Rigidity of Lagrangian submanifolds

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Abstract

We study the symplectic geometry and topology related to Lagrangian submanifolds. More specifically we study the following two problems. How do iterated Dehn twists affect the symplectic and Lagrangian topology (in particular Floer homology groups) and how does the existence of certain Lagrangian submanifolds depend on the symplectic structure?

First, we review the definition of symplectic Dehn twists and Lagrangian and fixed point Floer homologies. We then construct a chain complex that fits in long exact sequences generalizing Seidel's exact sequences for a single Dehn twist. This chain complex can be viewed as the difference between Floer homologies before and after the iterated Dehn twists.

Next, we study the second problem in the case of Lagrangian projective planes in rational symplectic 4-manifolds. Using previous results specific to dimension 4, we can reduce the problem to describing a finite-dimensional cone in $H^2(M, \mathbb{R})$ using finitely many inequalities. Finally, using rational blow-up we can relate Lagrangian projective planes in M with certain symplectic spheres in the blow-up of M, whose dependence on the symplectic structure is known when the blow-up is small.

Finally, we use almost toric fibrations to prove the existence of Lagrangian projective planes for many symplectic structures in arbitrarily large blow-ups of the complex projective plane.

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Chapter 1

Introduction

A **symplectic manifold** (M, ω) is a smooth manifold M together with a closed non-degenerate 2-form $\omega \in H^2(M, \mathbb{R})$. Clearly, such manifolds have to have even dimension $2n$. A **Lagrangian submanifold** of a symplectic manifold is a smooth submanifold of M of dimension n such that $\omega|_L = 0$. An embedding $i : M \to M'$ from (M, ω) to (M, ω') is a **symplectic embedding** if $\phi^*\omega' = \omega$. It is a **symplectomorphism** if, in addition, it is a diffeomorphism. A **Hamiltonian vector field** X_H of a smooth function $H : M \to \mathbb{R}$ is a vector field on M such that $dH = \omega(X_H, -)$. It can be easily verified that flows of such vector fields are symplectomorphisms from (M, ω) to itself, and are isotopic to identity by scaling H. The existence and uniqueness of Lagrangian submanifolds have been studied extensively for the past twenty years, yet most of the results are restricted to the cases of spheres and tori when M has dimension 4 and the cases of tori for higher dimensions. (Existence) Fixing diffeomorphism type L and a class $A \in H_n(M, \mathbb{Z})$, does there exist a Lagrangian embedding of L into a given symplectic manifold (M, ω) with $[L] = A$? Given Lagrangian embeddings $\iota, \iota' : L \to (M, \omega)$, does there exist a (Hamiltonian) symplectomorphism ϕ such that $\phi(L) = L'$?

Both problems are very hard. For the existence problem, systematic constructions often require extra structures like Lagrangian torus fibrations and Lefschetz fibrations. Other constructions rely on case-by-case studies of very specific symplectic manifolds like rational surfaces (blow-ups and downs of $\mathbb{CP}^2$). In this paper, we will study the existence of nonorientable Lagrangians in rational surfaces using rational blow-up and almost toric vibrations.

Another important problem in symplectic geometry is the study of the symplectic topology of singularities (also known as **Landau-Ginzburg models**). One of the first algebraic invariants introduced for the simplest nontrivial LG model, namely Lefschetz fibrations, is Seidel's *Fukaya-Seidel category*, which is very abstract. Another natural invariant is the **fixed point Floer homology** of the global monodromy map, which in the case of

Lefschetz fibrations are just composition of Dehn twists along all vanishing cycles. Seidel conjectured that the Hochschild homology of the Fukaya-Seidel category and the fixed point Floer homology of the global monodromy are almost the same with the difference being the topology of the ambient space. We will construct a long exact sequence that realizes this picture.

Chapter 2

Floer homologies and Dehn twists

2.1 Floer homologies

2.1.1 Lagrangian Floer homology

The open string analog of Hamiltonian Floer theory is Lagrangian Floer theory. Recall we are working with exact symplectic manifolds with boundaries and exact Lagrangians. Given two closed exact Lagrangian L, L' in (M, θ), we want to develop a cohomology theory where the generators are the points in $L \cap L'$ and the differentials are given by J-holomorphic strips $u : \mathbb{R} \times [0,1]$ asymptotic to two given intersection points. However, note that the intersection $L \cap L'$ may not be isolated and transverse. Therefore, one should use a Hamiltonian symplectomorphism ϕ_H so that $\phi_H(L) \cap L'$ is transverse. This is always possible and since L and L' are compact the ϕ_H can be chosen to be compactly supported. Note that $phi_H(L) \cap L'$ is in bijection to the space of hamiltonian chords

$$\mathcal{P}_H\left(L, L'\right) := \{\alpha : [0,1] \to (M, \theta) \mid \alpha(0) \in L, \alpha(1) \in L, \alpha \text{ tangent to } X_H\}$$

Now we can define out action functional to be

$$A_H(y) = \left(\int -y^*\theta + H(t, y(t))dt\right) + h_L(y(1)) - h_{L'}(y(0)) \tag{2.9}$$

and its critial points are precisely the hamiltonian chords. Pick any compactible almost complex structure J, the gradient flow equation is precisely the Floer equations again, with the domain being $u : \mathbb{R} \times [0,1]$:

$$\begin{cases} \partial_s u + J(t,u)\left(\partial_t u - X(t,u)\right) = 0 \\ u(s,0) \in L_0, \quad u(s,1) \in L_1 \end{cases} \tag{2.10}$$

Now if the transversally, compactness, and gluing issues are resolved like in the Hamiltonian Floer case, we have a well-defined ungraded cochain complex $CF(L, L')$ with coefficient in $\mathbb{Z}/2$ that is independent of (H, J). If in addition the grading and orientation issues are resolved, we will get a $\mathbb{Z}$-graded cochain complex with coefficients in $\mathbb{Z}$. We only present the problems that do not emerge in the definition of Hamiltonian Floer theory:

1. Problem of disc bubbling: The boundaries of the moduli space of trajectories consist of three kinds of elements: broken J-holomorphic strips, sphere bubbling in the interior of the strip and disc bubbling in the boundaries of the strip. The second one cannot exist in an exact symplectic manifold. The third one cannot exist for exact Lagrangians by Stokes theorem. However, in general disc bubbling is possible and since they are codimension 1 phenomenon, $\partial^2 \neq 0$ in general. There are several solutions to this issue but the simplest one is to restrict to exact Lagrangians.

2. Problem of orientation: Unlike Hamiltonian Floer theory or closed Gromov-Witten theories, open string invariants like Lagrangian Floer homology in general need additional data to be orienable. For example, a common assumption would be the Lagrangians should be spin. Then the path space hence $\mathcal{P}_H(L, L')$ will be oriented.

3. Problem of grading: In general $CF(L, L')$ has a relative $\mathbb{Z}/N$-grading where N is the minimal Maslov index of the Lagrangians. It is possible to have an absolute $\mathbb{Z}/N$-grading by replacing Lagrangians by their lifts to the N-fold cover of Lagrangian grassmanian bundles. In particular, when $2c_1(X) = 0$, one lifts the Lagrangian grassmanian bundles to their fiberwise universal cover and define an absolute $\mathbb{Z}$-grading on $CF(L, L')$

Fixed point Floer homology

Given a symplectomorphism $\phi : (M, \omega) \to (M, \omega)$, the fixed point Floer homology roughly speaking counts the number of essential fixed points that do not vanish under small perturbation by symplectomorphism (but could vanish under small perturbation by diffeomorphism). In nice cases, it can be defined as an infinite dimensional Morse homology on the ϕ-twisted loop space on M. In this paper, we will use Seidel's definition that uses pseudo-holomorphic sections which is more general.

Consider the mapping torus Y_ϕ defined using ϕ on M, the product $X_\phi := \mathbb{R} \times Y_\phi$ projects naturally to the cylinder $\mathbb{R} \times S^1$. X_ϕ has a natural symplectic structure $ds \wedge dt + \omega_\phi$ where ω_ϕ is the pullback of the closed two form on Y_ϕ that restricts to ω on each fiber. In fact, the map

$$X_\phi \xrightarrow{\pi} \mathbb{R} \times S^1 \tag{2.1}$$

is a symplectic fibration whose monodromy is ϕ.

Now there is a one-to-one correspondence between fixed points of ϕ and periodic orbits of (the pullback of) ∂t on Y_ϕ that covers S^1 once. The fixed point Floer chain complexes $CF(\phi)$ are generated by these orbits. For the differential, we count pseudo-holomorphic sections of the symplectic fibration $X_\phi \xrightarrow{\pi} \mathbb{R} \times S^1$ that are asymptotic to different orbits as $s \to \pm\infty$.

2.2 Fukaya categories

2.2.1 A_∞ categories

To properly model Floer theory for all (nice) Lagrangians simultaneously in a symplectic manifold (M, ω) requires the language of $\mathcal{A}_\infty$-algebra and categories:

Definition 2.2.1. An $\mathcal{A}_\infty$-category is a

1. A collection of objects obj $\mathcal{A}$, and
2. a collection of graded vector spaces $hom_{\mathcal{A}}(X_0, X_1)$ for every pair of objects (X_0, X_1), and
3. A collection of linear maps for $d \geqslant 1$:

$$\mu_{\mathcal{A}}^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \longrightarrow \hom_{\mathcal{A}}(X_0, X_d)[2-d] \tag{1.3}$$

That satisfy the $\mathcal{A}_\infty$-relations

$$\sum_{m,n} (-1)^{*_n} \mu_{\mathcal{A}}^{d-m+1}\left(a_d, \ldots, a_{n+m+1}, \mu_{\mathcal{A}}^m\left(a_{n+m}, \ldots, a_{n+1}\right), a_n, \ldots, a_1\right) = 0 \tag{1.4}$$

Where $*_n == |a_1| + \cdots + |a_n| - n$

Now we explain the definitions. For simplicity let's forget about the grading and signs and use μ^d for $\mu_{\mathcal{A}}^d$

1. An $\mathcal{A}_\infty$ category need not be a category in the ordinary sense. First note that it is not required in the definition that there are identity morphisms. Moreover, the composition between two morphisms $\mu_{\mathcal{A}}^2$ need not be associative. Instead according to (1.3) they satisfy the equation:

$$\begin{aligned} &\mu^2\left(\mu^2(\cdot,\cdot),\cdot\right) + \mu^2\left(\cdot,\mu^2\cdot,\cdot\right) \\ =&\mu^1\left(\mu^3(\cdot,\cdot,\cdot)\right) + \mu^3\left(\mu^1(\cdot),\cdot,\cdot\right) + \mu^3\left(\cdot,\mu^1(\cdot)\cdot\right),\cdot) + \mu^3\left(\cdot,\cdot,\mu^1(\cdot)\right) \end{aligned}$$

So μ^2 is associative up to homotopy given by μ^3, and those homotopies are associative up to higher homotopies given by higher μ^d. This is exactly the content of the $\mathcal{A}_\infty$-relations.

2. The first two equations in (1.3) are

$$\begin{gathered}\mu^1 \circ \mu^1 = 0 \\ \mu^1\left(\mu^2(\cdot,\cdot)\right) = \mu^2\left(\mu^1(\cdot),\cdot\right) + \mu^2\left(\cdot,\mu^1(\cdot)\right)\end{gathered} \tag{1.7}$$

These showes that $\mathcal{A}_\infty$-categories can be viewed as generaliztion to dg-categories, with differential given by μ^1 and multiplication given by μ^2, except that the multiplication is no longer associative. In particular, one can define the cohomology category $H(\mathcal{A})$ of $\mathcal{A}$ by replacing morphisms by cohomology classes of morphisms. This is a genuine dg-category except that there may be no identities.

Examples of $\mathcal{A}_\infty$ categories includes $\mathcal{A}_\infty$ algebras ($\mathcal{A}_\infty$ categories with one object), dg categories (as $\mathcal{A}_\infty$ categories with $\mu^d = 0\big)$ for $d \geqslant 3$, and more generality any graded abelian categories $\left(\mu^1 = 0, \mu^d = 0\right)$ for $d \geqslant 3$ and μ^2 is ordinary composition of morphisms).

For $\mathcal{A}_\infty$-categories over $\mathbb{K}$ with finitely many objects $\{L_i\}$, one can consider the equivalent $\mathcal{A}_\infty$-algebra $\mathcal{C}$ over $R = \mathbb{K}e_1 \oplus \cdots \oplus \mathbb{K}e_k$ with $\mathcal{C} = \oplus_{i,j} \hom\left(X_i, X_j\right)$. Since many categories we consider are finitely generated (e.g. The Fukaya-Seidel category of a Lefschetz Fibrations $\mathcal{FS}(\pi)$ is generated by Lefschetz thimbles), it is often harmless to make this change of view.

Also, the data of an $\mathcal{A}_\infty$-algebra $\mathcal{A}$ can be packaged into a dg-coalgebra structure on $T\mathcal{A}[1] = \oplus_{i>0}\mathcal{A}[1]^{\otimes i}$. All the μ^i fits into a map $\mu : T\mathcal{A}[1] \to \mathcal{A}[1]$, and can be extended to a map $\hat{\mu} : T\mathcal{A}[1] \to T\mathcal{A}[1]$ by:

$$\hat{\mu}\left(x_k \otimes \cdots \otimes x_1\right) := \sum_{i,j} (-1)^{\mathbf{2}_i} x_k \otimes \cdots \otimes x_{i+j+1} \otimes \mu^j\left(x_{i+j}, \ldots, x_{i+1}\right) \otimes x_i \otimes \cdots \otimes x_1 \tag{1.8}$$

Now the $\mathcal{A}_\infty$ relations are equivalent to

$$\hat{\mu}^2 = 0 \tag{1.9}$$

Definition 2.2.2. An $\mathcal{A}_\infty$ functor $\mathcal{F}$ between two $\mathcal{A}_\infty$ categories $\mathcal{A}$ and $\mathcal{B}$ is:

1. For every object $X \in \mathcal{A}$, an objects $\mathcal{F}X \in \mathcal{B}$

2. For every finite collection of objects in $\mathcal{A}$, a map of graded vector spaces $\mathscr{F}^d$: $\hom_{\mathcal{A}}\left(X_{d-1}, X_d\right) \otimes \cdots \otimes \hom_{\mathcal{A}}\left(X_0, X_1\right) \longrightarrow \hom_{\mathscr{B}}\left(\mathcal{F}X_0, \mathcal{F}X_d\right)[1-d]$ that satisfy

$$\sum_{r}\sum_{s_1,\ldots,s_r}\mu_{\mathcal{B}}^{r}\left(\mathcal{F}^{s_r}\left(a_d,\ldots,a_{d-s_r+1}\right),\ldots,\mathcal{F}^{s_1}\left(a_{s_1},\ldots,a_1\right)\right) \tag{1.10}$$

$$=\sum_{m,n}(-1)^{*_n}\mathcal{F}^{d-m+1}\left(a_d,\ldots,a_{n+m+1},\mu_{\mathcal{A}}^{m}\left(a_{n+m},\ldots,a_{n+1}\right),a_n,\ldots,a_1\right) \tag{1.11}$$

Same as the $\mathcal{A}_\infty$ relations, this formula should be interpreted as saying $\mathcal{F}$ respects composition of morphisms up to coherent higher homotopies.

The first relation says that $\mathcal{F}$ commutes with $\mu_{\mathcal{A}}^1$ and $\mu_{\mathcal{B}}^1$, hence induces a functor $H(\mathcal{F})$ between the cohomology categories $H(\mathcal{A})$ and $H(\mathcal{B})$.

Isomorphisms between $\mathcal{A}_\infty$-algebras and equivalences between two $\mathcal{A}_\infty$ categories can be defined in the obvious way. However the more useful notion are quasi-isomorphism and quasi-equivalences. Two $\mathcal{A}_\infty$-algebras $\mathcal{A}$ and $\mathcal{B}$ are quasi-isomorphic if there is a morphism between them that induces isomorphism between $H(\mathcal{A})$ and $H(\mathcal{B})$. Two $\mathcal{A}_\infty$-categories are quasi-equivalent if the induced functor $H(\mathcal{F})$ is an equivalence.

For the applications in Fukaya category and Mirror symmetry, it is often convienent to work in some larger categories of $\mathcal{A}_\infty$-modules

Definition 2.2.3. An $\mathcal{A}_\infty$ (right) module over an $\mathcal{A}_\infty$ category $\mathcal{A}$ is an $\mathcal{A}_\infty$ functor $\mathcal{M}$: $\mathcal{A}^{op} \to \mathrm{Ch}$ into the category of chain complexes.

Here is a more illuminating and concrete definiton:

An $\mathcal{A}_\infty$ (right) module over an $\mathcal{A}_\infty$ category $\mathcal{A}$ is:

A chain complex $\mathcal{M}(X)$ for every objects $X \in \mathcal{A}$ and a set of map for $d \geqslant 1$:

$$\begin{aligned}\mu_{\mathcal{M}}^{d}: \mathcal{M}\left(X_{d-1}\right) \otimes \hom_{\mathcal{A}}\left(X_{d-2}, X_{d-1}\right) \otimes \cdots \otimes \hom_{\mathcal{A}}\left(X_0, X_1\right) \\ \longrightarrow \mathcal{M}\left(X_0\right)[2-d]\end{aligned} \tag{1.12}$$

that satisfy:

$$\begin{aligned}\textstyle\sum_n \mu_{\mathcal{M}}^{n+1}\left(\mu_{\mathcal{M}}^{d-n}\left(b, a_{d-1}, \ldots, a_{n+1}\right), \ldots, a_1\right) \\ +\textstyle\sum_{m,n} \mu_{\mathcal{M}}^{d-m+1}\left(b, a_{d-1}, \ldots, a_{n+m+1}, \mu_{\mathcal{A}}^{m}\left(a_{n+m}, \ldots, a_{n+1}\right), \ldots, a_1\right)=0\end{aligned} \tag{1.14}$$

An $\mathcal{A}_\infty$ left module is defined in a similar way. An $\mathcal{A}_\infty$ bimodule is defined with both right and left action of $\mathcal{A}$:

Definition 2.2.4. An $\mathcal{A}_\infty$ bimodule $\mathcal{M}$ over an $\mathcal{A}_\infty$ category $\mathcal{A}$ is:

1. A chain complex $\mathcal{M}(A, B)$ for every pair of objects in $\mathcal{A}$.

2. For every finite collection of objects V_i and W_i a map of chain complexes

$$\mu_{\mathcal{M}}^{r|1|s} : \hom_{\mathcal{A}}(V_{r-1}, V_r) \times \cdots \hom_{\mathcal{A}}(V_0, V_1) \times \mathcal{M}(V_0, W_0) \times \tag{1.15}$$

$$\times \hom_{\mathcal{A}}(W_1, W_0) \times \cdots \times \hom_{\mathcal{A}}(W_s, W_{s-1}) \longrightarrow \mathcal{M}(V_r, W_s) \tag{1.16}$$

that satisfy:

$$\begin{aligned} &\sum \mu_{\mathcal{M}}^{r-i|1|s-j}\left(v_r, \ldots, v_{i+1}, \mu_{\mathcal{M}}^{i|1|j}(v_i, \ldots, v_1, \mathbf{b}, w_1, \ldots, w_j), w_{j+1} \ldots, w_s\right) \\ +&\sum \mu_{\mathcal{M}}^{r-i+1|1|s}(v_r, \ldots, v_{k+i+1}, \mu_{\mathcal{A}}^{i}(v_{k+i}, \ldots, v_{k+1}), v_k, \ldots, v_1, \mathbf{b}, w_1, \ldots, w_s) \\ +&\sum \mu_{\mathcal{M}}^{r|1|s-j+1}\left(v_r, \ldots, v_1, \mathbf{b}, w_1, \ldots, w_l, \mu_{\mathcal{A}}^{j}(w_{l+1}, \ldots, w_{l+j}), w_{l+j+1} \ldots, w_s\right) \\ &= 0 \end{aligned} \tag{1.20}$$

Again, this formula should be interpreted as saying the action of $\mathcal{A}$ on $\mathcal{M}$ is associative up to coherent higher homotopies.

The most important examples of bimodules include the diagnal bimodule $\mathcal{A}_\Delta$ and its dual. $\mathcal{A}_\Delta$ is defined by

$$\mathcal{A}_\Delta(X, Y) = \hom_{\mathcal{A}}(X, Y)$$

and

$$\mu_{\mathcal{A}}^{r|1|s} = \mu_{\mathcal{A}}^{r+1+s}$$

Therefore $\mathcal{A}_\Delta$ is essentially just the algebra $\mathcal{A}_\infty$ considered as a module over itself.

Pre-morphisms between $\mathcal{A}$-(bi)modules can be simply be defined as the natural transformations between functors from $\mathcal{A}$ to Ch. The category of (bi)modules with pre-morphisms has a natural dgcategory structure by pulling back the dg-structure from Ch. Now morphisms between (bi)-modules are defined to be the pre-morphisms that are closed and a quasi-isomorphism is defined to be a morphism that induce isomorphism on cohomologies.

Most of the operations one can do to bimodules over associative algebra can be carried over to bimodules over $\mathcal{A}_\infty$ algebras. The most important ones here are the degree shifts $\mathcal{M}[1]$, tensor product and dual. The shifts $\mathcal{M}[1]$ of a module $\mathcal{M}$ is defined by

$$\mu_{\mathcal{M}[1]}^{s;1;r}\left(a'_s, \ldots, a'_1; \mathbf{m}; a_r, \ldots, a_1\right) = (-1)^{\|a_1\| + \cdots + \|a_r\| + 1} \mu_{\mathcal{M}}^{s;1;r}\left(a'_s, \ldots, a'_1; \mathbf{m}; a_r, \ldots, a_1\right)$$

. The dual $\mathcal{M}^\vee$ of $\mathcal{M}$ is defined by

$$\left\langle \mu_{\mathcal{M}}^{s;1;r}\left(a_s,\ldots,a_1;\pi;a'_r,\ldots,a'_1\right),\mathbf{m}\right\rangle = (-1)^{|p|+1}\left\langle \pi,\mu_{\mathcal{M}}^{r;1;s}\left(a'_r,\ldots,a'_1;\mathbf{m};a_s,\ldots,a_1\right)\right\rangle$$

Given $\mathcal{A}_\infty$-bimodules $\mathcal{M}$ and $\mathcal{N}$ over $\mathcal{A}$, their tensor product is defined to be the vector space

$$\mathcal{M}\otimes T(\mathcal{A}[1])\otimes\mathcal{N}$$

with structure maps

$$\begin{aligned}\mu^{0|1|0}\left(\mathbf{m},d_1,\ldots,d_k,\mathbf{n}\right) = \mu_{\mathcal{M}}^{0|1|t}\left(\mathbf{m},d_1,\ldots,d_t\right)\otimes d_{t+1}\otimes\cdots\otimes d_k\otimes\mathbf{n}\\ \sum\mathbf{m}\otimes d_1\otimes\cdots\cdots\otimes d_{k-s}\otimes\mu_{\mathcal{N}}^{s|1|0}\left(d_{k-s+1},\ldots,d_k,\mathbf{n}\right)\\ +\sum(-1)^{W_{-(k+1)}^{-(j+i+1)}}\mathbf{m}\otimes d_1\otimes\cdots\otimes d_j\otimes\mu_{\mathcal{A}}^i\left(d_{j+1},\ldots,d_{j+i}\right)\otimes\\ d_{j+i+1}\otimes\cdots\otimes d_k\otimes\mathbf{n}\end{aligned}\tag{1.24}$$

and

$$\mu^{r|1|0}\left(c_1,\ldots,c_r,\mathbf{m},d_1,\ldots,d_k,\mathbf{n}\right) = \tag{1.25}$$

$$\sum_t(-1)^{\mathbf{W}_{-(k+1)}^{-(t+1)}}\mu_{\mathcal{M}}^{r|1|t}\left(c_1,\ldots,c_r,\mathbf{m},d_1,\ldots,d_t\right)\otimes d_{t+1}\otimes\cdots\otimes d_k\otimes\mathbf{n}\tag{1.26}$$

$$\mu^{0|1|s}\left(\mathbf{m},d_1,\ldots,d_k,\mathbf{n},e_1,\ldots,e_s\right) = \tag{1.27}$$

$$\sum_j\mathbf{m}\otimes d_1\otimes\cdots\otimes d_{k-j}\otimes\mu_{\mathcal{N}}^{j|1|s}\left(d_{k-j+1},\ldots,d_k,\mathbf{n},e_1,\ldots,e_s\right)\tag{1.28}$$

and

$$\mu^{r|1|s} = 0 \text{ if } r>0 \text{ and } s>0\tag{1.29}$$

The upshot is that, the coherent higher homotopies $\mu^{r|1|s}$ does not mix the left and the right actions.

One would expect tensoring with the diagonal bimodule $\mathcal{A}_\Delta$ gives the identity functor on the category of $\mathcal{A}$-bimodules. This is true up to quasi-equivalences. Similarly tensoring with the dual diagonal bimodule gives the Serre functor up to quasi-equivalences.

Bimodule $\mathcal{P}$ that is invertible can be defined as have a bimodule $\mathcal{P}^{-1}$ such that the following modules are quasi-equivalent

$$\mathcal{P}^{-1}\otimes_{\mathcal{A}}\mathcal{P}\simeq\mathcal{A}\simeq\mathcal{P}\otimes_{\mathcal{A}}\mathcal{P}^{-1}\tag{1.30}$$

This is important in the definition of noncommutative divisors since classically line bundles corresponds to invertible modules.

Perhaps the most important invariant of an $\mathcal{A}_\infty$ category are its Hochschild homology and cohomology. They govern the deformation of the $\mathcal{A}_\infty$ structures.

Definition 2.2.5. The Hochschild homology of an $\mathcal{A}_\infty$ category $\mathcal{A}$ with coefficients in an $\mathcal{A}$ bimodule $\mathcal{B}$ is the homology of the chain complex:

$$CC_*(\mathcal{A},\mathcal{B}) := \oplus \mathcal{M}\left(X_k, X_0\right) \times \hom_{\mathcal{A}}\left(X_{k-1}, X_k\right) \times \cdots \times \hom_{\mathcal{A}}\left(X_0, X_1\right) \tag{1.31}$$

where the direct sum is over all k and $k+1$-tuples of objects X_i. The differential is:

$$\begin{aligned} d_{\mathrm{CC}*}\left(\mathbf{b} \otimes x_1 \otimes \cdots \otimes x_k\right) = \\ \sum \mu_{\mathcal{B}}\left(x_{k-j+1}, \ldots, x_k, \mathbf{b}, x_1, \ldots, x_i\right) \otimes x_{i+1} \otimes \cdots \otimes x_{k-j} \\ + \sum \mathbf{b} \otimes x_1 \otimes \cdots \otimes \mu_{\mathcal{A}}^j\left(x_{s+1} \otimes \cdots \otimes x_{s+j}\right) \otimes x_{s+j} \otimes \cdots \otimes x_k \end{aligned} \tag{1.32}$$

Hochschild cohomology can be defined in a similar way on the vector space $\mathrm{CC}^*(\mathcal{A},\mathcal{B}) := \hom_{\mathrm{Vect}}\left(T\mathcal{A},\mathcal{B}\right)$. However the differential is harder to describle. So we choose a definition that is more concise and useful.

Definition 2.2.6. The Hochschild cochain complex is defined as

$$\mathrm{CC}^*(\mathcal{A},\mathcal{B}) := \hom_{\mathcal{A}-\text{ bimod }}\left(\mathcal{A}_\Delta, \mathcal{B}\right) \tag{2.2}$$

That is, it is the cochain complex of morphisms in the category of bimodules between $\mathcal{A}$ and $\mathcal{B}$ Like Hamiltonian Floer homologies, one can define product structures on Lagrangian Floer homologies. However, the details will be more complicated because the Floer datum (J,H) used for different pairs of Lagrangians L, L' are different. Given lagrangians L_1, L_2, L_3 and elements a,b,c in $CF\left(L_1, L_2\right)$, $CF\left(L_2, L_3\right)$ and $CF\left(L_1, L_3\right)$ respectively, they are defined by using three different set of Floer datums $\left(J_{12}, H_{12}\right), \left(J_{23}, H_{23}\right)$ and $\left(J_{13}, H_{13}\right)$. We want to define the moduli space $\mathcal{M}(a,b,c;J,H)$ of maps of discs $\mathbb{D}$ with 3 punctures on $\partial\mathbb{D}$ into M. To achieve transversality of this moduli space we also need a pertubation datum. Roughly speaking it is a family of J and H that is both domain-dependent and direction dependent such that near the boundary punctures they agree with the Floer datum. Moreover, we need to make sure that these datum are compatible with the gluing maps. We won't give a precise definition of perturbation datum here but instead refer to [Sei08], which also contains a proof of the existence of a compatible choice of perturbation datum.

Now $\mathcal{M}(a,b,c;J,H)$ can be defined as moduli space of maps that:

1. satisfy the perturbed Floer equations w.r.t the pertubation datum.

2. The three arcs of $\mathbb{D}$ divided by the three punctures are being mapped to L, L' and L'' respectively.

3. Near the punctures it is asymptopic to the three hamiltonian chords a, b and c.

Then the product structure can be defined as:

$$\mu^2(a,b) := \sum_{c \in CF(L,L'')} |\mathcal{M}(a,b,c;J,H)| \cdot c \tag{2.1}$$

By looking at the boundaries of the moduli spaces $\mathcal{M}(a,b,c;J,H)$ we see that this product satisfy the Leibniz rule. However it is not associative since $\coprod_d \mathcal{M}(a,b,d) \times \mathcal{M}(d,c,e)$ is not cobordant to $\coprod_d \mathcal{M}(b,c,d) \times \mathcal{M}(a,d,e)$ for every e. Indeed if we look at the moduli space of maps of discs with four boundary punctures and define the operations μ^3 using it, we see there are four more ways the map could degenerate, leading to the four types of boundaries that gives the four terms $\mu^3(d\cdot,\cdot,\cdot) + \mu^3(\cdot,d\cdot,\cdot)+$ $\mu^3(\cdot,\cdot,d\cdot) + d\mu^3(\cdot,\cdot,\cdot)$. This is exactly equation (1.9). So Floer theoretic operations are only associative up to coherent higher homotopies.

