{n+1}, \vec{g}^{n+1} - \vec{g}^n) + \alpha_2((\vec{g}^{n+1}, \vec{g}^{n+1} - \vec{g}^n))}{\alpha_1 \left|\vec{g}^n\right|^2 + \alpha_2 \left\|\vec{g}^n\right\|^2}, \qquad \text{and}$$
$$\vec{z}^{n+1} = \vec{g}^{n+1} + \gamma_{n+1}\vec{z}^n.$$

Stop condition: $J(\vec{u}^{n+1}) \leq \varepsilon \ll 1$.

In order to implement this method for the solution of the Regularized Burgers step in the Θ-scheme we have to modify the state equation. Now, $\vec{\xi} \in H_0$ is the solution of the following problem:

$$\frac{1}{\Theta_b \triangle t}(\vec{\xi}, \vec{\varphi}) + \beta\nu((\vec{\xi}, \vec{\varphi})) = \frac{1}{\Theta_b \triangle t}(\vec{w}, \vec{\varphi}) + \beta\nu((\vec{w}, \vec{\varphi})) + b(\vec{w}, \vec{w}, \vec{\varphi})$$
$$-\frac{1}{\Theta_b \triangle t}(\vec{u}^{n+\Theta}, \vec{\varphi}) + \beta\nu((\vec{u}^{n+\Theta}, \vec{\varphi})) - (p^{n+\Theta}, \nabla.\vec{\varphi}) - (\vec{f}, \vec{\varphi}), \quad \forall \vec{\varphi} \in H_0.$$

We set $\vec{u}^{n+(1-\Theta)} \equiv \vec{u}^n$ if $J(\vec{u}^n) \leq \varepsilon \ll 1$.

3. Spatial approximation of the solution

The squeezing property of the solution of (WNS) indicates that the flow is essentially characterized by a finite number of parameters. Taking into account that also the number of determinig modes is finite (c.f. P.Constantin *et al.* [CFT]), there is some theoretical justification to the use of the Galerkin approximation of the Navier–Stokes equations in the spaces H_h and V_h, with $\dim H_h = \dim V_h = nN(h)$.

When the solution of (WNS) converge to a permanent regime this approach does not always leads to acceptable results. Permanent regime is usually characterized by the presence of the small scale structures which are difficult to capture by the Galerkin approximation.

The existence of some kind of attracting sets for the solution of the Navier–Stokes equations may lead to the construction of a nonlinear system for the functions representing different modes (scales) of the solution. Capturing the interactions of the different modes can help to deal with difficult flows.

Yet, the Galerkin approximation remains the most reliable approximation method so far, as it is neither clear which modes determine the qualitative properties of the solution (e.g. the time of the transition to turbulence) nor it is established in general how to determine them.

3.1 Mixed finite element approximation.
The choice of finite element spaces is basically driven by the approximation of the Generalized Stokes problem. Let $H_h(\Omega)$ and $L^2_h(\Omega)$, with $\dim H_h(\Omega) = nN(h)$, $\dim L^2_h(\Omega) = M(h)$, are the finite element approximations of the spaces $[H^1(\Omega)]^n, L^2(\Omega)$, where h is parameter of the mesh size. We define H_{gh}, V_{gh}, H_{0h}, V_{0h} by

$$\begin{aligned} H_{gh}(\Omega) &= \{\vec{v}_h \in H^1_h(\Omega) \mid \vec{v}_h|_{\partial\Omega} = \vec{g}\}, \\ V_{gh}(\Omega) &= \{\vec{v}_h \in H^1_h(\Omega) \mid (\nabla.\vec{v}_h, q) = 0, \forall q_h \in L^2_h(\Omega), \vec{v}_h|_{\partial\Omega} = \vec{g}\}, \\ V_{0h}(\Omega) &= \{\vec{v}_h \in H^1_h(\Omega) \mid (\nabla.\vec{v}_h, q) = 0, \forall q_h \in L^2_h(\Omega), \vec{v}_h|_{\partial\Omega} = 0\}, \end{aligned}$$

and $H_{0h} = H_h(\Omega) \cap [H^1_0(\Omega)]^n$.

Replacing the spaces V, H, L^2 by their finite dimensional equivalents in (WNS) and (SNS) we obtain the Galerkin approximations $\vec{u}_h \in L^\infty(0,T;V_{gh})$ and $\{\vec{u}_h, p_h\} \in L^\infty(0,T;H_{gh}) \times L_h(\Omega)$ of the weak solution of (WNS) and (SNS).

3.2 Properties of the approximate spaces. Let $\{Q_h\}_h$ be a family of quadrilaterals so that $\bar{\Omega} = \bigcup_{Q\in Q_h} Q$. We set $h(Q)$ = length of the longest diagonal of Q, and $h = \max_{Q\in Q_h} h(Q)$. We suppose

$$\text{(A)} \qquad \frac{h}{\min\limits_{Q\in Q_h} h(Q)} \le \beta, \quad \forall Q_h.$$

The decompositions Q_h are assumed to be "face to face".

Finite element spaces approximating the "velocity space" $[H^1(\Omega)]^n$ and the "pressure space" $L^2(\Omega)$ have, in general, the forms

$$H_h(\Omega) = \{\vec{v}_h \in [L^2(\Omega)]^n \mid \vec{v}_h|_Q \in [P(Q)]^n, \forall Q \in Q_h\},$$
$$L_h(\Omega) = \{\vec{q}_h \in [L^2(\Omega)]^n \mid \vec{q}_h|_Q \in [R(Q)]^n, \forall Q \in Q_h\},$$

where $P(Q)$ and $R(Q)$ are spaces of polynomials, the maximum degree of which is bounded uniformly with respects to $Q \in Q_h$ and h. We refer to the approximation H_h as conforming (and we denote this space H_h^1) if the polynomial space $P(Q)$ is such that $H_h \subset [H^1(\Omega)]^n$ and nonconforming otherwise. Moreover, we assume:

(B) Each $\vec{v}_h \in H_h(\Omega)$ satisfies $\int_\Gamma(\vec{v}_h|_Q - \vec{v}_h|_{Q'})dS = 0, \Gamma = Q \cap Q', Q, Q' \in Q_h$ and $\int_\Gamma \vec{v}_h|_Q dS = 0, \Gamma = \partial\Omega \cap Q, Q \in Q_h$.

(C) For each $\vec{v} \in [H_0^1(\Omega)]^n \cap [H^2(\Omega)]^n$ and $q \in H^1(\Omega)$ there exist approximations $\Pi_h\vec{v} \in H_h(\Omega)$ and $\pi_h q \in L_h(\Omega)$ so that $\|\nabla_h(\vec{v} - \Pi_h\vec{v})\| \leq Ch\,\|\vec{v}\|_2$ and $|q - \pi_h q|_{L^2(\Omega)/\mathbb{R}} \leq Ch\,\|q\|_{L^2(\Omega)/\mathbb{R}}$.

The "discrete gradient" ∇_h is understood in the piecewise sense with respect to the decomposition of Q_h.

To ensure that the discrete pressure $p_h(t) \in L_h(\Omega)$ depends continuously on the discrete velocity $\vec{u}_h \in V_{gh}(\Omega)$ uniformly in h, we assume classical "inf–sup" condition for the mixed finite element methods (Babuška–Brezzi–Ladyzhenskaya condition):

For every $q_h \in L_h(\Omega)$, there exists a nontrivial function $\vec{\phi}_h \in V_{gh}(\Omega)$ such that

$$\textbf{(D)} \qquad \inf_{q_h \in L_h(\Omega)} \sup_{\phi_h \in V_{gh}(\Omega)} \frac{(\nabla_h.\vec{\phi}_h, q_h)}{||\vec{\phi}_h||.\,|q_h|} \geq \gamma > 0.$$

4. Stability and convergence of the fractional step Θ-scheme

The proof of the stability and convergence of the scheme follow the framework developed in [T I,II]. The difficulty in obtaing the a priory estimates lies in the presence of the trilinear form on the right-hand side of the Stokes steps.

Technical preliminaries. We assume that the gradient operator has an extension ∇_h on the algebraic sum $[H_0^1]^n + H_h$ so that $\nabla = \nabla_h$ on $[H_0^1]^n$ and $((.,.))_h \equiv (\nabla_h\cdot, \nabla_h\cdot)_{L^2}$ gives the norm on H_h. This extension ∇_h of ∇ is understood in the nonconforming case in the piecewise sense with respect to the decomposition Q_h of Ω. Thus for $\phi \in [H_0^1]^n + H_h$ the term $\nabla_h\phi$ is the L^2-function over Ω which coincides with$\nabla\phi|_Q, \forall Q \in Q_h$.

Since the spaces H_h, V_h are finite dimensional they can be equipped with equivalent norms induced by $[L^2(\Omega)]^n$ and $[H_0^1(\Omega)]^n$. Moreover the condition (A) ensures that this equivalence can be written in the form:

$$D_1\,|u_h| \leq \|u_h\|_h \leq D_2 h^{-1}\,|u_h|\,. \tag{4.1}$$

The constant D_2 depends on the degree of polynomial approximation. In the following we will assume that $D_1 = D_2 = 1$.

For each $\vec{u}_h, \vec{v}_h, \vec{w}_h \in H_h$ we define

$$b_h(\vec{u}_h, \vec{v}_h, \vec{w}_h) \equiv \frac{1}{2}\Big(((\vec{u}_h \cdot \nabla_h)\vec{v}_h, \vec{w}_h) - ((\vec{u}_h \cdot \nabla_h)\vec{w}_h, \vec{v}_h)\Big). \tag{4.2}$$

It holds

$$b_h(\vec{u}_h, \vec{v}_h, \vec{v}_h) = 0, \quad \forall \vec{u}_h, \vec{v}_h \in H_h. \tag{4.3}$$

There exists a function $S(h)$ so that

$$|b_h(\vec{u}_h, \vec{v}_h, \vec{w}_h)| \leq S(h)\, |\vec{u}_h|\, \|\vec{v}_h\|_h\, |\vec{w}_h|\,, \tag{4.4}$$

and $S(h) = D_3 h^{-1}$, in the conforming case, and, $S(h) = D_3 h^{-(1+\epsilon)}$, in the non-conforming case, in 2D, and $S(h) = D_4 h^{-\frac{3}{2}}$ in 3D, where $\epsilon > 0$, $D_3 > 0$ and $D_4 > 0$.

We assume homogeneous boundary conditions in what follows. To approximate the initial condition we project $\vec{u}_0$ to V_{0h} by finding $u_{0h} \in V_{0h}$ such that

$$((\vec{u}_{0h}, \vec{\varphi}_h))_h = ((\vec{u}_0, \vec{\varphi}_h))_h, \quad \forall \vec{\varphi}_h \in V_{0h}. \tag{4.5}$$

From now on we consider the following fully discrete problems ($m \equiv n + (1 - \Theta)$).

