Highly Structured Multiplication & The Miller Spectral Sequence

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Daniel Hess

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Advisor: Tyler Lawson

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Abstract

We examine the Miller spectral sequence for determining the mod-p homology of a connective spectrum X from the mod-p homology of its associated infinite loop space, $\Omega^\infty X$, considered as an algebra over the mod-p Dyer-Lashof algebra. For each prime p, we give a Koszul complex for computing the E^2 page of this spectral sequence, recovering a result of Miller (at $p = 2$) [35] and Kraines and Lada (at odd primes) [22]. As applications, we determine $H^*(H\mathbb{Z}; \mathbb{F}_p)$ and $H^*(H\mathbb{F}_p; \mathbb{F}_p)$ at all primes, recovering well-known results.

As an original application of the Miller spectral sequence, we study the relationship between $H_*(\Omega^\infty X; \mathbb{F}_p)$ and $H_*(X; \mathbb{F}_p)$ when X is an E_∞-ring spectrum. We show that the Miller spectral sequence can be used to detect nonzero "multiplicative" k-invariants of X at all primes. We also prove that for any integer $n \geq 1$, the underlying spectrum of a commutative $H\mathbb{F}_p$-algebra R is equivalent to its strict unit spectrum, $\mathrm{sl}_1(R)$, in a range that is wider than the stable range: $[n, pn - 1]$. This is a special case of a conjecture by Mathew and Stojanoska [29].

Contents

Chapter 1

Introduction

Abstract algebra studies sets that are equipped with one, or more, binary operations. Properties that these operations may satisfy, such as associativity or commutativity, will either hold or not hold. For example, given a group G and any two elements $a, b \in G$, either ab is the same as, or *equal* to, ba for all $a, b \in G$ or it is not. In homotopy theory, two objects are considered "the same" not only when they are equal, but also, more generally, when they are *homotopic*. As a result, one encounters algebraic structures that may satisfy properties lying "in between" strict associativity or commutativity. For example, a (based) loop space ΩX carries a binary operation, $*$, given by loop concatenation. With this multiplication, ΩX is a "group", but only up to homotopy; in particular, there is a homotopy between the loops $a * (b * c)$ and $(a * b) * c$—so that $*$ is *homotopy* associative—but the two loops are not literally equal. Similarly, the multiplication on iterated loop spaces $\Omega^n X$ (for $n \geq 2$) is homotopy commutative (and homotopy associative), but not commutative in the strict sense.

There are often several different homotopies that can witness the homotopy associativity or commutativity of a binary operation on a space. In some cases, however, there are homotopies between these homotopies; this expresses the fact that although these homotopies may not be unique, they are coherent in some way. One may also have homotopies between these "higher" homotopies, and so on. Such a binary operation is referred to as a *highly structured multiplication*. To organize and describe all of this homotopical information, one uses a bookkeeping tool called an *operad*. These were introduced in the late 1960s/early 1970s by Boardman and Vogt [9] and May [33], who were motivated by the study of iterated loop spaces. In this language, an object that carries a homotopy associative multiplication, with the higher homotopies being coherent up to level n, is an algebra over an A_n*-operad*. Similarly, an object that carries a homotopy associative and commutative multiplication that is coherent up to level n is an algebra over an E_n*-operad*.[1]

As is usual in algebraic topology, extra structure on a space X is reflected in extra structure on its homology, $H_*(X) = H_*(X; \mathbb{F}_p)$. In particular, an E_n-algebra structure on X produces *power operations*, analogous to Steenrod squares, on $H_*(X)$. As n grows, there are more operations defined on $H_*(X)$. These operations were originally studied at the prime $p = 2$ by Araki and Kudo [4]; the odd primary case was studied by Dyer and Lashof [15]. As a result, these operations are called *(Araki-Kudo-)Dyer-Lashof operations* and the $\mathbb{F}_p$-algebra of all such operations is called the *Dyer-Lashof algebra*. Properties of these operations, and many different applications, were studied extensively by Cohen, Lada, and May [14].

The Dyer-Lashof operations on the homology of a space over an E_∞-operad (i.e., an E_∞*-space*) is a particularly important special case. Such a space is necessarily an *infinite loop space*, which are important objects of study in stable homotopy theory: they are viewed as spaces that underlie, and are in some sense equivalent to, *spectra* [2, 28]. A spectrum is an object that represents a generalized cohomology theory and resembles both a topological space and a derived abelian group. If a given generalized cohomology theory has products, then it is represented by a *ring spectrum*. Many problems in modern stable homotopy theory involve determining the structure of the homotopy, homology, or cohomology of a (ring) spectrum.

One approach to calculating the homology groups, in particular, of a spectrum X is to make use of the *Miller spectral sequence*. This is a first-quadrant spectral sequence that takes the homology of the infinite loop space $\Omega^\infty X$ that is associated to X, together with its structure as an algebra over the Dyer-Lashof algebra, as its input, and converges to the homology of X. It was first constructed by Miller in 1978 [35] and was further studied by Kraines and Lada [21]. To compute the E^2 page of this spectral sequence, one uses the fact that the Dyer-Lashof algebra is an example of a homogeneous *Koszul algebra*, and thus admits a theory of *Koszul resolutions*. These resolutions, and the

[1] The letter "E" here stands for "homotopy everything".

resulting Koszul complexes, were introduced by Priddy [38] and generalize the classical Koszul complex for symmetric algebras, as originally constructed by Koszul [20]. In the classical setting, the Koszul complex is built out of exterior algebras; in the case of the Dyer-Lashof algebra, it is built out of the Steenrod algebra (see Theorem 5.3.1, Proposition 5.3.6).

In this thesis, we use the Miller spectral sequence to study the relationship between an E_∞-ring spectrum R and its associated spectrum of strict units, $\mathrm{sl}_1 R$. In general, the spectra R and $\mathrm{sl}_1 R$ can be quite different. For example, Postnikov towers exist in the category of spectra, and the k-invariants of R may be trivial while the k-invariants of $\mathrm{sl}_1 R$ are not (see Examples 6.0.3 and 6.0.4). Our main result is that if R is an E_∞-ring spectrum over the Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$ (i.e., a *commutative $H\mathbb{F}_p$-algebra*), then the k-invariants of R and of $\mathrm{sl}_1 R$ must agree in a range, which depends on the prime p:

Theorem 6.0.5. *If R is a commutative $H\mathbb{F}_p$-algebra and $n \geq 1$ is any integer, then there is a functorial equivalence of spectra*

$$\tau_{[n,pn-1]} R \simeq \tau_{[n,pn-1]} \mathrm{sl}_1 R.$$

In [29], Mathew and Stojanoska conjecture (and plan to prove in upcoming joint work with Clausen and Heuts) a similar result, namely that if R is an E_∞-ring spectrum such that $(p-1)!$ is invertible in $\pi_0(R)$, then for any $n \geq 1$, there is an equivalence of spectra $\tau_{[n,pn-1]} R \simeq \tau_{[n,pn-1]} \mathrm{gl}_1 R$. Here, $\mathrm{gl}_1 R$ is the spectrum of (non-strict) units associated to R. This spectrum satisfies $\pi_0(\mathrm{gl}_1 R) \cong \pi_0(R)^\times$ and $\pi_k(\mathrm{gl}_1 R) \cong \pi_k(R)$ for all $k \geq 1$; this is in contrast to the spectrum $\mathrm{sl}_1 R$, which has $\pi_0(\mathrm{sl}_1 R) = 0$ and $\pi_k(\mathrm{sl}_1 R) \cong \pi_k(R)$ for all $k \geq 1$. In [29, Theorem 5.1.2], it is shown that this more general result is true when $p = 2$; in other words, if R is any E_∞-ring spectrum, then for any $n \geq 1$, there is a functorial equivalence of spectra $\tau_{[n,2n-1]} R \simeq \tau_{[n,2n-1]} \mathrm{gl}_1 R$. Their motivation for formulating an extension of this result is that these equivalences may be used to compare differentials in certain spectral sequences; in particular, the homotopy fixed point spectral sequences for R^{hG} and for $(\mathrm{pic}(R))^{hG}$, where $\mathrm{pic}(R)$ is a spectrum, called the *Picard spectrum of* R, which satisfies $\tau_{\geq 1}\mathrm{pic}(R) \simeq \Sigma \mathrm{gl}_1 R$. In their paper [29], Mathew and Stojanoska were particularly interested in using such comparisons to determine $\pi_*(\mathrm{pic}(TMF))$, where TMF is the spectrum of topological modular forms [19].

Our approach to proving Theorem 6.0.5 begins with the fact that the infinite loop space associated to the spectrum $\mathrm{sl}_1 R$, which is denoted $\mathrm{SL}_1 R$, is by definition the component of the space $\Omega^\infty R$ corresponding to the class $[1] \in H_0(\Omega^\infty R)$. Because R is an E_∞-ring spectrum, $\Omega^\infty R$ is an example of an *E_∞-ring space*. The homology of such a space has *two* Pontryagin products: one that is induced by the loop concatenation product, and one that is induced by the multiplication on R. The former is thought of as an "additive" Pontryagin product, $\#$, and the latter is thought of as a "multiplicative" Pontryagin product, $\circ$. These two Pontryagin products and their associated Dyer-Lashof operations interact in the ring $H_*(\Omega^\infty R)$, and the structure of the ring $H_*(\mathrm{SL}_1 R)$, which carries the $\circ$ product, is thus influenced by the structure of $H_*(\Omega^\infty R)$ as a ring under both the $\#$ and $\circ$ products. This allows us to compare the Miller spectral sequence for $H_*(R)$ and the Miller spectral sequence for $H_*(\mathrm{sl}_1 R)$.

In general, it is conjectured that the relationship between the k-invariants of an E_∞-ring spectrum R and the k-invariants of $\mathrm{sl}_1 R$ (as well as $\mathrm{gl}_1 R$, the spectrum of units of R) may be described by *topological André-Quillen (TAQ) cohomology.* This is a cohomology theory for E_∞-ring spectra that was constructed in the late 1990s by Basterra [8]. If R is an E_∞-ring spectrum, then its spectrum-level k-invariants lift to TAQ cohomology. These structured k-invariants may be mapped back down to the spectrum-level k-invariants of R and also to the k-invariants of $\mathrm{gl}_1 R$. As a result, the k-invariants of the spectra R and $\mathrm{gl}_1 R$ are related, but the nature of this relationship is not known.

1.1 Layout

Sections 2, 3, and 4 consist of background material. In §2, we describe multiplicative structures that arise in homotopy theory using the language of operads, with particular attention paid to the case of iterated loop spaces. In §3, the Dyer-Lashof operations on the homology of an E_∞-space, and their properties, are introduced (Definition 3.1.1). The construction of the Dyer-Lashof operations in this case is given in §3.1.1. Remark 3.1.2 describes the Dyer-Lashof operations on the homology of an E_n-space for $n < \infty$.

The structure carried by the homology of an E_∞-ring space is described in §3.2. A brief overview of the stable homotopy category and spectra is in §4. Two different models of the stable homotopy category are considered: the *Boardman category of spectra* (§4.1) and the category of $\mathbb{S}$-*modules* (§4.2). We describe ring spectra and the associated spectra of units, sl_1R and gl_1R, in §4.3 and §4.4, respectively.

In §5, we introduce the Miller spectral sequence (Theorem 5.0.1). The (co-)Koszul complex that is used to compute its E^2 page is described in Theorem 5.3.1. A formula for its differential is in Theorem 5.3.8 and Proposition 5.3.9 describes a basis for the Koszul complex. The relationship between this Koszul complex and the Steenrod algebra is described in Theorem 5.3.6. In Examples 5.3.11 and 5.3.12, we compute $H^*(H\mathbb{Z})$ and $H^*(H\mathbb{F}_p)$ (for any prime p) using the Miller spectral sequence, recovering familiar answers. Finally, in §6, we consider the Postnikov towers of an E_∞-ring spectrum R and its spectra of units, gl_1R and sl_1R. In Examples 6.0.3 and 6.0.4, we give explicit examples of $H\mathbb{F}_p$-algebras whose unit spectra have k-invariants that differ from those of the underlying spectrum of the $H\mathbb{F}_p$-algebra. In particular, we show that this phenomenon occurs at odd primes as well as at the prime 2. Finally, we prove Theorem 6.0.5, which ultimately relies on two Lemmas: Lemma 5.3.13, which states that the Miller spectral sequence for an Eilenberg-Mac Lane spectrum must collapse at E^2, and Lemma 6.0.9, which identifies the indecomposable elements in $H_*(\mathrm{SL}_1R)$ and $H_*(\Omega^\infty R)$ in a range via a truncated exponential map. This map is defined in Lemma 6.0.7. The possible connection between these results and topological André-Quillen cohomology is discussed in §6.1.