Similarly, we can define higher operations μ^d using $d+1$-boundary punctured discs. Then (under the assumption that issues 1-3 are resolved and Floer and perturbation datum are chosen accordingly) these define an $\mathcal{A}_\infty$ structure on $CF(L,L)$, and more generally an $\mathcal{A}_\infty$ category structure if we define the objects to be Lagrangian submanifolds (that are spin together with a lift to covers of Lagrangian grassmanian bundles) and morphism groups to be $CF(L,L')$.

Let us summarize the construction so far.

Definition 2.6. Given an exact symplectic manifold (M,θ), the Fukaya category $\mathcal{F}(M,\theta)$ as an $\mathcal{A}_\infty$ category has objects (L,s) where L is a closed spin Lagrangian submanifold, $s: L \to \mathcal{L}\tilde{G}r(TM)$ a lift of the Lagrangian Gauss map. The morphisms are Floer chain groups: $Mor(L,L') = CF(L,L')$ and μ^d is defined by counting perturbed Floer polygons as above

There are many other versions of Fukaya categories as listed in the introduction. Here we will only define two of them: The Fukaya-Seidel category $\mathcal{FS}(\pi)$ of a Lefschetz fibration π and the Wrapped Fukaya category. To motivate the definitions we recall a notion for their mirrors:

Lagrangian surgery as mapping cones

The mapping cone of a morphism $L_1 \xrightarrow{p} L_2$ often can be quasi-represented by a Lagrangian $L_1 \#_p L_2$ defined as the **Lagrangian surgery** of L_1 and L_2 at p. To be more precise, this quasi-representability means that for any Lagrangian T, there is a long exact sequence

$$HF(T, L_1) \xrightarrow{\mu_2(\cdot,[p])} HF(T, L_2) \to HF(T, L_1\#_p L_2) \tag{2.3}$$

We can avoid using the testing object T by using the language of exact triangles in the Fukaya category:

$$L_1 \xrightarrow{p} L_2 \to L_1\#_p L_2 \to \tag{2.4}$$

Lagrangian correspondence as bimodules

It turns out "closed string objects" like symplectomorphisms ϕ can also be characterized using Lagrangian Floer theory. Recall $\phi : M \to M$ is a symplectomorphism if and only if its graph $Gr(\phi) \subset (M \times M, \omega \oplus (-\omega))$ is Lagrangian. Wehrheim and Woodward proved the following isomorphism:

Theorem 2.2.7. Given Lagrangian submanifolds L_0, L_1, L_2, L_3 and symplectomorphism ϕ:

- $HF(L_1 \times L_2, L_3 \times L_4) \cong HF(L_1, L_2) \otimes HF(L_4, L_3)$
- $L_1 \times L_2, Gr(\phi^{-1}) \cong HF(L_1, \phi(L_2))$
- $HF(\Delta, Gr(\phi^{-1})) \cong HF(\phi)$

where $\Delta = Gr(id)$ is the diagonal

2.3 Dehn twist exact sequences

It is a classical result that for an isolated hypersurface singularity

$$f : \mathbb{C}^{n+1} \to \mathbb{C}$$

the nearby smooth fibers are homotopy equivalent to a bouquet of spheres S^n called *vanishing cycles*. Moreover, the monodromy around the singularity is a composition of Dehn twists along these vanishing cycles [1]. In fact, these structures are compatible with the Weinstein structure on the total space $\mathbb{C}^{n+1}$ (with some minor modifications). For example, the vanishing cycles are exact Lagrangian submanifolds, and the Dehn twists are exact symplectomorphisms. It is therefore natural to study their symplectic invariants, such as Lagrangian Floer homology, Fukaya categories, and fixed-point Floer homology.

Seidel's exact sequences and generalization

Seidel proved the following exact sequences regarding the effect of Dehn twists along a Lagrangian sphere L on the symplectic topology of (M, ω) and the Lagrangian topology of other Lagrangian submanifolds:

Theorem 2.3.1. (Seidel's sequence, open string version) Given Lagrangian sphere L and Lagrangians L_1, L_2, there exists a long exact sequence

$$\cdots \to HF(L_1, L) \otimes HF(L, L_2) \xrightarrow{\mu_2} HF(L_1, L_2) \to HF(L_1, \tau_L(L_2)) \to \cdots \tag{2.5}$$

where μ_2 is the map defined by counting pseudo-holomorphic triangles bounded by (L, L_1, L_2).

Theorem 2.3.2. (Seidel's sequence, closed string version) Given Lagrangian sphere L and a symplectomorphism ϕ, there exists a long exact sequence

$$\cdots \to HF(L, \phi(L)) \xrightarrow{\mu_2} HF(\phi) \to HF(\phi \circ \tau_L) \to \cdots \tag{2.6}$$

Given a finite collection of (monotone or exact) Lagrangian spheres $\{L_i\}$ in a (monotone or exact) symplectic manifold (M, ω), the (compact) Fukaya category $\mathcal{B}$ generated by $\{L_i\}$ is defined to have $\{L_i\}$ as objects and $cF(L_i, L_j)$ as morphisms. In the case of Lefschetz fibrations, the order of the vanishing cycles is also important, which motivates Seidel [21] to define the directed Fukaya category $\mathcal{A}$:

$$\hom_{\mathcal{A}}(L_j, L_k) = \begin{cases} CF(L_j, L_k) & j < k, \\ \mathbb{K}e_j & j = k, \\ 0 & j > k \end{cases}$$

In this paper, we work in a $\mathbb{Z}$-graded setting. Let (M, ω) be a symplectic manifold of dimension $2d$ with $2c_1(M, \omega) = 0$ and $\{L_i\}$ be graded Lagrangian spheres. Seidel conjectured that the fixed-point Floer homology of the composed Dehn twists $\tau := \tau_1 \circ \cdots \circ \tau_n$ can be computed from the topology of M and all of $\hom_{\mathcal{A}}(L_j, L_k)$. To be more precise:

Conjecture 2.3.1 ([20, Conjecture 3])**.** *There is a long exact sequence:*

$$\cdots \to H^*(D) \to HF^*(id) \to HF^*(\tau) \to \ldots \tag{2.7}$$

Where D is the following cochain complex:

$$\bigoplus_{1 \le k \le n} \bigoplus_{1 \le i_1 < \cdots < i_k \le n} CF(L_{i_k}, L_{i_1}) \otimes CF(L_{i_{k-1}}, L_{i_k}) \cdots \otimes CF(L_{i_1}, L_{i_2})[d+n] \tag{2.8}$$

with a differential analogous to the Hochschild differential.

Remark 2.3.3. The differential has a more natural description, again due to Seidel. Consider $\mathcal{A}$ and $\mathcal{B}$ as an A_∞ algebra over the semi-simple ring $R = \mathbb{K}e_1 \oplus \cdots \oplus \mathbb{K}e_n$ and $\mathcal{A}_+$ to be the kernel of the augmentation map $\mathcal{A} \to R$. Then (2.8) is the chain complex

$$(\mathcal{B}[d] \otimes_R T(\mathcal{A}_+[1])^{diag} \tag{2.9}$$

which is exactly $CC_*(\mathcal{A}, \mathcal{B}[d])$. See [11, Section 2.9]

Let $\mathcal{F} := Fuk(M)$ be the compact Fukaya category of M. In this paper, we construct an exact triangle in the category of $\mathcal{F}$-bimodules that we expect to give the exact sequence (2.7) when we apply the bimodule Hom functor. We fixed the notations. We use τ_i to denote τ_{L_i}. The left and right Yoneda modules are denoted as $\mathcal{Y}^l_X(Y) := \hom(X, Y)$ and $\mathcal{Y}^r_X(Y) := \hom(Y, X)$ respectively. $\mathcal{F}_\Delta$ and $\mathfrak{G}_\tau$ are the diagonal and graph bimodule of τ respectively. All chain complexes are over $\mathbb{Z}/2$, though we expect everything to work over $\mathbb{Z}$.

The following is the main theorem.

Theorem 2.3.2 (Main theorem)**.** *Let (M, ω) be a symplectic manifold with $2c_1(M) = 0$ with a collection of Lagrangian spheres $\{L_i\}$ such that their Floer cohomologies are well defined. There is an exact triangle of $Fuk(M)$-bimodules*

$$\mathcal{E}_n \xrightarrow{\widetilde{ev}} \mathcal{F}_\Delta \to \mathfrak{G}_\tau$$

where

$$\mathcal{E}_n(A, B) = \bigoplus_{1 \le k \le n} \bigoplus_{1 \le i_1 < \cdots < i_k \le n} CF(A, L_{i_1}) \otimes \cdots \otimes CF(L_{i_k}, B)[k-1]$$

is a bimodule version of the bar complex $T\mathcal{A}$. Its structure maps consists of all possible contractions by the A_∞ operations μ_i and $\widetilde{ev}$ is the map

$$\mu_{k+1} : CF(A, L_{i_1}) \otimes \cdots \otimes CF(L_{i_k}, B)[k-1] \to (A, B) \tag{2.10}$$

on each direct summand.

Example 2.3.3. When $n = 1$, the exact triangle reduces to the exact triangle

$$\mathcal{Y}^l_{L_1} \otimes_{\mathbb{K}} \mathcal{Y}^r_{L_1} \xrightarrow{ev} \mathcal{F}_\Delta \to \mathfrak{G}_{\tau_1}$$

which is the bimodule version of the exact triangle in [16]

Remark 2.3.4. We point out that Sikimeti Ma'u and Tim Perutz announced a proof of Conjecture 2.3.1 as early as 2011 using a direct approach via quilted Floer theory. Although our approach also relies on M'au-Wehrheim-Woodward's Lagrangian correspondence theory, the main tool we use is the Lagrangian cobordism due to Biran and Cornea [5][3], as well as techniques from [16]. It is also worth pointing out that the Biran and Cornea generalized their work to Lagrangian cobordism in Lefschetz fibrations [4].

To relate this exact triangle in the category of bimodules over $\mathcal{F}$, we consider the category $\mathcal{F}^*(M \times M^-)$ which is the pre-triangulated split closure of the compact Fukaya category of $M \times M^-$ together with the diagonal Δ. we use the subcategory $\mathcal{F}^2$ of the category $\mathcal{F}^*(M \times M^-)$ which is generated by products of compact Lagrangians.

Remark 2.3.5. When M is exact and therefore noncompact, the category $\mathcal{F}^*(M \times M^-)$ needs some clarification since the diagonal is not a compact Lagrangian. The morphism between a compact Lagrangian A and a non-compact Lagrangian B can be defined without difficulties. The morphism between two noncompact Lagrangians ($e.g. A = B = \Delta$) is defined by perturbing B using a small *positive* Hamiltonian flow that is equal to the Reeb flow near the boundary.

It is an algebraic fact that the functor $\mathbf{M} : \mathcal{F}^2 \to \mathcal{F}$–mod–$\mathcal{F}$ that maps $A \times B$ to $\mathcal{Y}^l_A \otimes_{\mathbb{K}} \mathcal{Y}^r_B$ is full [11, Proposition 1.2]. When M is nondegenerate, Δ (and hence the graph of any symplectomorphism) is also contained in $\mathcal{F}^2$, and $\mathbf{M}$ is full on these objects too. This implies:

$$Hom_{\mathcal{F}-\mathcal{F}}(\mathcal{F}_\Delta, \mathcal{F}_\Delta) \cong HF(\Delta, \Delta, +) \cong HF(id, +) \tag{2.11}$$

$$Hom_{\mathcal{F}-\mathcal{F}}(\mathcal{F}_\Delta, \mathcal{F}_\phi) \cong HF(\Phi, +) \tag{2.12}$$

$$hom_{\mathcal{F}-\mathcal{F}}(N \times N', \Delta) \cong HF(N, N') \tag{2.13}$$

Remark 2.3.6. We explain what the $\pm$ means. When M is exact, which means that it is not compact, one has to perturb the symplectomorphism ϕ near the boundary to define $HF(\phi, \pm)$. This is achieved by picking a Hamiltonian H such that $H_{|\partial M} \equiv 0$ and the Hamiltonian flow ψ^H_t equal to the Reeb flow. Then the perturbation $\phi \circ \psi^H_t$ defines $HF(\phi, +)$ when $t > 0$ and $HF(\phi, -)$ when $t < 0$. When M is non-exact and closed, then we can ignore the "+" and "−" as the Floer homologies do not depend on the perturbation.

Now taking $Hom_{\mathcal{F}-\mathcal{F}}(\mathcal{F}_\Delta, -)$ and the isomorphisms we have:

Corollary 2.3.4. *If M and $\{L_i\}$ are exact or monotone, and the compact Fukaya category is nondegenerate, then there's a long exact sequence*

$$\dots \to H^*(D) \to HF^*(id, +) \to HF^*(\tau, +) \to \dots \tag{2.14}$$

where D is the chain complex $Hom_{\mathcal{F}-\mathcal{F}}(\mathcal{E}_n, \mathcal{F}_\Delta)$. See [11, Definition 2.18].

Remark 2.3.7. If in our definition of $\mathcal{F}^*(M \times M^-)$ we use the negative Hamiltonian to perturb the Lagrangians near infinity, we would get similar results with "+" replaced by "−"

Given two Lagrangians N and N', taking $Hom_{\mathcal{F}-\mathcal{F}}(N \times N', -)$, we have the open version of the exact sequence.

Corollary 2.3.5. *Under the same hypothesis as Theorem 2.3.2, for every pair of Lagrangians N and N' there is a long exact sequence*

$$\cdots \to H^*(\mathcal{E}_n(N, N')) \to HF^*(N, N') \to HF^*(N, \tau(N')) \to \ldots \tag{2.15}$$

Remark 2.3.8. Theorem 2.3.2 works for any symplectic manifolds (X, ω) and graded Lagrangian spheres $\{L_i\}$ that satisfy $2c_1(X, \omega) = 0$ since our argument is purely algebraic and uses the $\mathbb{Z}$-grading (though when d is odd a $\mathbb{Z}/2$ grading is enough). It is possible to use our argument on the category of product Lagrangians directly (without going to the category of bimodules) to produce an object E_n in $\mathcal{F}^2$ corresponding to the bimodule $\mathcal{E}_n$. However, the chain complex $CF(E_n, \Delta)$ is harder to work with than $Hom_{\mathcal{F}-\mathcal{F}}(\mathcal{E}_n, \mathcal{F}_\Delta)$. Also one needs to prove a Lagrangian surgery version of Lemma 2.3.15.

2.3.1 Proof

First, we recall the definitions of diagonal bimodules, Yoneda modules, and algebraic twists in an A_∞ category $\mathcal{F}$, as defined in [19]. For simplicity, we use (A, B) to denote $\hom_{\mathcal{F}}(A, B)$ and μ_i to denote $\mu^i_{\mathcal{F}}$.

Definition 2.3.6. The **diagonal bimodule** is defined as $\mathcal{F}_\Delta(A, B) = (A, B)$ with structure maps:

$$\mu_\Delta^{k|1|s} : (A_r, A_{r-1}) \otimes \cdots \otimes (A_0, B_0) \otimes \cdots \otimes (B_{s-1}, B_s) \to (A_k, B_s) \tag{2.16}$$

given by $\mu_\Delta^{r|1|s} = \mu_{r+s+1}$

Definition 2.3.7. Given an object X, the **left Yoneda module** $\mathcal{Y}^l_X$ is defined as $\mathcal{Y}^l_X := (-, X)$ with structure maps:

$$\mu_X^{r|1} : (A_r, A_{r-1}) \otimes \cdots \otimes (A_0, X) \to (A_r, X) \tag{2.17}$$

given by $\mu_X^{r|1} = \mu_{r+1}$

Definition 2.3.8. Given an object X, the **right Yoneda module** $\mathcal{Y}^r_X$ is defined as $\mathcal{Y}^r_X := (X, -)$ with structure maps:

$$\mu^{1|s}_X : (X, B_0) \otimes \cdots \otimes (B_{s-1}, B_s) \to (X, B_s) \tag{2.18}$$

given by $\mu^{1|s}_X = \mu_{s+1}$

Definition 2.3.9. Given a left module $\mathcal{M}$ and a right module $\mathcal{N}$, there's a bimodule $\mathcal{M} \otimes \mathcal{N}$ defined as $(\mathcal{M} \otimes \mathcal{N})(A, B) := \mathcal{M}(A) \otimes \mathcal{N}(B)$ with structure maps:

$$\mu^{r|1|s}_{\mathcal{M}\times\mathcal{N}} = \begin{cases} \mu^{r|1}_{\mathcal{M}} \otimes id & s = 0, r > 0 \\ id \otimes \mu^{1|s}_{\mathcal{N}} & r = 0, s > 0 \\ \mu^{0|1}_{\mathcal{M}} \otimes id + id \otimes \mu^{1|0}_{\mathcal{N}} & r = 0, s = 0 \\ 0 & r > 0, s > 0 \end{cases} \tag{2.19}$$

Definition 2.3.10. Given a left module $\mathcal{M}$ and an object X, the **abstract twist** of $\mathcal{M}$ along X is defined as $(T_X\mathcal{M})(A) := \mathcal{M}(A) \oplus (A, X) \otimes \mathcal{M}(X)[1]$ with structure maps:

$$\mu^{r|1}_{T_X\mathcal{M}}(a, b \otimes c) = (\mu^{r|1}_{\mathcal{M}}(a), \mu^{r+1|1}_{\mathcal{M}}(b, c) + b \otimes \mu^{r|1}_{\mathcal{M}}(c)) \tag{2.20}$$

Definition 2.3.11. Given any covariant A_∞ functor $\phi : \mathcal{F} \to \mathcal{F}$, the graph bimodule $\mathfrak{G}_\tau$ is defined as $\mathcal{G}_\phi(A, B) := (\phi(B), A)$. The structure maps

$$\mu^{\phi}_{k|1|s} : (A_{k-1}, A_k) \otimes ... \otimes (\phi(B_0), A_0) \otimes ... \otimes (B_s, B_{s-1}) \to (\phi(B_s), A_k) \tag{2.21}$$

are given by the composition

$$\mu^{\phi}_{k|1|s} = \sum_{1 \le j \le s} \mu_{k+s-j+2} \circ (\sum_{i_1 + \cdots + i_j = j} id^{\otimes k+1} \otimes \phi_{i_1} \otimes \cdots \otimes \phi_{i_j}) \tag{2.22}$$

We can see that for the composed Dehn twist τ, $\mathfrak{G}_\tau$ has a complicated structure map to perform calculations, since its structure maps involve the functor τ. Here we construct using Lagrangian correspondences a bimodule that is quasi-isomorphic to $\mathfrak{G}_\tau$ and easier to work with.

The relation between Lagrangian correspondences $L_{01} \subset M_0 \times M_1$ and functors from $H^0(Fuk(M_0))$ to $H^0(Fuk(M_1))$ was first written by Wehrheim and Woodward in [25]. Then together with Ma'u they were able to enhance it to produce chain-level functors from $Fuk(M_0)$ to $Fuk(M_1)$ [15]. Recall $\mathcal{F}^2$ is the subcategory of $Fuk(M \times M^-)$ generated by product Lagrangians and graphs of compactly supported exact symplectomorphism. From [16] (also see [18], [10], and [12]) we have an exact triangle in $\mathcal{F}^2$:

$$L_n \times L_n \to \Delta \to Gr(\tau_n^{-1}) \tag{2.23}$$

Let $\mathcal{F}$–mod–$\mathcal{F}$ be the category of $\mathcal{F}-\mathcal{F}$-bimodules. There's an A_∞ functor $\Phi : \mathcal{F}^2 \to \mathcal{F}$–mod–$\mathcal{F}$. This functor is similar to the M'au-Wehrheim-Woodward functor in [15], except that the target is the category of bimodules instead of functors. It is also similar to the one constructed by Ganatra in [11], except that we are in the compact setting instead of the wrapped setting. The functor Φ takes the exact triangle above to the well-established exact triangle in $\mathcal{F}$-bimodule due to Seidel [19]:

$$\mathcal{Y}^l_{L_n} \otimes_{\mathbb{K}} \mathcal{Y}^r_{L_n} \xrightarrow{ev} \mathcal{F}_\Delta \to \mathfrak{G}_{\tau_n} \tag{2.24}$$

Where $ev : \mathcal{Y}^l_{L_n} \otimes_{\mathbb{K}} \mathcal{Y}^r_{L_n} \to \mathcal{F}_\Delta$ is the map $ev_{r|1|s} := \mu_{r+s+2}$.

There is an autoequivalence $(\tau_i \times id)_\#$ on the subcategory of $\mathcal{F}$-bimodules generated by $\Phi(\mathcal{F}^2)$ that are induced by the symplectomorphisms $\tau_i \times id$. We define

$$ev'_i := (\tau_1 \times id \circ \cdots \circ \tau_i \times id)_\#(ev) = (\tau_1 \times id)_\# \circ \cdots \circ (\tau_i \times id)_\#(ev)$$

Applying $(\tau_1 \times id)_\# \circ \cdots \circ (\tau_{n-1} \times id)_\#$ to (2.24) we get an exact triangle:

$$\mathcal{Y}^l_{\tau_1 \cdots \tau_{n-1} L_n} \otimes_{\mathbb{K}} \mathcal{Y}^r_{L_n} \xrightarrow{ev'_{n-1}} \mathfrak{G}_{\tau_1 \cdots \tau_{n-1}} \to \mathfrak{G}_{\tau_1 \cdots \tau_n} \tag{2.25}$$

We introduce the following notation:

$$\mathfrak{G}_i := \mathfrak{G}_{\tau_1 \cdots \tau_i} \tag{2.26}$$

$$\mathcal{L}_i := \mathcal{Y}^l_{\tau_1 \cdots \tau_{i-1} L_i} \otimes_{\mathbb{K}} \mathcal{Y}^r_{L_i} \tag{2.27}$$

The structure maps of $\mathcal{T}_n := \mathcal{Y}^l_{\tau_1 \cdots \tau_{n-1} L_n}$ can be computed easily using formulas for abstract twists in Definition 2.3.10 which come from [19] and the fact that abstract twists of Yoneda modules are quasi-isomorphic to Dehn twists.

Therefore by Definition 2.3.9 the structure maps of $\mathcal{L}_i = \mathcal{T}_i \otimes \mathcal{Y}^r_{L_i}$ is given by

$$\mu^{r|1|s}_{\mathcal{L}_i} = \begin{cases} \mu^{r|1}_{\mathcal{T}^i} \otimes id & s = 0, r > 0 \\ id \otimes \mu^{1|s}_{L_i} & r = 0, s > 0 \\ \mu^{0|1}_{\mathcal{T}_i} \otimes id + id \otimes \mu^{1|0}_{L_i} & r = 0, s = 0 \\ 0 & r > 0, s > 0 \end{cases} \tag{2.28}$$

Where $\mu^{r|1}_{\mathcal{T}_i}$ involves all possible contractions from the left.

We define

$$\mathcal{G}_i := Cone(\mathcal{L}_i \xrightarrow{ev'_{i-1}} Cone(\mathcal{L}_{i-1} \xrightarrow{ev'_{i-2}} Cone(\cdots Cone(\mathcal{L}_1 \xrightarrow{ev} \Delta)))) \tag{2.29}$$

$$= Cone(\mathcal{L}_i \xrightarrow{ev'_{i-1}} \mathcal{G}_{i-1}) \tag{2.30}$$

Then we have from (2.25)

$$\mathcal{G}_i \simeq \mathfrak{G}_i \tag{2.31}$$

$\mathcal{G}_i$ is a more explicit model for $\mathfrak{G}_i$ since we can now describe the underlying vector spaces clearly:

$$\mathcal{G}_n(A, B) = \bigoplus_{0 \le i \le n-1} \mathcal{L}_{i+1}(A, B)[1] \tag{2.32}$$

$$= \bigoplus_{0 \le k \le n} \bigoplus_{1 \le i_1 < \cdots < i_k \le n} CF(A, L_{i_1}) \otimes \cdots \otimes CF(L_{i_k}, B)[k] \tag{2.33}$$

However, the structure maps of $\mathcal{G}_n$ are still implicit, since they include the maps ev'_i, whose formula is not explicit from the definition. In the following section, we will introduce a new collection of explicit maps ev_i that represent ev'_i up to multiplication by a nonzero constant. So we can write down the structure maps for $\mathcal{G}_n$ explicitly using ev_i.

Remark 2.3.9. In principle, it is possible to compute the formula for ev'_n using Wehrheim-Woordward's result [24] on the fibered Dehn twist $\tau_i \times id$

To describe the domain and codomain of the maps accurately we introduce the following notations for any bimodule $\mathcal{B}$:

$$\mathcal{B}(\mathbf{A}_k, \mathbf{B}_s) := CF(A_k, A_{k+1}) \otimes \cdots \otimes \mathcal{B}(A_0, B_0) \otimes \cdots \otimes CF(B_{s-1}, B_s) \tag{2.34}$$

$$\mathcal{N}^{i_1 \cdots i_k}(A, B) := CF(A, L_{i_1}) \otimes CF(L_{i_2}, L_{i_1}) \otimes \cdots \otimes CF(L_{i_r}, B) \tag{2.35}$$

$${}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s} := \mu_{r+k+s+1} : \mathcal{N}^{i_1 \cdots i_k}(\mathbf{A}_k, \mathbf{B}_s) \to CF(A_k, B_s) \tag{2.36}$$

$${}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_1 \cdots i_k} := \sum_i id^{\otimes i} \otimes \mu_1 \otimes id^{\otimes k-i} : \mathcal{N}^{i_1 \cdots i_k}(A_0, B_0) \to \mathcal{N}^{i_1 \cdots i_k}(A_0, B_0) \tag{2.37}$$

$${}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_1 \cdots i_{k-l}} := id \otimes \mu_{l+s+1} : \mathcal{N}^{i_1 \cdots i_k}(A_0, \mathbf{B}_s) \to \mathcal{N}^{i_1 \cdots i_{k-l}}(A_0, B_s) \tag{2.38}$$

$${}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_{l'} \cdots i_k} := \mu_{l'+k+1} \otimes id : \mathcal{N}^{i_1 \cdots i_k}(\mathbf{A}_k, B_0) \to \mathcal{N}^{i_{l'} \cdots i_k}(A_k, B_0) \tag{2.39}$$

$${}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_1 \cdots i_{l-1} i_{l'+1} \cdots i_k} \tag{2.40}$$

$$:= id \otimes \mu_{l'-l+1} \otimes id : \mathcal{N}^{i_1 \cdots i_k}(A_0, B_0) \to \mathcal{N}^{i_1 \cdots i_{l-1} i_{l'+1} \cdots i_k}(A_0, B_0) \tag{2.41}$$

When $k \neq 0$, $\mathbf{n}^{r|1|s}_{i_1 \cdots i_k}$, ${}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_1 \cdots i_{k-l}}$ and ${}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_1 \cdots i_{l-1} i_{l'+1} \cdots i_k}$ are 0. When $s \neq 0$, $\mathbf{n}^{r|1|s}_{i_1 \cdots i_k}$, ${}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_{l'} \cdots i_k}$ and ${}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_1 \cdots i_{l-1} i_{l'+1} \cdots i_k}$ are 0

Definition 2.3.12. Under the identification of graded vector spaces

$$\mathcal{L}_{i+1}(A, B) = \tag{2.42}$$

$$\bigoplus_{1 \leq k \leq i+1} \bigoplus_{1 \leq i_1 < \cdots < i_k = i+1} \mathcal{N}^{i_1 \cdots i_k}(A, B)[k-1] \tag{2.43}$$

We define $ev_i \in hom_{\mathcal{F}-\mathcal{F}}(\mathcal{L}_{i+1}, \mathcal{G}_{\tau_1 \cdots \tau_i})$ as follows:

$$\sum_{1 < i \leq n+1} \sum_{1 \leq i_1 \leq \cdots \leq i_k = n+1} \left(\left(\sum_{0 < l < k} {}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_1 \cdots i_{k-l}} \right) + {}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s} \right) \tag{2.44}$$

The following is the key proposition of our argument, which we will prove using induction in the next section.