Find $\vec{u}_h^{n+\Theta} \in V_{0h}$ such that

$$\begin{aligned} &\frac{1}{\Theta \triangle t}(\vec{u}_h^{n+\Theta}, \vec{\varphi}_h) + \alpha\nu((\vec{u}_h^{n+\Theta}, \vec{\varphi}_h))_h = \\ &\frac{1}{\Theta \triangle t}(\vec{u}_h^{n}, \vec{\varphi}_h) - \beta\nu((\vec{u}_h^{n}, \vec{\varphi}_h))_h - b_h(\vec{u}_h^{n}, \vec{u}_h^{n}, \vec{\varphi}_h) + (\vec{f}^n, \vec{\varphi}_h), \quad \forall \vec{\varphi}_h \in V_{0h}. \end{aligned} \tag{4.6}$$

Find $\vec{u}_h^{m} \in H_{0h}$ such that

(4.7)
$$\begin{aligned} &\frac{1}{\Theta_b \triangle t}(\vec{u}_h^{m}, \vec{\varphi}_h) + \beta\nu((\vec{u}_h^{m}, \vec{\varphi}_h))_h + b_h(\vec{u}_h^{m}, \vec{u}_h^{m}, \vec{\varphi}_h) = \\ &\frac{1}{\Theta_b \triangle t}(\vec{u}_h^{n+\Theta}, \vec{\varphi}_h) - \alpha\nu((\vec{u}_h^{n+\Theta}, \vec{\varphi}_h))_h + (p_h^{n+\Theta}, \nabla . \vec{\varphi}_h) + (\vec{f}^n, \vec{\varphi}_h), \quad \forall \vec{\varphi}_h \in H_{0h}. \end{aligned}$$

Find $\vec{u}_h^{n+1} \in V_{0h}$ such that

(4.8)
$$\begin{aligned} &\frac{1}{\Theta \triangle t}(\vec{u}_h^{n+1}, \vec{\varphi}_h) + \alpha\nu((\vec{u}_h^{n+1}, \vec{\varphi}_h))_h = \\ &\frac{1}{\Theta \triangle t}(\vec{u}_h^{m}, \vec{\varphi}_h) - \beta\nu((\vec{u}_h^{m}, \vec{\varphi}_h))_h - b_h(\vec{u}_h^{m}, \vec{u}_h^{m}, \vec{\varphi}_h) + (\vec{f}^{n+1}, \vec{\varphi}_h), \quad \forall \vec{\varphi}_h \in V_{0h}. \end{aligned}$$

We define $\vec{f}^n \equiv \frac{1}{\triangle t}\int_{t_{n-1}}^{t_n} f(t)\,dt$ and the pressure $p_h^{n+\Theta} \in L^2(\Omega)/\mathbb{R}$ we define by

$$(4.9)\qquad \begin{aligned}(p_h^{n+\Theta}, \nabla_h \cdot \vec{\varphi}_h) = \frac{1}{\Theta\triangle t}(\vec{u}_h^{n+\Theta} - \vec{u}_h^n, \vec{\varphi}_h) + \alpha\nu((\vec{u}_h^{n+\Theta}, \vec{\varphi}_h))_h + \\ \beta\nu((\vec{u}_h^n, \vec{\varphi}_h))_h + b_h(\vec{u}_h^n, \vec{u}_h^n, \vec{\varphi}_h) - (\vec{f}^n, \vec{\varphi}_h), \quad \forall\vec{\varphi}_h \in H_{0h}.\end{aligned}$$

Next, we define the following three families of functions $\vec{u}_{h,\triangle t}^{(i)} : [0,T] \to V_h$ (resp. H_h for $i = 3$)

$$(4.10)\qquad \left.\begin{aligned}\vec{u}_{h,\triangle t}^{(1)}(t) &= \vec{u}_h^n,\\ \vec{u}_{h,\triangle t}^{(2)}(t) &= \vec{u}_h^{n+\Theta},\\ \vec{u}_{h,\triangle t}^{(3)}(t) &= \vec{u}_h^{n+(1-\Theta)},\\ \vec{f}_{\triangle t}(t) &= \vec{f}^n,\end{aligned}\right\} \qquad t \in [n\triangle t, (n+1)\triangle t),$$

for $n = 1, 2, ..., N$, $N \cdot \triangle t = T$.

Correspondingly, we use the functions $\vec{w}_{h,\triangle t} : [0,T] \to V_h$:

$$(4.11)\qquad \vec{w}_{h,\triangle t}(t) = \vec{u}_h^n + \frac{1}{\triangle t}(t - t_n)(\vec{u}_h^{n+1} - \vec{u}_h^n), \quad t \in [n\triangle t, (n+1)\triangle t).$$

The functions defined above in (4.10,12) satisfy

$$(4.12)\qquad \begin{aligned}\frac{1}{\Theta}\frac{d}{dt}(\vec{w}_{h,\triangle t}, \vec{\varphi}_h) + \alpha\nu((\vec{u}_{h,\triangle t}^{(1)}(t+\triangle t), \vec{\varphi}_h))_h + \beta\nu((\vec{u}_{h,\triangle t}^{(1)}(t), \vec{\varphi}_h))_h = \\ \frac{1}{\Theta\triangle t}(\vec{u}_{h,\triangle t}^{(3)}(t) - \vec{u}_{h,\triangle t}^{(2)}(t), \vec{\varphi}_h) - \beta\nu((\vec{u}_{h,\triangle t}^{(3)}(t), \vec{\varphi}))_h - \alpha\nu((\vec{u}_{h,\triangle t}^{(2)}(t), \vec{\varphi}))_h - \\ b_h(\vec{u}_{h,\triangle t}^{(2)}(t), \vec{u}_{h,\triangle t}^{(2)}(t), \vec{\varphi}_h) - b_h(\vec{u}_{h,\triangle t}^{(3)}(t), \vec{u}_{h,\triangle t}^{(3)}(t), \vec{\varphi}_h) + (\vec{f}^n + \vec{f}^{n+1}, \vec{\varphi}_h),\end{aligned}$$

for all $\vec{\varphi}_h \in V_{0h}, t \in [n\triangle t, (n+1)\triangle t)$.

The results concerning the stability and convergence of the scheme (4.6–8) are given by the following

Theorem. *Let us assume that there exists unique strong solution of the problem (WNS), with homogeneous Dirichlet boundary condition, on the time interval $(0,T)$. Let us assume that $\Theta \in (0, 1 - \frac{\sqrt{2}}{2})$, $\alpha \in (0,1)$, $\delta \in (0, \frac{1}{3})$, and, let us assume that the time step $\triangle t$ and mesh size h are connected with the data by the following stability conditions*

$$(4.13)\qquad D_5\left(\left(\nu\frac{\triangle t}{h^2}\right)^2 + (\triangle t S(h))^2 \Lambda_T\right) \le (1-\delta)\Theta\triangle t\alpha\nu, \qquad \textit{where}$$

$$\Lambda_T \equiv |\vec{u}_0|^2 + D_5\left(\left(\nu\frac{\triangle t}{h^2}\right)^2 \|\vec{u}_0\|_h^2 + (\triangle t S(h))^2 |\vec{u}_0|^2 \|\vec{u}_0\|_h^2 + \left\|\vec{f}\right\|_{L^2(0,T,L^2(\Omega))}^2\right),$$

$$D_5 = \left(\frac{\Theta_b}{\delta}\right)^2\left(1 + \left(\frac{\Theta_b}{\Theta}\right)^2\right).$$

Then:

(1) *The functions* $\vec{u}^{(i)}_{h,\triangle t}$ *belong to a bounded set in* $L^2(0,T;V_h)\cap L^\infty(0,T;[L^2(\Omega)]^n)$, *for* $i=1,2$ *and to a bounded set in* $L^2(0,T;H_h)\cap L^\infty(0,T;[L^2(\Omega)]^n)$, *for* $i=3$.

(2) *These functions converge strongly to the unique strong solution of (WNS) in* $L^2(0,T;[H^1_0(\Omega)]^n)\cap L^q(0,T;[L^2(\Omega)]^n)$, *for* $q\in[1,\infty)$, *as* $(h,\triangle t)\to 0_+$, *in the case of conforming approximations and in* $L^q(0,T;[L^2(\Omega)]^n)$, *for* $q\in[2,\infty)$, *in the nonconforming case.*

The proof of the Theorem is given in the sections 4.1 (stability) and 4.2 (convergence).

4.1 Stability of the scheme. First, we obtain a priori estimates.

Step 1: We take $\vec{\varphi}_h=\vec{u}^{n+\Theta}_h$ in (4.6) to get

$$\begin{aligned}(4.1.1)\qquad \frac{1}{\Theta\triangle t}\left|\vec{u}^{n+\Theta}_h\right|^2+\alpha\nu\left\|\vec{u}^{n+\Theta}_h\right\|^2_h&=\frac{1}{\Theta\triangle t}(\vec{u}^n_h,\vec{u}^{n+\Theta}_h)-\beta\nu((\vec{u}^n_h,\vec{u}^{n+\Theta}_h))_h\\&-b_h(\vec{u}^n_h,\vec{u}^n_h,\vec{u}^{n+\Theta}_h)+(\vec{f}^n,\vec{u}^{n+\Theta}_h)\end{aligned}$$

Since the right-hand side of (4.1.1) is equal to
(4.1.2)

$$\begin{aligned}&-\frac{1}{2\Theta\triangle t}\left|\vec{u}^{n+\Theta}_h-\vec{u}^n_h\right|^2+\frac{1}{2\Theta\triangle t}\left|\vec{u}^{n+\Theta}_h\right|^2+\frac{1}{2\Theta\triangle t}\left|\vec{u}^n_h\right|^2+\frac{\beta\nu}{2}\left\|\vec{u}^{n+\Theta}_h-\vec{u}^n_h\right\|^2_h\\&\quad-\frac{\beta\nu}{2}\left\|\vec{u}^n_h\right\|^2_h-\frac{\beta\nu}{2}\left\|\vec{u}^{n+\Theta}_h\right\|^2_h-b_h(\vec{u}^n_h,\vec{u}^n_h,\vec{u}^{n+\Theta}_h)+(\vec{f}^n,\vec{u}^{n+\Theta}_h),\end{aligned}$$

and, since by (4.1) we have $\left\|\vec{u}^{n+\Theta}_h-\vec{u}^n_h\right\|^2_h\le\frac{1}{h^2}\left|\vec{u}^{n+\Theta}_h-\vec{u}^n_h\right|^2$, we obtain
(4.1.3)

$$\begin{aligned}&\left|\vec{u}^{n+\Theta}_h\right|^2+(1+\alpha)\Theta\triangle t\nu\left\|\vec{u}^{n+\Theta}_h\right\|^2_h+\Theta\triangle t\beta\nu\left\|\vec{u}^n_h\right\|^2_h+\\&\left(1-\frac{\Theta\triangle t\beta\nu}{h^2}\right)\left|\vec{u}^{n+\Theta}_h-\vec{u}^n_h\right|^2\le\left|\vec{u}^n_h\right|^2-2\Theta\triangle t b_h(\vec{u}^n_h,\vec{u}^n_h,\vec{u}^{n+\Theta}_h)+2\Theta\triangle t(\vec{f}^n,\vec{u}^{n+\Theta}_h).\end{aligned}$$