1.2 Conventions

All spaces are compactly generated weak Hausdorff. All maps between spaces are based and continuous. All modules are of finite type over their ground ring/field. We will use $\otimes$ to denote the tensor product of modules, taken over the ground ring unless otherwise specified. All products are unital unless otherwise specified. Throughout, (co)homology groups will be taken with coefficients in $\mathbb{F}_p$.

1.3 List of symbols

Let p be a prime number.

- $\mathbb{F}_p$, the field with p elements
- Σ_n, the symmetric group on n letters
- $\mathcal{A}_p$, the mod-p Steenrod algebra
- $\mathcal{R}_p$, the mod-p Dyer-Lashof algebra
- $\mathrm{GL}_1 R$, the space of units of an E_∞-ring spectrum R
- $\mathrm{gl}_1 R$, the spectrum of units of an E_∞-ring spectrum R
- $\mathrm{SL}_1 R$, the space of strict units of an E_∞-ring spectrum R
- $\mathrm{sl}_1 R$, the spectrum of strict units of an E_∞-ring spectrum R
- $R^\times$, the group of units of a ring R
- $M^\vee$, the R-linear dual of an R-module M
- $Q(A)$, the module of indecomposable elements of a Hopf algebra A
- $\mathcal{C}(X, Y)$, the set of morphisms between objects X and Y in a category $\mathcal{C}$
- ∂, the boundary map in a chain complex C_*
- δ, the coboundary map in a cochain complex C^*

Chapter 2

Multiplicative Structures

Let X be a set equipped with a unital binary product, or *multiplication*, which we will denote by the symbol $*$. We can think of such a structure on X in "categorical" terms (i.e., in terms of maps) in the following way: the multiplication determines a map $m\colon X \times X \to X$, and for $(x, y) \in X \times X$, we view the element $m(x, y) \in X$ as "$x * y$". If $i\colon \{1\} \hookrightarrow X$ denotes the inclusion of the unit element and $\mathrm{id}\colon X \to X$ denotes the identity map on X, then unitality of $*$ is equivalent to the commutativity of the diagram

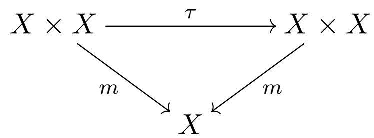
$$\begin{array}{ccccc} \{1\} \times X & \xrightarrow{i \times \mathrm{id}} & X \times X & \xleftarrow{\mathrm{id} \times i} & X \times \{1\} \\ & {\scriptstyle \mathrm{proj}_2} \searrow & \downarrow {\scriptstyle m} & \swarrow {\scriptstyle \mathrm{proj}_1} & \\ & & X & & \end{array}$$

Algebraic properties of such a product—namely, associativity and commutativity—can also be expressed in categorical terms:

- *Associativity* of $*$ is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{m \times \mathrm{id}} & X \times X \\ {\scriptstyle \mathrm{id} \times m} \downarrow & & \downarrow {\scriptstyle m} \\ X \times X & \xrightarrow{m} & X \end{array}$$

- Let $\tau\colon X \times X \to X \times X$ be the "twist map" sending the pair $(x, y) \in X \times X$ to $(y, x) \in X \times X$. *Commutativity* of $*$ is equivalent to the commutativity of the diagram



In classical algebra, a multiplication on a set either has one, or possibly both, of these properties, or it does not. In homotopy theory, it is possible to have a multiplication on a space that satisfies a property lying “in between” strict associativity/commutativity and no associativity/commutativity at all. In the following two sections, we will investigate such multiplicative structures. Our main references for the remainder of this section are [1] and [33].

2.1 Associativity & Loop Spaces

Let (X, x_0) be a based space.

Definition 2.1.1. The *loop space* of X, written ΩX, is the space of maps

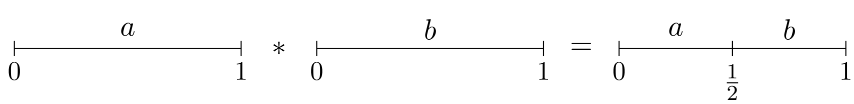
$$\Omega X = \mathrm{Map}((S^1, s_0), (X, x_0)) \cong \mathrm{Map}((I, \partial I), (X, x_0))$$

where S^1 is the circle (with basepoint $s_0 \in S^1$) and $I = [0, 1]$ is the unit interval, with boundary $\partial I = \{0, 1\}$.

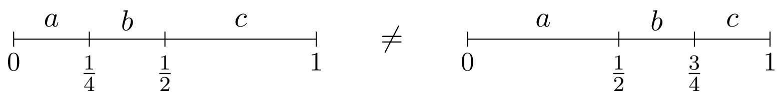
The space ΩX has a multiplication given by *loop concatenation*; i.e., by “following one loop and then another”. More precisely, let $a, b \in \Omega X$ be loops in X based at x_0, both to be thought of as maps $(I, \partial I) \to (X, x_0)$. We define their product, $a * b$, to be the loop

$$a * b = \begin{cases} a(2t) & 0 \leq t \leq \frac{1}{2} \\ b(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

This can be depicted in the following way:



One can check that this product is *not* strictly associative, as indicated below:

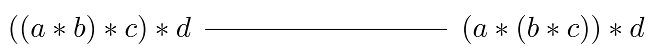


The image above also suggests that even though the loops $(a * b) * c$ and $a * (b * c)$ are not *equal*, they are *homotopic*. The homotopy is given by gradually re-parametrizing the loops a, b, and c. In other words, the product $*$ is *homotopy associative*. Similarly, one can check that the constant loop at x_0 is a unit for $*$ (up to homotopy) and that the loop $-a$, which is given by $(-a)(t) = a(-t)$, is an inverse (up to homotopy) of the element a. In other words, ΩX is a group up to homotopy.

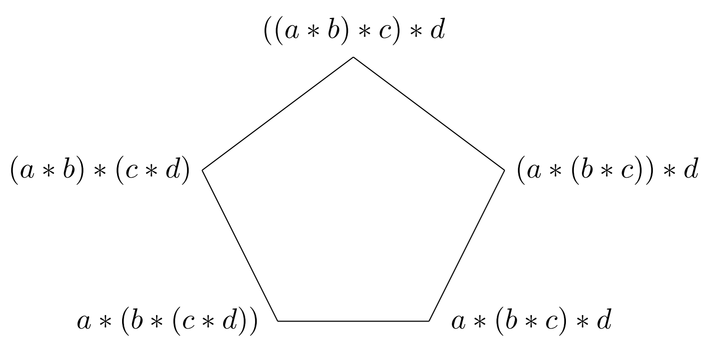
The homotopy associativity of the product $*$ on ΩX is particularly well-behaved, in a sense that is not shared by any other homotopy associative product. To begin describing this, consider the product of four loops $a, b, c, d \in \Omega X$. There are five ways of parenthesizing it:

$$((a * b) * c) * d, \quad (a * (b * c)) * d, \quad a * ((b * c) * d), \quad a * (b * (c * d)), \quad (a * b) * (c * d)$$

Note that these products are pairwise homotopic to one another. If we use a line segment to represent the homotopy interval, so that the fact that $((a*b)*c)*d$ and $(a*(b*c))*d$ are homotopic is denoted by



then we can glue these homotopies together to form a loop, as shown below:



This loop determines a map $S^1 \times (\Omega X)^4 \to \Omega X$. This map extends to a map $D^2 \times (\Omega X)^4 \to \Omega X$ (which we could represent by filling in the above pentagon), and this map gives a "homotopy between the homotopies" [33]. When such a "higher" homotopy exists, the original homotopies are said to be *coherent*.

This continues: if we consider all 14 ways of parenthesizing the product of five loops, then the homotopies between those products, together with the higher homotopies between those homotopies, can be glued together to give a map $S^2 \times (\Omega X)^5 \to \Omega X$. This extends to a map $D^3 \times (\Omega X)^5 \to \Omega X$, which expresses the fact that the homotopies are coherent. In general, if we consider all ways of parenthesizing a product of n loops, then the homotopies between those products assemble into a map $S^{n-3} \times (\Omega X)^n \to \Omega X$, and this map always extends to a map $D^{n-2} \times (\Omega X)^n \to \Omega X$. In other words, the homotopy associativity is *coherent up to level n*.

2.2 Commutativity & Iterated Loop Spaces

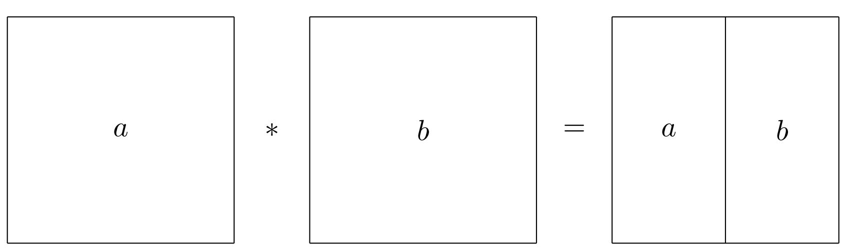
Just as with associativity, it is possible to have multiplicative structures that have properties lying "between" strict commutativity and no commutativity at all. As in the previous section, we begin by investigating such an example, which also arises in homotopy theory.

Definition 2.2.1. An *iterated loop space* is a space of the form

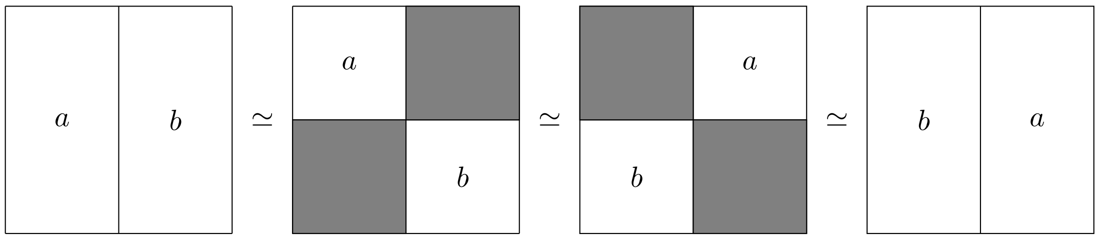
$$\Omega^n X = \mathrm{Map}((S^n, s_0), (X, x_0)) \cong \mathrm{Map}((I^n, \partial I), (X, x_0))$$

for some based space (X, x_0). The space $\Omega^n X$ is called the *n-fold loop space* of X.

First, consider a double loop space, $\Omega^2 X$. This space has a multiplication, which we also call $*$, that is similar to the multiplication on ΩX. Given two elements $a, b \in \Omega^2 X$, both to be thought of as maps $(I^2, \partial I) \to (X, x_0)$, we define their product $a * b$ by "concatenating" a and b. This is described in the image below:



This product is *not* strictly commutative, but the two products $a * b$ and $b * a$ are homotopic, as the following image indicates:



Here, the first homotopy vertically compresses the intervals on which a and b are traversed (the gray squares are mapped to the basepoint x_0 along with the boundary of the square), the second homotopy slides a and b over horizontally, and the third homotopy vertically decompresses the intervals on which a and b are traversed. In other words, the product on $\Omega^2 X$ is homotopy commutative, but not strictly commutative. If we consider iterated loop spaces $\Omega^n X$, then one may define a similar "concatenation product" on its elements, which are maps $(I^n, \partial I^n) \to (X, x_0)$. This product will also turn out to be homotopy associative and commutative, but not strictly so.

2.3 Operads

It is natural to ask whether the homotopy commutativity of the product on iterated loop spaces satisfies higher coherence conditions, just as the homotopy associative product on ΩX does. To begin answering this question, we will need the following bookkeeping tool, which we can use to describe and organize the multiplicative structure on these, and other, topological spaces.

To describe this tool in the most general setting possible, we first recall that a category $\mathcal{C}$ is called a *symmetric monoidal category* if it has a "tensor product" $\otimes$ that is

- symmetric, meaning that there is a natural isomorphism $A \otimes B \cong B \otimes A$ for all $A, B \in \mathcal{C}$,
- associative, meaning that there is a natural isomorphism $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ for all $A, B, C \in \mathcal{C}$, and

- unital, meaning that there is a "unit" $\mathbb{I} \in \mathcal{C}$ such that there are chosen isomorphisms $A \otimes \mathbb{I} \cong \mathbb{I} \otimes A \cong A$ for all $A \in \mathcal{C}$.

Moreover, the above symmetry, associativity, and unitality isomorphisms are subject to axioms that express the fact that they are related in "expected" ways. For example, for all $A, B \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} (A \otimes \mathbb{I}) \otimes B & \longrightarrow & A \otimes (\mathbb{I} \otimes B) \\ & \searrow \quad \swarrow & \\ & A \otimes B & \end{array}$$

is required to commute.