Proposition 2.3.13. *$[ev_i] = c[ev'_i]$ as elements in $Hom_{\mathcal{F}-\mathcal{F}}(\mathcal{L}_{i+1}, \mathcal{G}_i)$ for some nonzero constants c.*

Corollary 2.3.14. *There is a quasi-isomorphism*

$$\mathcal{G}_n \simeq Cone(\mathcal{L}_i \xrightarrow{ev_{i-1}} Cone(\mathcal{L}_{i-1} \xrightarrow{ev_{i-2}} Cone(\cdots Cone(\mathcal{L}_1 \xrightarrow{ev} \Delta)))) \tag{2.45}$$

The last step is to use the next lemma:

Lemma 2.3.15. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be A_∞-bimodules, $g : \mathcal{Y} \to \mathcal{Z}$ and $f : \mathcal{X} \to Cone(\mathcal{Y} \xrightarrow{g} \mathcal{Z})$ be bimodule morphisms. Then there exist unique maps $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ and $\tilde{g} : Cone(\mathcal{X} \xrightarrow{\tilde{f}} \mathcal{Y}) \to \mathcal{Z})$*

such that this is a quasi-isomorphism:

$$id : Cone(\mathcal{X} \xrightarrow{f} Cone(\mathcal{Y} \xrightarrow{g} \mathcal{Z})) \cong Cone(Cone(\mathcal{X}[-1] \xrightarrow{\tilde{f}} \mathcal{Y}) \xrightarrow{\tilde{g}} \mathcal{Z}) \tag{2.46}$$

Proof. Let X, Y, Z denote the underlying graded vector spaces of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ respectively. Clearly the underlying graded vector spaces of both side of (2.46) agree and is equal to

$$A := X[1] \oplus Y[1] \oplus Z$$

Under this identification of vector spaces, we have:

$$f = (f_1, f_2) : X[1] \to Y[1] \oplus Z$$

and

$$g : Y[1] \to Z$$

Now define

$$\tilde{f} := f_1$$

$$\tilde{g} := f_2 + g$$

We see

$$d_A(x[1], y[1], z) = (d_X x[1], d_Y y[1] + f_1 x[1], d_Z z + f_2 x[1] + g y[1])$$

$$d_B(x[1], y[1], z) = (d_X x[1], d_Y y[1] + \tilde{f} x[1], d_Z z + \tilde{g}(x[1], y[1]))$$

$$= (d_X x[1], d_Y y[1] + f_1 x[1], d_Z z + f_2 x[1] + g y[1])$$

Indeed they agree. The compatibility with higher operations can be verified likewise. □

Applying lemma 2.3.15 repeatedly on (2.45) we can rearrange the iterated cone:

$$\mathcal{G}_n \cong Cone(\cdots Cone(\mathcal{L}_n[-n+1] \xrightarrow{\widetilde{ev_{n-1}}} \cdots) \xrightarrow{\widetilde{ev_1}} \mathcal{L}_1) \xrightarrow{\widetilde{ev}} \Delta) \tag{2.47}$$

where all of $\widetilde{ev_i}$ combined is essentially the same as all of ev_i combined. For example, the component of ev_{i_k-1} in (2.45) that maps

$$CF(A, L_{i_1}) \otimes ... \otimes CF(L_{i_k}, B)[k]$$

to

$$CF(A, B)$$

is the same as the map $\widetilde{ev}$ in (2.47) restricted to $CF(A, L_{i_1}) \otimes ... \otimes CF(L_{i_k}, B)$: They are both just μ_{k+1}.

Now define

$$\mathcal{E}_n := Cone(\cdots Cone(\mathcal{L}_n[-n+1] \xrightarrow{\widetilde{ev_{n-1}}} \mathcal{L}_{n-1}[-n+2]) \xrightarrow{\widetilde{ev_{n-2}}} \cdots) \xrightarrow{\widetilde{ev_1}} \mathcal{L}_1).$$

Its structure map is explicit since ev_i and hence $\widetilde{ev_i}$ are explicit. By construction, we have that $\mathcal{E}_n$ sits in an exact triangle

$$\mathcal{E}_n \xrightarrow{\widetilde{ev}} \Delta \to \mathcal{G}_{\tau_1 \cdots \tau_n} \tag{2.48}$$

What's left to prove is Proposition 2.3.13, which we will prove in the next section using induction and a dimension argument.

Proof of Proposition 2.3.13 by induction

We explicitly describe the domain and codomain of ev_i' first. Recall by (2.32):

Definition 2.3.16. Let $\mathcal{G}_n$ be the bimodule whose underlying vector space is

$$\mathcal{G}_n(A,B) = \bigoplus_{0 \le i \le n-1} \mathcal{L}_{i+1}(A,B)[1] \tag{2.49}$$

$$= \bigoplus_{0 \le k \le n} \bigoplus_{1 \le i_1 < \cdots < i_k \le n} \mathcal{N}^{i_1 \cdots i_k}(A,B)[k] \tag{2.50}$$

where it is understood that when $k=0$, $\mathcal{N}^{\emptyset}(A,B) = CF(A,B)$. Its structure maps is given by A_∞ multiplications and ev_i':

$$\sum_{0 \le k \le n} \sum_{1 \le i_1 \le \cdots i_k \le n} (\mathbf{n}^{r|1|s}_{i_1 \cdots i_k} + \sum_{0 < l' < k} {}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_{l'} \cdots i_k} + \sum_{1 < l < l' < k} {}_{i_1 \cdots i_k}\mathbf{n}^{r|1|s}_{i_1 \cdots i_{l-1} i_{l'+1} \cdots i_k}) \tag{2.51}$$

$$+ \sum_{0 \le k \le n-1} ev_k' \tag{2.52}$$

Where it is understood that when $k=0$, $ev_0' = ev$ is the map $ev : \mathcal{Y}^l_{L_1} \otimes_{\mathbb{K}} \mathcal{Y}^r_{L_1} \to \mathcal{F}_\Delta$ in (2.24)

Definition 2.3.17. Under the identification of graded vector spaces

$$\mathcal{L}_{i+1}(A,B) = \tag{2.53}$$

$$\bigoplus_{1 \le k \le i+1} \bigoplus_{1 \le i_1 < \cdots < i_k = i+1} CF(A, L_{i_1}) \otimes \cdots \otimes CF(L_{i_k}, B)[k-1] \tag{2.54}$$

We define $ev_i \in hom_{\mathcal{F}-\mathcal{F}}(\mathcal{L}_{i+1}, \mathcal{G}_{\tau_1\cdots\tau_i})$ as follows:

$$\sum_{1<i\leq n+1} \sum_{1\leq i_1\leq\cdots\leq i_k=n+1} ((\sum_{0<l<k} {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}_{i_1\cdots i_{k-l}}) + {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}) \tag{2.55}$$

Now we proceed by induction to prove proposition 2.3.13. The case when $i = 1$ is exactly (2.24). Assuming for $i < n$, ev_i satisfy the hypothesis of the following lemma, then we can use the following dimension argument to conclude $[ev_i] = c[ev'_i]$ for $i < n$ for some nonzero constant c:

Lemma 2.3.18. *Let $\theta \in hom_{\mathcal{F}-\mathcal{F}}(\mathcal{L}_{i+1}, \mathcal{G}_i)$ be any bimodule pre-morphism satisfying*

1. *θ of degree zero.*
2. *θ is closed (i.e it is a bimodule morphism).*
3. *$[\theta] \neq 0$.*

then its class $[\theta]$ is sent under the isomorphism

$$H_*(L_{i+1}) \cong HF(L_{i+1}, L_{i+1}) \cong HF(L_{i+1} \times L_{i+1}, \Delta) \tag{2.56}$$

$$\cong HF(\tau_1 \cdots \tau_i(L_{i+1}) \times L_{i+1}, Gr(\tau_i^{-1} \cdots \tau_1^{-1})) \tag{2.57}$$

$$\cong Hom_{\mathcal{F}-\mathcal{F}}(\mathcal{L}_{i+1}, \mathcal{G}_i) \cong Hom_{\mathcal{F}-\mathcal{F}}(\mathcal{L}_{i+1}, \mathcal{G}_i) \tag{2.58}$$

to some nonzero elements in $H_0(L_i)$

Remark 2.3.10. Note that when d is odd this argument works fine with just $\mathbb{Z}/2$ grading. See Remark 2.3.8

This is because ev'_i also satisfies the hypothesis of Lemma 2.3.18 since it is the image of a degree preserving homologically nontrivial morphism under the fully faithful functor $(\tau_1 \times id \circ \cdots \circ \tau_i \times id)_\#$.

Therefore, by the induction hypothesis, when we are proving that ev_n satisfies the hypothesis of the above lemma, we can replace all the ev'_i in the structure maps of $\mathcal{G}_n$ by ev_i.

First of all, note that the degree of ev_n is 0 by construction. Indeed we have:

$$ {}_{i_1\cdots i_k}\mathbf{n}^{0|1|0} = \mu_{k+1} \tag{2.59}$$

which has degree $1 - k$ and maps

$$CF(A, L_{i_1}) \otimes \cdots \otimes CF(L_{i_k}, B)[k-1]$$

into (A, B). Also

$$ {}_{i_1\cdots i_k}\mathbf{n}^{0|1|0}_{i_1\cdots i_{k-l}} = \mu_{l+1} \tag{2.60} $$

Which has degree $1-l$ and maps

$$ CF(A, L_{i_1}) \otimes \cdots \otimes CF(L_{i_k}, B)[k-1] $$

into

$$ CF(A, L_{i_{k-l}}) \otimes \cdots \otimes CF(L_{i_k}, B)[k-l] $$

The proof that ev_n is closed is a long computation which we retain to the next section. Once we prove that ev_i is closed. The fact that it represents a nontrivial class follows by using the following basic algebraic lemma on ev_i restricting to the submodule $\mathcal{Y}^l_{L_{i+1}} \otimes_{\mathbb{K}} \mathcal{Y}^r_{L_{i+1}} \subset \mathcal{L}_i$, we see the map after the restriction is ev, which we know induces nontrivial maps in homology from [16].

Lemma 2.3.19. *Let $f : \mathcal{B} \to \mathcal{B}'$ be a bimodule morphism. Suppose that there exists a submodule $\bar{\mathcal{B}}$ of $\mathcal{B}$ such that $(f_{|\bar{\mathcal{B}}})_* : H_*(\bar{\mathcal{B}}) \to H_*(\mathcal{B}')$ is nontrivial, then f_* is nontrivial.*

Proposition 2.3.20. *The maps ev_i are homologically non-trivial.*

Proof. We just need to prove $(ev_i)^{0|1|0}$ induces a nontrivial map on homology. Note that $\mathcal{Y}^l_{L_{i+1}} \otimes_{\mathbb{K}} \mathcal{Y}^r_{L_{i+1}} \subset \mathcal{L}_i$ is a sub-module and the restriction of ev_i is the map ev in (2.24), which was shown to induce nontrivial map on homology □

proving ev_i is closed

Recall by the argument using induction hypothesis in the last section, we can replace the maps ev'_i in $\mu_{\mathcal{G}_n}$ by ev_i. This allows us to prove that ev_n is closed:

Lemma 2.3.21. *ev_n is closed, that is, the following equality holds*

$$ \sum_{i,j} \mu^{r-i|1|s-j}_{\mathcal{G}_n} \circ ev^{i|1|j}_n \tag{2.61} $$

$$ = \sum_{i,j} ev^{r-i|1|s-j}_n \circ id^{\otimes^{r-i}} \otimes \mu^{i|1|j}_{\mathcal{L}_{n+1}} \otimes id^{\otimes^{s-j}} \tag{2.62} $$

$$ + \sum_{i,j} ev^{r-i+1|1|s}_n \circ id^{\otimes^{r-i-j}} \otimes \mu^{i}_{\mathcal{A}} \otimes id^{\otimes^{s+j+1}} \tag{2.63} $$

$$ + \sum_{i,j} ev^{r|1|s-j+1}_n \circ id^{\otimes^{r+i+1}} \otimes \mu^{j}_{\mathcal{A}} \otimes id^{\otimes^{s-i-j}} \tag{2.64} $$

Proof. We verify this for $r = s = 0$ first, which is to prove the following equation:

$$\mu_{\mathcal{G}_n}^{0|1|0} \circ ev_n^{0|1|0} = ev_n^{0|1|0} \circ \mu_{\mathcal{L}_{n+1}}^{0|1|0} \tag{2.65}$$

We just need to prove this for any given pair of summands that is the domain and codomain. That is, given $1 \leq i_1 \cdots i_m \leq n$ and a subset $\{j_1 \cdots j_{m'}\} \subset \{i_1 \cdots i_m\}$, we just need to prove the LHS and RHS of (2.65) agree on each component:

$${}_{i_1\cdots i_m}(\mu_{\mathcal{G}_n}^{0|1|0} \circ ev_n^{0|1|0})_{j_1\cdots j_{m'}} = {}_{i_1\cdots i_m}(ev_n^{0|1|0} \circ \mu_{\mathcal{L}_{n+1}}^{0|1|0})_{j_1\cdots j_{m'}}$$

that maps the component $\mathcal{N}^{i_1\cdots i_m}$ to $\mathcal{N}^{j_1\cdots j_{m'}} \subset \mathcal{G}_n$.

Fixing index set $\{i_1 \cdots i_m\}$, a subset $\{j_1 \cdots j_{m'}\}$ and elements

$$(c_1, \cdots, c_{m+1}) \in \mathcal{N}^{i_1\cdots i_m}$$

we compute

$${}_{i_1\cdots i_m}(\mu_{\mathcal{G}_n}^{0|1|0} \circ ev_n^{0|1|0})_{j_1\cdots j_{m'}}(c_1, \cdots, c_{m+1}) \tag{2.66}$$

and

$${}_{i_1\cdots i_m}(ev_n^{0|1|0} \circ \mu_{\mathcal{L}_{n+1}}^{0|1|0})_{j_1\cdots j_{m'}}(c_1, \cdots, c_{m+1}) \tag{2.67}$$

First of all, observe that both (2.67) and (2.66) vanish unless $\{j_1 \cdots j_{m'}\} \subset \{i_1 \cdots i_m\}$ if of one of the following form:

1. $i_1 \cdots i_m = i_1 \cdots i_p j_1 \cdots j_{m'} i_q \cdots i_m$ for some $p \geq 1$ and $q \leq m$.

2. $i_1 \cdots i_m = j_1 \cdots j_p i_r \cdots i_{r+l} j_{p+1} \cdots j_l i_q \cdots i_m$ for some $p \geq 1, r > 1, l \geq 0, q \leq m$.

3. $i_1 \cdots i_m = j_1 \cdots j_{m'} i_q \cdots i_m$ for some q.

since ev_i only involves A_∞ multiplication from the right and $\mu_{\mathcal{L}_i}$ only involves A_∞ multiplication from the left. Therefore, for all other types of ordered subsets $j_1 \cdots j_{m'}$, there are no terms in $\mu_{\mathcal{G}_n}^{0|1|0} \circ ev_n^{0|1|0}$ or $ev_n^{0|1|0} \circ \mu_{\mathcal{L}_{n+1}}^{0|1|0}$ that map $\mathcal{N}^{i_1\cdots i_m}$ to $\mathcal{N}^{j_1\cdots j_{m'}}$.

Case 0: The empty subset corresponds to the summand $\mathcal{N}^{\emptyset} = \Delta$. In this case $\mu_{\mathcal{G}_n}^{0|1|0} \circ ev_n^{0|1|0} = ev_n^{0|1|0} \circ \mu_{\mathcal{L}_{n+1}}^{0|1|0}$ is equivalent to the A_∞ relation of order $m + 2$.

Case 1:

$$\mu_{\mathcal{G}_n}^{0|1|0} \circ ev_n^{0|1|0}(c_1, \cdots, c_{m+1}) \tag{2.68}$$

$$=({}_{i_1\cdots i_{q-1}}\mathbf{n}_{i_{p+1}\cdots i_{q-1}}^{0|1|0} \circ {}_{i_1\cdots i_m}\mathbf{n}_{i_1\cdots i_{q-1}}^{0|1|0})(c_1, \cdots, c_{m+1}) \tag{2.69}$$

$$=(\mu_{p+1}(c_1, \cdots, c_{p+1}), \cdots, \mu_{m-q+2}(c_q, \cdots, c_{m+1})) \tag{2.70}$$

$$ev_n^{0|1|0} \circ \mu_{\mathcal{L}_{n+1}}^{0|1|0}(c_1, \cdots, c_{m+1}) \tag{2.71}$$
$$=({}_{i_{p+1}\cdots i_m}\mathbf{n}_{i_{p+1}\cdots i_{q-1}}^{0|1|0} \circ {}_{i_1\cdots i_m}\mathbf{n}_{i_{p+1}\cdots i_m}^{0|1|0})(c_1, \cdots, c_{m+1}) \tag{2.72}$$
$$=(\mu_{p+1}(c_1, \cdots, c_{p+1}), \cdots, \mu_{m-q+2}(c_q, \cdots, c_{m+1})) \tag{2.73}$$

This finishes the verification for Case 1.

Case 2:

$$\mu_{\mathcal{G}_n}^{0|1|0} \circ ev_n^{0|1|0}(c_1, \cdots, c_{m+1}) \tag{2.74}$$
$$=({}_{i_1\cdots i_{q-1}}\mathbf{n}_{i_1\cdots i_r i_{r+l}\cdots i_{q-1}}^{0|1|0} \circ {}_{i_1\cdots i_m}\mathbf{n}_{i_1\cdots i_{q-1}}^{0|1|0})(c_1, \cdots, c_{m+1}) \tag{2.75}$$
$$=(\cdots, \mu_{l+2}(c_{p+1}, \cdots, c_{p+l+2}), \cdots, \mu_{m-q+2}(c_q, \cdots, c_{m+1})) \tag{2.76}$$

$$ev_n^{0|1|0} \circ \mu_{\mathcal{L}_{n+1}}^{0|1|0}(c_1, \cdots, c_{m+1}) \tag{2.77}$$
$$=({}_{i_1\cdots i_r i_{r+l}\cdots i_m}\mathbf{n}_{i_1\cdots i_r i_{r+l}\cdots i_{q-1}}^{r|1|s} \circ {}_{i_1\cdots i_m}\mathbf{n}_{i_1\cdots i_r i_{r+l}\cdots i_m}^{r|1|s})(c_1, \cdots, c_{m+1}) \tag{2.78}$$
$$=(\cdots, \mu_{l+2}(c_{p+1}, \cdots, c_{p+l+2}), \cdots, \mu_{m-q+2}(c_q, \cdots, c_{m+1})) \tag{2.79}$$

This finishes verification for Case 2.

Case 3:

$$\mu_{\mathcal{G}_n}^{0|1|0} \circ ev_n^{0|1|0}(c_1, \cdots, c_{m+1}) \tag{2.80}$$
$$=\sum_{0<l<m-q+1} {}_{i_1\cdots i_{m-l}}\mathbf{n}_{i_1\cdots i_{q-1}}^{0|1|0} \circ {}_{i_1\cdots i_m}\mathbf{n}_{i_1\cdots i_{m-l}}^{0|1|0}(c_1, \cdots, c_{m+1}) \tag{2.81}$$
$$+ {}_{i_1\cdots i_{q-1}}\mathbf{n}_{i_1\cdots i_{q-1}}^{0|1|0} \circ {}_{i_1\cdots i_k}\mathbf{n}_{i_1\cdots i_{q-1}}^{0|1|0} \tag{2.82}$$
$$=\sum_{i<q}(c_1, \cdots, \mu_1(c_i), \cdots, \mu_{m-q+2}(c_q, \cdots, c_{m+1}) \tag{2.83}$$
$$+ (c_1, \cdots, \mu_1(\mu_{m-q+2}(c_q, \cdots, c_i, \cdots, c_{m+1}))) \tag{2.84}$$
$$+ \sum_{0<l<m-q+1}(c_1, \cdots, \mu_{m-l-q+2}(c_q, \cdots, \mu_{l+1}(c_{m-l+1}, \cdots, c_{m+1}))) \tag{2.85}$$

$$
\begin{aligned}
& ev_n^{0|1|0} \circ \mu^{0|1|0}_{\mathcal{L}_{n+1}}(c_1, \cdots, c_{m+1}) && (2.86)\\
= & {}_{i_1\cdots i_m}\mathbf{n}^{0|1|0}_{i_1\cdots i_{q-1}} \circ {}_{i_1\cdots i_m}\mathbf{n}^{0|1|0}_{i_1\cdots i_m}(c_1, \cdots, c_{m+1}) && (2.87)\\
& + \sum_l {}_{i_1\cdots i_r i_{r+l}\cdots i_m}\mathbf{n}^{0|1|0}_{i_1\cdots i_{q-1}} \circ {}_{i_1\cdots i_m}\mathbf{n}^{0|1|0}_{i_1\cdots i_r i_{r+l}\cdots i_m}(c_1, \cdots, c_{m+1}) && (2.88)\\
= & \sum_{i<q}(c_1, \cdots, \mu_1(c_i), \cdots, \mu_{m-q+2}(c_q, \cdots, c_{m+1}) && (2.89)\\
& + \sum_{i\geq q}(c_1, \cdots, \mu_{m-q+2}(c_q, \cdots, \mu_1(c_i), \cdots, c_{m+1}) && (2.90)\\
& + \sum_l (c_1, \cdots, \mu_{m-l-q+1}(c_q, \cdots, \mu_{l+2}(\cdots), \cdots, c_{m+1})) && (2.91)
\end{aligned}
$$

We see that both sides agree by the A_∞ relations of order $m-q+3$.

For the case when $r>0$ and $s>0$ we proceed similarly. First, we list all the cases for each of the three maps $\mu^{r|1|s}_{\mathcal{G}_n}, \mu^{r|1|s}_{\mathcal{L}_{n+1}}$ and $ev_n^{r|1|s}$:

$$
ev_n^{r|1|s} = \begin{cases} \sum_{0\leq k\leq n+1}\sum_{1\leq i_1\cdots i_k=n+1} {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}, r\neq 0, s\neq 0 \\ \sum_{0\leq k\leq n+1}\sum_{1\leq i_1\cdots i_k=n+1} {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}, r\neq 0, s=0 \\ \sum_{0\leq k\leq n+1}\sum_{1\leq i_1\cdots i_k=n+1} ({}_{i_1\cdots i_k}\mathbf{n}^{r|1|s} + \sum_{0<l<k} {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}_{i_1\cdots i_{k-l}}), r=0, s\neq 0 \end{cases} \quad (2.92)
$$

$$
\mu^{r|1|s}_{\mathcal{L}_{n+1}} = \begin{cases} 0, r\neq 0, s\neq 0 \\ \sum_{0\leq k\leq n+1}\sum_{1\leq i_1\cdots i_k\leq n+1}\sum_{0\leq l<k} {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}_{i_l\cdots i_k}, r\neq 0, s=0 \\ \sum_{0\leq k\leq n+1}\sum_{1\leq i_1\cdots i_k\leq n+1} {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}_{i_1\cdots i_k}, r=0, s\neq 0 \end{cases} \quad (2.93)
$$

$$
\mu^{r|1|s}_{\mathcal{E}_n} = \begin{cases} \sum_{0\leq k\leq n}\sum_{1\leq i_1\cdots i_k\leq n} {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}, r\neq 0, s\neq 0 \\ \sum_{0\leq k\leq n}\sum_{1\leq i_1\cdots i_k\leq n} (\sum_{0\leq l<k} {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}_{i_l\cdots i_k} + {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}), r\neq 0, s=0 \\ \sum_{0\leq k\leq n}\sum_{1\leq i_1\cdots i_k\leq n} (\sum_{0\leq l<k} {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}_{i_1\cdots i_{k-l}} + {}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}), r=0, s\neq 0 \end{cases} \quad (2.94)
$$

Now fix any subset $\{j_1\cdots j_l\}\subset\{i_1\cdots i_m\}$ just like in the case when $r=s=0$, the maps in Lemma 2.3.21 vanishes unless $\{j_1\cdots j_l\}$ is one of the four cases we listed there. We verify the equality in 2.3.21 for these 4 cases one by one.

Case 0: $\emptyset \subset \{i_1 \cdots i_m\}$. This is equivalent to A_∞-relation of order $m+r+s+2$.

Case 1: $i_1 \cdots i_m = i_1 \cdots i_p j_1 \cdots j_{m'} i_q \cdots i_m$. In this case, all maps in Lemma 2.3.21 vanish unless $r = s = 0$. This is because when $\{j_1 \cdots j_{m'}\}$ is not empty, all maps with $r \neq 0$ and $s \neq 0$ vanish. But then neither (2.92.2)$\circ$ (2.93.2) nor (2.94.3)$\circ$ (2.92.3) maps $\mathcal{N}^{i_1\cdots i_m}$ to $\mathcal{N}^{j_1\cdots j_{m'}}$.

Case 2: $i_1 \cdots i_m = j_1 \cdots j_p i_r \cdots i_{r'} j_{p+1} \cdots j_{m'} i_q \cdots i_m$. In this case, all maps in Lemma 2.3.21 vanish unless $r = s = 0$. This is because ${}_{i_1\cdots i_k}\mathbf{n}^{r|1|s}_{i_1\cdots i_{l-1} i_{l'+1}\cdots i_k}$ are 0 whenever $r > 0$ or $s > 0$

Case 3: We have

$$\sum_j \mu^{0|1|s-j}_{\mathcal{G}_n} \circ ev^{0|1|j}_n(b_s, \cdots, b_{j+1})(b_j, \cdots, b_1)(x_m, \cdots, x_0) \tag{2.95}$$

$$= \sum_j \sum_{p+q=l} {}_{i_1\cdots i_{m-l}}\mathbf{n}^{r|1|s}_{i_1\cdots i_{m-p-q}} \circ {}_{i_1\cdots i_m}\mathbf{n}^{r|1|s}_{i_1\cdots i_{m-p}}(b_s, \cdots, b_{j+1})(b_j, \cdots, b_1)(x_m, \cdots, x_0) \tag{2.96}$$

$$= \sum_j \sum_{p+q=l} (\mu_{s-j+q+1}(b_s, \cdots, b_{j+1}, \mu_{j+p+1}(b_j, \cdots b_1, x_m, \cdots, x_{m-q}), x_{m-p-1}, \cdots, x_{m-p-q}), \tag{2.97}$$

$$x_{m-p-q-1}, \cdots, x_0) \tag{2.98}$$

Where the sum is over $p+q=l$ such that $i_{m-p-q} = j_{m'}$.

$$\sum_j ev^{0|1|s-j}_n \circ \mu^{0|1|j}_{\mathcal{L}_{n+1}}(b_s, \cdots, b_{j+1})(b_j, \cdots, b_1)(x_m, \cdots, x_0) \tag{2.99}$$

$$= \sum_j {}_{i_1\cdots i_m}\mathbf{n}^{r|1|s}_{i_1\cdots i_{m-l}} \circ {}_{i_1\cdots i_m}\mathbf{n}^{r|1|s}_{i_1\cdots i_m}(b_s, \cdots, b_{j+1})(b_j, \cdots, b_1)(x_m, \cdots, x_0) \tag{2.100}$$

$$= \sum_j (\mu_{s-j+l+1}(b_s, \cdots, b_{j+1}, \mu_{j+1}(b_j, \cdots, b_1, x_m)), x_{m-1}, \cdots, x_{m-l}), x_{m-l-1}, \cdots, x_0) \tag{2.101}$$

$$\sum_{i,j} ev^{0|1|s-j+1}_n(b_s, \cdots, \mu_j(b_{i+j}, \cdots, b_{i+1}), b_i, \cdots, b_1)(x_m, \cdots, x_0) \tag{2.102}$$

$$= \sum_{i,j} (\mu_{s-j+l+2}(b_s, \cdots, \mu_j(b_{i+j}, \cdots, b_{i+1}), b_i, \cdots, b_1, x_m, \cdots, x_{m-l}), x_{m-l-1}, \cdots, x_0) \tag{2.103}$$

We see that (2.98)+(2.101)+(2.103)=0 by applying the A_∞ relations separately for each l.

This finishes the proof for Lemma 2.3.21 and hence Theorem 2.3.2.

□

proof of theorem 2.3.2. The exact triangle is obtained by using lemma 2.3.15 on the right-hand side of (2.45) as indicated. The statement about the differential follows from our formula for ev_i. □

Chapter 3

Minimal Lagrangian genus of a rational surface

We fix some notations. Let $\mathcal{E}$ be the set of homology classes represented by smoothly embedded spheres of self-intersection -1.

$$\mathcal{E}_\omega = \{E \in \mathcal{E} |\ E \text{ is represented by an embedded } \omega-\text{symplectic sphere}\}.$$

we will denote by $H, E_1, E_2, \cdots, E_k$ a basis of $H_2(X_k, \mathbb{Z})$, under which the intersection matrix takes its standard form, i.e.,

$$H^2 = 1, \quad E_i^2 = -1, \quad H \cdot E_i = E_i \cdot E_j = 0, \quad \forall i \neq j.$$

In addition, each E_i is assumed to be in $\mathcal{E}$. Such a standard basis naturally arises from any explicit identification of X_k with $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$. There are many choices of standard bases, all related by the $D(X_k)-$action.

We fix a standard basis. If ω is given, E_i is further assumed to be in $\mathcal{E}_\omega$. Let $\beta \in H_2(X_k; \mathbb{Z}_2)$ and $\gamma_i \in H_2(X_k; \mathbb{Z}_2)$ be the mod 2 reductions of H and E_i. Let $A_k = H - \sum_{i=1}^k E_i$ and $\xi_k = \beta + \sum_{i=1}^k \gamma_i \in H_2(X_k; \mathbb{Z}_2)$ its mod 2 reduction. A_k is an integral characteristic class and ξ_k is a mod 2 characteristic class. In fact, ξ_k is the unique one by Poincáre duality and hence independent of the basis.

3.0.1 Cremona transformation and minimal genus for an ordinary class α

The Cremona transformation is an automorphism of the second homology group $H_2(X_k, \mathbb{Z})$ preserving the canonical class K_0 and realized by diffeomorphisms. Note that the Mod 2 Cremona transformations preserve the Pontriagin square, the characteristic class and zero class. Also that -1 reflection acts as identity on $H_2(X;\mathbb{Z}_2)$.

For $S^2 \times S^2$, the characteristic class is the zero class. The two fiber classes mod 2 are Cremona equivalent by the factor switching automorphism. So each nonzero class in $H_2(S^2 \times S^2;\mathbb{Z}_2)$ is Cremona equivalent to $b \pmod 2$ or $(b+f) \pmod 2$.