We estimate the right-hand side of (4.1.3) by the Young's inequality and (4.3,4). Because $0<\Theta<1-\frac{\sqrt{2}}{2}$ we have $2\Theta^2<\Theta_b^2$ and we obtain

$$(4.1.4)\quad \tfrac{1}{2}\left|\vec{u}^{n+\Theta}_h-\vec{u}^n_h\right|^2+(\Theta_b\triangle tS(h))^2\left|\vec{u}^n_h\right|^2\left\|\vec{u}^n_h\right\|^2_h+\Theta\triangle t\nu\left\|\vec{u}^{n+\Theta}_h\right\|^2_h+\frac{\Theta\triangle t}{\nu}\left|\vec{f}^n\right|^2.$$

The last two inequalities yield
(4.1.5)

$$\begin{aligned}&\left|\vec{u}^{n+\Theta}_h\right|^2+\Theta\triangle t\alpha\nu\left\|\vec{u}^{n+\Theta}_h\right\|^2_h+\Theta\triangle t\beta\nu\left\|\vec{u}^n_h\right\|^2_h+\left(\frac{1}{2}-\frac{\Theta\triangle t\beta\nu}{h^2}\right)\left|\vec{u}^{n+\Theta}_h-\vec{u}^n_h\right|^2\le\\&\qquad\left|\vec{u}^n_h\right|^2+(\Theta_b\triangle tS(h))^2\left|\vec{u}^n_h\right|^2\left\|\vec{u}^n_h\right\|^2_h+\frac{\Theta\triangle t}{\nu}\left|\vec{f}^n\right|^2.\end{aligned}$$

Step 2: We take $\vec{\varphi}_h = \vec{u}_h^m \in H_{0h}$ in (4.7) and using (4.3) we get

$$\frac{1}{\Theta_b \triangle t} |u_h^m|^2 + \beta\nu \|u_h^m\|_h^2 = \tag{4.1.6}$$
$$\frac{1}{\Theta_b\triangle t}(\vec{u}_h^{n+\Theta}, \vec{u}_h^m) - \alpha\nu((\vec{u}_h^{n+\Theta}, \vec{u}_h^m))_h + (p_h^{n+\Theta}, \nabla_h \cdot \vec{u}_h^m) + (\vec{f}^n, \vec{u}_h^m).$$

Taking $\vec{\varphi}_h = \vec{u}_h^m - \vec{u}_h^{n+\Theta} \in H_{0h}$ in (4.9) we obtain

(4.1.7)
$$(p_h^{n+\Theta}, \nabla_h \cdot \vec{u}_h^m) = (p_h^{n+\Theta}, \nabla_h \cdot (\vec{u}_h^m - \vec{u}_h^{n+\Theta})) = \frac{1}{\Theta\triangle t}(\vec{u}_h^{n+\Theta} - \vec{u}_h^n, \vec{u}_h^m - \vec{u}_h^{n+\Theta})$$
$$+\alpha\nu((\vec{u}_h^{n+\Theta}, \vec{u}_h^m - \vec{u}_h^{n+\Theta}))_h + \beta\nu((\vec{u}_h^n, \vec{u}_h^m - \vec{u}_h^{n+\Theta}))_h + b_h(\vec{u}_h^n, \vec{u}_h^n, \vec{u}_h^m - \vec{u}_h^{n+\Theta})$$
$$-(\vec{f}^n, \vec{u}_h^m - \vec{u}_h^{n+\Theta}).$$

The above equality and (4.1.6) give

(4.1.8)
$$\tfrac{1}{2}|\vec{u}_h^m|^2 + \Theta_b\triangle t\beta\nu\|\vec{u}_h^m\|_h^2 + \tfrac{1}{2}\left|\vec{u}_h^m - \vec{u}_h^{n+\Theta}\right|^2 \le \tfrac{1}{2}\left|\vec{u}_h^{n+\Theta}\right|^2 - \Theta_b\triangle t\alpha\nu((\vec{u}_h^{n+\Theta}, \vec{u}_h^m))_h$$
$$+\frac{\Theta_b}{\Theta}\left|\vec{u}_h^n - \vec{u}_h^{n+\Theta}\right|\left|\vec{u}_h^m - \vec{u}_h^{n+\Theta}\right| - \Theta_b\triangle t\alpha\nu\left\|\vec{u}_h^{n+\Theta}\right\|_h^2 + \Theta_b\triangle t\alpha\nu((\vec{u}_h^{n+\Theta}, \vec{u}_h^m))_h$$
$$+\Theta_b\triangle t\beta\nu((\vec{u}_h^n, \vec{u}_h^m - \vec{u}_h^{n+\Theta}))_h + \Theta_b\triangle t b_h(\vec{u}_h^n, \vec{u}_h^n, \vec{u}_h^m - \vec{u}_h^{n+\Theta}) + \Theta_b\triangle t(\vec{f}^n, \vec{u}_h^{n+\Theta}).$$

Using the Young's inequality with $0 < \delta < \frac{1}{3}$, estimate (4.4) and (4.1), we obtain

$$|\vec{u}_h^m|^2 + \Theta_b\triangle t\beta\nu\|\vec{u}_h^m\|_h^2 + \Theta_b\triangle t\alpha\nu\left\|\vec{u}_h^{n+\Theta}\right\|_h^2 + (1-3\delta)\left|\vec{u}_h^m - \vec{u}_h^{n+\Theta}\right|^2 \le \tag{4.1.9}$$
$$\left|\vec{u}_h^{n+\Theta}\right|^2 + \frac{1}{2\delta}\left(\frac{\Theta_b}{\Theta}\right)^2\left|\vec{u}_h^{n+\Theta} - \vec{u}_h^n\right|^2 + \frac{1}{2\delta}\left(\frac{\Theta_b\triangle t\beta\nu}{h^2}\right)^2\|\vec{u}_h^n\|_h^2 +$$
$$\frac{1}{2\delta}(\Theta_b\triangle tS(h))^2|\vec{u}_h^n|^2\|\vec{u}_h^n\|_h^2 + \frac{\Theta_b\triangle t}{\alpha\nu}\left|\vec{f}^n\right|^2.$$

The equation for the pressure (4.9) with $\vec{\varphi}_h = \vec{u}_h^{n+\Theta} - \vec{u}_h^n \in V_{0h}$ reads

(4.1.10)
$$0 = -(p_h^{n+\Theta}, \nabla_h \cdot (\vec{u}_h^{n+\Theta} - \vec{u}_h^n)) = \frac{1}{\Theta\triangle t}\left|\vec{u}_h^{n+\Theta} - \vec{u}_h^n\right|^2 + \alpha\nu((\vec{u}_h^{n+\Theta}, \vec{u}_h^{n+\Theta} - \vec{u}_h^n))_h$$
$$+\beta\nu((\vec{u}_h^n, \vec{u}_h^{n+\Theta} - \vec{u}_h^n))_h + b_h(\vec{u}_h^n, \vec{u}_h^n, \vec{u}_h^{n+\Theta} - \vec{u}_h^n) - (\vec{f}^n, \vec{u}_h^{n+\Theta} - \vec{u}_h^n),$$

and gives the inequality

$$\left|\vec{u}_h^{n+\Theta} - \vec{u}_h^n\right|^2 + \Theta\triangle t\alpha\nu\left\|\vec{u}_h^{n+\Theta} - \vec{u}_h^n\right\|_h^2 \le \tag{4.1.11}$$
$$\left(\frac{\Theta_b\triangle t\nu}{h^2}\right)^2\|\vec{u}_h^n\|_h^2 + (\Theta_b\triangle tS(h))^2|\vec{u}_h^n|^2\|\vec{u}_h^n\|_h^2 + \frac{\Theta\triangle t}{\alpha\nu}\left|\vec{f}^n\right|^2.$$

Using the last estimate, the inequality (4.1.9) reads

$$|\vec{u}_h^m|^2 + \Theta_b\triangle t\beta\nu\|\vec{u}_h^m\|_h^2 + \Theta_b\triangle t\alpha\nu\left\|\vec{u}_h^{n+\Theta}\right\|_h^2 + \tag{4.1.12}$$
$$(1-3\delta)\left|\vec{u}_h^m - \vec{u}_h^{n+\Theta}\right|^2 \le \left|\vec{u}_h^{n+\Theta}\right|^2 +$$
$$D_5\left(\left(\nu\frac{\triangle t}{h^2}\right)^2\|\vec{u}_h^n\|_h^2 + (\triangle tS(h))^2|\vec{u}_h^n|^2\|\vec{u}_h^n\|_h^2 + \frac{\triangle t}{\alpha\nu}\left|\vec{f}^n\right|^2\right).$$

Step 3: We take $\vec{\varphi}_h = \vec{u}_h^{n+1} \in V_{0h}$ in (4.8) and we get

$$\begin{aligned}(4.1.13)\qquad &\frac{1}{\Theta\triangle t}\left|\vec{u}_h^{n+1}\right|^2 + \alpha\nu\left\|\vec{u}_h^{n+1}\right\|_h^2 = \\ &\frac{1}{\Theta\triangle t}(\vec{u}_h^m, \vec{u}_h^{n+1}) - \beta\nu((\vec{u}_h^m, \vec{u}_h^{n+1}))_h - b_h(\vec{u}_h^m, \vec{u}_h^m, \vec{u}_h^{n+1}) + (\vec{f}^{n+1}, \vec{u}_h^{n+1}).\end{aligned}$$

As in the step 1 we obtain

$$\begin{aligned}(4.1.14)\qquad &\left|\vec{u}_h^{n+1}\right|^2 + \Theta\triangle t(1+\alpha)\nu\left\|\vec{u}_h^{n+1}\right\|_h^2 + \Theta\triangle t\beta\nu\left\|\vec{u}_h^m\right\|_h^2 + \\ &\left(1 - \frac{\Theta\triangle t\beta\nu}{h^2}\right)\left|\vec{u}_h^{n+1} - \vec{u}_h^m\right|^2 \le \left|\vec{u}_h^m\right|^2 - 2\Theta\triangle t b_h(\vec{u}_h^m, \vec{u}_h^m, \vec{u}_h^{n+1}) + \\ &2\Theta\triangle t(\vec{f}^{n+1}, \vec{u}_h^{n+1}).\end{aligned}$$