Common examples of symmetric monoidal categories are $(\mathbf{Set}, \times, \{*\})$, the category of sets together with the Cartesian product, whose unit is a singleton set; $(\mathbf{Top}, \times, \{*\})$, the category of topological spaces together with the Cartesian product, whose unit is a one-point space; and $(\mathbf{Vect}_k, \otimes_k, k)$, the category of vector spaces over a field k together with the tensor product over k, whose unit is k itself.

Definition 2.3.1. Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a symmetric monoidal category. An *operad* in $\mathcal{C}$ is a collection $\mathcal{O} = \{\mathcal{O}(k)\}_{k \geq 0}$ of objects of $\mathcal{C}$ such that

- there are morphisms

$$\gamma \colon \mathcal{O}(k) \otimes \Big(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k)\Big) \to \mathcal{O}(j_1 + \cdots + j_k)$$

  for each $k \geq 1$ and $j_1, \ldots, j_k \geq 0$,

- there is a right action of the symmetric group Σ_k on each term $\mathcal{O}(k)$, and

- $\mathcal{O}(0) = \mathbb{I}$.

The morphisms γ are also required to satisfy certain associativity, unitality, and equivariance conditions (see [33, Def 1.1] for details).

Remark 2.3.2. If we drop the requirement that there be an action of Σ_k on $\mathcal{O}(k)$, then we obtain a *non-symmetric operad.*

Definition 2.3.3. A *morphism of operads* $\mathcal{O} \to \mathcal{O}'$ is a collection of morphisms, $\{\mathcal{O}(k) \to \mathcal{O}'(k)\}_{k \geq 0}$, that preserves the structure of an operad.

Definition 2.3.4. Let $\mathcal{O}$ and $\mathcal{O}'$ be operads in the category **Top**. A morphism $\mathcal{O} \to \mathcal{O}'$ is a *homotopy equivalence* if for all $k \geq 0$, the map $\mathcal{O}(k) \to \mathcal{O}'(k)$ is a homotopy equivalence.

The conditions referenced in Definition 2.3.1 are reflections of the fact that an operad is supposed to represent a collection of "k-ary operations", parametrized by the objects $\{\mathcal{O}(k)\}_{k\geq 0}$. The symmetric group action then "permutes the inputs", and the morphisms

$$\gamma \colon \mathcal{O}(k) \otimes \Big(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k)\Big) \to \mathcal{O}(j_1 + \cdots + j_k)$$

describe composition of operations: Given $j_1 + \cdots + j_k$ inputs, they may be divided into k different "groups", with the first group having j_1 inputs, the second having j_2 inputs, etc. Those k groups can then act as inputs to a k-ary operation. The result is a $(j_1 + \cdots + j_k)$-ary operation.

The following are common examples of operads in **Top**:

Example 2.3.5 (Associative Operad)**.** Let $\mathrm{Assoc} = \{\mathrm{Assoc}(k)\}_{k\geq 0}$ be the *non-symmetric* operad defined by setting $\mathrm{Assoc}(k) = *$ for all k. In other words, Assoc is an operad that describes a multiplicative structure in which there is a single k-ary operation for all k. The associativity and unitality conditions in the definition of an operad ensure that this multiplicative structure is both associative and unital.

There is also a *symmetric* version of the associative operad, obtained by setting $\mathrm{Assoc}(k) = \Sigma_k$ for all k. Here, the composition

$$\Sigma_k \times \Big(\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}\Big) \to \Sigma_{j_1+\cdots+j_k}$$

sends a permutation $(\sigma; \tau_1, \ldots, \tau_k)$ to the following permutation: Consider the set of integers $\{1, 2, \ldots, j_1 + \cdots + j_k\}$. Let τ_1 act on the set $\{1, 2, \ldots, j_1\}$, let τ_2 act on the set $\{j_1 + 1, j_1 + 2, \ldots, j_2\}$, etc. The permutation σ may then act on these k "blocks" of letters. Note that this version of the symmetric operad distinguishes between different k-ary operations given by permuting the factors, unlike the non-symmetric version.

Example 2.3.6 (Commutative Operad)**.** Let $\mathrm{Comm} = \{\mathrm{Comm}(k)\}_{k\geq 0}$ be the operad defined by setting $\mathrm{Comm}(k) = *$ for all k, with trivial Σ_k action. As in the previous example, Comm is an operad that describes a multiplicative structure in which there is a single k-ary operation for all k, and this operation is both associative and unital. The added action of Σ_k on Comm(k) allows us to permute the inputs of any k-ary operation to obtain a new operation, but since Comm$(k) = *$, those k-ary operations are identical; in other words, the multiplicative structure described by Comm is also commutative.

Example 2.3.7 (Stasheff Associahedra)**.** Let $\mathcal{A} = \{\mathcal{A}(k)\}_{k\geq 0}$ be the *non-symmetric* operad whose terms are cell complexes that are defined in the following inductive way: Begin by letting $\mathcal{A}(0) = \mathcal{A}(1) = *$. For $k \geq 2$, let $\mathcal{A}(k)$ be a $(k-2)$-dimensional cell complex whose i-cells correspond to ways of inserting $(k-2)-i$ pairs of parentheses into a word on k letters. For example:

- $\mathcal{A}(2) = *$ (there is one way to parenthesize a word on two letters).
- $\mathcal{A}(3)$ is a cell complex with two vertices and one edge between them.
- $\mathcal{A}(4)$ is a cell complex with five vertices and five edges, arranged in a pentagon, and with one 2-cell filling in this pentagon.

The cell complexes $\mathcal{A}(1)$, $\mathcal{A}(2)$, etc. are called the *Stasheff associahedra*.

The composition maps

$$\gamma\colon \mathcal{A}(k) \times \Big(\mathcal{A}(j_1) \times \cdots \times \mathcal{A}(j_k)\Big) \to \mathcal{A}(j_1 + \cdots + j_k)$$

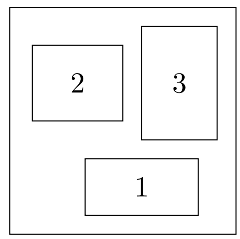
are defined in the following way: Given parenthesizations of a word on k letters and words on $j_1, \ldots, j_k$ letters, we can use them to parenthesize a word on $j_1 + \cdots + j_k$ letters. On the first j_1 letters of this word, use the given parenthesization of the word on j_1 letters. On the next j_2 letters of this word, use the given parenthesization of the word on j_2 letters, etc. Finally, use the parenthesization of the word on k letters to parenthesize these k blocks of letters. For example:

$$\mathcal{A}(3) \;\times\; \Big(\mathcal{A}(2) \;\times\; \mathcal{A}(3) \;\times\; \mathcal{A}(2)\Big) \;\to\; \mathcal{A}(2+3+2) = \mathcal{A}(7)$$

$$(a_1a_2)a_3 \quad (a_1a_2) \quad a_1(a_2a_3) \quad (a_1a_2) \qquad [(a_1a_2)a_3(a_4a_5)](a_6a_7)$$

This extends to a map of cell complexes, making $\mathcal{A}$ a non-symmetric operad [43].

Example 2.3.8 (Little n-Cubes)**.** Let $\mathcal{E}_n = \{\mathcal{E}_n(k)\}_{k\geq 0}$ be the operad defined by letting $\mathcal{E}_n(k)$ be the space of all k-tuples of rectilinear embeddings of the cube I^n into I^n with disjoint interiors and edges parallel to the edges of the "larger" I^n. For example,



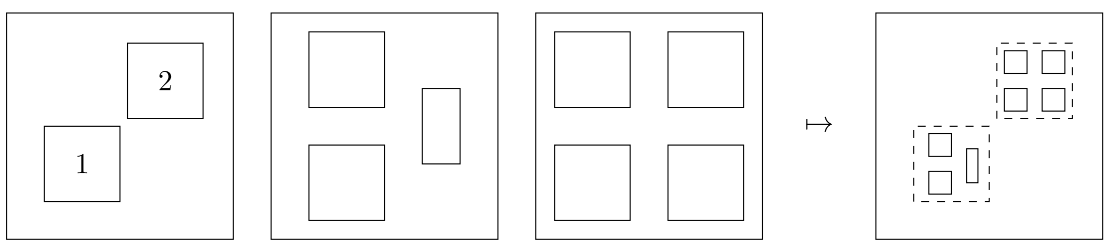
is an element of the space $\mathcal{E}_2(3)$. The symmetric group Σ_k acts (freely) on $\mathcal{E}_n(k)$ by permuting the cubes. The composition maps

$$\gamma\colon \mathcal{E}_n(k) \times \Big(\mathcal{E}_n(j_1) \times \cdots \times \mathcal{E}_n(j_k)\Big) \to \mathcal{E}_n(j_1 + \cdots + j_k)$$

are defined by "substitution". For example, a particular value of the composition map

$$\mathcal{E}_2(2) \times \Big(\mathcal{E}_2(3) \times \mathcal{E}_2(4)\Big) \to \mathcal{E}_2(3+4) = \mathcal{E}_2(7)$$

is pictured below:



We can also let $n = \infty$ and obtain a "little ∞-cubes" operad in the following way: there is a map of operads $\mathcal{E}_n \to \mathcal{E}_{n+1}$ given by "fattening up" (i.e., taking the Cartesian product with the interval $[0,1]$) a collection of little n-cubes inside a larger n-cube to produce a collection of little $(n+1)$-cubes inside a larger $(n+1)$-cube. The colimit of the sequence of maps

$$\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to \cdots$$

is the little ∞-cubes operad, $\mathcal{E}_\infty$.

There are relationships between the operads in the above examples. To see this, first note the following:

- Each of the Stasheff associahedra is contractible [43]. Therefore, there is a homotopy equivalence

$$\mathcal{A}(k) \to * = \mathrm{Assoc}(k)$$

  for each k, giving a homotopy equivalence of non-symmetric operads $\mathcal{A} \to \mathrm{Assoc}$.

- Because the image of $\mathcal{E}_n(k)$ is contractible in $\mathcal{E}_{n+1}(k)$ for all k, we have that $\mathcal{E}_\infty(k)$ is contractible for all k. Therefore, there is a weak equivalence

$$\mathcal{E}_\infty(k) \to * = \mathrm{Comm}(k)$$

  for each k, giving a homotopy equivalence of operads $\mathcal{E}_\infty \to \mathrm{Comm}$.

Definition 2.3.9. An operad $\mathcal{O}$ is an

- A_∞-*operad* if there is a map of operads $\mathcal{O} \to \mathrm{Assoc}$ that is a homotopy equivalence.
- E_n-*operad* if there is a map of operads $\mathcal{O} \to \mathcal{E}_n$ that is a homotopy equivalence.
- E_∞-*operad* if there is a map of operads $\mathcal{O} \to \mathrm{Comm}$ that is a homotopy equivalence.

In particular, $\mathcal{A}$ is an A_∞-operad, and $\mathcal{E}_\infty$ is an E_∞-operad. Moreover, notice that the little 1-cubes operad[1] , $\mathcal{E}_1$, is homotopy equivalent to the symmetric version of the associative operad, Assoc. To see this, contract each 1-cube in $\mathcal{E}_1(k)$ to a point, giving a configuration of k points in an interval. Therefore, $\mathcal{E}_1$ is also an A_∞-operad.

In particular, the family of E_n-operads describe multiplicative structures that possess more and more commutativity as n grows:

[1] This operad is also often called the *little intervals operad.*

Operad	Corresponding Multiplicative Structure
$\mathcal{A} = \mathcal{E}_1$	No commutativity
$\mathcal{E}_2$	Braided commutativity, homotopy commutative
$\mathcal{E}_3$	Homotopy commutative, coherent up to level 3
$\mathcal{E}_4$	Homotopy commutative, coherent up to level 4
$\vdots$	$\vdots$
$\mathcal{E}_\infty$	Homotopy commutative, coherent up to all levels
Comm	Strictly commutative

Remark 2.3.10. There is a similar hierarchy for associativity. Roughly, an A_n*-operad* in $\mathbf{Top}_*$ is an operad that is weakly equivalent to the suboperad of $\mathcal{A}$ generated by the terms $\mathcal{A}(k)$ for $k \leq n$ [43, 33].

Operad	Corresponding Multiplicative Structure
$\mathcal{A}_2$	No associativity
$\mathcal{A}_3$	Homotopy associative
$\mathcal{A}_4$	Homotopy associative, coherent up to level 4
$\vdots$	$\vdots$
$\mathcal{A}_\infty$	Homotopy associative, coherent up to all levels
Assoc	Strictly associative

The operad $\mathcal{A}_1$ fits into this hierarchy in the following way: by forgetting the "top" level of homotopy coherence, we see that, for any n, an A_n-operad is also an A_{n-1}-operad. In particular, an A_2-operad describes a unital multiplication (with no associativity/commutativity properties); by forgetting the multiplication itself, we are left with a choice of unit, which is the data described by an A_1-operad.