The group of Cremona transformation of X_k is generated by the reflections along L_{ijk} or $E_i - E_j$ (**[LiLi]**) with the action given by

$$R_{ijk}(A) = A + (A \cdot L_{ijk})L_{ijk}$$

$$R_{ij}(A) = A + (A \cdot (E_i - E_j))(E_i - E_j).$$

Note that there is a unique characteristic class by Poincare duality. For X_k, it is $\xi_k = \beta + \sum \gamma_i$.

Proposition 3.0.1. *For any rational surface, nonzero ordinary mod 2 classes with the same Pontriagin square are unique up to Cremona transformations. For X_k, any nonzero ordinary mod 2 class is Cremona equivalent to one of 4 model ordinary classes: $\beta, \beta + \gamma_1, \gamma_1, \gamma_1 + \gamma_2$, which have Pontriyagin square $1, 0, -1, -2$ respectively.*

Consequently,

$$\mathcal{C}_n^\alpha(X) \neq \emptyset \Longleftrightarrow n \equiv 2 - \mathcal{P}(\alpha) \mod 4.$$

In other words, $\eta^\alpha(X) = 2 - \mathcal{P}(\alpha)$.

Proof. We observe that it suffices to show that any nonzero ordinary mod 2 class is Cremona equivalent to a class with all components zero except two. If there are at most two nonzero terms, since the class is nonzero, it is of the form in the list by possibly applying the R_{ij}:

$$\beta = (1|0,0,\cdots), \beta + \gamma_1 = (1|1,0,\cdots), \gamma_1 = (0|1,0,\cdots), \gamma_1 + \gamma_2 = (0|1,1,\cdots).$$

If there are at least 3 nonzero terms, since there is at least one zero term we can assume the first 4 components are of the form $(1|0,1,1)$ or $(0|1,1,1)$ by possibly applying the R_{ij}. The action of R_{123} only change the first 4 components and the effect on the first 4 components is as follows:

$$R_{123}(0|1,1,1) = (1|0,0,0)$$

$$R_{123}(1|0,1,1) = (0|1,0,0).$$

So if there are at least 3 nonzero terms and at least one zero terms, the class is Cremona equivalent to one class with less nonzero terms. We can therefore reduce to the case there are at most 2 nonzero terms.

If $k = 0$ there are no such classes; if $k = 1$ the only such classes are H and E_1 which have Pontrjagin square 1 and -1 respectively. If $k \geq 2$, a nonzero ordinary class in $H_2(X_k;\mathbb{Z}_2)$

is Cremona equivalent to exactly one of the following four classes: $\beta, \beta+\gamma_1, \gamma_1, \gamma_1+\gamma_2$.

Now we apply the Lagrangian blowup construction to get a Lagrangian $2\mathbb{RP}^2$ in the class $\beta+\gamma_1$ from a Lag $\mathbb{RP}^2$ in the class β. Similarly, applying the Lag blowup construction to get a Lag $3\mathbb{RP}^2$ in the class $0+\gamma_1$ from a Lag Clifford torus. By applying the Lag blowup construction again to get a Lag $4\mathbb{RP}^2$ in the class $\gamma_1+\gamma_2$. □

We next compute the cardinality of each orbit. Recall that Pointryagin square is simply the lift of cup product mod $\mathbb{Z}_4$ if such a lift exists. For rational manifold since there's no torsion class such a lift always exists. Let $a_k(i)$ be the number of ordinary classes of X_k with Pointryagin square i.

Lemma 3.0.2. *When $i=1,2,3$, $a_k(i)=$*

$$\begin{cases} \bar{a}_k(i) & i+k\neq 1 \\ \bar{a}_k(i)-1 & i+k=1 \end{cases}$$

when $i=0$, $a_k(i)=$

$$\begin{cases} \bar{a}_k(i)-1 & i+k\neq 1 \\ \bar{a}_k(i)-2 & i+k=1 \end{cases}$$

Where $\bar{a}_k(i)=\delta_1(i)+\sum_{l\geq 1}\binom{k}{4l-i}+\sum_{l\geq 1}\binom{k}{4l+1-i}$

Proof. $\bar{a}_k(i)$ is simply the number of $\mathbb{Z}_2$-classes with Pontryagin square i. It is straightforward to obtain the formula for $\bar{a}_k(i)$ once we list all the classes. Let $P(i)$ denote the set of classes with Pontryagin square i. To simplify the notation, we use $E_{(m)}$ to denote the classes of the form $E_{j_1}+...+E_{j_m}$ with $j_1<...<j_m$.

$$P(1)=\{H, E_{(3)}, H-E_{(4)}, E_{(7)}, H-E_{(8)}...\} \tag{3.1}$$

$$P(2)=\{E_{(2)}, H-E_{(3)}, E_{(6)}, H-E_{(7)}...\} \tag{3.2}$$

$$P(3)=\{E_{(1)}, H-E_{(2)}, E_{(5)}, H-E_{(6)}...\} \tag{3.3}$$

$$P(0)=\{0, H-E_{(1)}, E_{(4)}, H-E_{(5)}, E_{(8)}...\} \tag{3.4}$$

Now we need to remove the zero class and characteristic classes. Note that the characteristic class is in $P(i)$ if and only if $i+k=1$. This finished the proof.

□

The numbers relevant to our studies of $\mathbb{RP}^2$ are $a_1(1)=a_2(1)=1, a_3(1)=2, a_4(1)=5, a_5(1)=16, a_6(1)=36, a_7(1)=72$ and $a_8(1)=135$.

Remarkably, it was shown by Shevchishin and Nemirovski ([**S**] and [**N**]) that the mod 2 class of a Lagrangian Klein Bottle in a uniruled manifold with $c_1 \cdot \omega > 0$ must be nonzero. On the other hand, Shevchishin gave an example of null-homological embedded Lagrangian Klein Bottle in T^2-bundle over T^2.

3.0.2 Symplectic -4 spheres from rational blow-up

The following important result comes from an observation in [6], which follows immediately from Dorfmeister's symplectic cut/sum formula for the sympletic Kodaira dimension [**D**].

Lemma 3.0.3. *Given a Lagrangian $\mathbb{RP}^2$ in $(X_k, \omega), k \geq 1$, the rational blow-up yields a symplectic rational surface (X_{k+1}, Ω) along with a symplectic -4 sphere.*

We will first classify their classes up to actions of $D(X_{k+1})$. We need a standard basis $H, E_1, \cdots, E_{k+1}$ for (X_{k+1}, Ω). Note that $X_{k+1}, k \geq 1$ is identified with $S^2 \times S^2 \# k\overline{\mathbb{C}}\mathbb{P}^2$. An explicit transformation of the basis $H, E_1, \cdots, E_{k+1}$ is given by

- $b = H - E_1, f = H - E_2$, $e_1 = H - E_1 - E_2$
- $e_i = E_{i+1}$, $2 \leq i \leq k$.

Here, b, f are the homology of the the first and the second factors of S^2, and e_i's form a basis for H_2^- consisting of exceptional classes. In terms of this basis we can classify symplectic (-4)-sphere classes coming from rational blow-ups up to actions of $D(X_{k+1})$. We need a similar result on ordinary blow-ups first:

We recall the classification of symplectic -4 sphere classes in [**DLW**]. Let (X_{k+1}, ω) be a rational symplectic surface and $\{H, E_1, ..., E_{k+1}\}$ a standard basis of $H_2(X_{k+1}; \mathbb{Z})$. Consider any class $A \in H_2(X_{k+1}, \mathbb{Z})$ with $A \cdot A = -4$ and represented by an $\omega-$symplectic sphere. Up to $D(X_{k+1})$-equivalence, the class A is one of the following:

1. If A is characteristic, then $k = 5$ and A is equivalent to $H - E_1 - ... - E_5$ $(e_1 - e_2 - e_3 - e_4)$
2. If A is not characteristic, then it is equivalent to one of the following:
   (a) $-H + 2E_1 - E_2$ $(b - 2f)$ or
   (b) $-a(-3H + \sum_{i=1}^{9} E_i) - 2E_{10}$ $(a \in \mathbb{N}^+)$.

Lemma 3.0.4. *If a symplectic -4 sphere Z in (X_{k+1}, Ω) arises from blowing up a symplectic -2 sphere in a rational surface, then $[Z]$ is $D(X_{k+1})-$equivalent to $-H + 2E_1 - E_2$ or $H - E_1 - ... - E_5$. If $[Z]$ is characteristic, then $k = 4$ and $[Z]$ is $D(X_{k+1})-$ equivalent to $H - E_1 - ... - E_5$.*

Proof. For $(X_k, \omega), k = 0, 1$, there are no symplectic -2 classes. When $k \geq 2$, symplectic -2 sphere classes of $(X_l, \omega), l \geq 2$ are $D(X_l)-$equivalent to $E_1 - E_2$ or $H - E_1 - E_2 - E_3$ ([**LiLi**]). The only possibility left is when $S^2 \times S^2$ blows up to X_3, but symplectic -2 sphere classes of $(S^2 \times S^2, \omega)$ are all equivalent to $b - f$.

□

Using this lemma we can prove the corresponding result for rational blow-ups:

Proposition 3.0.5. *Suppose $Z \subset (X_{k+1}, \Omega), k \geq 1$ is a symplectic -4 sphere that arises from rational blowing up a Lagrangian $\mathbb{RP}^2$ in (X_k, ω). Then the class $[Z]$ is $D(X_{k+1})-$ equivalent to $-H + 2E_1 - E_2$ or $H - E_1 - ... - E_5$.*

Proof. Note the rational blow down of $Z \subset (X_{k+1}, \Omega)$ is (X_k, ω), which is not minimal. By [**D13**],

- either there exists a symplectic exceptional sphere disjoint from Z,
- or there exist two disjoint symplectic exceptional spheres C_1 and C_2 each intersecting Z exactly once and positively.

Therefore we can assume after blowing down some number of exceptional spheres disjoint from Z that the resulting rational surface $\overline{W}$ is minimal or (ii) occurs for $Z \subset \overline{W}$.

If we reach a minimal rational surface then $\overline{W} = S^2 \times S^2$, since the other possibility, $\mathbb{CP}^2$, does not have a symplectic (-4) sphere. So $[Z]$ is either $b - 2f$ or $f - 2b$ in $H_2(\overline{W}; \mathbb{Z})$.

Assume now that (ii) occurs for $Z \subset \bar{W}$, where $\bar{W}$ has no exceptional spheres disjoint from Z. We can assume that Z intersects C_1 and C_2 normally up to a local isotopy of Z. After blowing down C_1 and C_2, we get a symplectic -2 sphere in a rational surface $\tilde{W}$. Then Z is obtained by blowing up a symplectic -2 sphere and thus we can apply Lemma 3.0.4 to finish the proof. □

3.0.3 Characteristic classes and $\eta(X, \omega)$

There is a unique characteristic class by Poincare duality for every rational surface: it is the zero class for $S^2 \times S^2$, and ξ_k for X_k. We first study the Lagrangian $\mathbb{RP}^2 \subset X_{4p}$ in the class ξ_{4p}. We'll also prove some results about higher genus non-orientable Lagrangian surfaces in X_k via the Rieser's result on real blowup of $(\mathbb{CP}^2, \mathbb{RP}^2)$. Finally, we derive the estimate of $\eta(X, \omega)$.

Existence of Lagrangian $\mathbb{RP}^2$

Proposition 3.0.6. *There are Lagrangian $\mathbb{RP}^2$ in the characteristic class ξ_{4p} of X_{4p} if and only if $p = 0, 1$.*

Proof. Suppose ξ_{4p} is represented by a Lagrangian $\mathbb{RP}^2$ L in X_{4p} for $p \geq 2$. Such a Lagrangian $\mathbb{RP}^2$ would give rise to a -4 symplectic sphere Z in X_{4p+1} from rational blow-up.

Assume $p \geq 2$. Then $4p+1 \geq 9 > 5$. By Proposition 3.0.5, $[Z]$ is $D(X_{4p+1})-$equivalent to $-H + 2E_1 - E_2$. By [**MO**], for the exceptional class E_9 which has trivial pairing with $-H + 2E_1 - E_2$, there is an exceptional sphere S disjoint from Z. When we reverse the (local) rational blow-up process S becomes an exceptional sphere in X_{4p} which is disjoint from L. But this contradicts to the fact that $[L]$ is characteristic.

The existence of Lagrangian $\mathbb{RP}^2$ in ξ_0 is obvious. In [**DHL**] a Lagrangian $\mathbb{RP}^2$ in ξ_4 is constructed from Lagrangian blowing up a characteristic Lagrangian S^2 in X^3.

□

Remark 3.0.7. The error in [**DHL**] occurred when claiming that A_{4p-1} for $p \geq 2$ is also represented by a Lagrangian sphere in X_{4p-1} for any p.

Real blow-up of $(\mathbb{CP}^2, \mathbb{RP}^2)$ and $\eta^{\xi_k}(X_k)$

Assume (X, ω) carries an involution $\phi : X \to X$ that is *anti-symplectic*. That is $\phi^*\omega = -\omega$. Then its fixed point set $Fix(\phi) = L$ is a Lagrangian surface in X. Now ψ is a *real packing* of (X, ϕ, L) if $\psi_i^{-1}(L)$ is the real part $B_{\delta_i}^{\mathbb{R}}$ of B_{δ_i} for any i and ψ commutes with complex conjugation ϕ_0 on B_{δ_i} and ϕ on X.

By [**Ri**] Theorem 1.21, the symplectic form $\tilde{\omega}$ on $\tilde{X} := X\#k\bar{\mathbb{CP}}^2$ can be constructed such that there exists involution $\tilde{\phi} : \tilde{X} \to \tilde{X}$ such that $\tilde{\phi}^*\tilde{\omega} = -\tilde{\omega}$, $\pi \circ \tilde{\phi} = \phi \circ \Pi$, $\tilde{L} = p^{-1}(L) = Fix(\tilde{\phi})$ and is $\tilde{\omega}$-Lagrangian. Here Π is the blow-down map. In this case, we call $(\tilde{X}, \tilde{\phi}, \tilde{L})$ a real blowup of (X, ϕ, L). Note that $\tilde{L} \cong L\#\mathbb{RP}P^2$ and $[\tilde{L}] = [L] + E$. On the other hand, ψ is a L-obstructed packing of (X, L) if $\psi_i^{-1}(L) = \emptyset$ for any i. Thus L still survives in $\tilde{X}$.

Rieser proved the following result regarding equal sized real blow ups:

Theorem 3.0.8. *[**Ri**] The real and absolute packing numbers of* $(\mathbb{CP}P^2, \mathbb{RP}P^2)$ *coincide.*

The technique he used can be applied to real blow ups with different sizes:

Lemma 3.0.9. *If there exist absolute packing of balls* $\psi_i : (B_{\delta_i}, \omega_0) \to (\mathbb{CP}P^2, \omega_0)$, *then there real packing of balls* $\psi_i' : (B_{\delta_i}, \phi_0, B_{\delta_i}^{\mathbb{R}}) \to (\mathbb{CP}P^2, \phi_0, \mathbb{RP}P^2)$ *where* ϕ_0 *is the complex conjugation.*

Proof. Let $(X_k, \tilde{\omega}_\epsilon, \tilde{\phi})$ denote the real blow up of $(\mathbb{CP}P^2, \omega_0, \phi_0)$ along k balls with size ϵ very small. Let $\tilde{J}$ denote the induced almost complex structure on $(X_k, \tilde{\omega}_\epsilon)$ such that $\tilde{\phi}_*\tilde{J} = -\tilde{J}\tilde{\phi}_*$. Let

$$a = [\Pi^*\omega_0] - \pi \sum_{j=1}^{k} \delta_j e_j,$$

where e_j are poincare duals to the exceptional divisors. By the existence of the absolute packing of balls ψ_i, we see a admits a symplectic form ρ. Therefore $a \cdot a > 0$, $a(E) > 0$ for all the exceptional classes E in X_k and $\tilde{\omega}_\epsilon \cdot a > 0$ by taking ϵ small enough. Now by inflation we have that for every $y > 0$, $\tilde{\omega_y} := \frac{1}{y}\tilde{\omega} + \rho$ is symplectic and tames $\tilde{J}$. By lemma 4.4 in [**Ri**], $\bar{\omega_y} := \frac{1}{2}(\tilde{\omega_y} - \tilde{\phi}^*\tilde{\omega_y})$ is also symplectic and $\phi^*\bar{\omega_y} = -\bar{\omega_y}$. Now by Corollary 4.3 in [**Ri**] (X, ω_0, ϕ_0) admits real symplectic embeddings of balls with sizes δ_i. Finally note that $Fix(\phi_0) = \mathbb{RP}P^2$. This finishes the proof. □

Theorem 3.0.10. *Consider the unique characteristic class ξ_k of $X_k, k \geq 1$. $\mathcal{P}(\xi_k) = -l + 1 \in \{-2, -1, 0, 1\}$ if $k = 4p - l, 0 \leq l \leq 3$.*

1. *ξ_{4p}, which has $\mathcal{P}(\xi_{4p}) = 1 \pmod 4$, admits a Lag $\mathbb{RP}^2$ for some symplectic form if and only if $p = 1$.*
2. *$\eta^{\xi_k}(X_k) = k + 1 \pmod 4$.*
   - *$\eta^{\xi_k}(X_k) = k + 1$ for $k \leq 3$,*
   - *$\eta^{\xi_k}(X_k) = k - 3$ for $4 \leq k \leq 8$,*
   - *$\eta^{\xi_k}(X_k) \leq k - 3$ for $k \geq 9$,*
3. *$\mathcal{C}^{\xi_k}_{k+1}(X_k) = \mathcal{C}^{\xi_k}(X_k) = \mathcal{C}(X_k)$.*

Proof. The first bullet is Audin's congruence since $A_k = H - \sum_{i=1}^k E_i$ an integral lift of ξ_k and hence $\mathcal{P}(\xi_k) = 1 - k \pmod 4$.

To prove the second bullet, note that there is a Lagrangian S^2 in the characteristic class $A_3 = H - E_1 - E_2 - E_3$ of X_3 (for any symplectic form pairing trivially with A_3). By performing the real blow-up of such a Lagrangian S^2 we have the upper bound $\eta(X_k, \xi_k) \leq k + 1$. Clearly, this lower bound is sharp for $k \leq 3$.

For $k \geq 4$, if performing the real blow-up of a characteristic Lagrangian S^2 in X_3, we get the upper bound $\eta(X_k, \xi_k) \leq k - 3$. Clearly, $\eta(X_k, \xi_k) = k - 3$ for $k = 4, 5, 6, 7$.

By Proposition 3.0.6 (i), $\eta(X_{4p}, \xi_{4p}) \geq 5$ for $p \geq 2$. It follows that $\eta(X_8, \xi_8) = 5$. These proves the third and fourth bullet.

By Lemma 3.0.9, the real and absolute packings of $(\mathbb{CP}^2, \mathbb{RP}^2)$ coincide. Each real packing with k balls induces a symplectic form in X_k which admits a Lagrangian $(k+1)\mathbb{RP}^2$ in the class ξ_k. This proves the last bullet. □

Remark 3.0.11. Theorem 3.0.10 (i) also corrects Lemma 4.6 in [6] for the case of characteristic classes. Lemma 4.6 in [6] is still valid for ordinary classes and hence the proof of Theorem 1.3 in [6] is still valid for ordinary classes.

3.0.4 Minimal genus of (X, ω)

Note by Audin's theorem $S^2 \times S^2$ does not contain Lagrangian $\mathbb{RP}^2$ since no class there has Pontryagin square 1. However, it contains many other interesting non-orientable Lagrangians like Klein bottles (see proposition [**Evans**]).

Corollary 3.0.12. *Every symplectic rational surface (X, ω) admits an embedded homology essential non-orientable surface of complexity $e(X)-2$. In other words, $\eta(X, \omega) \leq e(X)-2$.*

Proof. For X_k it follows from Theorem 3.0.10 (2) and (3).

For $S^2 \times S^2$, it follows from Evans [**Evans**] that

$$\eta(S^2 \times S^2, \omega) = \min_{\alpha} \eta^{\alpha}(S^2 \times S^2, \omega) = 2.$$

Explicitly, given any symplectic form, there is a Lag Klein bottle in one of the fiber classes. Up to scaling, the symplectic forms are of the form $B + \lambda F$ for $\lambda \in (0, \infty)$.

For $\lambda \in (0, 2)$, Lag Klein bottle in the F class.

For $\lambda \in (\frac{1}{2}, \infty)$, Lag Klein bottle in the F class. $\square$

Chapter 4

Lagrangian $\mathbb{RP}^2$ cone of a rational surface

Note that the $K-$symplectic cones are disjoint for distinct K. Let H, E_i be a standard basis of X_k and $K_0 = -3H + \sum_{i=1}^k E_i$. We will focus on $\mathcal{C}_1^\beta(X_k, K_0)$ and establish several properties. For $k \geq 7$, we calculate $\mathcal{C}_1^\beta(X_k, K_0)$ explicitly. Then we calculate $\mathcal{C}_1^\alpha(X_k, K_0)$ for other ordinary classes α with $\mathcal{P}(\alpha) = 1$ using Cremona transformations. We also calculate $\mathcal{C}_1^{\xi_4}(X_4, K_0)$.

Finally we show that $\mathcal{C}_1(X_k, K_0) \neq \mathcal{C}(X_k, K_0)$ for any $k \geq 1$..

4.0.1 A general rational blowup correspondence for β

Lemma 4.0.1. For a framed rational surface (X_k, ω) with $k \geq 1$, the sub-lattice $\mathcal{L}_\beta \subset H^2(X_k, \mathbb{Z})$ is generated by $2H, E_1, ..., E_k$.

For a β Lag L, there is a framing on X_{k+1} and a monomorphism $\iota_L : \mathcal{L}_\beta \hookrightarrow H_2(X_{k+1}, \mathbb{Z})$ with the following properties:

1. $\iota_L \otimes \mathbb{R} : H^2(X_k; \mathbb{R}) \to H^2(X_{k+1}; \mathbb{R})$ is an isomorphism onto $z^\perp$ for the -4 sphere class $z = -H + 2E_1 - E_2$,

2. The homology homomorphism ι_L is given by

$$v = (2a|c_1, ..., c_k) \mapsto B(v) = (3a - c_1|2a - c_1, a - c_1, c_2, ..., c_k) \tag{4.1}$$

3. ι_L preserves the $\mathbb{Z}$-intersection product. In particular, $B(v) \cdot B(v) = v \cdot v > 0$

4.
$$\mathcal{C}_1^L(M) = (\iota_L \otimes \mathbb{R})^{-1}(\mathcal{C}(X_{k+1}, z)),$$

where $\mathcal{C}(X_{k+1}, z)$ is the cone of $z-$relative symplectic forms on X_{k+1}.

Proof. Given an ordinary Lagrangian $\mathbb{RP}^2$ L in X_k, performing the rational blow up gives us a (-4) sphere Z in X_{k+1}.

Since β is ordinary, by Lemma we can pick a basis $\{H', E'_1, ..., E'_{k+1}\}$ for $H_2(X_{k+1}, \mathbb{Z})$ such that $[Z] = -H + 2E_1 - E_2$.

Consider the classes

$$3H' - 2E'_1 - E'_2, H' - E'_1 - E'_2, E'_3, ..., E'_{k+1}$$

which are orthogonal to z and pairwisely orthogonal. By corollary 3.3 in [6] and [**MO**] there exist disjoint symplectic spheres D'_i such that

$$[D'_1] = 3H' - 2E'_1 - E'_2, [D'_2] = H' - E'_1 - E'_2, [D'_3] = E'_3, ..., [D'_{k+1}] = E'_{k+1}$$

and are disjoint from Z.

Now performing rational blow-down on Z we get back to X_k and a (4)-sphere D_1 and k exceptional spheres $D_2, ...D_{k+1}$ in X_k. We can pick a basis $H, E_1, ..., E_k$ such that

$$[D_1] = 2H, [D_2] = E_1, ..., [D_{k+1}] = E_k.$$

In terms of these basis on $H^2(X_k, \mathbb{Z})$ and $H^2(X_{k+1}, \mathbb{Z})$ respectively, we can write down the homomorphism from $B : \Gamma \to H^2(X_{k+1}, \mathbb{Z})$:

$$2H \to 3H' - 2E'_1 - E'_2, E_1 \to H' - E'_1 - E'_2, E_i \to E'_{i+1}, i = 2, \cdots, k. \tag{4.2}$$

This gives the expression of $B(v)$. The second part is a straightforward computation. □

Note that $\mathcal{C}_1^{\beta}(X_k) = \cup_L \mathcal{C}_1^L(X_k)$.

Lemma 4.0.1. *For a framed X_k, $\mathcal{C}_1^{\beta}(X_k) = \mathcal{C}_1^L(X_k)$ and it is described by B.*

Proof. Back to X_k. We consider the subgroup of $D(X_k)$ fixing the class β. We need to show that a real class $(a|c_1, ..., c_k)$ is in the Lag cone with respect to one framing if and only if it is so with respect to any other framing.

The given framing of X_k and the $L-$framing are related by ϕ. ϕ induces Φ of X_{k+1}.

$B \circ \phi = \Phi \circ B$.

Such frames are related by the subgroup of $D(X_{k+1})$ fixing the class z. Note that the $z-$relative cone is invariant under such actions. Since X_{k+1} has $b^+ = 1$, for any Z with

$[Z]=z$, the relative cone $\mathcal{C}(X_{k+1},Z)$ is $\mathcal{C}(X_{k+1},z)$, which is the $z-$positive portion of $\mathcal{C}(X_{k+1})$.

□

We define $p: H_2(X_k;\mathbb{R})\to H_2(X_{k-1};\mathbb{R})$ and $\iota: H_2(X_{k-1};\mathbb{R})\to H_2(X_k;\mathbb{R})$ in terms of the basis. Recall $(a|c_1,...,c_k)$ represents the class $aH-c_1E_1-...-c_kE_k$ in either $H^2(X_k,\mathbb{Z})$ or $H^2(X_k,\mathbb{R})$. For our purpose we can always normalize so that $a=1$ when we are working over $\mathbb{R}$. However we cannot do this over $\mathbb{Z}$.

Proposition 4.0.2. *$v=(1|c_1,\cdots,c_k)$ is in $\mathcal{C}_1^{\beta}(X_k,K_0)$ if and only if*

$$B(v)=((\frac{3}{2}-c_1)|(1-c_1),(\frac{1}{2}-c_1),c_2,\cdots,c_k)$$

is in the K_0-symplectic cone of X_{k+1}.

(i) $\mathcal{C}_1^{\beta}(X_k,K_0)$ is symmetric in c_i.

(ii) If $v=(1|c_1,\cdots,c_k)\in\mathcal{C}_1^{\beta}(X_k,K_0)$ then $0<c_i<\frac{1}{2}$.

(iii) $\mathcal{C}_1^{\beta}(X_{k-1},K_0)=p(\mathcal{C}_1^{\beta}(X_k,K_0))$.

Proof. Let ω_v denote any symplectic form in the class v .If $v\in\mathcal{C}_1^{\beta}(X_k,K_0)$ then (X_k,ω_v) admits an Lagrangian $\mathbb{RP}^2$. Performing rational blow up along it gives us $(X_{k+1},\omega_{B(v)})$ where $\omega_{B(v)}$ is some symplectic form in the class $B(v)$. Conversely, given $(X_{k+1},\omega_{B(v)})$ with $k\geq 2$, performing rational blow down gives us a Lagrangian $\mathbb{RP}^2$ in the class β in (X_k,ω_v).

(i) is clear since β is symmetric in the γ_i and K_0 is symmetric in the E_i.

Now we the fact that $B(v)$ has to pair positively with the V_i':

It is obvious that $c_i>0$ from V_0'.

$c_1<\frac{1}{2}$ from $V_1'=E_2'$.

By the 4th bullet of Proposition 4.0.1 we have

$$\mathcal{C}_1^{\alpha}(X_{k-1},K_0)\subset p(\mathcal{C}_1^{\iota(\alpha)}(X_k,K_0)).$$

By Lemma 4.0.1, $\mathcal{C}_1^{\iota(\alpha)}(X_k,K_0)$ is determined by the area condition $B_{\iota(\alpha)}(v)(V')>0$ where $V'\in\mathcal{E}_{K_0}(X_{k+1})$.

Similarly, $v'=(1|c_1,\cdots,c_{k-1})$ is in $\mathcal{C}_1^{\alpha}(X_{k-1},K_0)$ if and only if $B_\alpha(v')$ is in the symplectic cone of X_k. Note that $p(v)^2=v^2-c_k^2>0$. We show that $p(v)$ satisfies the area conditions. For $V\in\mathcal{E}_{K_0}(X_k)$, $\iota(V)\in\mathcal{E}_{K_0}(X_{k+1})$.

$$(B_\alpha(p(v)),V)=(B_{\iota(\alpha)}(v),\iota(V))>0.$$

□

4.0.2 The cone $\mathcal{C}_1^\beta(X_k, K_0)$ for $k \leq 7$

We now apply Proposition 4.0.2 to describe the relative cone explicitly for $k = 7$.

Lemma 4.0.3. *A vector* $v = (1|c_1, \cdots, c_k)$ *with* $0 < v \cdot v$ *is in the cone* $\mathcal{C}_1^\beta(X_k, K_0), k \leq 7$ *if and only if :*

- $0 < c_i < \frac{1}{2}$

- $c_{i_1} + c_{i_2} + c_{i_3} + c_{i_4} + c_{i_5} < 2$ *if* $k \geq 5$

- $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 < \frac{5}{2}$ *if* $k = 7$

Proof. By Proposition 4.0.2 (ii), it suffices to assume that $k = 7$.