To evaluate $b_h(\vec{u}_h^m, \vec{u}_h^m, \vec{u}_h^{n+1})$ we use (4.7) with $\vec{\varphi}_h = \vec{u}_h^{n+1} - \vec{u}_h^m$

$$\begin{aligned}(4.1.15)\qquad &-b_h(\vec{u}_h^m, \vec{u}_h^m, \vec{u}_h^{n+1}) = -b_h(\vec{u}_h^m, \vec{u}_h^m, \vec{u}_h^{n+1} - \vec{u}_h^m) = \\ &\frac{1}{\Theta_b\triangle t}(\vec{u}_h^m - \vec{u}_h^{n+\Theta}, \vec{u}_h^{n+1} - \vec{u}_h^m) + \beta\nu((\vec{u}_h^m, \vec{u}_h^{n+1} - \vec{u}_h^m))_h + \\ &\alpha\nu((\vec{u}_h^{n+\Theta}, \vec{u}_h^{n+1} - \vec{u}_h^m))_h - (\vec{f}^n, \vec{u}_h^{n+1} - \vec{u}_h^m) - (p^{n+\Theta}, \nabla_h\cdot(\vec{u}_h^{n+1} - \vec{u}_h^m)).\end{aligned}$$

Since by the equation for the pressure (4.9)

$$\begin{aligned}(4.1.16)\qquad &-(p^{n+\Theta}, \nabla_h\cdot(\vec{u}_h^{n+1} - \vec{u}_h^m)) = -\frac{1}{\Theta\triangle t}(\vec{u}_h^{n+\Theta} - \vec{u}_h^n, \vec{u}_h^{n+1} - \vec{u}_h^m) \\ &-\alpha\nu((\vec{u}_h^{n+\Theta}, \vec{u}_h^{n+1} - \vec{u}_h^m))_h - \beta\nu((\vec{u}_h^n, \vec{u}_h^{n+1} - \vec{u}_h^m))_h - b_h(\vec{u}_h^n, \vec{u}_h^n, \vec{u}_h^{n+1} - \vec{u}_h^m) \\ &+(\vec{f}^n, \vec{u}_h^{n+1} - \vec{u}_h^m),\end{aligned}$$

we express $-2\Theta\triangle t b_h(\vec{u}_h^m, \vec{u}_h^m, \vec{u}_h^{n+1})$ in the form

$$\begin{aligned}(4.1.17)\qquad &2\frac{\Theta}{\Theta_b}(\vec{u}_h^m - \vec{u}_h^{n+\Theta}, \vec{u}_h^{n+1} - \vec{u}_h^m) - 2(\vec{u}_h^{n+\Theta} - \vec{u}_h^n, \vec{u}_h^{n+1} - \vec{u}_h^m) \\ &+2\Theta\triangle t\beta\nu((\vec{u}_h^m - \vec{u}_h^{n+\Theta}, \vec{u}_h^{n+1} - \vec{u}_h^m))_h - 2\Theta\triangle t b_h(\vec{u}_h^n, \vec{u}_h^n, \vec{u}_h^{n+1} - \vec{u}_h^m).\end{aligned}$$

Now, we obtain the following inequality from (4.1.14) using (4.1.17) and Schwarz' and Young's inequalities and (4.1)

$$\begin{aligned}(4.1.18)\qquad &\left|\vec{u}_h^{n+1}\right|^2 + \Theta\triangle t\alpha\nu\left\|\vec{u}_h^{n+1}\right\|_h^2 + \Theta\triangle t\beta\nu\left\|\vec{u}_h^m\right\|_h^2 \\ &+\left(\frac{1}{8} - \frac{\Theta\triangle t\beta\nu}{h^2}\right)\left|\vec{u}_h^{n+1} - \vec{u}_h^m\right|^2 \le \left|\vec{u}_h^m\right|^2 + 8\left(\frac{\Theta_b\triangle t\nu}{h^2}\right)^2\left\|\vec{u}_h^n\right\|_h^2 \\ &+2\left(\frac{\Theta}{\Theta_b}\right)^2\left|\vec{u}_h^m - \vec{u}_h^{n+\Theta}\right|^2 + 8\left|\vec{u}_h^{n+\Theta} - \vec{u}_h^n\right|^2 \\ &(2\Theta_b\triangle t S(h))^2\left|\vec{u}_h^n\right|^2\left\|\vec{u}_h^n\right\|_h^2 + \frac{\Theta\triangle t}{\nu}\left|\vec{f}^{n+1}\right|^2.\end{aligned}$$

For $0 < \Theta < 1 - \frac{\sqrt{2}}{2}$ there exists $\delta > 0$ such that $2\left(\frac{\Theta}{\Theta_b}\right)^2 < 1 - 3\delta$. The results of the step 1 and step 2, namely (4.1.12,11) and (4.1.5), provide the desired inequality

$$\begin{aligned} &\left|\vec{u}_h^{n+1}\right|^2 + \Theta\triangle t\alpha\nu \left\|\vec{u}_h^{n+1}\right\|_h^2 + \Theta\triangle t\beta\nu \left\|\vec{u}_h^{m}\right\|_h^2 \\ &+\left(\frac{1}{8} - \frac{\Theta\triangle t\beta\nu}{h^2}\right)\left|\vec{u}_h^{n+1} - \vec{u}_h^{m}\right|^2 \le \left|\vec{u}_h^{n}\right|^2 + D_5\frac{\triangle t}{2\alpha\nu}\left(\left|\vec{f}^n\right|^2 + \left|\vec{f}^{n+1}\right|^2\right) + \\ &D_5\left(\left(\frac{\triangle t\nu}{h^2}\right)^2 \left\|\vec{u}_h^{n}\right\|_h^2 + (\triangle tS(h))^2\left|\vec{u}_h^{n}\right|^2\left\|\vec{u}_h^{n}\right\|_h^2\right). \end{aligned} \tag{4.1.19}$$

Summing up the above inequality for $n = 1, 2, ..., r$ we obtain

$$\begin{aligned} &\left|u_h^{r+1}\right|^2 + \Theta\triangle t\alpha\nu \sum_{n=0}^{r}\left\|\vec{u}_h^{n+1}\right\|_h^2 + \Theta\triangle t\beta\nu\sum_{n=0}^{r}\left\|\vec{u}_h^{m}\right\|_h^2 \\ &+\left(\frac{1}{8} - \frac{\Theta\triangle t\beta\nu}{h^2}\right)\sum_{n=0}^{r}\left|\vec{u}_h^{n+1} - \vec{u}_h^{m}\right|^2 \le \left|\vec{u}_{0h}\right|^2 + D_5\frac{\triangle t}{\alpha\nu}\sum_{n=0}^{r+1}\left(\left|\vec{f}^n\right|^2\right) \\ &+D_5\left(\nu\frac{\triangle t}{h^2}\right)^2\sum_{n=0}^{r}\left\|\vec{u}_h^{n}\right\|_h^2 + D_5\left(\triangle tS(h)\right)^2\sum_{n=0}^{r}\left|\vec{u}_h^{n}\right|^2\left\|\vec{u}_h^{n}\right\|_h^2 \end{aligned} \tag{4.1.20}$$

To obtain the a priori estimates we proceed by induction in r. Since the left-hand side of (4.1.20) is bounded for $r = 0$ let us assume that

$$\begin{aligned} &\left|u_h^{r+1}\right|^2 + \delta\Theta\triangle t\alpha\nu \sum_{n=0}^{r}\left\|\vec{u}_h^{n+1}\right\|_h^2 + \Theta\triangle t\beta\nu\sum_{n=0}^{r}\left\|\vec{u}_h^{m}\right\|_h^2 \\ &+\left(\frac{1}{8} - \frac{\Theta\triangle t\beta\nu}{h^2}\right)\sum_{n=0}^{r}\left|\vec{u}_h^{n+1} - \vec{u}_h^{m}\right|^2 \le \left|\vec{u}_{0h}\right|^2 + D_5\left(\nu\frac{\triangle t}{h^2}\right)^2\left\|\vec{u}_{0h}\right\|_h^2 \\ &+D_5\left(\triangle tS(h)\right)^2\left|\vec{u}_{0h}\right|^2\left\|\vec{u}_{0h}\right\|_h^2 + D_5\frac{\triangle t}{\alpha\nu}\sum_{n=0}^{r+1}\left(\left|\vec{f}^n\right|^2\right). \end{aligned} \tag{4.1.21}$$

We show that this inequality is also true for $r+1$ and then the a priori estimate for $\vec{u}_h^n$ follows by recursion. We denote the right–hand side of the inequality (4.1.21) by Λ_r and we observe that by the induction assumption we have

$$\left|\vec{u}_h^{r+1}\right|^2 \le \Lambda_r \le \Lambda_T, \quad \text{for } n = 1, 2, ..., r. \tag{4.1.22}$$

This implies that the right–hand side of (4.1.20) for $r+1$ can be estimated by

$$\Lambda_{r+1} + D_5\left(\left(\nu\frac{\triangle t}{h^2}\right)^2 + \left(\triangle tS(h)\right)^2\Lambda_{r+1}\right)\sum_{n=1}^{r+1}\left\|\vec{u}_h^{n}\right\|_h^2.$$

The stability assumption (4.13) now implies that (4.1.21) is also true for $r+1$ and thus for arbitrary N we have

$$\begin{aligned} &\left|u_h^{n+1}\right|^2 + \delta\Theta\triangle t\alpha\nu \sum_{n=0}^{N}\left\|\vec{u}_h^{n+1}\right\|_h^2 + \Theta\triangle t\beta\nu\sum_{n=0}^{N}\left\|\vec{u}_h^{m}\right\|_h^2 \\ &+\left(\frac{1}{8} - \frac{\Theta\triangle t\beta\nu}{h^2}\right)\sum_{n=0}^{N}\left|\vec{u}_h^{n+1} - \vec{u}_h^{n+(1-\Theta)}\right|^2 \le \Lambda_T. \end{aligned} \tag{4.1.23}$$

Inequalities (4.1.5), (4.1.23) and the regularity requirement (4.13) in the form

$$\left(\Theta_b \triangle t S(h)\right)^2 \Lambda_T \leq D_5 \left(\triangle t S(h)\right)^2 \Lambda_T \leq \Theta \triangle t \alpha \nu$$

yield

$$\begin{aligned} \left|\vec{u}_h^{n+\Theta}\right|^2 \leq \left|\vec{u}_h^n\right|^2 + \Theta \triangle t \alpha \nu \sum_{n=0}^{N} \|\vec{u}_h^n\|_h^2 + \frac{\Theta \triangle t}{\nu} \sum_{n=0}^{N} \left|\vec{f}^n\right|^2 \leq \\ \Lambda_T \left(1 + \frac{1}{\delta} + \frac{\Theta}{\nu}\right). \end{aligned} \tag{4.1.24}$$