If an operad describes a type of multiplicative structure in the abstract, then the following gives a way to describe a multiplicative structure *on* an object of a symmetric monoidal category:

Definition 2.3.11. Given an object X in a symmetric monoidal category $\mathcal{C}$ and an operad $\mathcal{O}$ in $\mathcal{C}$, we say that X is an $\mathcal{O}$*-algebra* if there are maps $\theta_k \colon \mathcal{O}(k) \otimes X^{\otimes k} \to X$ that are compatible with the operadic structure on $\mathcal{O}$.

Remark 2.3.12. If $\mathcal{C} = \mathbf{Top}_*$, we will refer to an $\mathcal{O}$-algebra as an $\mathcal{O}$*-space*. (Similarly, if $\mathcal{C} = \mathbf{Set}$, then $\mathcal{O}$-algebras are $\mathcal{O}$*-sets*, etc.)

This essentially gives a way to parametrize k-ary operations on X which have the multiplicative structure defined by $\mathcal{O}$: if $\mathcal{C} = \mathbf{Top}$, for example, then each point of $\mathcal{O}(k)$ gives a specific map, or k-ary operation, $X^k \to X$ on any $\mathcal{O}$-space X. The relationships between these operations (i.e., the multiplicative structure on X) are encoded in $\mathcal{O}$ itself.

Example 2.3.13. An Assoc-algebra is a strictly associative monoid.

Example 2.3.14. A Comm-algebra is a strictly commutative monoid.

Example 2.3.15 (A_∞-spaces)**.** An A_∞-space is a homotopy associative monoid that is coherent up to all higher homotopies. The map

$$\theta_2\colon \underbrace{\mathcal{A}(2)}_{=*} \times (X \times X) = X \times X \to X$$

picks out a multiplication on X, the map

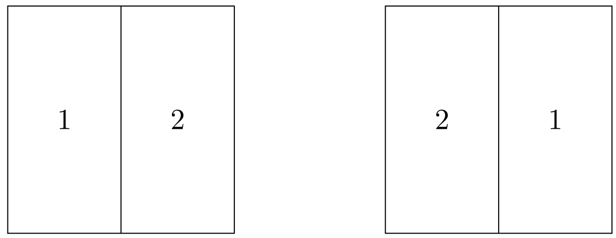
$$\theta_3\colon \underbrace{\mathcal{A}(3)}_{=[0,1]} \times X^{\times 3} \to X$$

gives a homotopy expressing the homotopy associativity of this multiplication, etc. The higher homotopy coherencies are encoded by the action maps θ_k for $k \geq 4$. As described in §2.1, loop spaces ΩX are naturally A_∞-spaces. The $\mathcal{A}$-algebra structure of these spaces expresses the homotopy associativity (and all higher coherencies) of the product on ΩX.

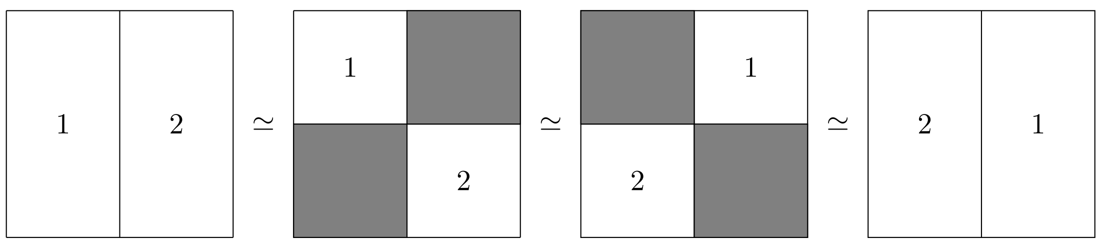
Example 2.3.16 (E_n-spaces)**.** An E_n-space is a homotopy commutative monoid that is coherent up to level n. For example, if X is any E_2-space, then there is a map

$$\theta_2\colon \mathcal{E}_2(2) \times (X \times X) \to X.$$

The diagrams



describe two points of $\mathcal{E}_2(2)$; if one point gives a choice of multiplication on X, call it $m\colon X \times X \to X$, then the other point corresponds to $m \circ \tau\colon X \times X \to X$, the same multiplication, but with the order of the inputs (labeled "1" and "2"), flipped. The (right) action of Σ_2 on $\mathcal{E}_2(2)$ carries one point to the other. A homotopy between these points is pictured below:



In terms of the multiplication on X, this says that $m \simeq m \circ \tau$; in other words, that m is a homotopy commutative product. Note this is precisely how we observed that the product on a double loop space is homotopy commutative in §2.2. This is not a coincidence; an iterated loop space $\Omega^n X$ is naturally an E_n-space [33]. The homotopy commutativity of the product on $\Omega^n X$ becomes more and more coherent as n grows.

Definition 2.3.17. We say that X is an *infinite loop space* if, for every $n \geq 1$, there exists a space Y such that $X \simeq \Omega^n Y$.

Infinite loop spaces naturally have the structure of an algebra over the little ∞-cubes operad, and so they are E_∞-spaces. (In fact, they are the *only* grouplike E_∞-spaces [33].) They are also very closely related to "stable" phenomena in homotopy theory, as we will see in §4.

2.3.1 Operad Pairs

There is also a way in which one may describe multiplicative structures on spaces with *two* products, one thought of as "additive" and the other as "multiplicative". Let $\mathcal{O}$ be an operad whose actions are thought of as "additive", let $\mathcal{G}$ be an operad whose actions are "multiplicative", and let X be a $\mathcal{G}$-space.

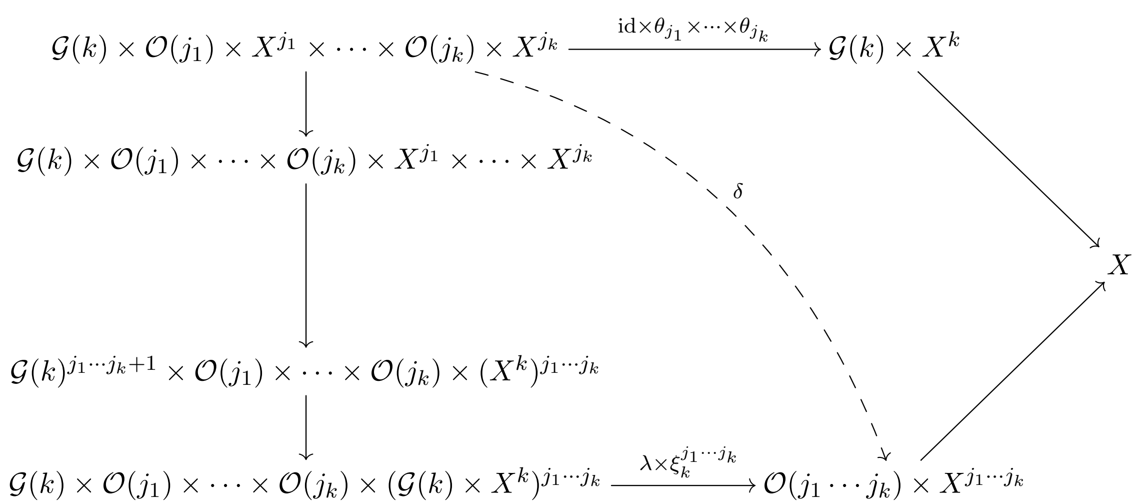
Definition 2.3.18. An *action* of an operad $\mathcal{G}$ on an operad $\mathcal{O}$ is a collection of maps

$$\lambda\colon \mathcal{G}(k) \times \Big(\mathcal{O}(j_1) \times \mathcal{O}(j_2) \times \cdot \times \mathcal{O}(j_k)\Big) \to \mathcal{O}(j_1 j_2 \cdots j_k)$$

satisfying certain unitality, equivariance, and distributivity conditions (see [30, Def. 4.2]). If $\mathcal{G}$ acts on $\mathcal{O}$, we call $(\mathcal{O}, \mathcal{G})$ an *operad pair*.

Definition 2.3.19. A $\mathcal{G}_0$*-space* is a $\mathcal{G}$-space X with a basepoint 1 and a second basepoint 0 (thought of as the "multiplicative" and "additive" units, respectively) such that the structure maps $\theta_k\colon \mathcal{G}(k) \times X^k \to X$ of the action of $\mathcal{G}$ on X satisfy $\theta_k(g, x_1, \ldots, x_k) = 0$ for all $g \in \mathcal{G}(k)$ if $x_i = 0$ for any $1 \leq i \leq r$.

Definition 2.3.20. An $(\mathcal{O}, \mathcal{G})$-space is a $\mathcal{G}_0$-space (with action maps $\xi_k\colon \mathcal{G}(k) \times X^k \to X$) and a $\mathcal{O}$-space (with action maps $\theta_k\colon \mathcal{O}(k) \times X^k \to X$) such that the following distributivity diagram commutes for all $k \geq 1$ and $j_1, \ldots, j_k \geq 1$:



The map δ is defined by $\delta(g, c_1, y_1, \ldots, c_k, y_k) = (\lambda(g, c_1, \ldots, c_k),\ \xi_k(g, y_I))$, where I runs over the set of all sequences $(i_1, \ldots, i_k)$ with $1 \leq i_r \leq j_r$, ordered lexicographically, and where $y_I = (x_{1\cdot i_1}, \ldots, x_{k\cdot i_k}) \in X^k$ if $y_r = (x_{r\cdot 1}, \ldots, x_{r\cdot j_r}) \in X^{j_r}$ [34, pg. 467].

Example 2.3.21. Let $\mathcal{G} = \mathcal{O} = \text{Comm}$, the commutative operad. Recalling that $\text{Comm}(k) = *$ for all k, we see that this operad acts on itself with trivial action maps, λ. An $(\mathcal{O}, \mathcal{G})$-space X is a (strictly) commutative semiring; i.e., a commutative ring without additive inverses. If we write $+$ for the additive product and $\bullet$ for the multiplicative product, then δ expresses the "usual" distributivity law:

$$(x_{1\cdot 1} + x_{1\cdot 2} + \cdots + x_{1\cdot j_1}) \bullet \cdots \bullet (x_{1\cdot 1} + x_{1\cdot 2} + \cdots + x_{1\cdot j_k}) = \sum_I (x_{1\cdot i_1} \bullet \cdots \bullet x_{k\cdot i_k})$$

2.4 Monads

Let $\mathcal{C}$ be a category.

Definition 2.4.1. A *monad* $T\colon \mathcal{C} \to \mathcal{C}$ is a monoid in the category of functors $\mathcal{C} \to \mathcal{C}$ (with multiplication given by comoposition, $\circ$). In other words, a monad is a functor together with two natural transformations, $\mu\colon T \circ T = T^2 \to T$ ("multiplication") and $\eta\colon \text{id}_{\mathcal{C}} \to T$ ("unit"), such that the following diagrams commute:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\circ\mu} & T^2 \\ {\scriptstyle \mu\circ T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad\qquad \begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ {\scriptstyle T\eta}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

Definition 2.4.2. An *algebra*, X, over a monad T is an object $X \in \mathcal{C}$ together with a map $\xi\colon T(X) \to X$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & T(X) \\ & {\scriptstyle \text{id}_{\mathcal{C}}}\searrow & \downarrow{\scriptstyle \xi} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} T(T(X)) & \xrightarrow{\mu} & T(X) \\ {\scriptstyle T\circ\xi}\downarrow & & \downarrow{\scriptstyle \xi} \\ T(X) & \xrightarrow{\xi} & X \end{array}$$

Remark 2.4.3. Dually, we have the notion of a *comonad* $S\colon \mathcal{C} \to \mathcal{C}$, which has a comultiplication $S \to S \circ S = S^2$ and a counit $S \to \text{id}_{\mathcal{C}}$.

If $\mathcal{C}$ is a symmetric monoidal category with coproducts, and the symmetric monoidal structure preserves coproducts in each variable, then a monad gives a convenient way to "package" all of the data of an operad action into a single map. To do this, let $\mathcal{O}$ be a given operad, and let X be an $\mathcal{O}$-algebra in a symmetric monoidal category $\mathcal{C}$.