We first prove these conditions are necessary. For this purpose, we list classes in $\mathcal{E}_{K_0}(X_8)$. Up to the permutations of E_i', the list is given by

$V_0' = (0\|\text{-}1)$	$(k' \geq 1)$
$V_1' = (1\|1, 1)$	$(k' \geq 2)$
$V_2' = (2\|1, 1, 1, 1, 1)$	$(k' \geq 5)$
$V_3' = (3\|2, 1, 1, 1, 1, 1, 1)$	$(k' \geq 7)$
$V_4' = (4\|2, 2, 2, 1, 1, 1, 1, 1)$	$(k' \geq 8)$
$V_5' = (5\|2, 2, 2, 2, 2, 2, 1, 1)$	$(k' \geq 8)$
$V_6' = (6\|3, 2, 2, 2, 2, 2, 2, 2)$	$(k' \geq 8)$

Now we use the fact Proposition 4.0.2 and the fact that $B(v)$ has to pair positively with the V_i':

It is obvious that $c_i > 0$ from V_0'.

$c_1 < \frac{1}{2}$ from $V_1' = E_2'$.

$c_1 + c_2 + c_3 + c_4 + c_5 < 2$ from $V_2' = 2H' - E_1' - E_3' - E_4' - E_5' - E_6'$

$c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 < \frac{5}{2}$ from $V_3' = 3H' - 2E_1' - E_3' - \cdots - E_8'$.

• Now we check these conditions are also sufficient.

To reduce the computations, we observe that it suffices to check that $B(v)$ pairs positively with the minimal area exceptional classes. By the symmetry in c_i, Proposition 4.0.2 (i), we may assume that

$$c_1 \geq c_2 \geq c_3 \geq \cdots \geq c_k.$$

In this case, we have

$$(1 - c_1) \geq c_2 \geq c_3 \geq \cdots c_k$$

since $c_i < \frac{1}{2}$.

Recall $B_{v'} = ((\frac{3}{2}-c_1)|(1-c_1), (\frac{1}{2}-c_1), c_2, \cdots, c_{k-1})$, we see the maximal area exceptional class is E'_1 with area $1-c_1$ while the minimal area exceptional class is either E'_2 with area $\frac{1}{2}-c_1$, or E'_k with area c_k. So for the V'_0 type, the minimal area one is either E'_2 or E'_k. We have $B_{v'} \cdot E'_2 = \frac{1}{2} - c_1 > 0$ and $B_{v'} \cdot E'_k = c_k > 0$ from (4.0.5).

For the other types, the E'_1 coefficient of a minimal area class should be the largest since $(1-c_1) \geq c_2 \geq c_3 \geq \cdots c_k$. So we look at the E'_2 coefficients. Depending on whether $\frac{1}{2}-c_1 > c_2$ there are two possibilities for a class to have minimal area: If $\frac{1}{2}-c_1 > c_2$ then E'_2 should have the second largest coefficient; If $\frac{1}{2}-c_1 < c_2$ then E'_3 should have the second largest coefficient. Moreover if $\frac{1}{2}-c_1 < c_k$ then $E'_3, ..., E'_{k+1}$ should have the 2nd to k-th largest coefficient.

V'_1 type: If $\frac{1}{2}-c_1 > c_2$ then the minimal area one is $(1|1,1)$ with area $c_1 > 0$, otherwise it is $(1|1,0,1)$ with area $(\frac{1}{2}-c_2) > 0$.

V'_2 type: If $\frac{1}{2}-c_1 > c_5$ then the minimal area one is $(2|1,1,1,1,1)$ with area $\frac{3}{2}-(c_2+c_3+c_4) > 0$, otherwise it is $(2|1,0,1,1,1,1)$ with area $2-(c_1+c_2+c_3+c_4+c_5) > 0$.

V'_3 type: If $\frac{1}{2}-c_1 > c_7$ then the minimal area one is $(3|2,1,1,1,1,1,1)$ with area $2-(c_2+c_3+c_4+c_5+c_6) > 0$, otherwise it is $(3|2,0,1,1,1,1,1,1)$ with area $\frac{5}{2}-(c_1+c_2+c_3+c_4+c_5+c_6+c_7) > 0$.

V'_4 type: If $\frac{1}{2}-c_1 > c_3$ then the minimal area one is $(4|2,2,2,1,1,1,1,1)$ with area

$$(1-2c_2)+2-(c_3+c_4+c_5+c_6+c_7) > 0,$$

otherwise it is $(4|2,1,2,2,1,1,1,1)$ with area

$$1-(c_2+c_3)+\frac{5}{2}-(c_1+c_2+c_3+c_4+c_5+c_6+c_7) > 0.$$

V'_5 type: If $\frac{1}{2}-c_1 > c_6$ then minimal area one is $(5|2,2,2,2,2,2,1,1)$ with area

$$2-(c_2+c_3+c_4+c_5)+\frac{5}{2}-(c_1+c_2+c_3+c_4+c_5+c_6+c_7) > 0,$$

otherwise it is $(5|2,1,2,2,2,2,2,1)$ with area

$$(\frac{1}{2}-c_1)+2-(c_2+c_3+c_4+c_5+c_6)+\frac{5}{2}-(c_1+c_2+c_3+c_4+c_5+c_6+c_7) > 0.$$

V'_6 type: The minimal area one is $(6|3,2,2,2,2,2,2,2)$ with area

$$(\frac{1}{2}-c_2)+2-(c_3+c_4+c_5+c_6+c_7)+\frac{5}{2}-(c_1+c_2+c_3+c_4+c_5+c_6+c_7) > 0.$$

□

The constraint $0 < v \cdot v$ in proposition 4.0.3 that comes from volumes of ball embeddings can actually be deduced from 4.0.5, 4.0.5 and 4.0.5 when $k \leq 7$:

Lemma 4.0.4. *Let $k \leq 7$ and $(1|c_1, \ldots, c_k)$ be class vector satisfying (4.0.5),(4.0.5) and (4.0.5). Then*

$$v \cdot v > 0$$

Proof. For $k \leq 5$, from (4.0.5) and (4.0.5) we have $c_1^2 + \cdots + c_k^2 < \frac{1}{2}(c_1 + \cdots + c_k) < 1$.

For $k = 6$, note that the set of points in $\mathbb{R}^6$ satisfying 4.0.3 is a convex polytope P^6 obtained from cutting the cube K^6 with width equal to $\frac{1}{2}$ by hyperplanes defined by $c_{i_1} + c_{i_2} + c_{i_3} + c_{i_4} + c_{i_5} < 2$. We want to prove the quadratic function $f = c_1^2 + ... + c_k^2$ when restricted to P^6 is smaller than or equal to 1. Since the f is convex, it only reaches maximum on vertices. There are two type of vertices to check: those of the cube K^6 that survived after the cuts and those that are created by the cuts.

The vertices of the cube are of the form $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6)$ with $\epsilon_i = 0$ or $\frac{1}{2}$. The ones that survives after the cuts have a a least 2 zeroes. Therefore for these vertices $\epsilon_1^2 + ... + \epsilon_6^2 \leq 1$.

The new vertices created by the cuts are contained in every hyperplane defined by $c_{i_1} + c_{i_2} + c_{i_3} + c_{i_4} + c_{i_5} = 2$. The only solution to these 6 equations is $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$. We see that $6(\frac{2}{5})^2 = \frac{24}{25} < 1$. This concludes the proof for $k = 6$.

For $k = 7$, the same argument applies to vertices of K^7 of the form $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)$ with at least three ϵ_i being 0. Similarly the hyperplanes $c_{i_1} + c_{i_2} + c_{i_3} + c_{i_4} + c_{i_5} = 2$ produce a single new vertex $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$. Call this polytope P^7. However in this case the inequality $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 < \frac{5}{2}$ is not redundant in the sense that the hyperplane $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 = \frac{5}{2}$ cuts off the vertex $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$ since $7 \cdot \frac{2}{5} > \frac{5}{2}$. So we really need to study the new vertices created by this new cut.

To do so we first need to look at edges of P^7. The new edges created by the hyperplanes $c_{i_1} + c_{i_2} + c_{i_3} + c_{i_4} + c_{i_5} < 2$ connects $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)$ with $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$. By symmetry we can just look at the edge connecting $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)$ with $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$. Its intersection with the hyperplane $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 = \frac{5}{2}$ is $(\frac{7}{16}, \frac{7}{16}, \frac{7}{16}, \frac{7}{16}, \frac{4}{16}, \frac{4}{16}, \frac{4}{16})$. Now $4 \cdot (\frac{7}{16})^2 + 3 \cdot (\frac{4}{16})^2 = \frac{244}{256} < 1$. This concludes the proof.

□

Using Lemma 4.0.4 to remove the volume constraint in Lemma 4.0.3 and we get:

Corollary 4.0.5. *(Theorem 4.0.3) A vector $v = (1|c_1, \cdots, c_k)$ is in the cone $\mathcal{C}_1^{\beta}(X_k, K_0), k \leq 7$ if and only if :*

- $0 < c_i < \frac{1}{2}$

- $c_{i_1} + c_{i_2} + c_{i_3} + c_{i_4} + c_{i_5} < 2$ *if* $k \geq 5$

- $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 < \frac{5}{2}$ *if* $k = 7$

Corollary 4.0.6. *Inside the normalized* K_0-*symplectic cone, the conditions for* $\mathcal{C}_1^{\beta}(X_7, K_0)$ *are the* 8 *inequalities* $c_i < \frac{1}{2}, i = 1, \cdots, 7$ *and* $c_1 + \cdots + c_7 < \frac{5}{2}$.

Inside the normalized reduced symplectic cone, the condition is $c_1 < \frac{1}{2}$.

Proof. Inside the normalized symplectic cone, we have $c_i > 0$ and $c_{i_1} + c_{i_2} + c_{i_3} + c_{i_4} + c_{i_5} < 2$.

If further inside the normalized reduced symplectic cone, then

$$c_{i_1} + c_{i_2} + c_{i_3} < 1, \quad c_i \leq c_1.$$

So $c_1 < \frac{1}{2}$ implies that $c_i < \frac{1}{2}$ and

$$c_1 + \cdots + c_7 < (c_1 + c_2 + c_3) + (c_4 + c_5 + c_6) + c_7 < 1 + 1 + \frac{1}{2} = \frac{5}{2}.$$

□

Remark 4.0.2. The case of $k = 5$ was computed in [**LLW2**].

4.0.3 The cone $\mathcal{C}_1^{\alpha}(X_k, K_0)$ for $k \leq 7$ and general α

We again restrict our attention to the K_0-cone. By Proposition 3.0.1, each ordinary Lagrangian $\mathbb{RP}^2$ class is cremona equivalent to β. We explicitly describe the transformation of $\mathcal{C}_1^{\alpha}(X_k, K_0)$.What's more, the only characteristic class admitting a Lagrangian $\mathbb{RP}^2$ is ξ_4 of X_4. Thus we have

Lemma 4.0.3. (i) For $k \neq 4$, the cone $\mathcal{C}_1(X_k, K_0)$ is the union

$$\mathcal{C}_1(X_k, K_0) = \cup_{\phi} \mathcal{C}_1^{\phi(\beta)}(X_k, K_0) = \cup_{\phi} \phi(\mathcal{C}_1^{\beta}(X_k, K_0))$$

for any Cremona transformation ϕ. For $k = 4$,

$$\mathcal{C}_1(X_4, K_0) = \cup_{\phi} \phi(\mathcal{C}_1^{\beta}(X_4, K_0)) \cup \mathcal{C}_1^{\xi_4}(X_4, K_0).$$

(ii) For any k we have

$$\mathcal{C}_1(X_{k-1}, K_0) = p(\mathcal{C}_1(X_k, K_0)).$$

For convenience, we abbreviate $\mathcal{C}_1^{\gamma_i + \gamma_j + \gamma_k}(X)$ as $\mathcal{C}_1^{ijk}(X)$ etc.

Lemma 4.0.4. For X_7, let $\{i,j,k,l,m,n,o\} = \{1,2,3,4,5,6,7\}$. There are $a_7(1) - 1 = 71$ other $\mathcal{C}_1^{\alpha}(X_7)$ except $\mathcal{C}_1^{\beta}(X_7)$:

$$\begin{aligned}
\mathcal{C}_1^{ijk}(X_7, K_0) &= \{\lambda u, \lambda > 0, u = (1|c_1, \cdots, c_7) \in \mathcal{C}(X_7, K_o)|[c_i + c_j > c_k]_{ijk}, \\
&\quad [2 - c_i - c_j - c_k - 2c_l > 0]_{lmno}, \\
&\quad 4 > c_i + c_j + c_k + 2c_l + 2c_m + 2c_n + 2c_o\} \\
\mathcal{C}_1^{L_{ijkl}}(X_7, K_0) &= \{\lambda u|[1 > c_i + c_j + c_k - c_l]_{ijkl}, \\
&\quad c_i + c_j + c_k + c_l > 1, \\
&\quad [3 > c_i + c_j + c_k + c_l + 2c_m + 2c_n]_{mno}\} \\
\mathcal{C}_1^{1234567}(X_7, K_0) &= \{\lambda u|2 > c_1 + c_2 + c_3 + c_4 + c_5 + c_6 - c_7]_{1234567}, \\
&\quad c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 > 2\}
\end{aligned}$$

Proof. • We begin with the $\binom{7}{3} = 35$ ijk type classes. By symmetry, it suffices to describe $\mathcal{C}_1^{123}(X_7, K_0)$ for

$$123 = R_{123}(\beta).$$

From Corollary 4.0.6, inside the K_0-symplectic cone, $\mathcal{C}_1^{\beta}(X_7, K_0)$ is determined by $\omega(T) > 0$ for the following 8 classes in 2 types

$$T = [(1|2,0,0,0,0,0,0)]_{1234567}, (5|2,2,2,2,2,2,2)$$

We can list all for actions of Cremona transformations on T for $k = 7$.

type of T	T	$R_{123}(T)$
$(1\|2)$	$[(1\|2,0,0,0,0,0,0)]_{123}$	$[(0\|1,\text{-}1,\text{-}1,0,0,0,0)]_{123}$
	$[(1\|0,0,0,2,0,0,0)]_{4567}$	$[(2\|1,1,1,2,0,0,0)]_{4567}$
$(5\|2,2,2,2,2,2,2)$	$(5\|2,2,2,2,2,2,2)$	$(4\|1,1,1,2,2,2,2)$

We thus get the 8 inequalities.

• We next consider the $\binom{7}{3} = 35$ L_{ijkl} type classes. By symmetry, it suffices to describe $\mathcal{C}_1^{L_{1245}}(X_7, K_0)$ for

$$L_{1245} = R_{123}(345).$$

From the table above, there are eight T in 3 types:

$$[(0|0,0,1,\text{-}1,\text{-}1,0,0)]_{345}, [(2|2,0,1,1,1,0,0)]_{1267}, (4|2,2,1,1,1,2,2)$$

type of T	T	$R_{123}(T)$
(0|0, 0, 1,-1,-1,0, 0)	(0|0, 0, 1,-1,-1, 0, 0) $[(0\|0, 0,\text{-}1,1,\text{-}1,0, 0)]_{45}$	(-1|-1,-1,0,-1,-1,0, 0) $[(1\|1, 1, 0, 1,\text{-}1,0, 0)]_{45}$
(2|2, 0, 1, 1, 1, 0, 0)	$[(2\|2, 0, 1, 1, 1, 0, 0)]_{12}$ $[(2\|0, 0, 1, 1, 1, 2, 0)]_{67}$	$[(1\|1,\text{-}1,0, 1, 1, 0, 0)]_{12}$ $[(3\|1, 1, 2, 1, 1, 2, 0)]_{67}$
(4|2, 2, 1, 1, 1, 2, 2)	(4|2, 2, 1, 1, 1, 2, 2)	(3|1, 1, 0, 1, 1, 2, 2)

- Finally, we treat the class

$$1234567 = R_{123}(L_{4567}).$$

From the table above, there are eight T in 3 types:

$$(\text{-}1|0, 0, 0, \text{-}1,\text{-}1,\text{-}1,\text{-}1), [(1|0, 0, 0, \text{-}1, 1, 1, 1)]_{4567}, [(3|0, 2, 2, 1, 1, 1, 1)]_{123}$$

type of T	T	$R_{123}(T)$
(-1|0, 0, 0, 0, -1,-1,-1,-1)	(-1|0, 0, 0, 0, -1,-1,-1,-1)	(-2|-1,-1,-1,-1,-1,-1,-1)
(1|0, 0, 0, -1, 1, 1, 1)	$[(1\|0, 0, 0\text{-}1, 1, 1, 1)]_{4567}$	$[(2\|1, 1, 1,\text{-}1,1, 1, 1)]_{4567}$
(3|0, 2, 2, 1, 1, 1, 1)	$[(3\|0, 2, 2, 1, 1, 1, 1)]_{123}$	$[(2\|\text{-}1, 1, 1, 1, 1, 1, 1)]_{123}$

Note that all the $[R_{ijk}(\mathcal{C}_1^{klm}(X_7, K_0))]_{ijlm}$ are equal to $\mathcal{C}_1^{L_{ijlm}}(X_7, K_0)$. Also we have $[R_{ijk}(\mathcal{C}_1^{L_{lmno}}(X_7, K_0))]_{ijklmno}$ are all equal to $\mathcal{C}_1^{1234567}(X_7, K_0)$.

For (ii), it is clear except when $k = 4$. For $k = 4$, we have $\mathcal{C}_1^{\xi_4}(X_4, K_0) = p(\mathcal{C}_1^{\xi_4}(X_5, K_0))$. It is interesting to note the inequalities for $\mathcal{C}_1^{\xi_4}(X_5, K_0))$ do not involve c_5. □

Now we can directly check that $\mathcal{C}_1(X_k) \subsetneq \mathcal{C}(X_k)$, $1 \le k \le 7$. By Lemma 4.0.3, it is enough to find a symplectic structure in $\mathcal{C}(X_k) - \mathcal{C}_1(X_k)$ for $k = 7$.

Moreover, we can restrict our attention to the K_0-cone since $K-$cones are disjoint for distinct K.

For $k = 7$, the class

$$u = H - \frac{4}{5}E_1 - \frac{1}{12}E_2 - \frac{1}{24}E_3 - \frac{1}{48}E_4 - \frac{1}{96}E_5 - \frac{1}{192}E_6 - \frac{1}{384}E_7$$

is in $\mathcal{C}(X_7, K_0) - \mathcal{C}_1(X_7, K_0)$:

$u \notin \mathcal{C}_1^{\beta}(X_7, K_0) : c_1 > \frac{1}{2}$

$u \notin \mathcal{C}_1^{ijk}(X_7, K_0) : c_j + c_k < c_i$ if $i < j < k$

$u \notin \mathcal{C}_1^{L_{ijkl}}(X_7, K_0) : c_i + c_j + c_k + c_l < 1$

$u \notin \mathcal{C}_1^{1234567}(X_7, K_0) : c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 < 2$

The cone $\mathcal{C}_1^{\xi_4}(X_4, K_0)$ revisited

We first calculate the cone $\mathcal{C}_1^{\xi_4}(X_4, K_0)$. In abbreviation, we denote $tH - c_1E_1 - c_2E_2 - \cdots - c_kE_k$ as $(t|c_1, c_2, \cdots, c_k)$. We will also use the notation $[\cdot]_{ijk}$ etc to denote the set of conditions given by all permutations of i, j, k. For example, $[c_i > 0]_{ijk}$ denotes the three conditions $c_i > 0, c_j > 0, c_k > 0$.

Lemma 4.0.7. *Up to changes of the standard bases, the homomorphism $\iota_r : H_2(X_4;\mathbb{Q}) \to H_2(X_5;\mathbb{Q})$ is of the following form: $U_i \in H_2(X_4;\mathbb{Z})$ to $V_i \in H_2(X_5;\mathbb{Z})$, , $1 \le i \le 5$, where*

$$U_1 = H - E_1, U_2 = H - E_2, U_3 = H - E_3, U_4 = H - E_4, U_5 = 2H - \sum_{i=1}^{5} E_i$$

$$V_1 = H' - E_1', V_2 = H' - E_2', V_3 = H' - E_3', V_4 = H' - E_4', V_5 = H' - E_5'.$$

It has the following properties:

- *ι_r is a ring homomorphism and a monomorphism.*
- *The image is the orthogonal complement of A_5'.*
- *$B \in H_2(X_4;\mathbb{Z})$ is represented by a symplectic surface disjoint from L if and only if $\iota_r(B)$ is represented by a symplectic surface disjoint from L.*
- *$v \in H^2(X_4;\mathbb{R})$ is in the cone $\mathcal{C}_1^{\xi_4}(X_4, K_0)$ if and only if $\iota(v)$ is in the $Z-$relative symplectic cone $\mathcal{C}(X_5, Z, K_0)$.*

Proof. We consider the set of square zero symplectic sphere classes of (X_5, Ω) that pair trivially with A_5'. It has five classes, $V_i = H' - E_i', i = 1, 2, 3, 4, 5$. They give rise to five square zero symplectic spheres S_i in (X_4, ω) that are disjoint from L.

Denote the same surfaces in X_5 by S_i'. The set of zero symplectic sphere classes of (X_4, ω) that pair trivially with ξ_4 also has exactly five classes, $U_i = H - E_i, i = 1, 2, 3, 4$ and $U_5 = 2H - \sum_{i=1}^5 E_i$. Therefore, up to a permutation of E_i, we can assume that U_i correspond to V_i under the homomorphism ι_r.

Note that the $\{U_i\}$ is a basis of the vector space $H^2(X_4, \mathbb{Q})$, and the $\{V_i\}$ is a basis of the orthogonal complement of A_5' in $H^2(X_5, \mathbb{Q})$.

For a symplectic surface $\Sigma \subset X_4$ disjoint from L, denote the surface by Σ' in X_5. Note that $[S] \cdot U_i = [S'] \cdot V_i$ since we have the configuration of symplectic surfaces in X_4 and X_5.

Note that such a rational surface is identified with $S^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}}^2$. An explicit transformation of homology classes from $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}}^2$ is given by

- $b = H - E_1, f = H - E_2$, $e_1 = H - E_1 - E_2$

- $e_i = E_{i+1}$, $i = 2, 3, 4$.

Here, b, f are the homology of the two factors of S^2, and e_i's form a basis for H_2^- consisting of exceptional classes.

We claim that the exceptional -4 sphere obtained from the rational blowup has the (integral) homology class of $e_1 - \cdots - e_4$, with some choices of homological basis $\{b, f, e_1, \cdots, e_4\}$ in $H_2(S^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}}^2)$. This can be extracted from [6, Lemma 4.10, 4.11] as follows.

Consider a standard basis of $X_4 := \mathbb{CP}^2 \# 4\overline{\mathbb{C}\mathbb{P}}^2 = S^2 \times S^2 \# \overline{\mathbb{C}\mathbb{P}}^2$, consisting of $\{b', f', e'_1, e'_2, e'_3\}$. Note that the $\mathbb{Z}/2\mathbb{Z}$ class of the $\mathbb{RP}^2$ is $e'_1 + e'_2 + e'_3$ under any such standard basis of $S^2 \times S^2 \# 3\overline{\mathbb{C}\mathbb{P}}^2$ because the Cremona transforms preserves this $\mathbb{Z}/2\mathbb{Z}$-class. In particular, this is a *characteristic class*, i.e. a homology class which pairs non-trivially with any exceptional class in $\mathbb{Z}/2\mathbb{Z}$-pairing.

On the other hand, from the main theorem of [**DLW**]there are two equivalence classes of -4 spherical classes that admits symplectic representatives in $S^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}}^2$: $b - 2f$ and $e_1 - e_2 - e_3 - e_4$ up to Cremona transforms, which is an equivalent change of basis. If the exceptional divisor from the rational blow-down has homology class $b - 2f$ up to Cremona transform, there e_i would be an exceptional class with zero integral intersection number with this class. From [**MO**], there exists an exceptional curve that is disjoint from the -4 divisor. Such an exceptional curve will become an exceptional curve that is disjoint from the given Lagrangian $\mathbb{RP}^2$ after reversing the rational blow-up, hence a contradiction.

□

Lemma 4.0.8. *The cone* $\mathcal{C}_1^{\xi_4}(X_4, K_0)$ *is*

$$\{\lambda(1|c_1, c_2, c_3, c_4)\ |\lambda > 0, c_1 + c_2 + c_3 + c_4 > 1, \quad [1 > c_i + c_j + c_k - c_l]_{1234}, \}.$$

Proof. Note first the inequalities

$$[1 > c_1 + c_2 + c_3 - c_4]_{1234}, c_1 + c_2 + c_3 + c_4 > 1$$

imply that $c_i > 0$ and $c_i + c_j < 1$ so the vector is indeed a K_0-symplectic vector of X_4.

$$2c_1 = (c_1 + c_2 + c_3 + c_4) + (c_1 - c_2 - c_3 - c_4) > 1 + (-1) = 0$$

$$2(c_1 + c_2) = (c_1 + c_2 - c_3 + c_4) + (c_1 + c_2 + c_3 - c_4) < 1 + 1 = 2$$

Recall that

$$2H \to 3H' - (E'_1 + E'_2 + E'_3 + E'_4) + E'_5$$

Since $2H - 2E_i$ is sent to $2H' - 2E'_i$, the homology monomorphism has the effect

$$2E_i \to 3H' - (E'_1 + E'_2 + E'_3 + E'_4) + E'_5 - (2H' - 2E'_i) = H' + E'_5 + E'_i - (E'_j + E'_k + E'_l)$$

So a vector $2(1|c_1, c_2, c_3, c_4)$ is sent to $(a'|c'_1, c'_2, c'_3, c'_4, c'_5)$ with

$$\begin{aligned}
a' &= 3 - (c_1 + c_2 + c_3 + c_4) \\
c'_1 &= 1 - (-c_1 + c_2 + c_3 + c_4) \\
c'_2 &= 1 - (c_1 - c_2 + c_3 + c_4) \\
c'_3 &= 1 - (c_1 + c_2 - c_3 + c_4) \\
c'_4 &= 1 - (c_1 + c_2 + c_3 - c_4) \\
c'_5 &= -1 + (c_1 + c_2 + c_3 + c_4)
\end{aligned}$$

Note that by making the Weinstein neighborhood arbitrarily small, we can assume that

$$\Omega(A'_5) = a' - (c'_1 + c'_2 + c'_3 + c'_4 + c'_5)$$

is arbitrarily close to zero. In fact, for our purpose we can assume that it is zero.

E'_i: The conditions $\iota(v)(E'_i) = c'_i > 0$ correspond to the inequalities above.

$H' - E'_1 - E'_2$: $\quad a' - c'_1 - c'_2 = c'_3 + c'_4 + c'_5 > 0$.

$H' - E'_1 - E'_5$: $\quad a' - c'_1 - c'_5 = c'_2 + c'_3 + c'_4 > 0$.

$2H' - E'_1 - E'_2 - E'_3 - E'_4 - E'_5$: $\quad 2a' - \sum_{i=1}^{5} c'_i = \sum_{i=1}^{5} c'_i > 0$.

□

Remark 4.0.9. In [6] we computed the homology homomorphism over $\mathbb{Q}$ from orthogonal classes in X_k to X_{k+1} in terms of a different basis. A $\mathbb{Z}_2$-*orthogonal system* for A is a set $\{F_i\}_{i=1}^k$ of pairwise orthogonal exceptional classes in $H_2(M;\mathbb{Z})$ such that $A \cdot F_i = 0$ in $\mathbb{Z}_2$-homology for all i. If A admits a $\mathbb{Z}_2$-orthogonal system, together a sphere in $2H$ they form a basis in the space of class that is orthogonal to $[L]$. The basis used for X_k was

$$2H, H - U_2 - U_3, H - U_1 - U_3, H - U_2 - U_3, E_4, ..., E_k$$

where U_i and E_i are exceptional classes. The Lagrangian class is $[L] = U_1 + U_2 + U_3 \bmod 2$.

The basis used for X_{k+1} was

$$2H', E'_0, E'_1, E'_2, E'_3, ...E'_k$$

where E'_i are exceptional classes. The (-4)-sphere is in the class $[Z] = E'_0 - E'_1 - E'_2 - E'_3$.

In terms of these basis, the homology homomorphism is as follows: if $[\alpha] = aH - t_1U_1 -$

$t_2U_2 - t_3U_3$ then

$$\iota([\alpha]) = aH' - \frac{-t_1+t_2+t_3}{2}E'_0 - \frac{t_1-t_2+t_3}{2}E'_1 - \frac{t_1+t_2-t_3}{2}E'_2 - \frac{t_1+t_2+t_3}{2}E'_3$$

and $\iota(E_i) = E'_i$ for $i \geq 4$.

The relation between the the basis used in [6] and in the paper is as follows: First we can identify U_1, U_2 and U_3 with E_1, E_2 and E_3. So $[L]$ in [6] is $E_1 + E_2 + E_3$ mod 2. Now the classes in this paper is related to the classes in [6] by reflection along $H - E_1 - E_2 - E_3$.:

$$[L] = H \mapsto 2H - U_1 - U_2 - U_3 \equiv U_1 + U_2 + U_3 \tag{4.3}$$

$$2H \mapsto 4H - 2E_1 - 2E_2 - 2E_3 \tag{4.4}$$

$$E_1 \mapsto H - E_2 - E_3 \tag{4.5}$$

$$E_2 \mapsto H - E_1 - E_3 \tag{4.6}$$

$$E_3 \mapsto H - E_1 - E_2 \tag{4.7}$$

$$E_i \mapsto E_i, i > 3 \tag{4.8}$$

Upstairs on X_{k+1}, the classes in this paper is related to the classes in [6] by reflection along $H' - E'_0 - E'_2 - E'_3$:

$$[Z] = -H' + 2E'_0 - E'_1 \mapsto E'_0 - E'_1 - E'_2 - E'_3 \tag{4.9}$$

$$3H' - 2E'_0 - E'_1 \mapsto 4H' - 3E'_0 - E'_1 - E'_2 - E'_3 \tag{4.10}$$

$$H' - E'_0 - E'_1 \mapsto H' - E'_0 - E'_1 \tag{4.11}$$

$$E'_2 \mapsto H' - E'_0 - E'_3 \tag{4.12}$$

$$E'_3 \mapsto H' - E'_0 - E'_2 \tag{4.13}$$

$$E'_i \mapsto E'_i, i > 3 \tag{4.14}$$

Now one can verify that the left hand sides are related by the map computed in the proof above. Also the right hand sides are related by the map ι.