The next estimate comes from (4.1.12), (4.1.24) and (4.13). We have

$$\begin{aligned} \left|\vec{u}_h^m\right|^2 \leq \\ \left|\vec{u}_h^{n+\Theta}\right|^2 + D_5 \left(\left(\nu \frac{\triangle t}{h^2}\right)^2 + \left(\triangle t S(h)\right)^2 \Lambda_T\right) \sum_{n=0}^{N} \|\vec{u}_h^{n+1}\|_h^2 + \frac{\triangle t}{\alpha \nu} \left|\vec{f}^n\right|^2 \leq \\ \Lambda_T \left(1 + \frac{2}{\delta} + \frac{(1+\alpha)\Theta}{\alpha \nu}\right). \end{aligned} \tag{4.1.25}$$

Summing up (4.1.11) for $n = 0, 1, ..., N$, we obtain by (4.1.23)

$$\begin{aligned} \sum_{n=0}^{N} \left|\vec{u}_h^{n+\Theta} - \vec{u}_h^n\right|^2 + \Theta \triangle t \alpha \nu \sum_{n=0}^{N} \left\|\vec{u}_h^{n+\Theta} - \vec{u}_h^n\right\|_h^2 \leq \\ \Theta \triangle t \alpha \nu \sum_{n=0}^{N} \|\vec{u}_h^n\|_h^2 + \frac{\Theta \triangle t}{\alpha \nu} \sum_{n=0}^{N} \left|\vec{f}^n\right|^2 \leq \Lambda_T \left(\frac{1}{\delta} + \frac{\Theta}{\alpha \nu}\right), \end{aligned} \tag{4.1.26}$$

and, also, from (4.1.26) and (4.1.5), we see that

$$\begin{aligned} \Theta \triangle t \alpha \nu \sum_{n=0}^{N} \left\|\vec{u}_h^{n+\Theta}\right\|_h^2 \leq \\ \Theta \triangle t \alpha \nu \sum_{n=0}^{N} \left\|\vec{u}_h^{n+\Theta} - \vec{u}_h^n\right\|_h^2 + \Theta \triangle t \alpha \nu \sum_{n=0}^{N} \|\vec{u}_h^n\|_h^2 \leq \Lambda_T \left(1 + \frac{2}{\delta} + \frac{\Theta}{\alpha \nu}\right). \end{aligned} \tag{4.1.27}$$

Expressing the results of (4.1.23-27) in terms of the functions $\vec{u}_{h,\triangle t}^{(i)}, i = 1, 2, 3$, we have

$$\begin{aligned} \sup_{t \in [0,T]} \left|\vec{u}_{h,\triangle t}^{(i)}(t)\right|^2 \leq \Lambda_T \left(1 + \frac{2}{\delta} + \frac{2\Theta}{\alpha \nu}\right), \qquad \text{for } i = 1, 2, 3, \\ \int_0^T \left\|\vec{u}_{h,\triangle t}^{(i)}(t)\right\|_h^2 dt = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\|\vec{u}_{h,\triangle t}^{(i)}(t)\right\|_h^2 = \triangle t \sum_{n=0}^{N-1} \left\|\vec{u}_h^{(k)}(t)\right\|_h^2 \leq \\ \frac{\Lambda_T}{\Theta \alpha \beta \nu} \left(\frac{2}{\delta} + \frac{\Theta}{\alpha \nu}\right), \quad \text{for } i = 1, 2, 3, \end{aligned} \tag{4.1.28}$$

where $k = n$ for $i = 1$, $k = n + \Theta$ for $i = 2$ and $k = n + (1 - \Theta)$ for $i = 3$.

The a priori bound for $\vec{w}_{h,\triangle t}$ is obtained using (4.1.23) and the stability assumption (4.13), namely

$$\begin{aligned} &\left|\vec{u}^{(1)}_{h,\triangle t} - \vec{w}_{h,\triangle t}\right|^2_{L^2(0,T;[L^2(\Omega)]^n)} = \\ (4.1.29) \quad &\frac{1}{(\triangle t)^2} \sum_{n=0}^{N-1} \left|\vec{u}^{n+1}_h - \vec{u}^n_h\right|^2 \int_{t_n}^{t_{n+1}} (t - t_n)^2\, dt = \frac{1}{3} \triangle t \sum_{n=0}^{N-1} \left|\vec{u}^{n+1}_h - \vec{u}^n_h\right|^2 \le \\ &2\triangle t \sum_{n=0}^{N} \left|\vec{u}^{n+1}_h - \vec{u}^{n+(1-\Theta)}_h\right|^2 \le 2\triangle t \frac{\Lambda_T}{\delta}. \end{aligned}$$

We infer from (4.1.28) and (4.1.29) that $\{\vec{w}_{h,\triangle t}\}_{(h,\triangle t)} \subset V_h$ is a bounded set in $L^2(0,T;[L^2(\Omega)]^n)$ as it follows by the inequality

$$(4.1.30) \qquad \int_0^T |\vec{w}_{h,\triangle t}|^2\, dt \le \int_0^T \left\|\vec{u}^{(1)}_{h,\triangle t}\right\|^2 dt + \int_0^T \left|\vec{u}^{(1)}_{h,\triangle t} - \vec{w}_{h,\triangle t}\right|^2 dt.$$

4.2 Convergence of the scheme. The results (4.1.28) and (4.1.29) justify the existence of functions $\vec{u}^{(i)}_*(t)$ and $\vec{w}_*$ such that as $(h, \triangle t) \to 0_+$ we have

$$(4.2.1) \quad \vec{u}^{(i)}_{h,\triangle t} \rightharpoonup \vec{u}^{(i)}_* \quad \begin{cases} \text{weakly in } L^2(0,T;V), \text{ conforming case,} \\ \text{weakly in } L^2(0,T;[L^2(\Omega)]^n), \text{nonconforming case,} \\ \text{weakly-}* \text{ in } L^\infty(0,T;[L^2(\Omega)]^n), \quad \text{for } i = 1, 2, \end{cases}$$

$$(4.2.2) \quad \vec{u}^{(3)}_{h,\triangle t} \rightharpoonup \vec{u}^{(3)}_* \quad \begin{cases} \text{weakly in } L^2(0,T;[H^1_0(\Omega)]^n), \text{ conforming case,} \\ \text{weakly in } L^2(0,T;[L^2(\Omega)]^n), \text{ nonconforming case,} \\ \text{weakly-}* \text{ in } L^\infty(0,T;[L^2(\Omega)]^n). \end{cases}$$

Since $\{\vec{w}_{h,\triangle t}\}_{(h,\triangle t)}$ is bounded in $L^2(0,T;[L^2(\Omega)]^n)$ by (4.1.30), we have

$$(4.2.3) \qquad \vec{w}_{h,\triangle t} \rightharpoonup \vec{w}_*, \qquad \text{weakly in } L^2(0,T;[L^2(\Omega)]^n).$$

The inequality (4.1.26) yields

$$\begin{aligned} &\left|\vec{u}^{(1)}_{h,\triangle t} - \vec{u}^{(2)}_{h,\triangle t}\right|^2_{L^2(0,T;[L^2(\Omega)]^n)} = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left|\vec{u}^{(1)}_{h,\triangle t} - \vec{u}^{(2)}_{h,\triangle t}\right|^2 dt = \\ (4.2.4) \quad &\triangle t \sum_{n=0}^{N-1} \left|\vec{u}^n_h - \vec{u}^{n+\Theta}_h\right|^2 \longrightarrow 0, \qquad \text{as } (h, \triangle t) \longrightarrow 0_+. \end{aligned}$$

Similarly, from (4.1.23), we have

$$(4.2.5) \qquad \left|\vec{u}^{(1)}_{h,\triangle t} - \vec{u}^{(3)}_{h,\triangle t}\right|^2_{L^2(0,T;[L^2(\Omega)]^n)} \longrightarrow 0, \qquad \text{as } (h, \triangle t) \longrightarrow 0_+.$$

Now, from (4.1.29) and (4.2.3-5), we see that for almost all $t \in (0, T)$

$$(4.2.6) \qquad \vec{u}^{(i)}_* = \vec{w}_* = \vec{u}_*, \quad \text{a.e. in } \Omega, \quad \text{for } i = 1, 2, 3.$$

We show that $\vec{w}_*$ is the solution of (WNS) by proving that the family $\{\vec{w}_{h,\triangle t}\}$ is relatively compact in, at least, $L^2(0,T;[L^2(\Omega)]^n)$. We treat the conforming and nonconforming approximations separately.

Conforming finite element approximations. We note that in this case the spaces V_h and H_h are subspaces of $[H_0^1(\Omega)]$. Integrating (4.12) over the time interval $[t, t+\gamma]$, for some $0<\gamma<1$, we obtain

$$\begin{aligned}
&\frac{1}{\Theta}(\vec{w}_{h,\triangle t}(t+\gamma)-\vec{w}_{h,\triangle t}(t),\vec{\varphi}_h)+\alpha\nu\int_t^{t+\gamma}((\vec{u}^{(1)}_{h,\triangle t}(s+\triangle t),\vec{\varphi}_h))\,ds+\\
&\beta\nu\int_t^{t+\gamma}((\vec{u}^{(1)}_{h,\triangle t}(s),\vec{\varphi}_h))\,ds=\frac{1}{\Theta\triangle t}\int_t^{t+\gamma}(\vec{u}^{(3)}_{h,\triangle t}(s)-\vec{u}^{(2)}_{h,\triangle t}(s),\vec{\varphi}_h)\,ds-\\
&\beta\nu\int_t^{t+\gamma}((\vec{u}^{(3)}_{h,\triangle t}(s),\vec{\varphi}))\,ds-\alpha\nu\int_t^{t+\gamma}((\vec{u}^{(2)}_{h,\triangle t}(s),\vec{\varphi}))\,ds-\\
&\int_t^{t+\gamma}\left(b(\vec{u}^{(2)}_{h,\triangle t}(s),\vec{u}^{(2)}_{h,\triangle t}(s),\vec{\varphi}_h)+b(\vec{u}^{(3)}_{h,\triangle t}(s),\vec{u}^{(3)}_{h,\triangle t}(s),\vec{\varphi}_h)\right)ds+\\
&\int_t^{t+\gamma}(\vec{f}_{\triangle t}(s)+\vec{f}_{\triangle t}(s+\triangle t),\vec{\varphi}_h)\,ds.
\end{aligned}\tag{4.2.7}$$