Associated to this, we define a monad $O\colon \mathcal{C} \to \mathcal{C}$ by letting $O(X)$ be the coequalizer of the diagram

$$\coprod_{k\geq 0} \mathcal{O}(k) \otimes X^{\otimes(k-1)} \underset{\coprod \mathrm{id}\otimes s_i}{\overset{\coprod \sigma_i \otimes \mathrm{id}}{\rightrightarrows}} \coprod_{k\geq 0} \mathcal{O}(k) \otimes X^{\otimes k}$$

To describe the maps in this diagram, we first need some notation: For $0 \leq i \leq k-1$, let $\sigma_i\colon \mathcal{O}(k) \to \mathcal{O}(k-1)$ be the map given by the "composition" map

$$\gamma\colon \mathcal{O}(k) \otimes \Big(\mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1)\Big) \to \mathcal{O}(k-1),$$

where $\mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1)$ contains k factors in total, with the i'th equal to the unit, $\mathcal{O}(0)$. We also let $s_i\colon X^{\otimes(k-1)} \to X^{\otimes k}$ be the map that inserts the unit, $\mathbb{I} \in \mathcal{C}$ into the i'th position:

$$s_i(x_1 \otimes \cdots \otimes x_{k-1}) = x_1 \otimes \cdots \otimes \mathbb{I} \otimes \cdots \otimes x_{k-1}$$

If $\mathcal{C}$ is the category of sets, then this is the same as setting

$$O(X) = \coprod \mathcal{O}(k) \otimes X^{\otimes k} / \sim$$

where the equivalence relation $\sim$ is defined by requiring $(c\sigma_i, x) \sim (c, s_i x)$ and, for any permutation $\sigma \in \Sigma_k$, we must have $(c\sigma, x) \sim (c, \sigma x)$.

The idea behind the above construction is that each of the Σ_k-equivariant action maps $\theta_k\colon \mathcal{O}(k) \otimes X^{\otimes k} \to X$ defining the $\mathcal{O}$-algebra structure on X is a component of the single O-algebra map $O(X) \to X$. As a result, there is a one-to-one correspondence between algebras over the operad $\mathcal{O}$ and algebras over its associated monad, O.

As an application of these ideas, let $\mathbf{E}_n$ be the monad associated to the E_n-operad of little n-cubes. In the early 1970s, J.P. May used this monad to prove the following fundamental result, which shows that iterated loop spaces are the *only* examples of E_n-algebras in $\mathbf{Top}_*$:

Theorem 2.4.4 (The Recognition Principle). *If Y is any E_n-space such that $\pi_0(Y)$ is a group (i.e., Y is "grouplike"), then there exists a space X such that there is an equivalence $Y \simeq \Omega^n X$ of E_n-spaces.*

We refer to such a space X as an *n-fold delooping* of Y.

Proof. The geometric realization of the two-sided monadic bar construction $B_\bullet(\Sigma^n, \mathbf{E}_n, X)$ (Appendix B) is an explicit model for the delooping of X; in other words, there is a weak equivalence $X \simeq \Omega^n |B_\bullet(\Sigma^n, \mathbf{E}_n, X)|$. See [33, §13] for details. □

Chapter 3

Homology & Dyer-Lashof Operations

In this section, homology will be taken with coefficients in $\mathbb{F}_p$, where p can be any prime, unless otherwise specified.

3.1 The Homology of E_∞-Spaces

Recall that the homology of any space $H_*(X)$ is a graded coalgebra, with coproduct

$$\Delta\colon H_*(X) \to H_*(X) \otimes H_*(X)$$

induced by the diagonal map $X \to X \times X$. If X is an E_n-space, then the multiplicative structure on X gives additional structure on the homology of X. As n increases, more and more structure is added to $H_*(X)$. For example, if X is an E_1-space (equivalently, an A_∞-space), then X has a product $m\colon X \times X \to X$. In homology, there is then an induced product

$$m_*\colon H_*(X \times X) \cong H_*(X) \otimes H_*(X) \to H_*(X)$$

called the *Pontryagin product*. In this case, $H_*(X)$ has the structure of a graded *Hopf algebra*, which is an algebraic structure that includes being simultaneously a graded, unital, associative algebra and a counital, coassociative coalgebra. If X is an E_2-space, then the homotopy commutativity of the product on X is reflected in the fact that $H_*(X)$ is a graded *commutative* Hopf algebra.

In the case that X is an E_n-space for $3 \leq n \leq \infty$, the higher homotopy coherences produce *power operations* in $H_*(X)$. These homology operations are analogous to Steenrod operations in cohomology and were first studied in the $p = 2$ case by Araki and Kudo [4] in 1956, who were studying the homology of iterated loop spaces of spheres. In 1962, Dyer and Lashof [15] studied these homology operations at all primes. As a result, they are typically referred to as *Dyer-Lashof operations* or, if $p = 2$, as *(Araki-Kudo)-Dyer-Lashof operations.*

We will only need these operations in the case $n = \infty$, which are described below:

Theorem 3.1.1. *[11, pg 57, Theorem 1.1] Let X be an E_∞-space and let p be a prime. (Modifications in the case that $p = 2$ are in brackets.) Then there are natural operations*

$$Q^r \colon H_k(X;\mathbb{F}_p) \to H_{k+2r(p-1)}(X;\mathbb{F}_p) \qquad (r \geq 0),$$

$[H_k(X;\mathbb{F}_2) \to H_{k+r}(X;\mathbb{F}_2)]$ called Dyer-Lashof operations*. These operations satisfy the following properties:*

- *(Unit): $Q^r(1) = 0$ for $r \geq 1$.*
- *(Additivity): $Q^r(x + y) = Q^r(x) + Q^r(y)$.*
- *(Cartan Formula): $Q^r(xy) = \sum_{i+j=r} Q^i(x)Q^j(y)$.*
- *(p'th Power): $Q^r(x) = x^p$ if $2r = |x|$ [if $r = |x|$].*
- *(Instability): $Q^r(x) = 0$ if $2r < |x|$ [if $r < |x|$].*
- *(Adem Relations): If $p \geq 2$ and $r > ps$, then*

$$Q^rQ^s = \sum_{j=\lceil r/p \rceil}^{r-(p-1)s-1} (-1)^{r+j} \binom{(p-1)(j-s)-1}{pj-r} Q^{r+s-j}Q^j$$

If $p > 2$, $r \geq ps$, and β is the mod-p Bockstein, then

$$Q^r \beta Q^s = \sum_{j=\lceil r/p \rceil}^{r-(p-1)s} (-1)^{r+j} \binom{(p-1)(j-s)}{pj-r} \beta Q^{r+s-j} Q^j$$
$$+ \sum_{j=\lceil (r+1)/p \rceil}^{r-(p-1)s} (-1)^{r+j-1} \binom{(p-1)(j-s)-1}{pj-r-1} Q^{r+s-j} \beta Q^j$$

Remark 3.1.2. In the case that X is an E_n-space, where $n < \infty$, the Dyer-Lashof operation $Q^r(x)$ is now only defined if $2r - |x| \leq n-1$ (or, if $p = 2$, these are defined in the range $r - |x| \leq n-1$). In addition, the "top" operation $Q^{(|x|+n-1)/2}$ is no longer additive. The operations Q^r also satisfy an *extension property* in this case: if the E_n-structure on X happens to extend to an E_{n+1}-structure, then the operations on $H_*(X)$ produced by the E_n-structure will agree with the corresponding operations on $H_*(X)$ produced by the E_{n+1}-structure.

Definition 3.1.3. We have the following terminology, defined differently depending on the prime:

- (if $p = 2$): Given a sequence $I = (i_1, i_2, \ldots, i_n)$ of positive integers, we will use the notation Q^I for the operation $Q^{i_1} Q^{i_2} \cdots Q^{i_n}$. We define

  - the *degree of* I, written $|I|$, as $i_1 + i_2 + \cdots + i_n$,
  - the *length of* I, written $\ell(I)$, as n, and
  - the *excess of* I, written $e(I)$, as

  $$i_n - [(2i_2 - i_1) + \cdots + (2i_n - i_{n-1})] = i_1 - (i_2 + \cdots + i_n).$$

  We say that I is an *admissible sequence* if $i_j \leq 2i_{j+1}$ for all $1 \leq j \leq n-1$. If I is the empty sequence, then I is admissible and satisfies $|I| = 0$, $\ell(I) = 0$, and $e(I) = \infty$; it corresponds to the identity homology operation.

- (if $p \geq 3$): If we are instead given a sequence $J = (\epsilon_1, i_1, \epsilon_2, i_2, \ldots, \epsilon_n, i_n)$, where each ϵ_j is 0 or 1 and $i_j \geq \epsilon_j$, then Q^J is the operation $\beta^{\epsilon_1} Q^{i_1} \beta^{\epsilon_2} Q^{i_2} \cdots \beta^{\epsilon_n} Q^{i_n}$. In this case, we define

– the *degree of* J, written $|J|$, as

$$[2i_1(p-1)-\epsilon_1]+\cdots+[2i_n(p-1)-\epsilon_n]$$

– the *length of* J, written $\ell(J)$, as n,

– the *excess of* J, written $e(J)$, as

$$i_1-\epsilon_1-2(p-1)[i_2+i_3+\cdots+i_n].$$

In this case, we say that J is an *admissible sequence* if $i_j \leq pi_{j+1}-\epsilon_{j+1}$ for all $1 \leq j \leq n-1$. We also let $b(J)=\epsilon_1$.

Note that if Q^I is admissible, then we are unable to apply Adem relations to decompose Q^I into a sum of other Dyer-Lashof operations. The excess of I is a measurement of how much a given sequence "exceeds" the requirement to be admissible. It is also the lowest possible degree of an element x such that $Q^I(x)$ does not vanish due to the instability relations.

Definition 3.1.4. Let $\mathcal{R}_p(-\infty)$ denote the unital, associative graded $\mathbb{F}_p$-algebra generated by

- (if $p=2$): the symbols Q^r (for $r \geq 0$) of degree r
- (if $p \geq 3$): the symbols Q^r (for $r \geq 0$) and βQ^r (for $r \geq 1$) of degree $2r(p-1)$ and $2r(p-1)-1$, respectively,

and with relations generated by the Adem relations (and if $p \geq 3$, by the relations obtained by applying β to the Adem relations, together with the relation $\beta^2=0$.) For $n \geq 0$, let $B(n)$ be the two-sided ideal generated by the set $\{Q^I \mid e(I) < n\}$ and let $\mathcal{R}_p(n)=\mathcal{R}_p(-\infty)/B(n)$. Then $\mathcal{R}_p=\mathcal{R}_p(0)$ is called the *mod-p Dyer-Lashof algebra*.

Remark 3.1.5. The Dyer-Lashof algebra does *not* actually contain β as an element; moreover, the element βQ^r is not to be viewed as a composite.

Definition 3.1.6. An $\mathcal{R}_p$-algebra A is a unital, associative, graded algebra with a left action of $\mathcal{R}_p$. An $\mathcal{R}_p$-algebra is said to be *allowable* if $Q^{|x|}(x)=x^2$ for all $x \in A$. More generally, A is said to be *n-allowable* if, for any $x \in A$, we have $Q^i(x)=0$ for $i < |x|+n$

(so that "allowable" is the same as "0-allowable"). Note that if A is any n-allowable, nonnegatively graded $\mathcal{R}_p$-algebra, then the action of $\mathcal{R}_p(-\infty)$ on A factors through an $\mathcal{R}_p(n)$-action.[1]

At all primes, the set $\{Q^I \mid I \text{ is admissible}\}$ is a basis for $\mathcal{R}_p$. Note that we also do not require $Q^0 = 1$; as a result, the elements of $\mathcal{R}_2$, for example, look like:

- (deg 0): 1, Q^0, Q^0Q^0, $\ldots$
- (deg 1): Q^1
- (deg 2): Q^2, Q^1Q^1
- (deg 3): Q^3, Q^2Q^1
- (deg 4): Q^4, Q^2Q^2, $Q^2Q^1Q^1$

$$\vdots$$

Example 3.1.7. Starting with any connected, based space X, we may produce an infinite loop space by forming the colimit of the sequence of maps

$$X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \Omega^3\Sigma^3 X \to \cdots$$

which we will denote by $\Omega^\infty\Sigma^\infty X$, the "free infinite loop space on X".[2] Here, the first map $X \to \Omega\Sigma X$ is the unit of the suspension-loop adjunction $h\mathbf{Top}(\Sigma X, Y) \cong h\mathbf{Top}(X, \Omega Y)$. For $n \geq 1$, we get an inclusion $\Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1} X$ defined by taking a map $S^n \to \Sigma^n X$ (i.e., an element of $\Omega^n\Sigma^n X$) and sending it to its suspension $\Sigma S^n = S^{n+1} \to \Sigma^{n+1} X$, which is an element of $\Omega^{n+1}\Sigma^{n+1} X$. In [14, pg. 40, Theorem 4.2], there is a computation of the homology of the free infinite loop space on X, due to work of J.P. May and F. Cohen: the conclusion is that $H_*(\Omega^\infty\Sigma^\infty X)$ is the free graded commutative Hopf algebra over $\mathbb{F}_p$, with an action of both the mod-p Dyer-Lashof algebra and the mod-p dual Steenrod algebra (Appendix A), on the elements $Q^I(x)$ and $\beta Q^I(x)$, where x ranges over an additive basis for $H_*(X)$ and I is an admissible sequence. (If X is not connected, then inverses for the elements in degree 0 need to be adjoined.)