4.0.4 $\mathcal{C}_1(X_k) \subsetneq \mathcal{C}(X_k)$ for all $k \geq 1$.

Theorem 4.0.10. $\mathcal{C}_1(X_k) \subsetneq \mathcal{C}(X_k)$ *for all* $k \geq 1$.

Proof. We can normalize so that the coefficient of H is 1. Define

$$\mathcal{P}^1_k(\beta) := \{(c_1, ..., c_k) | (a, c_1, ..., c_k) \in \mathcal{C}^\beta_1(X_k) \forall a \in \mathbb{R}^+\}$$

and the problem then becomes proving the union of polytopes $\cup_\phi \phi(\mathcal{P}_k^1(\beta))$ where ϕ ranges over Cremona transforms is strictly contained in the polytope

$$\mathcal{P}_k := \{(c_1, ..., c_k) | (a, c_1, ..., c_k) \in \mathcal{C}(X_k) \forall a \in \mathbb{R}^+\}$$

Also, from the proof of proposition 4.0.3, we can see that

$$\mathcal{P}_k^1(\beta) \subset \mathcal{I}_{\frac{1}{2}}^k = \{(c_1, ..., c_k) | 0 < c_i < \frac{1}{2}\} \subset \mathcal{P}_k$$

We wish to prove that

$$u := H - \frac{4}{5}E_1 - \sum_{l=2}^{k} \frac{1}{3 \cdot 2^l} E_l \in \mathcal{C}(X_k) - \mathcal{C}_1(X_k) \tag{4.15}$$

First we list all the classes that contains Lagrangian $\mathbb{RP}^2$ and Cremona transforms that maps β to the those. There are two types of classes:

- $\gamma_{i_1 \cdots i_{4n-1}} := \gamma_{i_1} + ... + \gamma_{i_{4n-1}}$ for $n > 0$
- $L_{i_1 \cdots i_{4n}} := \beta + \gamma_{i_1} + ... + \gamma_{i_{4n}}$ for $n > 0$

Note that for $k = 4n > 4$, $L_{i_1 \cdots i_{4n}}$ does not contain Lagrangian $\mathbb{RP}^2$ and for $k = 4$ Lemma 4.0.8 proves $u \notin \mathcal{C}_1^\beta(X_k, K_0)$. So whenever we $L_{i_1 \cdots i_{4n}}$ appears we assume without mentioning that it is in X_k for some $k > 4n$.

It is straightforward to find Cremona transforms that maps β to the two types of classes above using induction:

$$\gamma_{i_1 \cdots i_{4n-1}} = R_{(4n-1)(4n-2)(4n-3)}(L_{i_1 \cdots i_{4n-4}})$$

$$L_{i_1 \cdots i_{4n}} = R_{(4n+1)(4n)(4n-1)}(\gamma_{i_1 \cdots i_{4n-2} i_{4n+1}})$$

Call the resulting transforms R_{γ_n} and R_{L_n}.

Now it is clear that for the class u in (4.15), the transforms $R_{(abc)}(u)$ preserves the coefficient of E_1 as long as $1 \notin \{a, b, c\}$. Let a_1' be the coefficient of E_1 in $R_{(1bc)}(u)$ respectively, the transforms $R_{(1bc)}(u)$ affects the coefficient of E_1 by:

$$a_1' = \frac{1 - \frac{1}{3 \cdot 2^a} - \frac{1}{3 \cdot 2^b}}{2 - \frac{4}{5} - \frac{1}{3 \cdot 2^a} - \frac{1}{3 \cdot 2^b}} > \frac{1 - \frac{1}{3 \cdot 2^a} - \frac{1}{3 \cdot 2^b}}{\frac{6}{5}} > \frac{1 - \frac{1}{12} - \frac{1}{24}}{\frac{6}{5}} > \frac{1}{2}$$

Therefore we have that R_{γ_n} and R_{L_n} maps u to the complement of $\mathcal{I}_{\frac{1}{2}}^k$ which is clearly in the complement of $\mathcal{C}_1^\beta(X_k)$. Since $\mathcal{P}_k^1(\beta) \subsetneq \mathcal{I}_{\frac{1}{2}}^k$ this finishes the proof. □

Remark 4.0.11. A simpler but less explicit way to proof is to use the fact that $\mathcal{I}^k_{\frac{1}{2}}$ is disjoint from the vertices of $\mathcal{P}_k$. Since the vertices are preserved under the Cremona transform and the effective action of Cremona group on the set of $\mathbb{Z}_2$-classes are finite as there are only finitely many $\mathbb{Z}_2$-classes, the union over all the orbit of $\mathcal{I}^k_{\frac{1}{2}}$ under this effective action is still disjoint from the vertices.

Chapter 5

Visible Lagrangians in almost toric fibration

Given a Lagrangian fibration $\pi : (M,\omega) \to B$, roughly speaking, a visible Lagrangian $L \subset M$ is a Lagrangian submanifold that projects to a nice low dimensional subset of B. A lot of information about such Lagrangians is contained in their image and they can be easily manipulated to construct various types of Lagrangian submanifolds in (M,ω). A more precise definition of visible Lagrangian submanifold will be given in section 5.0.3. We apply this idea to visualize as well as probe the existence of non-orientable Lagrangian surfaces in symplectic 4-manifolds. For the backgrounds, we mainly follow the book of Evans [9] for the exposition of Lagrangian fibrations and visible Lagrangians, though our definition of "visible" is slightly more general than [9].

Recall that any closed non-orientable surface L is diffeomorphic to $l\mathbb{RP}^2$ for a unique natural number l. We call l the **complexity** of L. The existence of non-orientable Lagrangian surfaces was observed in [**DHL**] while Audin's theorem ([2]) restricts the complexity of non-orientable Lagrangian submanifold.

Theorem 5.0.1. Let $\alpha \in H^2(M,\mathbb{Z}_2)$ be a class in a symplectic 4-manifold. Then (the Poincare dual of) α is represented by an embedded non-orientable Lagrangian surface L whose complexity $l(L)$ is congruent to $\mathcal{P}(\alpha) - 2$ modulo 4.

Here $\mathcal{P}$ is the Pointrjagin square operation $\mathcal{P} : H^2(X,\mathbb{Z}_2) \to H^4(X,\mathbb{Z}_2)$, which is a lift of the mod 2 cup product. It is particularly easy to calculate when $H^2(X,\mathbb{Z}_2)$ has no torsion. This is the case when M is a rational surface. For the next theorem recall that a *Characteristic class* $\alpha \in H^2(X^4,R)$ that represents cup square: $\alpha\cup\beta = \beta\cup\beta$ for all classes β over R.

Theorem 5.0.2. [8] Given a rational surface M, a mod 2 *non-characteristic* class α of a rational surface M and a positive integer l satisfying $l \equiv \mathcal{P}(\alpha) - 2 (\text{mod } 4)$, for some symplectic form on M, there exists a Lagrangian $l\,\mathbb{RP}^2$ in the class α. In particular, there exists a a Lagrangian $l\,\mathbb{RP}^2$ in the class α with $1 \leq l \leq 4$ for some symplectic form.

The importance of this theorem is that the minimal possible complexity l is achieved at least for some symplectic form. In contrast, it is shown in [7] that this is false for the characteristic class of many rational surfaces.

Any positive symplectic rational surface (M, ω) admits a Lagrangian fibration [14]. Now a natural question is whether there is a Lagrangian fibration on the rational surface M so that we can represent the *non-characteristic* class α by a visible Lagrangian submanifold L and if so, whether we can choose L to have minimal complexity. The answer is yes:

Theorem 5.0.3. The minimal complexity Lagrangians in Theorem 5.0.2 can be chosen to be visible for some symplectic form.

In addition, every complexity satisfying $l \equiv \mathcal{P}(\alpha) - 2 (\text{mod } 4)$ can be realized by a weakly visible Lagrangian for some symplectic form.

We also study in some detail visible Lagrangian $\mathbb{RP}^2$. In particular, we show the following results.

Theorem 5.0.4. For Markov triple $(a, b, c) \neq (1, 1, 1)$ and any k, every class vector $(1|c_1, ..., c_k)$ satisfying

1. $0 < c_i < \frac{b^2}{abc}$
2. $c_1 + ... + c_k < \frac{a^2}{abc}$ admits a visible Lagrangian $\mathbb{RP}^2$.

Theorem 5.0.5. Every Lagrangian $\mathbb{RP}^2$ in a symplectic rational surface with Euler number up to 8 is visible.

The structure of this section is as follows. In section 5.0.1 we give the definition of regular Lagrangian fibrations. These are Lagrangian fibrations without singularities. We will also explain Arnold-Liouville theorem and the flux map that allows us to pass to combinatorial geometries in $\mathbb{R}^n$. Next, we will introduce the *toric fibrations* as Lagrangian fibrations with mild singularities along with its applications and limitations. Then a generalization called *almost toric fibrations* will be introduced along with the operations of nodal slide, nodal trade, mutation, and almost toric blow-up. In section 5.0.3, we introduce visible Lagrangians. These are Lagrangians that are compatible with Lagrangian fibration. With all the tools the existence of certain Lagrangian submanifolds becomes apparent in section 5.0.5. We will follow [9] closely but try to give many examples and intuitions.

5.0.1 Lagrangian torus fibrations

We follow the exposition in [9], [22] and [13]. We begin with complete integrable systems which provide many Lagrangian torus fibrations, especially local models.

Complete integrable systems and local action coordinates around a regular fiber

Let (M, ω) be a symplectic manifold of dimension $2n$. Just as a Riemannian metric assigns to every smooth function H a canonical vector field ∇H, a symplectic structure assigns to every smooth function H (called Hamiltonian) a Hamiltonian vector field X_H whose flow ϕ_H^t preserves H. X_H is uniquely defined by:

$$\iota_{X_H}\omega = -dH. \tag{5.1}$$

In the spirit of Noether's theorem this is a correspondence between continuous symmetry (ϕ_H^t-invariance) and conserved quantity (the function H). Assuming the flow ϕ_H^t exists for all time, this gives a smooth $\mathbb{R}$-action on (M, ω).

Example 5.0.6. Consider the unit sphere (S^2, ω_0) in $\mathbb{R}^3$ with the standard area form. Let $H : S^2 \to \mathbb{R}$ be the projection to the z-coordinate, then the Hamiltonian flow is the rotation about the z-axis, which clearly preserves ω_0 and H. Note that the $\mathbb{R}$-action descends to an S^1 action in this example. We will see later that this is very often the case.

We can consider several Hamiltonians at the same time: $\mathbf{H} = (H_1, \ldots, H_k) : M \to \mathbb{R}^k$. We call this a Hamiltonian $\mathbb{R}^k$-action if

$$0 = \{H_i, H_j\} := \omega(X_{H_i}, X_{H_j}) \tag{5.2}$$

for all i, j. This implies the flows $\phi_{H_i}^t$ and $\phi_{H_j}^t$ commute for all i, j.

Definition 5.0.1. The tuples $(M, \omega, H_1, \cdots H_k)$ of Hamiltonian $\mathbb{R}^k$-action is called a **Hamiltonian system**. It is called **complete** if $k = \frac{1}{2}dim(M) = n$. It is called **integrable** if

1. $(H_1, \cdots, H_n) : M \to \mathbb{R}^n$ contains a **dense open** set of regular values.
2. $(H_1, \cdots, H_n)$ is **proper** with **connected fibers**.

The prototypical example is the following

Example 5.0.7. Consider $(\mathbb{C}^*)^n$ with the standard symplectic form and $H_i : (\mathbb{C}^*)^n \to \mathbb{R}$ defined by $H_i(z_1, \cdots, z_n) := \pi|z_i^2|$. Then the image of $(H_1, \cdots, H_n)$ is $(\mathbb{R}^+)^n$ and there are no critical values. The fiber over $(r_1, \cdots, r_n)$ is a Clifford torus, which is the product torus $S^1(r_1) \times \cdots S^1(r_n) \subset (\mathbb{C}^*)^n$ and is clearly Lagrangian

Example 5.0.7 is the local model near a regular fiber in any integrable system. This is the famous Arnold-Liouvile theorem.

Theorem 5.0.8. (Arnold-Liouvile theorem) Suppose $(M, \omega, H_1, \cdots, H_n)$ is a complete integrable Hamiltonian system and F_b is a regular fiber over $b \in \mathbb{R}^n$. Then F_b is a torus and there is a disc neighborhood B of b and a local coordinate change $\alpha : B \to C \subset \mathbb{R}^n$ such that $\mathbf{G} = \alpha \circ \mathbf{H}$ generates a **free** Hamiltonian T^n-action on $\mathbf{H}^{-1}(B)$. Consequently, the system $\mathbf{H}^{-1}(B)$ is fibred symplectomorphic to the product system $C \times T^n$, where the symplectic form on $C \times T^n$ is given by $\omega = \sum_i dc_i \wedge d\theta_i$ where c_i are local coordinates in C and θ_i are coordinates in T^n.

Locally, a neighborhood of a regular fiber of complete integrable Hamiltonian system is equipped with a free Hamiltonian torus action.

Under the local symplectomorphism to $C \times T^n$, the projection to $C \subset \mathbb{R}^n$ gives rises the so called (local) **action coordinates** $\alpha \circ H_i$.

We next introduce two types of singularities.

Example 5.0.9. (standard corank k elliptic system) The standard corank k elliptic system is the tuple of functions on $(\mathbb{R}^{n+k}, \omega_0)$

$$\mathbf{H}(x_1, \cdots, x_n, y_1, \cdots, y_k) = (x_1^2 + y_1^2, \cdots, x_k^2 + y_k^2, x_{k+1}, \cdots, x_n). \tag{5.3}$$

A critical point modeled on Example 5.0.9 is called an **elliptic singularity of corank k**.

Example 5.0.10. (standard 4-dimensional nodal system) The standard 4-dimensional nodal system is the pair of functions on $(\mathbb{R}^4, \omega_0)$

$$\mathbf{H}(x_1, x_2, y_1, y_2) = (F_1, F_2) = (-x_1y_1 - x_2y_2, \quad x_2y_1 - x_1y_2). \tag{5.4}$$

A critical point of a $4-$dimensional system modeled on Example 5.0.10 is called a **nodal singularity**.

Regular Lagrangian fibrations and the flux map

Now we take another point of view and look at Lagrangian fibrations directly.

Definition 5.0.2. A **regular Lagrangian fibration** is a smooth proper submersion $\pi : M \to B$ such that each fiber is a connected Lagrangian submanifold of (M, ω).

Example 5.0.7 is also a local model for regular Lagrangian fibrations near any fiber.

Example 5.0.11. Consider the 4-torus T^4 with symplectic form induced by the standard quotient map from $(\mathbb{R}^4, \sum_i dx_i \wedge dy_i)$. Then the projection to the (x_1, x_2) coordinates and projection to the (y_1, y_2) coordinates are both Lagrangian fibrations.

Surprisingly, all regular Lagrangian fibrations have torus as their fibers. This is a consequence of Theorem 5.0.8. To see this, consider local coordinates $b_1, \cdots, b_n$ on an open set $U \subset B$, then we have the following lemma.

Lemma 5.0.3. *The functions $\pi \circ b_1, \cdots, \pi \circ b_n$ form a complete integrable Hamiltonian system on $\pi^{-1}(U)$*

Proof. Denote $\pi \circ b_i$ as f_i, We simply need to prove they Poisson commutes. Fixing a fiber F_b over $b \in B$ we have $F_b \subset f_i^{-1}(c)$ for some constant c. Therefore for any tangent vector $v \in TL$ and i, $0 = df_i(v) = \omega(v, X_{f_i})$. So X_{f_i} is symplectic orthogonal to TL, whose symplectic orthogonal is itself. As a consequence X_{f_i} is tangent to TL. Now for every i, $\{f_j, f_i\} = \omega(X_{f_j}, X_{f_i}) = 0$ since both vector fields are tangent to L. □

Corollary 5.0.4. *The fibers of any regular Lagrangian fibration are tori.*

One of the most important features of a regular Lagrangian fibration is that it induces a map from the universal cover $\tilde{B}$ to $\mathbb{R}^n$ called the **flux map** and a great deal of the geometry of the Lagrangian fibration can be seen from the image of the flux map in $\mathbb{R}^n$ (which is used to define the **base diagram**).

The flux map is a more geometric and global way to present the action coordinates. When $\omega = d\lambda$ is exact the flux map $I : \tilde{B} \to \mathbb{R}^n$ is defined by

$$I(\tilde{b}) = \left(I_1(\tilde{b}), \ldots, I_n(\tilde{b})\right) := \left(\frac{1}{2\pi}\int_{c_1(\tilde{b})} \lambda, \ldots, \frac{1}{2\pi}\int_{c_n(\tilde{b})} \lambda\right). \tag{5.5}$$

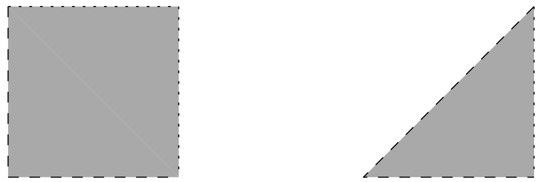
Here $c_i(\tilde{b})$ are elements of $H_1(\pi^{-1}(b), \mathbb{Z})$ where $b = p(\tilde{b})$ constructed as follows. Consider the local system $\xi \to B$ whose fibre over b is the abelian group $H_1(\pi^{-1}(b), \mathbb{Z}) \cong \mathbb{Z}^n$. Let $p : \tilde{B} \to B$ be the universal cover and let $\tilde{\xi} = p^*\xi$. Since $\tilde{B}$ is simply-connected, $\tilde{\xi}$ is trivial. Now let $c_1, \ldots, c_n$ be a $\mathbb{Z}$-basis of continuous sections of $\tilde{\xi} \to \tilde{B}$.

Example 5.0.12. ($(\mathbb{C}^*)^2$) Consider Example 5.0.7 with $n = 2$. In this case, $\tilde{B} = B$ is contractible. We can trivialize the local system by picking the standard basis $[1, 0]^T, [0, 1]^T$ for $H_1(\pi^{-1}(b), \mathbb{Z})$ and extending over B. Then the flux map is exactly (H_1, H_2) up to a factor of π. The flux image is Figure 5.1a.

Example 5.0.13. ($(\mathbb{C}^*)^2$ with a different basis) Consider the same example but with a different basis $[1, 0]^T, [1, 1]^T$ for $H_1(\pi^{-1}(b), \mathbb{Z})$ instead. The flux image would then be Figure 5.1b.

Integral affine structures

If $f : X \to B$ is a regular Lagrangian fibration then B inherits an integral affine structure. This is also a consequence of Theorem 5.0.8. An integral affine transformation is a map $T : \mathbb{R}^n \to \mathbb{R}^n$ of the form $T(b) = bA + C$ where $A \in GL(n, \mathbb{Z})$ and $C \in \mathbb{R}^n$.



(a) Standard flux image (b) Another flux image

Figure 5.1: Flux image of $(\mathbb{C}^*)^2$ under different basis of $H_1(\pi^{-1}(b), \mathbb{Z})$. The dotted lines indicate extension to infinity while the dashed line indicates pre-compact boundary components.

Definition 5.0.5. An integral affine structure $\mathcal{A}$ on a manifold without boundary B is an atlas for B whose transition functions are integral affine transformations. Equivalently, an integral affine structure on an $n-$manifold B is a lattice Λ in its tangent bundle.

A $k-$dimensional submanifold $P \subset (B, \mathcal{A})$ is affine planar if at each point $b \in P$ the subspace T_bP is spanned by k vectors in Λ. In particular, an embedded curve is affine linear if at every point it has a tangent vector in Λ.

For a manifold with boundary and corners, an integral affine structure is an integral affine structure on a thickening where the boundary and corners are affine planar.

The standard lattice Λ_0 generated by the unit vectors tangent to the coordinate axes in $\mathbb{R}^n$ defines the standard integral affine structure $\mathcal{A}_0$ on $\mathbb{R}^n$. A rational convex polytope in $\mathbb{R}^n$ inherits an integral affine structure.

Given any integral affine manifold B, there is a developing map, that is a (globally-defined) local diffeomorphism $I : \tilde{B} \dashrightarrow \mathbb{R}^n$ from the universal cover into Euclidean space such that the integral affine structure inherited by $\tilde{B}$ from the covering map agrees with the pullback of the integral affine structure along the developing map. In our context, the flux map is the developing map.

We pull back the integral affine structure from $\mathbb{R}^n$ along I to get an integral affine structure on $\tilde{B}$. This integral affine structure on $\tilde{B}$ is invariant under the action of deck transformations and descends to one on B. Hence we can define a map

$$AM : \pi_1(B) \to GL(n, \mathbb{Z}),$$

which is called the affine monodromy with respsect to the choice of the $\mathbb{Z}-$basis. The affine monodromy of the base determines explicitly the topological monodromy of the bundle and vice versa.

The affine monodromy of $\pi_1(B, b)$ is an automorphism in $Aut(\Lambda_b)$. If we choose a $\mathbb{Z}-$basis then the affine monodromy is in $GL(n, \mathbb{Z})$.

Many interesting symplectic manifolds, like $\mathbb{CP}^2$ do not admit regular Lagrangian fibrations. In fact even the simplest example of $\mathbb{C}^n$ does not admit a regular Lagrangian fibration. It is therefore necessary to allow certain types of singularities.

Toric fibrations and the piecewise linearity of the toric boundary

The simplest type of singularities are **elliptic singularities**. The most well-studied examples of Lagrangian fibrations with elliptic singularities are toric fibrations. These arise as moment maps of Hamiltonian T^n action on (M, ω).

Definition 5.0.6. A complete integrable system $(M, \omega, H_1, \cdots, H_n)$ is called a **toric fibration** if the Hamiltonian $\mathbb{R}^n$ action descends to a Hamiltonian T^n action on (M, ω). In this case the map $\mu = (H_1, \cdots, H_n) : (M, \omega) \to \mathbb{R}^n$ is called the **moment map** and its image $B \subset \mathbb{R}^n$ is called the moment image.

Recall that for a complete integrable system the map to $\mathbb{R}^n$ is assumed to be proper with connected fibers. Hence, if we let $B^{reg} \subset B$ be the set of regular values of μ, then $\mu^{-1}(B^{reg}) \to B^{reg}$ is a regular Lagrangian fibration. Therefore B^{reg} has an integral affine structure. Moreover, by the following local piecewise linearity result, B will have the structure of an integral affine manifold with piecewise linear boundary and corners, extending the integral affine structure on B^{reg}.

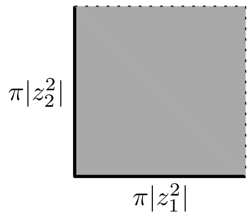
Proposition 5.0.7. *(Proposition 3.3 in [9]). Let $\mu : X \to \mathbb{R}^n$ be the moment map of a Hamiltonian T^n-action and $\partial\mu(X)$ the peicewise smooth boundary of the moment image. Pick local smooth embeddings $\delta_i : (0,1)^{n-1} \to \partial\mu(X) \subset \mathbb{R}^n$ parametrizing the smooth pieces of $\partial\mu(X)$ and assume that there are smooth lifts $\gamma_i : (0,1)^{n-1} \to X$ such that $\delta_i = \mu \circ \gamma_i$.*

Then the image of each δ_i is contained in an affine hyperplane Π_i with rational slopes, that is $\Pi_i = \{x \in \mathbb{R}^n | \alpha \cdot x = c\}$ for some intege rvector α. If $z \in \mu^{-1}(\delta_i)$ then the stabiliser of z is precisely the 1-dimensional subtorus $s_i^T(\mathbb{R}) \subset T^n$ where $s_i(x) = \alpha \cdot x$.

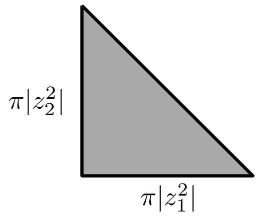
Remark 5.0.8. Notice that this is a local result and hence applies in situations where there are only locally Hamiltonian torus action (not necessarily free).

In the case at hand of a global torus action on a closed symplectic manifold the Atiyah-Guillemin- Sternberg convexity theorem in fact tells that the moment image $B \subset \mathbb{R}^n$ is a **rational convex** polytope (and hence globally convex) and B^{reg} is the interior. Each codimension k face P of B is affine planar in the sense that the tangent space $T_b P$ is spanned by a codimension k sublattice of Λ_0. The preimage of each point in the interior of a codimension k face of B is an isotropic torus of dimension $n-k$ and the points in the preimage are elliptic singularities of corank k.

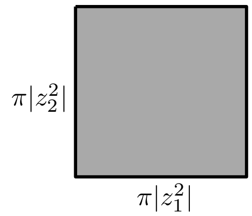
In particular, B^{reg} is simply connected and $\widetilde{B^{reg}} = B^{reg}$ and the moment map is the flux map. A change of homology basis for the flux map corresponds to a linear transformation of the H_i, or an equivalent T^n action.



(a) $\mathbb{C}^2$



(b) $\mathbb{CP}^2$



(c) $\mathbb{CP}^1 \times \mathbb{CP}^1$

Figure 5.2: Standard moment images

Not surprisingly the prototypical example is $\mathbb{C}^n$. In fact it provides the local model near any corank k singular fiber.

Example 5.0.14. ($\mathbb{C}^n$) Consider $\mathbb{C}^n$ with the standard symplectic form and $H_i : (\mathbb{C}^*)^n \to \mathbb{R}$ defined by $H_i(z_1, \cdots, z_n) := \pi|z_i^2|$. Then the image of $(H_1, \cdots, H_n)$ is $\mathbb{R}^n_{\geq 0}$ and the set of critical values is the union of all coordinate planes. See Figure 5.2a. The fiber over $(r_1, \cdots, r_n)$ when $r_i > 0$ for every i is a Clifford torus. The singular fibers over $(r_1, \cdots, r_k, 0, \cdots, 0)$ when $r_i > 0$ are all $k-$dimensional isotropic tori contained in subspaces of $\mathbb{C}^n$. In particular, the fiber over $(0, \cdots, 0)$ is a single point. Another moment map corresponding to a different but equivalent T^2 action has the image in Figure 5.3a.

From the toric point of view $\mathbb{C}^n$ is a partial compactification of $(\mathbb{C}^*)^n$. There are many interesting complete compactifications.

Example 5.0.15. ($\mathbb{CP}^2$) The standard T^2 action $\left(e^{i\theta_1}, e^{i\theta_2}\right) \cdot [z_1 : z_2 : z_3] = \left[e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : z_3\right]$ on $\mathbb{CP}^2$ with the Fubini-Study form corresponds to the moment map given by

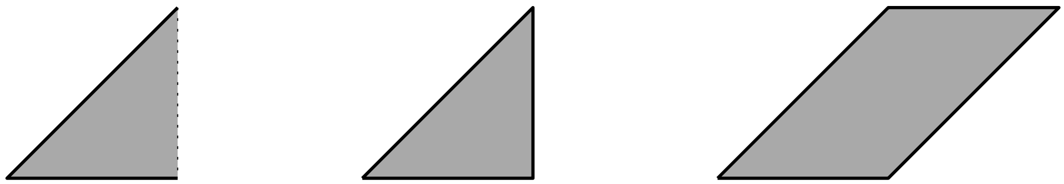
$$(H_1, H_2)([z_1, z_2, z_3]) = (\frac{|z_1|^2}{|z|^2}, \frac{|z_2|^2}{|z|^2}) \tag{5.6}$$

Its moment image in $\mathbb{R}^n$ is in Figure 5.2b. Another moment map corresponding to a different but equivalent T^2 action has the image in Figure 5.3b.

Example 5.0.16. ($(\mathbb{CP}^1)^2$) The standard T^2 action $\left(e^{i\theta_1}, e^{i\theta_2}\right) \cdot ([z_1 : z_2], [z_3 : z_4]) = ([e^{i\theta_1} z_1 : z_2], [e^{i\theta_2} z_3 : z_4])$ on $(\mathbb{CP}^1)^2$ with product area form corresponds to the moment map

$$(H_1, H_2)([z_1, z_2], [z_3, z_4]) = (\frac{|z_1|^2}{|z_1|^2 + |z_2|^2}, \frac{|z_3|^2}{|z_3|^2 + |z_4|^2}) \tag{5.7}$$

Its moment image in $\mathbb{R}^n$ is in Figure 5.2c. Another moment map corresponding to a different but equivalent T^2 action has the image in Figure 5.3c.



(a) $\mathbb{C}^2$ **(b)** $\mathbb{CP}^2$ **(c)** $\mathbb{CP}^1 \times \mathbb{CP}^1$

Figure 5.3: Different moment images

Remark 5.0.9. Similar to Example 5.0.13 if we choose a different basis of $H_1(T^2, \mathbb{Z})$, then the moment image in $\mathbb{R}^n$ will undergo an $SL(n, \mathbb{Z})$ transform. Figure 5.3 shows the moment images of the toric fibration of $\mathbb{C}^2$, $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ under the basis $\{[1, 0]^T, [1, 1]^T\}$.