Now, we estimate the terms in (4.2.7). Recalling the estimates in (4.1.28), we get for $i=1,2,3$

$$\begin{aligned}
&\left|\int_t^{t+\gamma}((\vec{u}^{(i)}_{h,\triangle t}(s),\vec{\varphi}_h))\,ds\right|\le\|\vec{\varphi}_h\|\int_t^{t+\gamma}\left\|\vec{u}^{(i)}_{h,\triangle t}(s)\right\|ds\le\\
&\sqrt{\gamma}\,\|\vec{\varphi}_h\|\left(\int_t^{t+\gamma}\left\|\vec{u}^{(i)}_{h,\triangle t}(s)\right\|^2ds\right)^{\frac12}\le\\
&\sqrt{\gamma}\,\|\vec{\varphi}_h\|\left(\int_0^{T}\left\|\vec{u}^{(i)}_{h,\triangle t}(s)\right\|^2ds\right)^{\frac12}\le\sqrt{\frac{\Lambda_T}{\Theta\alpha\beta\nu}\left(\frac{2}{\delta}+\frac{\Theta}{\alpha\nu}\right)}\sqrt{\gamma}\,\|\vec{\varphi}_h\|.
\end{aligned}\tag{4.2.8}$$

We majorize the trilinear forms by using the estimate (4.1.28), and we get

$$\begin{aligned}
&\left|\int_t^{t+\gamma}b(\vec{u}^{(i)}_{h,\triangle t}(s),\vec{u}^{(i)}_{h,\triangle t}(s),\vec{\varphi}_h)\,ds\right|\le\\
&C\,\|\vec{\varphi}_h\|\int_t^{t+\gamma}\left|\vec{u}^{(i)}_{h,\triangle t}(s)\right|^{\frac12}\left\|\vec{u}^{(i)}_{h,\triangle t}(s)\right\|^{\frac32}ds\le C\sqrt{\Lambda_T}\,\|\vec{\varphi}_h\|\int_t^{t+\gamma}\left\|\vec{u}^{(i)}_{h,\triangle t}(s)\right\|^{\frac32}ds\le\\
&C\sqrt{\Lambda_T}\,\|\vec{\varphi}_h\|\,\sqrt[4]{\gamma}\left(\int_0^{T}\left\|\vec{u}^{(i)}_{h,\triangle t}(s)\right\|^2ds\right)^{\frac34}\le C\Lambda_T^{\frac54}\left(\frac{1}{\Theta\alpha\beta\nu}\left(\frac{2}{\delta}+\frac{\Theta}{\alpha\nu}\right)\right)^{\frac34}\sqrt{\gamma}\,\|\vec{\varphi}_h\|.
\end{aligned}\tag{4.2.9}$$

The first term on the right-hand side of (4.2.7) can be controled by integrating (4.8) over the time interval $[0,T]$ with we $\vec{\varphi}_h\in V_h$. We have

$$\begin{aligned}
&\left|\frac{1}{\Theta\triangle t}\int_t^{t+\gamma}(\vec{u}^{(3)}_{h,\triangle t}(s)-\vec{u}^{(2)}_{h,\triangle t}(s),\vec{\varphi}_h)\,ds\right|\le\\
&\beta\nu\left|\int_t^{t+\gamma}((\vec{u}^{(3)}_{h,\triangle t}(s),\vec{\varphi}_h))\,ds\right|+\alpha\nu\left|\int_t^{t+\gamma}((\vec{u}^{(2)}_{h,\triangle t}(s),\vec{\varphi}_h))\,ds\right|+\\
&\left|\int_t^{t+\gamma}b_h(\vec{u}^{(3)}_{h,\triangle t}(s),\vec{u}^{(3)}_{h,\triangle t}(s),\vec{\varphi}_h)\,ds\right|+\left|\int_t^{t+\gamma}(\vec{f}_{\triangle t}(s),\vec{\varphi}_h)\,ds\right|.
\end{aligned}\tag{4.2.10}$$

Applying the estimates (4.2.8-10) and (4.1.28) to (4.2.7) we obtain

$$|(\vec{w}_{h,\Delta t}(t+\gamma)-\vec{w}_{h,\Delta t}(t),\vec{\varphi}_h)|\le C\sqrt{\gamma}\,\|\vec{\varphi}_h\|,\qquad \forall\vec{\varphi}_h\in V_h, \tag{4.2.11}$$

where the constant $C>0$ is independent of $\gamma,\Delta t$ and h. Integrating the above inequality for $t\in[0,T-\gamma]$ we obtain with $\vec{\varphi}_h=\vec{w}_{h,\Delta t}(t+\gamma)-\vec{w}_{h,\Delta t}(t)\in V_h$ the inequality
(4.2.12)

$$\int_0^{T-\gamma}|\vec{w}_{h,\Delta t}(s+\gamma)-\vec{w}_{h,\Delta t}(s)|^2\,ds\le C'\sqrt{\gamma\int_0^T\left\|\vec{u}^{(1)}_{h,\Delta t}(s)\right\|^2 ds}\le C''\sqrt{\gamma},$$

where again the constant $C''>0$ does not depend on $\gamma,\Delta t$ and h. We have $\{\vec{w}_{h,\Delta t}\}_{(h,\Delta t)}\subset[H^1_0(\Omega)]^n\subset\subset[L^2(\Omega)]^n$. Application of the compactness criterion [N, Theorem 2.3.1] and the inequality (4.2.12) imply that the set $\{\vec{w}_{h,\Delta t}\}_{(h,\Delta t)}$ is relatively compact in $L^q(0,T;[L^2(\Omega)]^n)$, for $q\in[1,2]$. Because the sequance $\{\vec{w}_{h,\Delta t}\}_{(h,\Delta t)}$ converges weakly to $\vec{w}_*$ in $L^2(0,T;[L^2(\Omega)]^n)$ by (4.2.3), it converges strongly to this function in $L^q(0,T;[L^2(\Omega)]^n)$, for $q\in[1,2]$, due to its relative compactness in this space. The strong convergence of the sequence $\{\vec{w}_{h,\Delta t}\}_{(h,\Delta t)}$ in the space $L^2(0,T;[L^2(\Omega)]^n)$, its uniform bound in $L^\infty(0,T;[L^2(\Omega)]^n)$, and, the Lebesque dominated convergence theorem, yield finally the strong convergence in $L^q(0,T;[L^2(\Omega)]^n)$, for $q\in[1,\infty)$.

To show the strong convergence in $L^2(0,T;[H^1_0(\Omega)]^n)$, we assume, for a moment, that $\vec{w}_*$ is the unique strong solution of (WNS). Integrating (WNS) over the time interval $[0,T]$ and taking $\vec{\varphi}=\vec{w}_*-\vec{w}_{h,\Delta t}$, we obtain
(4.2.13)

$$\int_0^T\|\vec{w}_*-\vec{w}_{h,\Delta t}\|^2\,dt=\int_0^T((\vec{w}_{h,\Delta t},\vec{w}_*-\vec{w}_{h,\Delta t}))\,dt-$$

$$\int_0^T(\frac{d\vec{w}_*}{dt},\vec{w}_*-\vec{w}_{h,\Delta t})-\int_0^T b(\vec{w}_*,\vec{w}_*,\vec{w}_*-\vec{w}_{h,\Delta t})\,dt+\int_0^T(\vec{f},\vec{w}_*-\vec{w}_{h,\Delta t})\,dt.$$

Due to the strong convergence $\vec{w}_*-\vec{w}_{h,\Delta t}\to 0$ in $L^2(0,T;[L^2(\Omega)]^n)$ the last three terms converge to zero. We estimate similarly as in (4.1.28)

$$\int_0^T\|\vec{w}_{h,\Delta t}\|^2\,dt\le\frac{\Lambda_T}{\Theta\alpha\beta\nu}\left(\frac{2}{\delta}+\frac{\Theta}{\alpha\nu}\right). \tag{4.2.14}$$

Because of the compact imbedding of $L^2(0,T;[H^1_0(\Omega)]^n)$ into $L^2(0,T;[L^2(\Omega)]^n)$ and the already known strong convergence in $L^2(0,T;[L^2(\Omega)]^n)$, we conclude that

$$\vec{w}_{h,\Delta t}\rightharpoonup\vec{w}_*,\qquad\text{weakly in } L^2(0,T;[H^1_0(\Omega)]^n). \tag{4.2.15}$$

Thus, for the first term on the right–hand side of (4.2.13) we have

$$\limsup_{(h,\Delta t)\to 0_+}\int_0^T((\vec{w}_{h,\Delta t},\vec{w}_*-\vec{w}_{h,\Delta t}))=-\limsup_{(h,\Delta t)\to 0_+}\int_0^T\|\vec{w}_{h,\Delta t}\|^2+\int_0^T\|\vec{w}_*\|^2.$$

The previous strong convergence results and (4.1.28) yield

$$\limsup_{(h,\triangle t)\to 0_+} \int_0^T \|\vec{w}_{h,\triangle t}\|^2 \geq \limsup_{(h,\triangle t)\to 0_+} \int_0^T \left\|\vec{u}^{(1)}_{h,\triangle t}\right\|^2 \geq \int_0^T \|\vec{u}_*\|^2 = \int_0^T \|\vec{w}_*\|^2 ,$$

which implies together with (4.2.13) that

$$\vec{w}_{h,\triangle t} \longrightarrow \vec{w}_*, \qquad \text{strongly in } L^2(0,T;[H_0^1(\Omega)]^n). \quad \square \tag{4.2.16}$$

Nonconforming finite element approximation. The relative compactness of the set $\{\vec{w}_{h,\triangle t}\}_{(h,\triangle t)}$ is again essential to obtain the strong convergence. Extending the function $\vec{w}_{h,\triangle t}$ by zero outside the interval $[0,T]$, we show first that there exists a constant $C(\gamma) > 0$, independent of h and $\triangle t$, and, such that for $\gamma \in (0,\frac{1}{4})$, it holds

$$\int_{-\infty}^{+\infty} \left|\tau^{2\gamma}\right| \left|\hat{\vec{w}}_{h,\triangle t}(\tau)\right| d\tau \leq C(\gamma)\frac{\Lambda_T}{\Theta\alpha\beta\nu}\left(\frac{2}{\delta}+\frac{\Theta}{\alpha\nu}\right) < \infty, \tag{4.2.17}$$

where $\hat{\vec{w}}_{h,\triangle t}$ is the Fourier transform of $\vec{w}_{h,\triangle t}$. To show that the above inequality holds, we define a function $\vec{g}_{h,\triangle t} : (0,T) \to V_h$ by