---

[1] The definition of n-allowable also makes sense for any $\mathcal{R}_p$-module M.

[2] This is also called $Q(X)$ in some sources.

3.1.1 Construction of the Dyer-Lashof Operations in the Case $n = \infty$

To see exactly how an E_∞-structure on X leads to the existence of Dyer-Lashof operations on $H_*(X)$, we recall the construction of these operations, closely following [14]: Suppose we have an E_∞-space X. This provides us with Σ_k-equivariant action maps $\mathcal{E}_\infty(k) \times X^k \to X$ for all $k \geq 0$. In particular, if p is any prime, then there is a Σ_p-equivariant map $\mathcal{E}_\infty(p) \times X^p \to X$. We may also write this as a map $E\Sigma_p \times X^p \to X$, since $\mathcal{E}_\infty(p)$ is a contractible space with a free action of the group Σ_p and is therefore a model for $E\Sigma_p$. The Borel construction[3] $E\Sigma_p \times_{\Sigma_p} X^p$ is called the *p'th extended power of* X, written D_pX.

Consider the homology of the space D_pX, which is the homology of the chain complex $C_*(E\Sigma_p) \otimes_{\Sigma_p} C_*(X)^{\otimes p}$. The augmented chain complex $C_*(E\Sigma_p) \to \mathbb{F}_p$ is a Σ_p-free resolution of $\mathbb{F}_p$. Since the cyclic group C_p is a subgroup of Σ_p, there is a map $j\colon W_* \to C_*(E\Sigma_p)$ of chain complexes over $\mathbb{F}_p$ with an action of C_p, where (W_*, ∂) is the standard C_p-free resolution of $\mathbb{F}_p$. This is a chain complex with $W_i = \mathbb{F}_p[C_p]\{e_i\}$ for each $i \geq 0$ and with boundary map defined as follows: let α be a generator of the group C_p, and let $N = 1+\alpha+\alpha^2+\cdots+\alpha^{p-1}$ and $T = \alpha - 1$ be elements of $\mathbb{F}_p[C_p]$. The boundary map ∂ is then determined by the formulas $\partial(e_{2i}) = Ne_{2i-1}$ and $\partial(e_{2i+1}) = Te_{2i}$ [32].

Consider the composite

$$H_*(W_* \otimes_{\Sigma_p} C_*(X)^{\otimes p}) \xrightarrow{j\otimes\eta} H_*(C_*(E\Sigma_p) \otimes_{\Sigma_p} C_*(X^p)) \xrightarrow{\sim} H_*(E\Sigma_p \times_{\Sigma_p} X^p) \to H_*(X),$$

where η is the shuffle map $C_*(X)^{\otimes p} \to C_*(X^p)$ (which is chain homotopy inverse to the Alexander-Whitney map $C_*(X^p) \to C_*(X)^{\otimes p}$), and the last map is induced by the action of the operad $\mathcal{E}_\infty$ on X. If $x \in C_*(X)$, then we may consider the element

$$e_i \otimes x^{\otimes p} \in W_* \otimes C_*(X)^{\otimes p}.$$

The image in $H_*(X)$ of the homology class of $e_i \otimes x^{\otimes p}$ is denoted $Q_i(x)$. Using these elements, we define the Dyer-Lashof operations as follows [14]:

- If $p = 2$: $Q^r(x) = \begin{cases} 0 & \text{if } r < |x| \\ Q^r(x) = Q_{r-|x|}(x) & \text{if } r \geq |x| \end{cases}$

[3] The group Σ_p acts diagonally on the product $E\Sigma_p \times X^p$; the Borel construction is the quotient of this product by this action. It is also known as the *homotopy quotient* of X^p by the action of Σ_p.

- If $p \geq 3$: $Q^r(x) = \begin{cases} 0 & \text{if } 2r < |x| \\ cQ_{(2r-|x|)(p-1)}(x) & \text{if } 2r \geq |x| \end{cases}$

  Here, the constant c is given by

  $$c = (-1)^{r+[|x|(|x|-1)(p-1)]/4}\left[\left(\frac{p-1}{2}\right)!\right]^{|x|}.$$

  In addition, we have $\beta Q^r(x) = cQ_{(2r-|x|)(p-1)-1}(x)$.

From these definitions, one may prove all of the properties of the Dyer-Lashof operations listed in Theorem 3.1.1, as in [14].

3.2 The Homology of E_∞-Ring Spaces

As mentioned in §2.4, we may use operad pairs to describe "ring" structures on spaces. If each product on such a space is described by the action of an E_∞-operad, then, given the results of the previous section, it is natural to expect that its homology will have *two* different Pontryagin products and *two* different corresponding sets of Dyer-Lashof operations. One set will be thought of as "additive" and the other as "multiplicative". In this section, we will describe this type of structure on homology, using [14] as our main reference.

Definition 3.2.1. A space X is an E_∞*-ring space* if it is an $(\mathcal{O}, \mathcal{G})$-space for some pair of E_∞-operads $\mathcal{O}$ and $\mathcal{G}$.

Example 3.2.2. If Y is an E_∞-space, then $X = \Omega^\infty\Sigma^\infty Y$ is an E_∞-ring space. Here, we think of the "additive" E_∞-structure as coming from the fact that X is an infinite loop space, and we think of the "multiplicative" E_∞-structure as coming from the fact that Y itself is an E_∞-space.

Example 3.2.3. Let BO be the classifying space (see Example B.1.5) of the infinite orthogonal group, $O = \varinjlim O(n)$, where $O(n)$ is the group of $n \times n$ orthogonal matrices. Then space $\mathbb{Z} \times BO$ is the classifying space for real vector bundles and is an E_∞-ring space; the "additive" E_∞-structure comes from the direct sum of vector bundles and the "multiplicative" E_∞-structure comes from the tensor product of vector bundles.

Given an E_∞-ring space X, we let

$$\#\colon H_*(X)\otimes H_*(X)\to H_*(X)\qquad\text{and}\qquad \circ\colon H_*(X)\otimes H_*(X)\to H_*(X)$$

denote the two Pontryagin products on its homology. The first is the "additive" product, and the second is the "multiplicative" product. We also let Q^r denote the corresponding additive Dyer-Lashof operations, and we let $\widetilde{Q}^r$ denote the multiplicative Dyer-Lashof operations. We will use the notation $\#^n$ and $\circ^n$ for the n-fold $\#$ and $\circ$ products $H_*(X)^{\otimes(n+1)}\to H_*(X)$, respectively. Thus the map $\#^n$, for example, sends the class $x_1\otimes\cdots\otimes x_{n+1}\in H_*(X)^{\otimes(n+1)}$ to $x_1\#\cdots\#x_{n+1}\in H_*(X)$.

Remark 3.2.4. If X is an E_∞-ring space, then $H_*(X)$, together with the $\#$ and $\circ$ products (forgetting the Dyer-Lashof operations Q^r and $\widetilde{Q}^r$), forms what Ravenel and Wilson call a graded *Hopf ring* [40, §1]. Miller observed [40, pg. 244] that a graded Hopf ring is equivalently a ring object in the category of cocommutative graded coalgebras, **CCGrCoalg**. In other words, if R is a Hopf ring, then the functor $\mathrm{Hom}_{\mathbf{CCGrCoalg}}(-,R)$ takes values in the category of (commutative) rings.

To begin describing the structure of $H_*(X)$ for an E_∞-ring space X, we first consider how these products/operations act on the components of X. For an element $i\in\pi_0(X)$, we write X_i for the corresponding component of X and we write $[i]\in H_0(X)$ for the corresponding homology class. The product $\#$ takes $X_i\times X_j$ to X_{i+j}, so that $[i]\#[j]=[i+j]$. The Dyer-Lashof operation Q^n takes X_i to X_{pi}, and so $Q^0[i]=[pi]$. On the other hand, the product $\circ$ takes $X_i\times X_j$ to X_{ij}, so that $[i]\circ[j]=[ij]$. The Dyer-Lashof operation $\widetilde{Q}^n$ takes X_i to X_{i^p}, and so $\widetilde{Q}^0[i]=[i^p]$.

Next, as described in [14], there are a number of identities that encode the interaction between these two sets of products/operations. We list two particularly important identities below, where the following notation is used: for an element $x\in H_*(X)$, we write its coproduct as $\Delta(x)=\sum x'\otimes x''$; similarly, we write the value of the iterated coproduct[4] $\Delta^{n-1}\colon H_*(X)\to H_*(X)^{\otimes n}$ on x as $\sum x^{(1)}\otimes\cdots\otimes x^{(n)}$. We also let $\epsilon\colon H_*(X)\to\mathbb{F}_p$ be the augmentation map. (Note that $\epsilon[i]=1$ for any $[i]\in H_0(X)$.)

[4] Since the coproduct on $H_*(X)$ is coassociative, we have

$$\sum\Delta(x')\otimes x''=\sum x'\otimes\Delta(x'').$$

We write the common value as $\Delta^2(x)=\sum x^{(1)}\otimes x^{(2)}\otimes x^{(3)}$.

Theorem 3.2.5. *We have the following formulas, which hold at any prime p:*

- (Distributive Law): $(x\#y)\circ z = \sum(-1)^{|y||z'|}(x\circ z')\#(y\circ z'')$
- (Mixed Cartan Formula):

$$\widetilde{Q}^n(x\#y) = \sum_{n_0+\cdots+n_p=n}\sum \widetilde{Q}_0^{n_0}(x^{(0)}\otimes y^{(0)})\#\cdots\#\widetilde{Q}_p^{n_p}(x^{(p)}\otimes y^{(p)})$$

where $x\otimes y\mapsto \sum x^{(0)}\otimes y^{(0)}\otimes\cdots\otimes x^{(p)}\otimes y^{(p)}$ *under the iterated coproduct of* $H_*(X)\otimes H_*(X)$ *and we have:*

$$\widetilde{Q}_0^n(x\otimes y) = [\epsilon(y)]\widetilde{Q}^n(x)$$
$$\widetilde{Q}_i^n(x\otimes y) = \tfrac{1}{p}\tbinom{p}{i}Q^n\left(\sum\sum x^{(1)}\circ\cdots\circ x^{(p-i)}\circ y^{(1)}\circ\cdots\circ y^{(i)}\right) \text{ for } 1\le i\le p-1.$$
$$\widetilde{Q}_p^n(x\otimes y) = [\epsilon(x)]\widetilde{Q}^n(y)$$

In particular, at the prime $p=2$*, these formulas reduce to:*

- (Distributive Law): $(x\#y)\circ z = \sum(x\circ z')\#(y\circ z'')$
- (Mixed Cartan Formula): $\widetilde{Q}^n(x\#y) = \sum_{i+j+k=n}\sum \widetilde{Q}^i(x')\#Q^j(x''\circ y')\#\widetilde{Q}^k(y'')$

Proof. Both formulas essentially follow from considering the distributivity diagram in the definition of an operad pair (Definition 2.3.20). For details, see [14]. □

Remark 3.2.6. In [14, Theorem 3.3, pg. 96], there are also formulas for the *mixed Adem relations*, which allow for the computation of $\widetilde{Q}^r\beta^\epsilon Q^s(x)$ in terms of the additive product, #, and its Dyer-Lashof operations. These formulas for a general prime p are complicated and are omitted here; for $p=2$, we have

$$\widetilde{Q}^rQ^s(x) = \sum_{i+j+k+\ell=r+s}\sum\binom{r-i-2\ell-1}{j+s-i-\ell}Q^i\widetilde{Q}^j(x')\#Q^k\widetilde{Q}^\ell(x'')$$

The above formula is originally due to Tsuchiya. A proof remained unpublished until N. Kuhn [24, Prop. 1.6] provided further details on its derivation.

The following formulas will also be useful in §6:

- $[0] \circ x = \epsilon(x)[0]$.
- $([i]\#x) \circ ([j]\#y) = \sum (-1)^{|y'|\cdot|x''|}(x' \circ y')\#(x'' \circ [j])\#(y'' \circ [i])\#[ij]$
- $Q^r[0] = 0$ for all p and all $r \geq 0$.
- $\widetilde{Q}^r[0] = 0$ and $\widetilde{Q}^r[1] = 0$ for all p and all $r \geq 1$.