Toric blow-up

Since we are mainly interested in symplectic rational surfaces it is necessary to introduce the operation of **symplectic blow-up**. This operation is compatible with the toric structure and is especially easy to see using base diagrams. We give a brief definition following [McDuff-Salamon] and several toric examples here.

Consider a manifold Z diffeomorphic to $\mathbb{CP}^{n-1}$ sitting inside its tautological bundle $\mathcal{O}(-1)$. Let $\mathcal{N}(Z)$ denote its tubular neighbor in the tautological bundle. Note that $\partial\mathcal{N}(Z) \cong S^{2n-1}$. Topologically speaking, blowing up a point $p \in M$ simply means removing a ball centered at p and sewing in $\mathcal{N}(Z)$ along the boundary S^3. Note that this is equivalent to removing a ball and quotienting the boundary S^3 by the Hopf S^1 action. Conversely, blowing down an exceptional divisor $\mathbb{CP}^{n-1} \cong \tilde{Z} \subset M$ means removing a tubular neighborhood of $\tilde{Z}$ and sewing in a ball. Note that if $\tilde{Z}$ is an exceptional divisor then its self-intersection number is -1. Hence its tubular neighborhood has a boundary diffeomorphic to S^3.

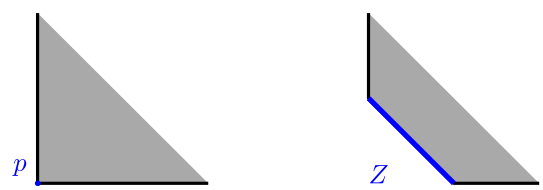
Symplectic blow-ups and blow-downs are not much more complicated than above. We simply need to keep track of the symplectic form during the cut-and-paste process. The symplectic structure on the blow-up $\tilde{M}$ is unique up to symplectomorphism as long as M is simply connected and the symplectic embeddings of balls are isotopic. Instead of giving a detailed construction here, we claim that these two operations are compatible with toric structures and use toric base diagram to illustrate the idea. Indeed, we can choose the ball and the tubular neighborhood $\mathcal{N}(Z)$ to be T^n-invariant (note this means the ball has to be centered at a fixed point). So they admit a toric fibration on their own:

Example 5.0.17. (B^n) We can simply restrict the moment map in example 5.0.14 to the open ball $B^n \subset \mathbb{C}^n$ centered at the origin. The moment image for B^4 is Figure 5.4a. The difference between moment images for B^4 and $\mathbb{C}^2$ is the moment image for B^4 is a finite area triangle.

Example 5.0.18. ($\mathcal{O}(-1)$) For simplicity, we restrict to dimension 2. We use the model $\mathcal{O}(-1) = \{(z_1, z_2, [z_3, z_4]) \in \mathbb{C}^2 \times \mathbb{CP}^1\}$. The Hamiltonian T^2 action is given by $(z_1, z_2, [z_3, z_4]) \to (e^{i\theta_1} z_1, e^{i\theta_2} z_2, [e^{i\theta_1} z_3, e^{i\theta_2} z_4])$. The corresponding moment map is

$$(H_1, H_2)(z_1, z_2, [z_3, z_4]) = (\frac{1}{2}|z_1|^2 + \frac{|z_3|^2}{|z_3|^2 + |z_4|^2}, \frac{1}{2}|z_2|^2 + \frac{|z_4|^2}{|z_3|^2 + |z_4|^2}) \tag{5.8}$$

The image is Figure 5.4b, where the exceptional divisor is the preimage of the slant blue edge.



(a) $\boldsymbol{B^4}$ **(b)** $\boldsymbol{\mathcal{N}(Z) \subset \mathcal{O}(-1)}$

Figure 5.4

Now a toric blow-up of size R is depicted in Figure 5.5

Example 5.0.19. (Thrice blow-up of $\mathbb{CP}^2$ of equal sizes) Figure 5.6 shows the toric base diagram of $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$ where the blow-up sizes are equal. It is clear from the picture that even though it is possible to perform more toric blow-ups (for example along the blue dotted line), the fourth blow-up has to have a smaller size than the first three. This shows that the monotone symplectic form on $\mathbb{CP}^2 \# 4\overline{\mathbb{CP}^2}$ cannot be obtained from toric blow-up.

Toric fibrations are nice since the base B is always simply connected and admits embedding into $\mathbb{R}^n$. In fact most problems about toric manifolds and varieties can be reduced to essentially combinatorial problems about the image of B. However, symplectic manifolds that admit a toric fibration (or equivalently a Hamiltonian T^n action) still form a very restricted class. All closed toric symplectic manifolds are rational. That is, they are blow-ups

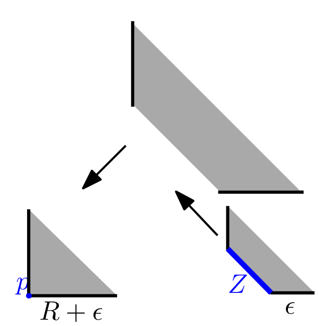


Figure 5.5: Toric blow up of size $\boldsymbol{R}$ **at** $\boldsymbol{p}$

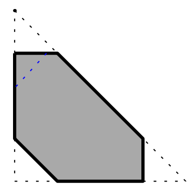


Figure 5.6: Monotone $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$

and blow-downs of $\mathbb{CP}^2$ or $S^2 \times S^2$. Even if we only look at rational surfaces, the symplectic forms that come from toric blow-ups are rather restrictive (Example 5.0.19) and are hard to describe explicitly. So the next step is to allow singularities other than elliptic singularities.

4−**dimensional Lagrangian torus fibrations**

Let (X, ω) be a symplectic manifold without boundary and B a topological space.

Definition 5.0.10. A Lagrangian torus fibration $f : X \to B$ is a proper continuous map with fibers compact, connected and immersed isotropic submanifolds without boundary such that f has the structure of a regular Lagrangian torus fibration over an open dense subset $B^{reg} \subset B$.

In other words, a regular Lagrangian fibration is a Lagrangian torus fibration with $B = B^{reg}$. Clearly, a toric fibration is a Lagrangian torus fibration with elliptic singuarities and B has a stratified integral affine structure as discussed in .

Observe that a hypersurface in $(\mathbb{R}^n \times T^n, \omega_0)$ that projects to a hypersurface in $\mathbb{R}^n$ is fibered by circles in the kernel of the symplectic form only if the hypersurface in $\mathbb{R}^n$ is affine planar. In such a situation we can perform the boundary reduction by collapsing the circle fibers to get elliptic singularities of corank 1. Elliptic singularities of arbitrary corank can be constructed by performing along different smooth hyersurfaces simultaneously. Therefore there is a local Hamiltonian torus action near any fiber with an ellitpic singularity.

Elliptic singularities are nice enough for toric fibrations to be tractable in any dimension. When we restrict to real dimension 4 we can include nodal singularity where the base is still a manifold with boundary and corners.

From now on we will assume the total space is a closed symplectic 4−manifold (M, ω) and B is two-dimensional manifold possibly with boundary and corners where B^{reg} is in the interior. Note that B^{reg} has an induced integral affine structure. We will further assume that the base B has an integral affine structure $\mathcal{A}$ away from a finite set $S \subset B \setminus B^{reg}$ of points that extends the integral affine structure on B^{reg}. This is the case for regular Lagrangian fibrations and toric fibrations where $S = \emptyset$.

Suppose further that there is an affine immersion $\Phi : B \setminus S \to \mathbb{R}^2$. Then a base diagram for such a Lagrangian torus fibration on a closed symplectic 4−manifold (M, ω) is the subset

$\Delta = \Phi(B \setminus S, \mathcal{A}) \subset \mathbb{R}^2$, The affine immersion $\Phi : B \setminus S \to \mathbb{R}^2$ may fail to exist. For a regular Lagrangian torus fibration, Φ exists if B is simply connected and is given by a flux map. For toric fibrations, Φ is given by the moment map.

In practice, we either consider the universal covering $\widetilde{B \setminus S}$ or a simply connected fundamental domain of $\widetilde{B \setminus S}$ under the action of $\pi_1(B \setminus S)$ to get a base diagram. Since B is of dimension 2, as in complex analysis, a fundamental domain is picked often by specifying a branch cut.

When there is a base diagram, following Symington, we will be able to viasulize a large class of Lagrangian surfaces via $\Delta \subset \mathbb{R}^2$.

We will describe the base diagrams of almost toric fibrations in some detail.

5.0.2 Almost toric fibrations and based diagrams

Definition 5.0.11. An almost toric fibration on a symplectic 4-manifold (M, ω) is a Lagragian fibration $\pi : M \to B$ such that any critical point of π has either an elliptic or a nodal singularity Darboux chart.

Note the functions $F_1 := -x_1y_1 - x_2y_2$ and $F_2 := x_2y_1 - x_1y_2$ poisson commute so this is a complete integrable system. However this is not a toric fibration since F_1 only generates a Hamiltonian $\mathbb{R}$-action.

The nodal singularity of an almost toric fibration is topologically identical to the nodal singularity of Lefschetz fibrations by a smooth change of coordinate that preserves neither the symplectic nor the complex structure on $\mathbb{R}^4 \cong \mathbb{C}^2$.

This turns out to be a successful generalization. Many symplectic K3 surfaces, Enriques surfaces, T^2 bundles and blow-ups of ruled surface admits almost toric fibrations. For a complete classification of closed almost toric manifolds see [13].

Geometry of a nodal fiber

The fiber containing exactly one nodal singularity is called a nodal fiber. The nodal fiber is a pinched torus, ie. a sphere with a self-intersection point.

Regular fibers have the canonical local model in Example 5.0.7 and fibers containing an elliptic singularity have canonical local models in Example 5.0.14 (or via boundary reduction). A complication regarding nodal singularity is that there is no canonical local model of a nodal fiber up to fibered symplectomorphism.

Let $\pi : U \to V$ be a local almost toric fibration around a nodal fiber with nodal singularity at x. V is diffeomorphic to the 2 disc. Set $V^0 = V \setminus \pi(x)$ and parametrized the punctured disc V^0 by the polar coodinate (r, θ). Parametrize the cylinder $\widetilde{V^0}$ by (r, θ) as well where θ is now considered as a real-valued.

Theorem 5.0.20. The action map $I : \widetilde{V^0} \to \mathbb{R}^2$ has the form

$$\frac{1}{2\pi}(S(b) + b_2\theta - b_1(\log r - 1), \quad 2\pi b_2), \tag{5.9}$$

where $b = b_1 + ib_2 = re^{i\theta}$ is the local coordinate on B and $S(b)$ is a smooth function. In particular, there is a well defined limit in $\mathbb{R}^2$ as $r \to 0$, which is called the base node.

The germ of the function $S(b)$ determines the germ of the nodal fiber neighborhood up to a fibred symplectomorphism.

The affine monodromy of the clockwise generator of $\pi_1(V^0)$, $(r, \theta) \to (r, theta + w\pi)$ is $AM = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, with $(1, 0)$ as an eigenvector with eigenvalue 1. If we post compose I by an integral affine transformation $A \in GL(2, \mathbb{Z})$, then the line of eigenvector of the affine monodromy of IA points in the direction the primitive integral eigenvector $(p, q) = (1, 0)A$.

Note this is the same as the monodromy of Lefschetz fibrations (Dehn twists) which makes sense since topologically they are the same singularity.

The infinitely many local models were classified by Ngoc in [23]. Nevertheless, Symington [22] proved that, for any two local models of nodal fibers, there exist a symplectomorphism between the total spaces that restricts to fibered symplectomorphism outside a small neighborhood of the singular fiber. Therefore we can work with any local model since we are interested only in the symplectic topology of the total space.

Almost toric base diagrams

Since the domain and target are both 2-dimensional, we can view I as a multi-valued map as in complex analysis and use a branch cut to get a single valued map.

Definition 5.0.21. Fixing a branch cut $D \subset B \setminus S$, an **almost toric base diagram** is its image under an affine immersion I, with the position of the each critical value marked by a cross and the images of the cut curves marked by dotted lines, and toric boundary marked by heavy lines.

Theorem 5.0.22. ([22], Corollary 5.4) Almost toric base diagrams determine the total space up to symplectomorphism when B is a punctured surface.

Note that a base diagram generally does not 'close up' unless all the cut curves are along the eigendirection of the affine monodromy. We illustrate based diagrams by the following **semi-global** model near a nodal singularity. This model also makes operations like nodal trade and nodal slide easy to construct.

Example 5.0.23. (Auroux system on $\mathbb{C}^2$) Fixing a real number $c > 0$, the Auroux system is defined by the Hamiltonoians

$$\mathbf{H} = (H_1, H_2)(z_1, z_2) = (|z_1 z_2 - c|^2, \quad \frac{1}{2}(|z_1|^2 - |z_2|^2)). \tag{5.10}$$

The image $B = \mathbf{H}(\mathbb{C}^2)$ is the closed right half-plane. This system has a nodal singularity at $(0, 0)$ with image $(c^2, 0)$. It also has elliptic singularities along the conic $z_1 z_2 = c$ with image the vertical boundary of the half-plane.

There exists a branch cut $D \subset B \backslash \{(c^2, 0)\}$ such that the image of the action coordinates has the form

$$I(D) = \{(x_1, x_2) : 0 \leq x_1 \leq \phi(x_2)\} \backslash \{(x_1, 0) : x_1 \geq m\},$$

where $\phi : \mathbb{R} \to (0, \infty)$ is some function and $m > 0$ is some number. It is hard to compute ϕ and m precisely.

We focus on the local picture Figure 5.7a near the singular fiber to understand the picture. Note the dotted line $\{(x_1, 0) : x_1 \geq m\}$ is the image of the cut curve we remove to obtain a well-defined map. The monodromy across the branch cut in Figure 5.7a is given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Figure 5.7b shows the images of a different branch cut. Note the base diamgram opens up since the cut curve is not along the eigenline of the monodromy map. The dotted lines are related by the affine monodromy.

Figure 5.7c shows the images of a different branch cut. We can post-compose by an integral affine transformation of $\mathbb{R}^2$ to get a new base diagram. Figure 5.7d is obtained by applying the integral affine transformation $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ to Figure 5.7c. This base diagram is especially useful in the "nodal trade" construction.

Operations on almost toric base diagrams

We have seen that there are operations that do not change the almost toric fibrations:

(i) changing branch cut, as in Figure 5.7b and Figure 5.7c,

(ii) integral affine transformation, as going from Figure 5.7c to 5.7d.

There are operations that change the almost toric fibrations but keep the symplectomorphism type.

Nodal trade Note that the local model in Figure 5.7d is very similar to the local model near a corank 2 elliptic singularity (Figure 5.4a) away from the singularity. We can then "trade" corank 2 elliptic singularities for nodal singularities. Figure 5.8 shows this process. The neighborhood of the blue dotted curve in both local models are fibered

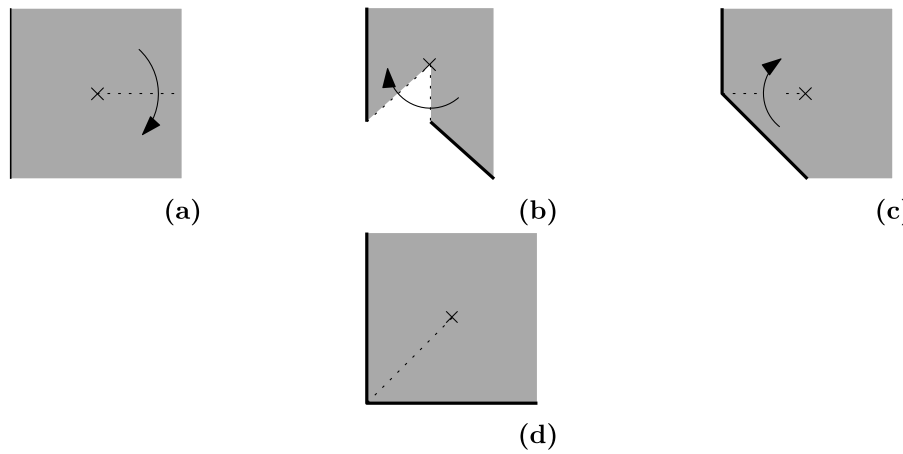


Figure 5.7: Base diagram for Auroux system under different cuts

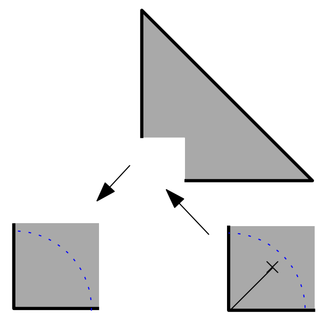


Figure 5.8: Nodal trade

symplectomorphic, allowing us to perform a "cut and paste" procedure and trade corank 2 elliptic singularities for nodal singularities.

Such an operation on almost toric fibrations is called a nodal trade. It obviously alters the almost toric fibration, there are in fact many different resulting fibrations, one for each Vü Ngoc model, nonetheless symplecotomorphic by Theorem 5.0.22.

Nodal slide By changing the parameter $c > 0$ in the Auroux system we obtain a family

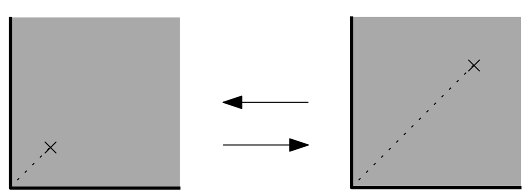
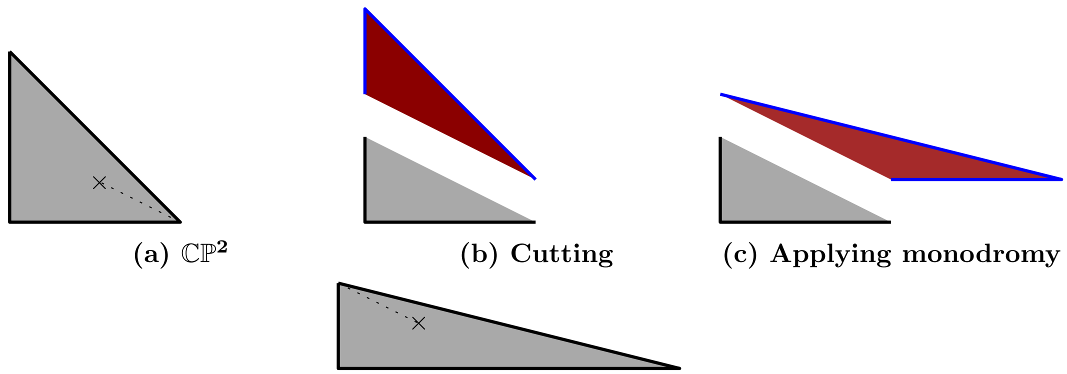


Figure 5.9: Nodal slide

of almost toric fibrations such that the base node moves along the eigenline. Like nodal trade, this operation is local and thus can be applied to any nodal singularity in any almost toric fibration.

Theorem 5.0.24. If two symplectic manifolds admit almost toric fibration with base diagrams related by nodal slides, and the base diagrams have the homotopy type of a punctured 2-dimensional surface, then they are symplectomorphic.

Mutation Given a branch cut $D \subset B \setminus S$, equivalently a fundamental domain of



[h!] **(a)** $\mathbb{CP}^2$ **(b) Cutting** **(c) Applying monodromy**

(d) $\mathbb{CP}^2$

Figure 5.10: After mutation

$widetilde B \setminus S$, $I(D)$ gives an almost toric base diagram. Changing the fundamental domain by applying $g \in \pi_1(B^{reg})$ will affect the base diagram by the monodromy map associated with g. There is a way of modifying the fundamental domain such that the effect on the base diagram is particularly easy to compute. Figure 5.10a shows the example of an almost toric fibration on $\mathbb{CP}^2$. The covering map $B^{\tilde{reg}} \to B^{reg}$ is homotopy equivalent to the universal cover of circle $t \to e^{it}$. Starting with a fundamental domain D (Figure 5.10a), we can "shift up" or "shift down" by 180 degrees and obtain another fundamental domain D'. Figure 5.10b and 5.10c show the process of "shifting up". Alternatively, we can "shift down" by applying inverse monodromy to the lower part of Figure 5.10b. For $\mathbb{CP}^2$, the set of triangles that can be obtained by performing mutations from the standard triangle is in bijection with **Markov triples** (a, b, c). These are solutions of the equation:

$$a^2 + b^2 + c^2 = 3abc \tag{5.11}$$

There are operations that change the symplectic structures.

There are operations that change the diffeomorphism type.

Blow-up Symplectic blow-up is compatible with almost toric blow-up. Similar to a toric blow-up, we need to look at the base diagrams of almost toric models of a neighborhood of the exceptional divisor $\mathbb{CP}^1 \subset \mathcal{O}(-1)$ and "implant" it into the base diagrams of general almost toric manifolds.

We use the same identification $\mathcal{O}(-1) = \{(z_1, z_2, [z_3, z_4]) \in \mathbb{C}^2 \times \mathbb{CP}^1\}$ as we did in the toric case. The Hamiltonians

$$(H_1, H_2)(z_1, z_2, [z_3 : z_4]) = (|z_1 - c|^2, \frac{1}{2}|z_2|^2 - \frac{|z_3|^2}{|z_3|^2 + |z_4|^2}) \tag{5.12}$$

defines an almost toric fibration with one nodal singularity and a toric boundary. There is a fundamental domain whose image under the flux map is Figure 5.11a. The image of the exceptional divisor lies inside the blue region. To be able to "implant" this local model into blow-ups of general almost toric manifolds we need to move the branch cut toward the toric boundary. The result is Figure 5.11b. Now we can perform almost toric blow-ups at any corank 1 elliptic singular point in an almost toric manifold removing a neighborhood of the point and gluing in the local model in Figure 5.11b.

(a) $\mathcal{O}(-1)$ **(b) A different branch cut**

Figure 5.11

An important example that is related to section 5.0.5 is the following. This is a 5 point

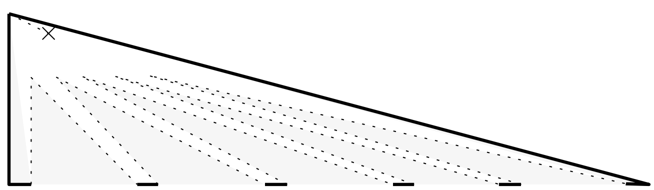


Figure 5.12: An almost toric fibration on monotone $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$

monotone blow-up of $\mathbb{CP}^2$. Note that the monotone symplectic forms $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$ on cannot be obtained by toric blowups as we observed earlier. Some of the blow-ups in the figure do not look like the local model in Figure 5.11b but are valid almost toric blow-ups since they can be $SL(2,\mathbb{Z})$-transformed to Figure 5.11b.

5.0.3 Visible Lagrangians

We again follow the exposition in [9]. We will first define *regular visible Lagrangians*. These are Lagrangians that behave very well (as fiber bundle) with respect to *pi*. They serve as the local model near any regular point on *non-regular visible Lagrangians*, which admits many kinds of non-regular points with respect to π.

5.0.4 Regular visible Lagrangians

In general, a Lagrangian L does not have to be compatible with a fixed Lagrangian fibration in any sense. In particular, the image of L under the flux map I does not need to be a nice (e.g. equidimensional) subset of the base diagram. In this section, we introduce a class of Lagrangians that behave nicely with respect to the projection. They are called **visible Lagrangian** in the sense that their images under flux map I are nice subsets of the base diagram and many properties of L (e.g. intersection numbers) are visible from the base diagram.

For our application, it is useful to consider Lagrangians that are slightly more general than the visible Lagrangians defined in [9]. The following definition is from [9].

Definition 5.0.25. Given a Lagrangian fibration $\pi : M \to B$ and a Lagrangian submanifold $L \subset M$, L is called **regular visible with respect to** π if $\pi_{|L^0}$ factors as $\iota \circ f$ where $L^0 := L \cap M^0$ is in the complement of singular fibers, $f : L^0 \to K$ is a fiber bundle and $\iota : K \to B$ is an embedding.

Remark 5.0.26. The visible Lagrangians used by Evans were more general than the above definition since over the vertices and edges the circle fibers degenerate to a point or a two-to-one cover. See 5.0.4 and 5.0.4.

First of all, we have that the images of such Lagrangians in the base diagrams are rational line segments. It suffices to prove this locally:

Theorem 5.0.27. (Theorem 5.1 of [9]) Consider the local model $(H_1, \cdots, H_n) : \mathbb{R}^n \times T^n \to \mathbb{R}^n$ defined by $(\mathbf{p}, \mathbf{q}) \mapsto \mathbf{p}$ where $q_1, \ldots, q_n$ are taken modulo 2π and the symplectic form is $\sum dp_i \wedge dq_i$. Let L be a Lagrangian submanifold satisfying definition 5.0.25, then $\iota(K)$ is an affine linear subspace that is rational with respect to the lattice $(2\pi\mathbb{Z})^n$

Sketch of proof. The Lagrangian condition implies the tangent space $\iota_*(T_x K)$ is orthogonal (with respect to Euclidean metric in $\mathbb{R}^n$) to tangent fibers of f and by comparing dimensions they are equal. Now fibers of f are integral submanifolds of $\iota_*(TK)^\perp$, which implies $\iota_*(TK)^\perp$ and hence $\iota * (TK)$ has to be rational. Finally, $\iota_*(TK)$ depends smoothly on points in K so it is necessarily constant.

From the sketch proof above we can spot the most important property of such Lagrangians: Their intersections with Lagrangian torus fibers are isotropic subtori of dimension $n - \dim(K)$ whose homology class in $H^{n-\dim(K)}(T^n, \mathbb{Z})$ is given by the slope of $\iota(K)$.

For simplicity, we focus on the case $dim(L) = 2$. Then the intersection numbers of two such Lagrangians L_1 and L_2 can be easily calculated from their images in the base diagram. It is simply the determinant $|v_1 \wedge v_2|$ where v_i is the primitive vector indicating the slope of images of L_i.

It is often useful to include Lagrangians that do not fiber nicely with respect to π. For example, the image line segment of a Lagrangian could hit an edge or a vertex, so that the fiber circle degenerates to a point or a covering. Tropical Lagrangians whose images are thickening of trivalent rational graphs in $\mathbb{R}^2$ are also important. Finally, there is a way of constructing Lagrangian torus locally so that π_L is only slightly worse than a fibration. We will call *all* of these Lagrangians visible with respect to π.

Non-regular visible Lagrangians arising from degenerations

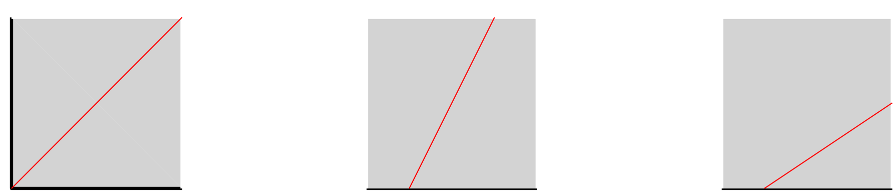
Regular visible Lagrangians are all circle bundles over line segments $l \subset B$. To construct more interesting examples we need to include degenerations of the circle fibers. This could happen when l hits a vertex, an edge, or a base node in an almost toric base diagram.

Local disc near a vertex Consider a line segment l with rational slope with one endpoint being a vertex in the standard toric base diagram for $\mathbb{C}^2$ with the fibration given by $\mu(z_1, z_2) = (\frac{1}{2}|z_1|^2, \frac{1}{2}|z_2|^2)$. By an integral affine transformation, we may assume the local picture near the vertex is Figure 5.13a, where the slope of l is $\frac{n}{m}$.

There is a Lagrangian subspace (not necessarily a submanifold) with image l called the *Schoen-Wolfson cone.* It can be parametrized by:

$$(s,t) \mapsto \frac{1}{\sqrt{m+n}}\left(t\sqrt{m}e^{is\sqrt{n/m}}, it\sqrt{n}e^{-is\sqrt{m/n}}\right), \quad s \in [0, 2\pi\sqrt{mn}],\ t \in [0,\infty)$$

It is singular unless $m = n = 1$, in which case it is a disc.



(a) A disc **(b) A (m,n)-pinwheel core** **(c) Another pinwheel core**

Figure 5.13

Local Mobius band near an edge Consider a line segment l with rational slope with one endpoint lying on the edge in the standard toric base diagram for $\mathbb{R} \times S^1 \times \mathbb{C}$ with the fibration given by $\mu(p, q, z) = (p, \frac{1}{2}|z|^2)$. By an integral affine transformation, we may assume the local picture near the vertex is Figure 5.13b and 5.13c, where the slope of l is $\frac{n}{m}$. There is an immersed Lagrangian submanifold with image l called the (m,n)-*pinwheel core.* It can be parameterized by

$$(s,t) \mapsto \left(ms, -nt, \sqrt{2ns}e^{imt}\right), \qquad (s,t) \in [0,\infty) \times S^1$$

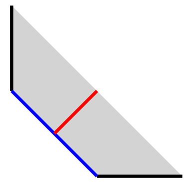


Figure 5.14: A Mobius band in $\mathcal{O}(-1)$

It is an embedding away from $s = 0$ and an n to 1 cover on $s = 0$. By integral affine transformations we may adjust m by adding or subtracting n and get equivalent pictures.

When $n = 2$ and $m \equiv 1$ mod n, the pinwheel core is a Mobius band. In all other cases the image of the immersion are not smooth. The following Figure 5.26a semi-local Mobius band in $\mathcal{O}(-1)$ will be useful to us.

Lagrangian toric blow-up The toric blow-up described in section 5.0.1 is compatible with the local Lagrangian disc in B^4 in Figure 5.13a and the local Lagrangian Mobius band in $\mathcal{O}(-1)$ in Figure 5.14. The process is shown in the following figure. It removes a Lagrangian disc and glues in a Lagrangian Mobius band, which is equivalent to a connected sum with $\mathbb{RP}^2$.