(4.2.18)

$$\begin{gathered}((\vec{g}_{h,\triangle t},\vec{\varphi}_h))_h = \frac{1}{\triangle t}(\vec{u}^{(3)}_{h,\triangle t}(t) - \vec{u}^{(2)}_{h,\triangle t}(t),\vec{\varphi}_h) - \\ \Theta\alpha\nu((\vec{u}^{(1)}_{h,\triangle t}(t+\triangle t),\vec{\varphi}_h))_h - \Theta\beta\nu((\vec{u}^{(1)}_{h,\triangle t}(t),\vec{\varphi}_h))_h - \Theta\beta\nu((\vec{u}^{(3)}_{h,\triangle t}(t),\vec{\varphi}_h))_h - \\ \Theta\alpha\nu((\vec{u}^{(2)}_{h,\triangle t}(t),\vec{\varphi}_h))_h - \Theta b_h(\vec{u}^{(2)}_{h,\triangle t}(t),\vec{u}^{(2)}_{h,\triangle t}(t),\vec{\varphi}_h) - \Theta b_h(\vec{u}^{(3)}_{h,\triangle t}(t),\vec{u}^{(3)}_{h,\triangle t}(t),\vec{\varphi}_h) + \\ \Theta(\vec{f}_{\triangle t}(t)+\vec{f}_{\triangle t}(t+\triangle t),\vec{\varphi}_h), \qquad \forall\vec{\varphi}_h \in V_h,\end{gathered}$$

and $t \in [n\triangle t,(n+1)\triangle t)$. Then, (4.12) can be interpreted as

$$\frac{d}{dt}(\vec{w}_{h,\triangle t}(t),\vec{\varphi}_h) = ((\vec{g}_{h,\triangle t}(t),\vec{\varphi}_h))_h, \qquad \forall\vec{\varphi}_h \in V_h, \quad t \in (0,T).$$

We estimate

(4.2.19)

$$\begin{gathered}\frac{1}{\Theta}\int_0^T \|\vec{g}_{h,\triangle t}(t)\|_h \, dt = \frac{1}{\Theta}\int_0^T \sup_{\|\vec{\varphi}_h\|_h=1} ((\vec{g}_{h,\triangle t}(t),\vec{\varphi}_h))_h \, dt \leq \\ \frac{1}{\triangle t}\int_0^T \left|\vec{u}^{(3)}_{h,\triangle t}(t) - \vec{u}^{(2)}_{h,\triangle t}(t)\right| dt + 2\nu\int_0^T \left(\left\|\vec{u}^{(1)}_{h,\triangle t}(t)\right\|_h^2 + \left\|\vec{u}^{(3)}_{h,\triangle t}(t)\right\|_h\right) dt + \\ \beta\nu\int_0^T \left\|\vec{u}^{(3)}_{h,\triangle t}(t)\right\|_h dt + \int_0^T \left(\left\|\vec{u}^{(2)}_{h,\triangle t}(t)\right\|_h^2 + \left\|\vec{u}^{(3)}_{h,\triangle t}(t)\right\|_h^2\right) dt + 2\int_0^T \left|\vec{f}_{\triangle t}(t)\right| dt.\end{gathered}$$

We conclude from (4.1.28), (4.2.4,5) and (4.2.19) that

$$\int_0^T \|\vec{g}_{h,\triangle t}(t)\|_h \, dt \leq C, \tag{4.2.20}$$

where the constant $C > 0$ does not depend on h and $\triangle t$.

Extending the function $\vec{g}_{h,\triangle t}$ from $[0,T]$ to $\mathbb{R}$ by zero and using (4.12), we get

$$\begin{aligned}\frac{d}{dt}(\vec{w}_{h,\triangle t}(t),\vec{\varphi}_h) &= \\ ((\vec{g}_{h,\triangle t}(t),\vec{\varphi}_h))_h + (\vec{u}_{0h},\vec{\varphi}_h)\delta_0 - (\vec{u}_h^N,\vec{\varphi}_h)\delta_T, &\qquad \forall\vec{\varphi}_h \in V_h.\end{aligned} \tag{4.2.21}$$

Here, δ_0, δ_T are the Dirac distributions at 0 and T, respectively, describing possibly ocuring discontinuities at these points. Taking the Fourier transform of (4.2.21) we obtain

$$\begin{aligned}-2\pi i\tau(\hat{\vec{w}}_{h,\triangle t}(\tau),\vec{\varphi}_h) &= \\ ((\hat{\vec{g}}_{h,\triangle t}(\tau),\vec{\varphi}_h))_h + (\vec{u}_{0h},\vec{\varphi}_h) - (\vec{u}_h^N,\vec{\varphi}_h)\exp(-2\pi i\tau T).&\end{aligned} \tag{4.2.22}$$

This equation implies

$$\begin{aligned}2\pi\left|\tau\right|\left|\hat{\vec{w}}_{h,\triangle t}(\tau)\right|^2 &\le \left\|\hat{\vec{g}}_{h,\triangle t}(\tau)\right\|_h \left\|\hat{\vec{w}}_{h,\triangle t}(\tau)\right\|_h + \left|\vec{u}_{0h}\right|\left|\hat{\vec{w}}_{h,\triangle t}(\tau)\right| \le \\ &\int_0^T \left\|\hat{\vec{g}}_{h,\triangle t}(\tau)\right\|_h d\tau \left\|\hat{\vec{w}}_{h,\triangle t}(\tau)\right\|_h + \left|\vec{u}_{0h}\right|\left|\hat{\vec{w}}_{h,\triangle t}(\tau)\right|.\end{aligned} \tag{4.2.23}$$

This estimate and (4.2.20) imply the exitence of a constant $C > 0$, independent of $h, \triangle t$ and γ, such that

$$\left|\tau\right|\left|\hat{\vec{w}}_{h,\triangle t}\right|^2 \le C\left\|\hat{\vec{w}}_{h,\triangle t}\right\|_h.$$

Since $\gamma < \frac{1}{4}$, it holds

$$\left|\tau^{2\gamma}\right| \le \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \qquad \forall\tau\in\mathbb{R},$$

and thus, by Parceval equality and the two above inequalities, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|\tau^{2\gamma}\right|\left|\hat{\vec{w}}_{h,\triangle t}(t)\right| dt &\le C(\gamma)\int_0^T \left\|\vec{w}_h(\tau)\right\|_h^2 d\tau \le \\ &C(\gamma)\frac{\Lambda_T}{\Theta\alpha\beta\nu}\left(\frac{2}{\delta}+\frac{\Theta}{\alpha\nu}\right).\end{aligned} \tag{4.2.24}$$

The last inequality follows from (4.1.28) and (4.1.29).

The relative compactness of the set $\{\vec{w}_{h,\triangle t}\}_{(h,\triangle t)}$ in the space $L^2(0,T;[L^2(\Omega)]^n)$ is now assured by [T I, Theorem 2.2, Ch. III, §2], because of (4.1.30) and (4.2.24). Moreover, because $\{\vec{w}_{h,\triangle t}\}_{(h,\triangle t)}$ is bounded in $L^\infty(0,T;[L^2(\Omega)]^n)$, what follows by (4.1.28), the convergence is shown in the space $L^q(0,T;[L^2(\Omega)]^n)$, for $q\in[2,\infty)$. □

Applying (4.2.4,5) and (4.1.29) we finally see that

$$\vec{u}^{(i)}_{h,\triangle t} \longrightarrow w_*, \qquad \text{strongly in } L^q(0,T;[L^2(\Omega)]^n), \quad i=1,2,3, \tag{4.2.25}$$

for $q\in[1,\infty)$ in the conforming case and for $q\in[2,\infty)$ in the nonconforming case. Moreover, this convergence holds in $L^2(0,T;[H_0^1(\Omega)]^n)$ for the conforming case, because

$$\int_0^T \left\|\vec{u}^{(i)}(t)-\vec{w}_*(t)\right\|_h^2 dt \le \int_0^T\left(\left\|\vec{u}^{(i)}(t)-\vec{w}_{h,\triangle t}(t)\right\|_h^2 + \left\|\vec{w}_{h,\triangle t}(t)-\vec{w}_*(t)\right\|_h^2\right) dt,$$

provided $\vec{w}_*$ is the unique solution of (WNS). This will be shown next.

Passage to the limit. We take $\psi \in C^1[0,T]$, with $\psi(T) = 0$, $\vec{\varphi}_h = \Pi_h \vec{v}$, for some $\vec{v} \in V_0 \cap [\mathcal{D}(\Omega)]^n$ and integrate (4.12) over the time interval $[0,T]$. Using the Green's theorem we obtain
(4.2.26)

$$-\frac{1}{\Theta}\int_0^T (\vec{w}_{h,\Delta t}, \psi'(t)\Pi_h\vec{v})\,dt + \alpha\nu\int_0^T ((\vec{u}^{(1)}_{h,\Delta t}(t+\Delta t), \psi(t)\Pi_h\vec{v}))_h\,dt +$$
$$\beta\nu\int_0^T ((\vec{u}^{(1)}_{h,\Delta t}(t), \psi(t)\Pi_h\vec{v}))_h\,dt = \frac{1}{\Theta\Delta t}\int_0^T (\vec{u}^{(3)}_{h,\Delta t}(t) - \vec{u}^{(2)}_{h,\Delta t}(t), \psi(t)\Pi_h\vec{v})\,dt$$
$$-\beta\nu\int_0^T ((\vec{u}^{(3)}_{h,\Delta t}(t), \psi(t)\Pi_h\vec{v}))_h\,dt - \alpha\nu\int_0^T ((\vec{u}^{(2)}_{h,\Delta t}(t), \psi(t)\Pi_h\vec{v}))_h\,dt -$$
$$\int_0^T b_h(\vec{u}^{(2)}_{h,\Delta t}(t), \vec{u}^{(2)}_{h,\Delta t}(t), \psi(t)\Pi_h\vec{v})\,dt - \int_0^T b_h(\vec{u}^{(3)}_{h,\Delta t}(t), \vec{u}^{(3)}_{h,\Delta t}(t), \psi(t)\Pi_h\vec{v})\,dt +$$
$$\int_0^T (\vec{f}_{\Delta t}(t) + \vec{f}_{\Delta t}(t+\Delta t), \psi(t)\Pi_h\vec{v})\,dt - \frac{1}{\Theta}(\vec{u}_{0h}, \Pi_h\vec{v})\psi(0).$$

Since, by (4.7), we have

$$\frac{1}{\Theta\Delta t}\int_0^T (\vec{u}^{(3)}_{h,\Delta t}(t) - \vec{u}^{(2)}_{h,\Delta t}(t), \psi(t)\Pi_h\vec{v})\,dt = -\frac{\Theta_b}{\Theta}\beta\nu\int_0^T ((\vec{u}^{(3)}_{h,\Delta t}(t), \psi(t)\Pi_h\vec{v}))_h\,dt -$$
$$\frac{\Theta_b}{\Theta}\int_0^T b_h(\vec{u}^{(3)}_{h,\Delta t}(t), \vec{u}^{(3)}_{h,\Delta t}(t), \psi(t)\Pi_h\vec{v})\,dt - \frac{\Theta_b}{\Theta}\alpha\nu\int_0^T ((\vec{u}^{(2)}_{h,\Delta t}(t), \psi(t)\Pi_h\vec{v}))_h\,dt -$$
$$\frac{\Theta_b}{\Theta}\int_0^T (\vec{f}_{\Delta t}(t), \psi(t)\Pi_h\vec{v})\,dt,$$

we substitute the above into (4.2.26) and pass directly to the limit due to the strong convergence results (4.2.25). Using the condition (C) (section 3.2) and (4.5) for the approximation of the initial velocity, we obtain

$$-\int_0^T (\vec{w}_*, \psi'(t)\vec{v})\,dt + \int_0^T ((\vec{w}_*, \psi(t)\vec{v}))\,dt + \int_0^T b(\vec{w}_*, \vec{w}_*, \psi(t)\vec{v})\,dt =$$
$$(\vec{u}(0), \vec{v})\psi(0) - \int_0^T (\vec{f}(t), \psi(t)\vec{v})\,dt.$$

Consequently, (WNS) is satisfied in the distributional sense. Using the approximation of the initial velocity field and the Green's theorem it follows that the initial condition is also satisfied.