Chapter 4

The Stable Homotopy Category

In [2], Adams refers to an invariant in homotopy theory as *stable* if it is essentially independent of suspension. To make this idea more precise, let $F\colon h\mathbf{Top}_* \to \mathbf{GrAb}$ be a functor from the homotopy category of spaces to the category of graded abelian groups. For a graded abelian group A_*, we will temporarily abuse notation and let $(\Sigma A)_*$ be the *shift suspension* of A_*, the graded abelian group whose $(k+1)$'st graded piece is given by $(\Sigma A)_{k+1} = A_k$. We say that F is a *stable invariant* if, for any based space (X, x_0), we have a chosen natural isomorphism $F(\Sigma X) \cong \Sigma F(X)$. Roughly, F is a stable invariant if it "sees" ΣX as a shifted copy of X.

For example, reduced homology is a stable invariant: for any space X and integer $k \geq 0$, there is an isomorphism $\tilde{H}_k(X) \cong \tilde{H}_{k+1}(\Sigma X)$. In contrast to this, homotopy groups are not stable invariants; for example, $\pi_0(S^0) \cong \mathbb{Z}/2$ while $\pi_1(S^1) \cong \mathbb{Z}$. However, the following theorem implies that homotopy groups *are* stable invariants when restricted to a certain range. We will refer to this range as the *stable range* because of this. Before the statement of the theorem, recall that a space X is said to be *s-connected* if $\pi_k(X)$ is trivial for $0 \leq k \leq s$ and that the *dimension* of a CW-complex Y, written $\dim(Y)$, is the largest value of k (which could be $+\infty$) such that $H_k(Y) \neq 0$.

Theorem 4.0.1 (Freudenthal Suspension Theorem)**.** *If X is s-connected and $\dim(Y) \leq 2s$, then the map $\Sigma\colon h\mathbf{Top}_*(Y, X) \to h\mathbf{Top}_*(\Sigma Y, \Sigma X)$ induced by suspension is an isomorphism.*

To see the connection to homotopy groups, fix an integer $k \geq 0$, let $Y = S^k$, and let

X be an s-connected space. Consider the direct system

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X) \xrightarrow{\Sigma} \pi_{k+2}(\Sigma^2 X) \xrightarrow{\Sigma} \cdots \qquad (*)$$

Since X is s-connected, $\Sigma^n X$ is at least $(s+n)$-connected; as a consequence of this and Theorem 4.0.1, the maps $\pi_{k+n}(\Sigma^n X) \to \pi_{k+n+1}(\Sigma^{n+1} X)$ in $(*)$ are all isomorphisms once n is large enough to make the inequality $k + n \leq 2(s+n)$ hold. In other words, they are all isomorphisms once $n \geq k - 2s$.

Definition 4.0.2. The colimit of $(*)$ is the k'th *stable homotopy group* of the space X, denoted $\pi_k^{\mathrm{st}}(X)$. By the above, $\pi_k^{\mathrm{st}}(X)$ is isomorphic to $\pi_{k+n}(\Sigma^n X)$ for any $n \geq k - 2s$.

Note that $\pi_k^{\mathrm{st}}(X)$ is always an abelian group and that stable homotopy groups are, in fact, stable invariants: $\pi_k^{\mathrm{st}}(X) \cong \pi_{k+1}^{\mathrm{st}}(\Sigma X)$.

In addition to being able to interpret both homotopy and reduced homology groups as stable invariants, another important advantage of working in the stable range is that cofiber and fiber sequences coincide. This is implied by the following corollary of Theorem 4.0.1.

Corollary 4.0.3. *Suppose that* $X \xrightarrow{i} Y \xrightarrow{j} Z$ *is a cofiber sequence and that* U *satisfies* $\dim(U) \leq 2c$, *where* c *is the minimum of the connectivities of* X, Y, *and* Z. *Then the sequence*

$$h\mathbf{Top}_*(U, X) \xrightarrow{i_*} h\mathbf{Top}_*(U, Y) \xrightarrow{j_*} h\mathbf{Top}_*(U, Z)$$

(where the maps given by postcomposition with i *and* j*) is exact. In other words, the image of* i_* *is equal to the preimage under* j_* *of the map* $U \to Z$ *sending* U *to the basepoint of* Z.

Proof. See [28, pg. 8-10]. □

The idea of "working in the stable range" plays a central role in modern algebraic topology and is made precise by working in the *stable homotopy category* instead of the ordinary homotopy category of spaces. This is a category — whose objects are called *spectra* — that is constructed as the homotopy category of some model category $\mathcal{C}$ obtained by "stabilizing" the category $\mathbf{Top}_*$. Before discussing the problem of constructing $\mathcal{C}$, we list some properties that the stable homotopy category, $h\mathcal{C}$, ought to have:

1. For each space X, there should be a spectrum $\Sigma^\infty X$ in $h\mathcal{C}$ (to be thought of as a "free stabilization" of X) and our definition of homotopy in $\mathcal{C}$ should imply that $\pi_k(\Sigma^\infty X) \cong \pi_k^{\mathrm{st}}(X)$. Moreover, the association $X \mapsto \Sigma^\infty X$ should be functorial and admit a right adjoint $\Omega^\infty \colon h\mathcal{C} \to h\mathbf{Top}_*$.

2. The suspension functor Σ should be an equivalence in $h\mathcal{C}$ and be defined in a way that ensures that $\Sigma^\infty \circ \Sigma$ and $\Sigma \circ \Sigma^\infty$ are naturally isomorphic.

3. The set $h\mathcal{C}(E, F)$ of homotopy classes of maps between any two spectra $E, F \in h\mathcal{C}$ should have a natural abelian group structure. Composition of functions should also be bilinear. Intuitively, this property implies that spectra should be thought of as homotopy-theoretic versions of abelian groups.

4. Continuing with the theme of the previous property and the result of Corollary 4.0.3, the category $h\mathcal{C}$ should have properties similar to that of the derived category of abelian groups. In particular, like the category of abelian groups, $\mathcal{C}$ should have

   - a biproduct $\vee$, the *wedge sum* (analogous to the direct sum $\oplus$);
   - a zero object $* = \Sigma^\infty(\mathrm{pt})$ (analogous to the abelian group 0); and
   - a symmetric monoidal product[1] $\wedge$, the *smash product*, with unit $\mathbb{S} := \Sigma^\infty S^0$ (analogous to the tensor product $\otimes$ with unit $\mathbb{Z}$).

   that all pass to the homotopy category of $\mathcal{C}$. In addition to this, $h\mathcal{C}$ should be a triangulated category with the shift operation given by Σ and distinguished triangles given by fiber/cofiber sequences

$$E \to F \to G \to \Sigma E \to \cdots$$

5. Within $h\mathcal{C}$, it should be possible to define mapping spectra $F(E, E')$ as well as homotopy fibers, $F(f)$, and homotopy cofibers, $C(f)$, of maps of spectra $f \colon E \to F$ that fit into fiber/cofiber sequences of the form

$$F(f) \to E \xrightarrow{f} F \to \Sigma F(f) \to \cdots \qquad \text{and} \qquad E \xrightarrow{f} F \to C(f) \to \Sigma E \to \cdots,$$

   respectively. Moreover, the functor $F(E, -)$ should be right adjoint to $(-) \wedge E$.

[1] In other words, a unital, associative, and commutative product.

It turns out that it is indeed possible to construct a stable homotopy category with the above properties. A 1991 theorem of Lewis' [27], however, states that it is impossible to construct the "point-set" level category $\mathcal{C}$ so that it has all of the properties that one might hope for. Fortunately, there are several possible choices of $\mathcal{C}$, each having different properties, but determining the *same* stable homotopy category. J.M. Boardman constructed the first relatively modern version of this category, called the *Boardman category of spectra*, which we will denote by **Sp**, in his 1964 PhD thesis. The objects of this category are easily defined and it enjoys several nice properties, but, as we will see, it has a flaw that makes it unsuitable for the applications we have in mind. We discuss this category in §3.1 mainly for the purpose of developing intuition. In §3.2 we will discuss a suitable choice for $\mathcal{C}$, the category of $\mathbb{S}$*-modules* of [16].

4.1 The Boardman Category of Spectra

The objects and morphisms in the category **Sp** are defined in [2, Pt. III] by Adams as follows:

Definition 4.1.1. A *spectrum* is a sequence $E = \{E_n\}_{n\in\mathbb{Z}}$ of based CW-complexes together with CW-inclusions $\sigma_n \colon \Sigma E_n \to E_{n+1}$ (alternatively, CW-inclusions $\omega_n \colon E_n \to \Omega E_{n+1}$) for all n.

Example 4.1.2. For any space X, there is the *suspension spectrum* $\Sigma^\infty X$ with n'th term

$$(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X & \text{if } n \geq 0 \\ * & \text{if } n < 0 \end{cases}$$

and the obvious structure maps. A particularly important suspension spectrum is the *sphere spectrum*, $\mathbb{S} = \Sigma^\infty S^0$.

Example 4.1.3. For any abelian group G, there is the *Eilenberg-Mac Lane spectrum* HG with n'th term

$$(HG)_n = \begin{cases} K(G,n) & \text{if } n \geq 0 \\ * & \text{if } n < 0 \end{cases}$$

The structure maps for $n \geq 0$ given by weak equivalences $K(G, n) \to \Omega K(G, n + 1)$.

Example 4.1.4. Given any spectrum E and space X, there is a spectrum $E \wedge X$ whose n'th term is $E_n \wedge X$. The structure maps of $E \wedge X$ are obtained by applying the functor $(-) \wedge X$ to the structure maps of E.

With Example 4.1.4 in mind, note that if X is a space, then the spectrum $(\Sigma^\infty X) \wedge S^1$, which we would like to call the suspension of $\Sigma^\infty X$, has n'th term

$$(\Sigma^n X) \wedge S^1 \cong \Sigma^{n+1} X = (\Sigma^\infty X)_{n+1}.$$

Taking this as inspiration, we define the suspension of a general spectrum as follows:

Definition 4.1.5. Let k be an integer. The k'th *suspension* of a spectrum E, denoted $\Sigma^k E$, is the spectrum whose n'th term is given by $(\Sigma^k E)_n = E_{n+k}$.

The morphisms in **Sp** may be thought of in terms of the following definition.

Definition 4.1.6. A *function* $f\colon E \to F$ between spectra E and F is a collection of maps $f_n\colon E_n \to F_n$ for each $n \in \mathbb{Z}$ such that the diagram

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\Sigma f_n} & \Sigma F_n \\ {\scriptstyle \sigma_n}\downarrow & & \downarrow{\scriptstyle \tau_n} \\ E_{n+1} & \xrightarrow[f_{n+1}]{} & F_{n+1} \end{array}$$

commutes. (Here, τ_n is the n'th structure map of F.) A *map* of spectra is an "eventually defined function" in the sense of [2, pg. 140-142].

Remark 4.1.7. In more modern categories of spectra (for example, the category of $\mathbb{S}$-modules, to be introduced in section §4.2), there is no need to distinguish between functions and "eventually defined functions".

As one might hope, homotopies in **Sp** can be defined by imitating the usual definition in $\mathbf{Top}_*$. To do this, let $I_+ = [0, 1] \sqcup \{*\}$ be the union of the unit interval with a disjoint basepoint. Note that there are maps $i_0, i_1\colon E \to E \wedge I_+$ which send E to $E \wedge (\{0\} \sqcup \{*\})$ and $E \wedge (\{1\} \sqcup \{*\})$, respectively.

Definition 4.1.8. We say that two maps $f, g\colon E \to F$ of spectra are *homotopic* if there is a map $h\colon E \wedge I_+ \to F$ such that $h \circ i_0 = f$ and $h \circ i_1 = g$.

As in $\mathbf{Top}_*$, homotopy is an equivalence relation. We can also define homotopy groups of spectra by imitating the usual definition in $\mathbf{Top}_*$.

Definition 4.1.9. The k'th *homotopy group* of the spectrum E is

$$\pi_k(E) := \mathbf{Sp}(\Sigma^k \mathbb{S}, E)/\sim,$$

where $f \sim g$ if and only if f and g are homotopic.

Remark 4.1.10. By [2, Pt. III, Theorem 3.7], the set of homotopy classes of maps between any two spectra has an abelian group structure. In particular, $\pi_k(E)$ is always an abelian group. Also note that Definition 4.1.9 makes sense for negative integers. Spectra with no nontrivial negative dimensional homotopy groups are called *connective*.

Remark 4.1.11. For any two spectra E and F, it is possible to define a spectrum of maps $E \to F$, written $\operatorname{Map}(E, F)$. We can use this together with Definition 4.1.9 to define the ordinary *homology* and *cohomology* groups of a spectrum E with coefficients in the abelian group A as

$$H_*(E; A) := \pi_*(E \wedge HA) \qquad H^*(E; A) := \pi_{-*} \operatorname{Map}(E, HA).$$

The spectrum $\operatorname{Map}(E, F)$ satisfies $\Sigma \operatorname{Map}(E, F) = \operatorname{Map}(E, \Sigma F)$. A consequence of this is that, for any integer n, we have $H^n(E; A) \cong h\mathbf{Sp}(E, \Sigma^n HA)$. Note that this is similar to the fact that, if X is a *space*, then $H^n(X; A) \cong h\mathbf{Top}_*(X, K(A, n))$.