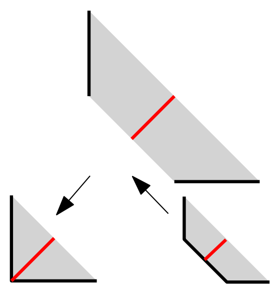


Figure 5.15: A Lagrangian toric blow-up

Local disc model a node Almost toric fibrations involve nodal singularities that map to *nodes* under the flux map I. As a Lagrangian fiber approaches the singular fiber, a circle of a certain class shrinks to a point (think of vanishing cycles in a Lefschetz fibration). Thus by taking the union of the circles, we obtain a disc. Under careful examination, we can see when is it Lagrangian:

Theorem 5.0.28. ([9] Lemma 6.15) Let $b \in \mathbb{R}^2$ be a node in a base diagram for an almost toric fibration and l a ray emanating from b in the eigen-direction. Then there is a Lagrangian disc living over l that is regular visible on l except at b.

Example 5.0.29. Consider the Auroux system Example 5.0.23, there are Lagrangian discs living over both sides of the eigenline. Figure 5.16 shows two Lagrangians discs over two sides of eigenline labeled red and blue. Note Lagrangian living over the red segment is not a

Lagrangian sphere (There is no Lagrangian sphere in $\mathbb{CP}^2$). This does not contradict section 5.0.4 as the lower left corner is not a vertex. Indeed the monodromy maps the horizontal edge to the vertical edge so they should be treated as one toric divisor (a straight line).

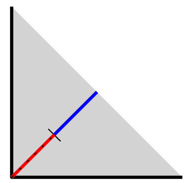


Figure 5.16: Two Lagrangians discs over two sides of eigenline

Some global visible Lagrangians

Now we can glue the local models along the isotropic circles in the fiber to produce global examples of visible Lagrangians

Example 5.0.30. (Klein bottle in $\mathbb{CP}^1 \times \mathbb{CP}^1$) Figure 5.17 shows two Lagrangian Klein bottles in $\mathbb{CP}^2$. Both of them are obtained by gluing two Mobius bands along their boundary circles.

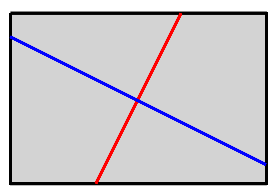


Figure 5.17: Two Lagrangian Klein bottles in $\mathbb{CP}^1 \times \mathbb{CP}^1$

Example 5.0.31. ($\mathbb{RP}^2 \subset \mathbb{CP}^2$)

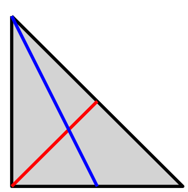
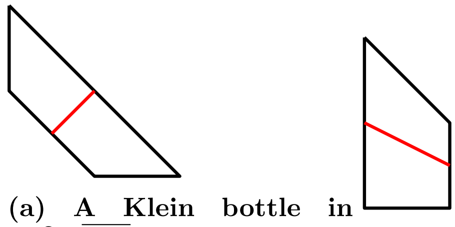


Figure 5.18: Three visible Lagrangian $\mathbb{RP}^2$ intersecting on the central torus

Figure 5.18 shows three Lagrangian $\mathbb{RP}^2$ in $\mathbb{CP}^2$. All of them are obtained by gluing a Lagrangian disc near the vertex and a Lagrangian Mobius band near the edge. All of them are in the class H mod 2. $v_1 = (1,1), v_2 = (1,-2), v_3 = (-2,1)$, so $|v_i \wedge v_j| = 3$.

Example 5.0.32. (Lagrangian toric blow-up of $\mathbb{RP}^2 \subset \mathbb{CP}^2$)



(a) A Klein bottle in $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ (b) An equivalent picture

Figure 5.19: Lagrangian toric blow-up

Other Non-regular visible Lagrangians

In this section, we present some Lagrangians that do not arise from degenerations of circle fibers of regular visible Lagrangians. Nevertheless, these Lagrangians behave nicely with respect to π and are flexible enough for us to construct interesting examples. So we include them in our class of visible Lagrangians.

Local model near an intersection point

Lemma 5.0.33. ([9] Lemma H.6) Suppose we have several straight lines of rational slope in $\mathbb{R}^2$ incident on a point $b \in B$. Let $v_1, \cdots, v_k$ be primitive integer vectors pointing along these lines. The visible Lagrangian cylinders above these lines have a total of $\delta(b)$ transversal intersections, where

$$\delta(b) = \sum_{i<j} |v_i \wedge v_j|.$$

Example 5.0.34. (Klein bottle in $\mathbb{CP}^1 \times \mathbb{CP}^1$ revisited) Figure 5.17 shows two Lagrangian Klein bottles in $\mathbb{CP}^2$. The intersection number is 2.

Example 5.0.35. ($\mathbb{RP}^2 \subset \mathbb{CP}^2$ revisited) All of them are in the class H mod 2. $v_1 = (1,1), v_2 = (1,-2), v_3 = (-2,1)$, so the pairwise intersection number is $|v_i \wedge v_j| = 3$.

Tropical Lagrangians

These Lagrangians were introduced by [17] and are also nicely behaved with respect to π. Their images are small thickening of *tropical subvariety*. Here we focus on the case $dim(L) = 2$ and look at Lagrangians that project to (thickenings of) tropical curves.

The prototypical example is the **Lagrangian pair-of-pants**. Mikhalkin proved the following:

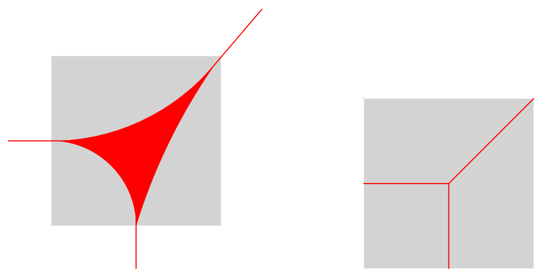
Theorem 5.0.36. (Mikhalkin) Let R_1, R_2, R_3 be three rays with rational slope in the p_1, p_2-plane emanating from the origin, and let v_1, v_2, v_3 be the primitive integer vectors pointing along these rays. Suppose that the *balancing condition*

$$v_1 + v_2 + v_3 = 0 \tag{5.13}$$

holds and that any two of these vectors form a $\mathbb{Z}$-basis for the integer lattice. Let L_1, L_2, L_3 be the visible Lagrangian half-cylinders living over R_1, R_2, R_3 and fix $\epsilon > 0$. Let $U := \{p_1, p_2 : |p_1, p_2| > \epsilon\}$. There is an embedded Lagrangian submanifold $L \subset \mathbb{R}^2 \times T^2$, diffeomorphic to the pair-of-pants, such that $U \cap L = U \cap (L_1 \cup L_2 \cup L_3)$.

Proof. See [17] and also [9] □

The standard case is when $v_1 = (-1, 0)$, $v_2 = (0, -1)$, $v_3 = (1, 1)$. See Figure 5.20. All other cases are equivalent to the standard one since they are related by a $GL(2, \mathbb{Z})$-transformation that is covered by a fibered symplectomorphism. Since the thickening around the trisection point can be made arbitrarily small we will just use trivalent graphs to denote the images of such Lagrangians.



(a) Image of pair-of pants **(b) A tropical curve**

Figure 5.20

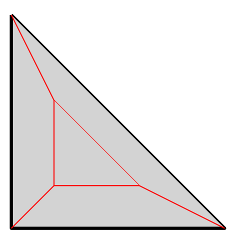


Figure 5.21: A tropical torus in $\mathbb{CP}^2$

If the condition that any two of the vectors $v_1 = (-1, 0)$, $v_2 = (0, -1)$, $v_3 = (1, 1)$ is dropped, then the result still holds but with immersed Lagrangians instead of embedded Lagrangians. In fact, the self-intersection number can be read from the slopes of the rays.

Theorem 5.0.37. (Mikhalkin) Let Δ be the absolute value of the determinant of the matrix whose rows are v_1 and v_2. The number of self-intersections of the Lagrangian pair-of-pants is $\delta = \frac{\Delta - 1}{2}$, the absolute value of the determinant of the matrix formed by v_1 and v_2.

We can perform surgeries to remove the self-intersections. The resulting Lagrangian will not be tropical. Nevertheless, the surgery could be performed in arbitrarily small neighborhoods of the immersed point. So it is still valid to use tropical curves to represent such Lagrangians.

Complexity of visible Lagrangians

Definition 5.0.38. We call a Lagrangian submanifold L visible with respect to a Lagrangian fibration π if L is a finite union of various types of Lagrangians described in this section:

- regular
- disc near a vertex
- Mobius band near an edge
- disc near a node
- tropical

Lemma 5.0.39. Suppose we have a Lagrangian graph Γ with T components L_i, where

- The component L_i has complexity η_i and k_i double points,
- The components L_i and L_j have intersection number b_{ij}.

The complexity is

$$\sum \eta_i + 2[\sum k_i + \sum b_{ij} - (T-1)]$$

Proof. intersection number: determinant □

Weakly visible torus over a local segment and stabilization

This is a local construction. Consider the standard Liouville coordinate (p_1, p_2, q_1, q_2) on the local chart $\mathbb{R}^2 \times T^2$ with the standard symplectic form. There is an obvious visible Lagrangian cylinder $i : \mathbb{R} \times S^1 \to \mathbb{R}^2 \times T^2, i(s,t) = (s, 0, 0, t)$ for $\pi(p_1, p_2, q_1, q_2) = (p_1, p_2)$. This cylinder can be modified into a Lagrangian torus by forcing i to be periodic in s:

$$j : S^1 \times S^1 \to \mathbb{R}^2 \times T^2 \tag{5.14}$$

$$j(s,t) = (sin(s), 0, 0, t) \tag{5.15}$$

This torus is projected to the line segment $[-1, 1] \times \{0\}$ (Figure 5.22.), where the preimage of interior points are two disjoint circles. This torus is not visible according to the definition in [9]. Nevertheless, its intersection with other visible Lagrangians in the interior can be read easily using Lemma 5.0.33. So we include this type of Lagrangian in our class of visible Lagrangians.

Figure 5.22: A rational slope segment local torus

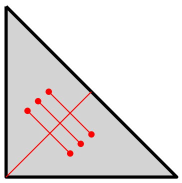


Figure 5.23: A 13$\mathbb{RP}^2$: $\mathbb{RP}^2$ stabilized 3 times

Definition 5.0.40. We call a Lagrangian submanifold L weakly visible with respect to a Lagrangian fibration π if L is a finite union of visible Lagrangians and rational slope local tori.

Lemma 5.0.41. Suppose we have a straight lines of rational slope and a local line segment in $\mathbb{R}^2$ incident on an interior point b of the local line segment. Let v_1 and v_2 be primitive integer vectors pointing along these lines. The visible Lagrangian cylinder above the v_1 line and the local torus over the local v_2-segment have a total of $2|v_1 \wedge v_2|$ transversal intersections.

Adding a weakly visible Lagrangian handle

Lemma 5.0.42. Every non-orientable visible Lagrangian Σ can be stabilized to a weakly visible one with complexity increased by 4.

Example 5.0.43. We pick line segments with slope 1. Each local torus intersects the $\mathbb{RP}^2$ at 2 points.

Example 5.0.44. (Stabilization: local connected sum with $4\mathbb{RP}^2$) Recall in example 5.22 we can construct Lagrangian tori that project to arbitrarily short line segments. Over any interior point of the line segment, the fiber consists of two disjoint circles. Figure 5.23 shows a visible Lagrangian $\mathbb{RP}^2$ together with three tori of this type. Each of these local torus intersect with $\mathbb{RP}^2$ in two points. Each intersection point can be resolved at the price of increasing the complexity by 2. Therefore the stabilization process can be done visibly in this way.

5.0.5 Existence of visible non-orientable Lagrangians in rational surfaces

In this section, we will use almost toric blow-up, nodal trade, nodal slide, and mutations to visualize Lagrangian $\mathbb{RP}^2$ in many rational symplectic 4-manifolds. A **rational symplectic manifold** (M, ω) is a symplectic manifold obtained from performing blow-ups and blow-downs from $(\mathbb{CP}^2, \omega_{FS})$. More specifically these manifolds are $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ for $k \geq 0$ and $S^2 \times S^2$. The terminology comes from algebraic geometry where blow-up is a type of birational transform.

Existence of minimal complexity Lagrangians

In this section, we prove theorem 5.0.3 using the various visible Lagrangians from the previous section. We need two more constructions using Lagrangian surgery and blow-up.

Lemma 5.0.45. The 0 class is represented by a visible $6\mathbb{RP}^2$.

Proof. A pair of $\mathbb{RP}^2$ in $\mathbb{CP}^2$ intersecting at 3 points as in Figure 5.18 gives rise to a visible $6\mathbb{RP}^2$. Toric blowing up around the remaining corner.

This visible $6\mathbb{RP}^2$ also exists in $S^2 \times S^2$. We just need two corners. □

Let $M_k := \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$. We denote $H, E_1, \cdots, E_k$ as the basis of $H_2(M_k, \mathbb{Z}_2)$ and their dual basis $h, e_1, \cdots, e_k$ of $H^2(M_k, \mathbb{Z}_2)$ where $H^2 = 1$ and $E_i^2 = -1$ are the plane class and exceptional classes.

Theorem 5.0.46. Every nonzero mod 2 ordinary class of a rational surface is represented by a visible Lagrangian $l\mathbb{RP}^2$ with $1 \leq l \leq 4$

Proof. Recall a rational manifold has $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ or $S^2 \times S^2$ as topological types. First we focus on $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$. It is shown in [7] that, up to the action of $Diff^+(M_k)$ on $H^2(M_k, \mathbb{Z}_2)$, every nonzero class in $H^2(M_k, \mathbb{Z}_2)$ can be transformed into one of 5 classes $\{\beta, \beta+\gamma_1, \gamma_1, \gamma_1+\gamma_2, \xi_k\}$, where $\xi_k := \beta + \gamma_1 + \cdots + \gamma_k$ is the unique *characteristic class* since $\xi_k \cdot \alpha \equiv \alpha^2 \mod 2$ for any class α.

- $\mathbb{RP}^2$ in the class h. This is trivial since we can start at any visible $\mathbb{RP}^2$ in $\mathbb{CP}^2$ and perform blow-up away from it.
- $2\mathbb{RP}^2$ in the class $h + \gamma_1$. This can be obtained by performing a Lagrangian toric blow-up on visible $\mathbb{RP}^2$ in $\mathbb{CP}^2$ as in Figure 5.19a and then performing subsequent blow-ups away from it.
- $3\mathbb{RP}^2$ in the class γ_1. This can be obtained from the tropical torus in Figure 5.21 by a Lagrangian toric blow-up and subsequent blow-ups away from the Lagrangian.
- $4\mathbb{RP}^2$ in the class $\gamma_1 + \gamma_2$. This can be obtained from the tropical torus in Figure 5.21 by two Lagrangian toric blow-ups and subsequent blow-ups away from the Lagrangian.

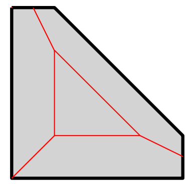


Figure 5.24: A tropical $4\mathbb{RP}^2$ in $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$

Now we turn to the topological type $S^2 \times S^2$.

For $S^2 \times S^2$ we use h_1 and h_2 to denote the basis of $H_2(S^2 \times S^2, \mathbb{Z}_2)$. Every class in $H_2(S^2 \times S^2, \mathbb{Z}_2)$ can be transformed into one of $\{h_1, h_2, h_1 + h_2\}$.

• $2\mathbb{RP}^2$ in the class h_1. This the blue Lagrangian Klein bottle in Figure 5.17.

• $2\mathbb{RP}^2$ in the class h_2. This is the red Lagrangian Klein bottle in Figure 5.17.

• $4\mathbb{RP}^2$ in the class $h_1 + h_2$. This can be obtained by performing Lagrangian surgeries on the two Lagrangian Klein bottle in Figure 5.17.

□

Dependence of Lagrangian $\mathbb{RP}^2$ on ω

By Theorem 5.0.1, $S^2 \times S^2$ does not admit any symplectic form that admits a Lagrangian $\mathbb{RP}^2$: (Since $H^*(S^2 \times S^2, \mathbb{Z})$ contain no torsion, the Pontrjagin square is just cup square mod 4). So we can focus on the manifolds $M_k := \mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$. The following result greatly simplifies the problem:

Theorem 5.0.47. If a 4-manifold M is rational and ω_0 and ω_1 are symplectic forms on M, then (M, ω_0) is symplectomorphic to (M, ω_1) if $[\omega_0] = [\omega_1]$ in $H^2(M, \mathbb{R})$.

This result allows us to reformulate any problems about (M, ω) to problems about (M, a) where $a \in H^2(M, \mathbb{R})$ is a cohomology class that admits a symplectic form. For example, if we have found a Lagrangian $\mathbb{RP}^2$ in (M, ω), then theorem 5.0.47 says there is a Lagrangian $\mathbb{RP}^2$ in (M, ω') where ω' is any symplectic form cohomologous to ω. This greatly simplifies the problem since the moduli space of symplectic form is in general very complicated and infinite dimensional, but the set of classes $a \in H^2(M, \mathbb{R})$ admitting a symplectic form is just a finite-dimensional cone. We denote it as $\mathcal{C}(M)$. The set of classes admitting a Lagrangian $\mathbb{RP}^2$ is then a subcone of $\mathcal{C}(M)$, which we denote as $\mathcal{C}_1(M)$

Now we can formulate the problem, we choose a basis $H, E_1, \cdots, E_k$ of $H_2(M_k)$ and their dual basis $h, e_1, \cdots, e_k$ of $H^2(M_k)$ where $H^2 = 1$ $E_i^2 = -1$. Now we can denote write any class a as $a = bh + c_1 e_i + \cdots + c_k e_k$ for $b, c_i \in \mathbb{R}^+$. Since the problem we are considering is insensitive to scaling for symplectic form, we can normalize and only consider classes whose leading coefficient is 1: $a = h + c_1 e_i + \cdots + c_k e_k$. For simplicity we denote the class vector as $(1|c_1, \cdots, c_k) \in H^2(M_k, \mathbb{R})$

Problem. *Find explicit conditions on c_i so that $(1|c_1, \cdots, c_k)$ is in $\mathcal{C}_1(M_k)$ if and only if those conditions are true.*

A complete answer to this problem is given for $k \leq 7$ in [7]:

Theorem 5.0.48. $a = h + c_1 e_i + \cdots + c_k e_k$ is in $\mathcal{C}_1(M_k)$ if and only if:

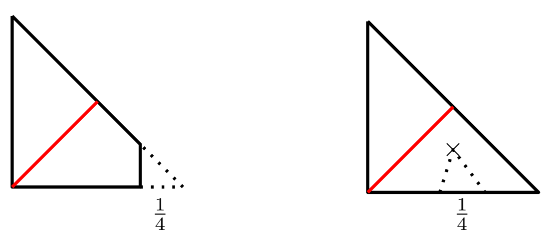
$$0 < c_i < \frac{1}{2} \tag{5.16}$$

$$c_{i_1} + c_{i_2} + c_{i_3} + c_{i_4} + c_{i_5} < 2 \tag{5.17}$$

$$c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 < \frac{5}{2} \tag{5.18}$$

The method used in the proof is hard to generalize as it used the classification of symplectic forms admitting symplectic self-intersection (-4)-spheres in M_k for $k \leq 8$.

However, if we only focus on the existence part of the problem, the theory of almost toric fibration and visible Lagrangians produces many examples for arbitrary k. The ideas are simple: fix a Lagrangian $\mathbb{RP}^2$ in $\mathbb{CP}^2$ and blow-up away from it. Recall in the almost toric setting an almost toric blow-up is easy to describe. It is simply packing triangles along the edges of the base diagram. In figure (5.25a) and (5.25b) the Lagrangian $\mathbb{RP}^2$ is living over the red segment. We denote this Lagrangian as L.



(a) Toric blow up **(b) Almost toric blow up**

Figure 5.25: $c_1 = \frac{1}{4}$

As we can see from the figure (5.25a) and (5.25b), almost toric blow-up is more flexible than toric blow-up. Now the problem becomes:

Problem. *How many (and what are the size of) triangles can we place along the boundary (performing almost toric blow-up) without touching the red segment (the Lagrangian $\mathbb{RP}^2$)*

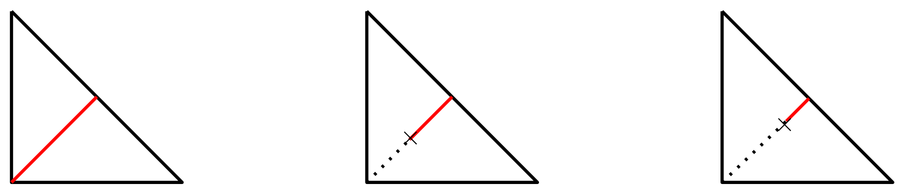
Recall in an (almost) toric blow-up, the image of the ball in the base diagram has to be a triangle whose base length and height are the same. We can see the condition

$$0 < c_i < \frac{1}{2}$$

already from Figure 5.25b. If the blow-up size c_1 exceeds $\frac{1}{2}$ then L will not be preserved. We can see the almost toric perspective is not more useful than the toric perspective at this stage. Nevertheless, we have the following:

Lemma 5.0.49. $(1|c_1)$ is in $\mathcal{C}_1(M_1)$ if $\frac{c_1}{b} < \frac{1}{2}$

If we perform a nodal trade on the lower left vertex, we can arrange it so that the red segment becomes shorter. Then by applying a nodal slide, we can make the red segment arbitrarily short. This brings us closer to the goal. However, with the cut in figure (5.26c)



(a) **(b) Nodal trade** **(c) Nodal slide**

Figure 5.26

bisecting the base diagram it is still hard to place triangles. Although touching the cut is permitted, the monodromy around the cut messes up the triangle (Figure 5.27). The solution to this is mutation.

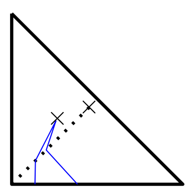


Figure 5.27: Monodromy affects the triangle

Visible Lagrangian $\mathbb{RP}^2$ from Markov triples

We can transfer the cut to the opposite side at the price of applying monodromy to half of the base diagram. For aesthetic reasons we apply a $SL(2,\mathbb{Z})$ transform to the base diagram in figure (5.26c) to get figure (5.28). Note this does not change the almost toric fibration.

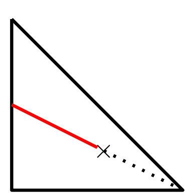
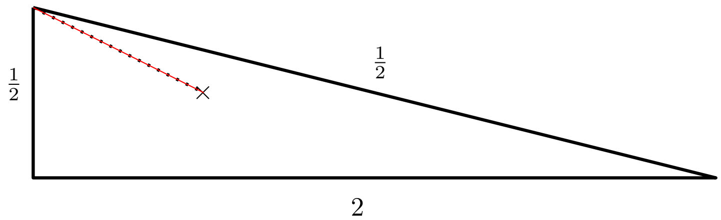
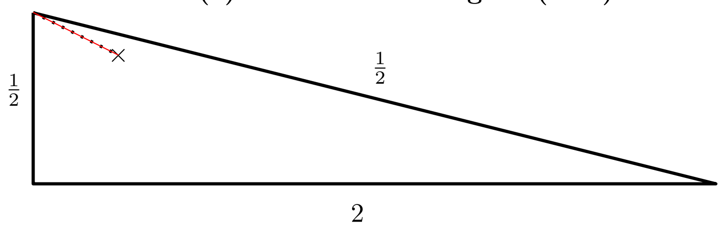


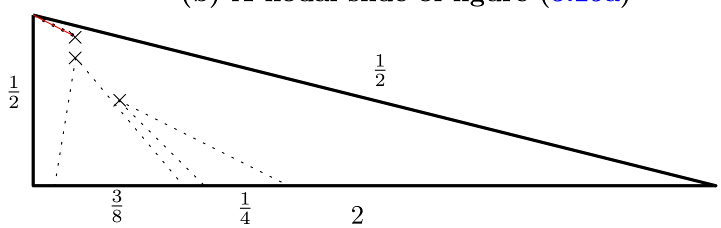
Figure 5.28: An ATBD equivalent to (5.26c)



(a) A Mutation of figure (5.28)



(b) A nodal slide of figure (5.29a)



(c) Two-time blow up

Figure 5.29

Now if we apply mutation to figure (5.28), we will get another base diagram representing the same almost toric fibration. However, the branched cut is moved to the other side figure (5.29a), and more importantly, L is now living over the branched cut, which can be made arbitrarily short by nodal slide. This proves the following:

Lemma 5.0.50. For all $k > 0$, $(1|c_1, \cdots, c_k)$ is in $\mathcal{C}_1^{visible}(M_k)$ if

$$c_i < \frac{1}{2} \tag{5.19}$$

$$c_1 + \cdots + c_k \leq 2 \tag{5.20}$$

This recovers the "if" part of theorem 5.0.48 when $k \leq 5$ and proves the existence of Lagrangian $\mathbb{RP}^2$ for many symplectic forms in arbitrarily large blow-up. In fact we can do much more by performing more nodal trade and mutations:

Now we see we can perform almost toric blow-ups along the toric boundary whose image is the vertical line with length $\frac{5}{2}$ as many times as we like as long as the sizes of each blow-up are smaller than the height $\frac{2}{5}$, and the total size is smaller than $\frac{5}{2}$. This proves:

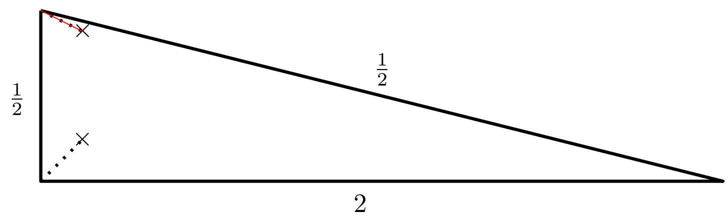


Figure 5.30: A new nodal trade

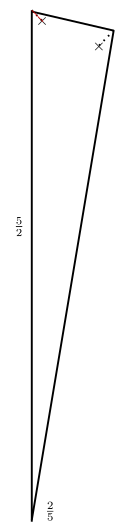


Figure 5.31: Mutation

Lemma 5.0.51. For all $k > 0$, $(1|c_1, \cdots, c_k)$ is in $\mathcal{C}_1^{visible}(M_k)$ if

$$c_i < \frac{2}{5} \tag{5.21}$$

$$c_1 + \cdots + c_k \leq \frac{5}{2} \tag{5.22}$$

This only partially proves the "if" part of theorem 5.0.48 when $k \leq 7$, but again the upshot is that this lemma works for arbitrary k.

We can keep going forever. Mutations can always be performed on one of the two branch cuts that are disjoint from the image of L. This allows us to embed balls of sizes smaller than the second-longest edge with a total size smaller than the longest edge.

As we have seen in section 5.0.2, the set of almost toric base diagrams of $\mathbb{CP}^2$ is in bijection with the set of Markov triples. This proves the existence of Lagrangian $\mathbb{RP}^2$ in infinitely many rational manifolds (M_k, ω):

Theorem 5.0.52. For Markov triple $(a, b, c) \neq (1, 1, 1)$ and any k, every class vector $(1|c_1, ..., c_k)$ satisfying

1. $0 < c_i < \frac{b^2}{abc}$

2. $c_1 + ... + c_k < \frac{a^2}{abc}$ admits a visible Lagrangian $\mathbb{RP}^2$.

Visible Lagrangian $\mathbb{RP}^2$ from one sided ATBD

Returning to Figure 5.29c we can notice that if we put the larger blow-ups to the right side of the triangle and smaller ones to the left, there will be some place left for us to blow up along the left edge. For example Figure 5.32 shows the embedding of three almost $\frac{1}{2}$ balls and four almost $\frac{1}{4}$ balls. This kind of example is very special and hard to find patterns of. We suspect it has something to do with divisibility by 2.

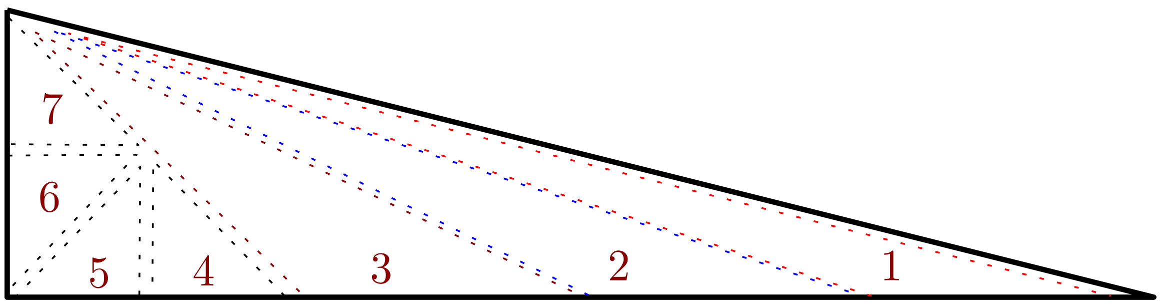


Figure 5.32: three $\frac{1}{2}$ balls and four $\frac{1}{4}$ balls

Finally, even though most Lagrangian $\mathbb{RP}^2$ in rational symplectic manifold (M_k, ω) are not visible for a fixed Lagrangian fibration, when $k \leq 8$, cohomologous Lagrangian $\mathbb{RP}^2$ are isotopic by [6]. Since the ordinary Lagrangian $\mathbb{RP}^2$ classes are related by ambient diffeomorphism when $k \leq 5$, we have

Theorem 5.0.53. Every Lagrangian $\mathbb{RP}^2$ is visible when $k \leq 5$.

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