Convergence of the whole sequence $\{w_{h,\Delta t}\}_{(h,\Delta t)}$ to the solution of (WNS) is garanteed by the assumed unicity of the strong solution on the given time interval $[0,T]$. This concludes the last step of the proof. □

5. Lid–driven cavity flow

In this section we use a benchmark problem originally described by O.Burggraf [B] to display some of the computational properties of the method. The test problem consists in the solution of the Navier–Stokes equations in the square duct with prescribed shear and body forces. The shear force is non–zero on one face of the duct and zero on the other three. Since the velocity is set to be zero at the corners with the given shear it is a type of a regularized cavity problem. The details of this benchmark problem can be found in T.Shih *et al.* [S].

We use four groups of the two dimensional finite elements that satisfy the conditions (B)–(E). We restrict to the polynomials up to the order ≤ 2 and we use the following notation: P_k, k non-negative integer, for the space of polynomials up to order $\leq k$ and Q_k is the space of polynomials up to order $\leq k$ in each variable.

(I) Q_2/Q_1 element: Conforming piecewise biquadratic velocities and continues piecewise bilinear pressure. The element is referred to as Taylor–Hood finite element.

(II) Q_2/P_1 element: Conforming piecewise biquadratic velocities and discontinues piecewise linear pressure. We refer to this element also as Crouzeix–Raviart element.

(III) rQ_2/P_1 element: Conforming piecewise reduced biquadratic (serendipity family) velocities and discontinues piecewise linear pressure.

(IV) Q_1/P_0 element: Nonconforming piecewise bilinear velocities (continuity at midpoints of sides only) and piecewise constant pressure.

The accuracy for Taylor–Hood, Crouzeix–Raviart and $rQ_1/P1$ elements is $\mathcal{O}(h^3)\times\mathcal{O}(h^2)$, and the accuracy is $\mathcal{O}(h^2)\times\mathcal{O}(h)$ for Q_1/P_0 element.

5.1 Description of the test problem. We consider $\Omega = [0,1]\times[0,1]$. We define the shear $\vec{g} = (g_1, g_2)$ by

$$
\begin{aligned}
g_1(x,y) &= \begin{cases} 16(x^4 - 2x^3 + x^2), & x \in \partial\Omega, y = 1 \\ 0, & \text{elsewhere} \end{cases} \\
g_2(x,y) &= 0, \quad x, y \in \partial\Omega.
\end{aligned}
$$

The body force $\vec{f} = (\vec{f_1}, \vec{f_2})$ is given by

$$
\begin{aligned}
\vec{f_1}(x,y) =& 0, \quad x, y \in \Omega \\
\vec{f_2}(x,y) =& \nu \cdot 8\Big(24S(x) + 2s'(x)v''(y) + s'''(x)v(y)\Big) + \\
& \nu \cdot 64\Big(S_2(x)V(y) - s(y)s'(y)S_1(x)\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
s(x) &= x^4 - 2x^3 + x^2, \\
v(y) &= y^4 - y^2, \\
S(x) &= \int s(x)\,dx = \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3,
\end{aligned}
$$

$$S_1(x) = s(x)s''(x) - (s'(x))^2 = -4x^6 + 12x^5 - 14x^4 + 8x^3 - 2x^2,$$
$$S_2(x) = \int s(x)s'(x)\,dx = \frac{1}{2}(s(x))^2,$$
$$V(x) = v(y)v'''(y) - v'(y)v''(y) = -24y^5 + 8y^3 - 4y.$$

In the above expressions the primes denote the differentiation with respect to x and y, respectively. The solution $\{(u_1^\infty, u_2^\infty), p^\infty\}$ to the stationary Navier–Stokes equations with the above given data is given by:

$$u_1^\infty = 8s(x)v'(y) = 8(x^4 - 2x^3 + x^2)(4y^3 - 2y),$$
$$u_2^\infty = -8s'(x)v(y) = -8(4x^3 - 6x^2 + 2x)(y^4 - y^2),$$
$$p^\infty = \nu \cdot 8\left(S(x)v'''(y) + s'(x)v'(y)\right) + 64\left(S_2(x)(v(y)v''(y) - (v'(y))^2\right).$$

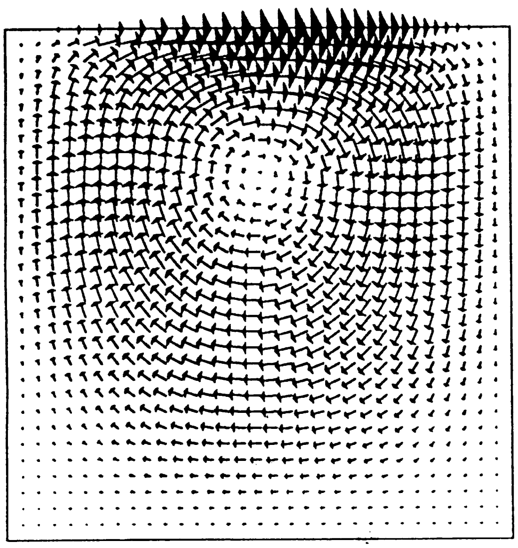

Fig. 1: Velocity field, $\nu = 0.01[m^{-2}s^{-1}]$

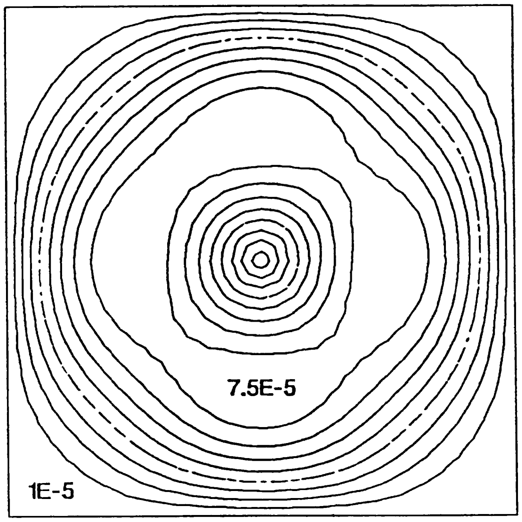

Fig. 2: Maximum error distribution, $\nu = 0.01[m^{-2}s^{-1}]$

Numerical results. We have computed the test problem for $\nu = 0.01[m^{-2}s^{-1}]$ and $\nu = 0.001[m^{-2}s^{-1}]$. The later case is far beyond the validity range of the analytical solution of the test problem because there is presented permanent time-dependent regime. We display the plots of the velocity field, pressure and error distribution for $\nu = 0.01[m^{-2}s{-}1]$. Moreover, we indicate the behavior of various errors and $|\vec{u}_t(t)|$. For the case with lower viscosity, the computed pressure seems to be in a good agreement with the analytical solution and this is true even for $\nu = 0.0001[m^{-2}s^{-1}]$, where the velocity field is different from the analytical solution. The numerical results suggest the use of this method for the values of up to $Re \sim 3000$, behind this limit the convergence namely of the Burgers step is very slow (rate about .95) because of oscillations produced by a very weak dumping of the method.

In all the computations, we use the finite element approximations of the group (I,II) and only for a lower viscosity $\sim 0.1[m^{-2}s^{-1}]$ the other approximations seems to be

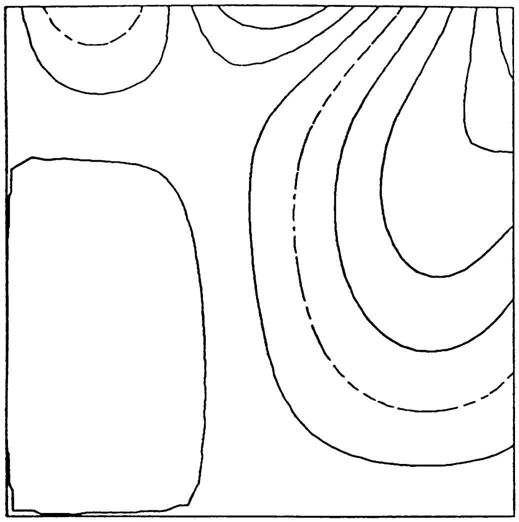

Fig. 3: Pressure isolines, $\nu = 0.01[m^{-2}s^{-1}]$

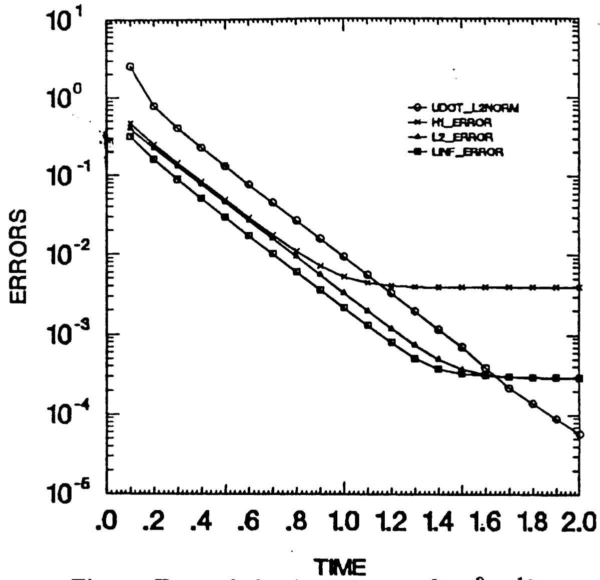

Fig. 4: Errors behavior, $\nu = 0.01[m^{-2}s^{-1}]$

acceptable. We did the computations on the scale resolution $h = 1/64$ and we used the time step $\triangle t = 0.1s$, fractional parameter $\theta = 0.25$, spliting parameter $\alpha = 0.5$ (for this sort of problems) and the integration was performed by the 4×4 Gaussian quadrature formula.

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