To see that Definition 4.1.9 is "correct", we note that, by [2, Pt. III, Prop. 2.8], the abelian group $\pi_k(E)$ can be alternatively viewed as the colimit of the sequence

$$\begin{array}{ccccccccc}
\pi_k(E_0) & & \dashrightarrow & & \pi_{k+1}(E_1) & & \dashrightarrow & & \pi_{k+2}(E_2) \dashrightarrow \cdots \\
& \searrow^{\Sigma} & & \nearrow_{\sigma_0} & & \searrow^{\Sigma} & & \nearrow_{\sigma_1} & \\
& & \pi_{k+1}(\Sigma E_0) & & & & \pi_{k+2}(\Sigma E_1) & &
\end{array}$$

In particular, if we take $E = \Sigma^\infty X$, then we do have

$$\pi_k(\Sigma^\infty X) = \varinjlim \pi_{k+n}(\Sigma^n X) = \pi_k^{\text{st}}(X).$$

In addition to the above, it can be shown that $h\mathbf{Sp}$ has all five desired properties of the stable homotopy category listed just before §3.1. In a bit more detail:

- From the above definitions, it is straightforward to check that the process of taking suspension spectra determines a functor $\Sigma^\infty \colon h\mathbf{Top}_* \to h\mathbf{Sp}$. The functor $\Omega^\infty \colon h\mathbf{Sp} \to h\mathbf{Top}_*$ that is right adjoint to Σ^∞ sends a connective spectrum E to the colimit of the sequence

$$E_0 \xrightarrow{\omega_0} \Omega E_1 \xrightarrow{\Omega\omega_1} \Omega^2 E_2 \xrightarrow{\Omega^2\omega_2} \cdots$$

  Note that if each ω_n is a homeomorphism (or weak equivalence), then $\Omega^\infty E \simeq E_0$, the *infinite loop space* associated to the spectrum E.

  Note that if $E = \Sigma^\infty X$ is the suspension spectrum of a space X, then $\Omega^\infty\Sigma^\infty X$ is exactly the free infinite loop space on X discussed in Example 3.1.7.

- The wedge sum of two spectra E and F is the spectrum $E \vee F = \{E_n \vee F_n\}$. This coproduct is also a product by [2, Pt. III, Prop. 3.11].

- The smash product $E \wedge F$ of two spectra is defined in [2, Pt. III, §4] and its unit is $\mathbb{S}$. It is important to point out here that this turns out to only be a symmetric monoidal product *after* passage to $h\mathbf{Sp}$.

- The homotopy cofiber of a map $f \colon E \to F$ is the spectrum

$$C(f) := \{Y_n \cup_{f_n} (I_+ \wedge X_n)\}$$

  (see [2, pg. 154] for details on this construction). In $h\mathbf{Sp}$, $C(f)$ can be defined via the pushout square

$$\begin{array}{ccc} E & \longrightarrow & * \\ {\scriptstyle f}\downarrow & & \downarrow \\ F & \longrightarrow & C(f). \end{array}$$

  Cofiber sequences in $h\mathbf{Sp}$ coincide with fiber sequences by [2, Pt. III, Prop. 3.10]. As a result, the homotopy fiber of f could be defined as $F(f) := \Sigma^{-1}C(f)$; alternatively, $F(f)$ can be defined via the following pullback square in $h\mathbf{Sp}$:

$$\begin{array}{ccc} F(f) & \longrightarrow & * \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & F. \end{array}$$

4.2 The Category of $\mathbb{S}$-modules

The interpretation of spectra as homotopy-theoretic versions of abelian groups is important in stable homotopy theory and naturally leads to the following question: Can rings, modules, and algebras be constructed in the stable homotopy category as well? As one might hope, this can be done using the smash product in $h\mathbf{Sp}$ in a way that is completely analogous to the usual definitions in ordinary algebra.

Definition 4.2.1. An (associative) *ring spectrum* is a monoid in $h\mathbf{Sp}$. In other words, a ring spectrum is a spectrum R together with a multiplication map $\mu\colon R \wedge R \to R$ and a two-sided unit map $\eta\colon \mathbb{S} \to R$ such that the following diagrams commute *up to homotopy*:

$$\begin{array}{ccc} R \wedge R \wedge R & \xrightarrow{\mathrm{id}\wedge\mu} & R \wedge R \\ {\scriptstyle \mu\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mu} \\ R \wedge R & \xrightarrow{\mu} & R \end{array} \qquad \begin{array}{ccccc} \mathbb{S} \wedge R & \xrightarrow{\eta\wedge\mathrm{id}} & R \wedge R & \xleftarrow{\mathrm{id}\wedge\eta} & R \wedge \mathbb{S} \\ & {\scriptstyle \simeq}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \simeq} & \\ & & R & & \end{array}$$

If, in addition, the diagram

$$\begin{array}{ccc} R \wedge R & \xrightarrow{\text{swap factors}} & R \wedge R \\ & {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & R & \end{array}$$

commutes, so that R is a commutative monoid in $h\mathbf{Sp}$, then we say that R is a *commutative ring spectrum.*

Given a ring spectrum R, it is possible to define R-module spectra and R-algebra spectra in the stable homotopy category in a similar way. In order to study these objects seriously, it is necessary to make sense of them on the point-set level—this is analogous to the idea that the study of ordinary rings, modules, and algebras should not be done solely on the level of derived categories. In particular, one can make *constructions* in the category $\mathbf{Sp}$, such as limits, colimits, etc., that cannot be done at the level of the homotopy category. Unfortunately, the fact that the smash product is not a symmetric monoidal product in $\mathbf{Sp}$ itself makes this difficult. For example, as pointed out in [7, pg. 8], a particular problem that arises is the following: If R is a ring spectrum and

$M, N \in \mathbf{Sp}$ pass to R-module spectra in the stable homotopy category, then the cofiber of a map $M \to N$ in $\mathbf{Sp}$ could fail to be an R-module spectrum in $h\mathbf{Sp}$.

To fix this, the category $\mathbf{Sp}$ must be replaced with a point-set level category of spectra that has a symmetric monoidal smash product *before* passage to the stable homotopy category. A suitable replacement is the category $\mathcal{M}_{\mathbb{S}}$ of $\mathbb{S}$*-modules*, introduced in 1993 and described in [16]. Its smash product is denoted by $\wedge_{\mathbb{S}}$ and its unit is $\mathbb{S}$. As a preliminary step to constructing this symmetric monoidal category, one must consider the category of $\mathbb{L}$*-spectra*. Roughly, this is a point-set level category of spectra in which an action by a certain E_∞-operad of spaces (the *linear isometries operad*) is "built in" so as to produce a well-behaved smash product, $\wedge_{\mathbb{L}}$. The action of this operad on a spectrum is defined using the fact that the category of $\mathbb{L}$-spectra is constructed so that it is *tensored* over spaces, meaning that it makes sense to take the smash product of a space and a spectrum to produce another spectrum. One issue with the smash product of $\mathbb{L}$-spectra is that the sphere spectrum does *not* act as the unit. To correct for this, we simply make the following definition:

Definition 4.2.2. An $\mathbb{S}$*-module* is an $\mathbb{L}$-spectrum M for which the map $\lambda\colon \mathbb{S} \wedge_{\mathbb{L}} M \to M$ is an isomorphism. We will write $\mathcal{M}_{\mathbb{S}}$ for the category of $\mathbb{S}$-modules. The term "$\mathbb{S}$-module" is justified by the commutativity of the following associativity and unitality diagrams

$$\begin{array}{ccc} \mathbb{S} \wedge_{\mathbb{S}} \mathbb{S} \wedge_{\mathbb{S}} M & \xrightarrow{\lambda\wedge \mathrm{id}} & \mathbb{S} \wedge_{\mathbb{S}} M \\ {\scriptstyle \mathrm{id}\wedge\lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ \mathbb{S} \wedge_{\mathbb{S}} M & \xrightarrow{\lambda} & M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\lambda^{-1}} & \mathbb{S} \wedge_{\mathbb{S}} M \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \lambda} \\ & & M \end{array}$$

for any $\mathbb{S}$-module M.

For two $\mathbb{S}$-modules M and N, their smash product $M \wedge_{\mathbb{S}} N := M \wedge_{\mathbb{L}} N$ and function spectrum $F_{\mathbb{S}}(M, N) := \mathbb{S} \wedge_{\mathbb{L}} F_{\mathbb{L}}(M, N)$ are both $\mathbb{S}$-modules. As would be expected, there is also an adjunction $(-) \wedge_{\mathbb{S}} N \leftrightarrows F_{\mathbb{S}}(-, N)$ [16, Theorem 1.6, pg. 37].

Definition 4.2.3. An $\mathbb{S}$*-algebra* is a monoid in $\mathcal{M}_{\mathbb{S}}$. Similarly, a *commutative* $\mathbb{S}$*-algebra* is a commutative monoid in $\mathcal{M}_{\mathbb{S}}$. We will write $\mathcal{C}_{\mathbb{S}}$ for the category of commutative $\mathbb{S}$-algebras.

Example 4.2.4. If X is any E_∞-space, then the suspension spectrum $\Sigma^\infty_+ X$ is a commutative $\mathbb{S}$-algebra. The multiplication on $\Sigma^\infty_+ X$ is induced by the multiplication on X. In particular, $\mathbb{S}$ is a commutative $\mathbb{S}$-algebra, with multiplication given by the composition product.

Example 4.2.5. If R is any associative ring, then HR is an $\mathbb{S}$-algebra. If R is a commutative ring, then HR is a commutative $\mathbb{S}$-algebra. The multiplication is induced by the multiplication on the ring R.

Example 4.2.6. [31, Ch. VIII] The topological real and complex K-theory spectra, KO and KU, respectively, are commutative $\mathbb{S}$-algebras. The multiplication is induced by the tensor product of vector spaces.

Just as there is a notion of module over any commutative ring in ordinary algebra, there is a notion of module over any commutative $\mathbb{S}$-algebra in stable homotopy theory:

Definition 4.2.7. Let R be a commutative $\mathbb{S}$-algebra. An *R-module* is an $\mathbb{S}$-module M together with $\mathbb{S}$-module maps $\eta\colon R \to M$ and $\alpha\colon R \wedge_{\mathbb{S}} M \to M$ such that the following associativity and unitality diagrams commute:

$$\begin{array}{ccc} R \wedge_{\mathbb{S}} R \wedge_{\mathbb{S}} M & \xrightarrow{\mu \wedge \mathrm{id}} & R \wedge_{\mathbb{S}} M \\ {\scriptstyle \mathrm{id}\wedge\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ R \wedge_{\mathbb{S}} M & \xrightarrow{\alpha} & M \end{array} \qquad \begin{array}{ccc} \mathbb{S} \wedge_{\mathbb{S}} M & \xrightarrow{\eta \wedge \mathrm{id}} & R \wedge_{\mathbb{S}} M \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \alpha} \\ & & M. \end{array}$$

Here, as in Definition 4.2.1, η and μ are the unit and multiplication maps, respectively, of R. We will denote the category of R-modules by $\mathcal{M}_R$.

Example 4.2.8. [16, pg. 88] If M is a module over a ring R, then the Eilenberg-Mac Lane spectrum HM is an HR-algebra.

As in [16], the smash product and mapping R-module can be defined for two R-modules M and N using the fact that $\mathcal{M}_{\mathbb{S}}$ is bicomplete (i.e., all small limits and colimits exist; see [16, Prop. 1.4, pg. 36] for a proof of this fact). If α and β denote the R-module structure maps of M and N, respectively, then the smash product $M \wedge_R N$ is defined as the coequalizer of

$$M \wedge_{\mathbb{S}} R \wedge_{\mathbb{S}} N \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M \wedge_{\mathbb{S}} N$$

in $\mathcal{M}_{\mathbb{S}}$ and the function R-module $F_R(M,N)$ is defined as the equalizer of

$$F_{\mathbb{S}}(M,N) \underset{\delta}{\overset{\gamma}{\rightrightarrows}} F_{\mathbb{S}}(R \wedge_{\mathbb{S}} M, N)$$

in $\mathcal{M}_{\mathbb{S}}$, where $\gamma = F(\alpha, \mathrm{id})$ and δ is the adjoint to the composite

$$R \wedge_{\mathbb{S}} (M \wedge_{\mathbb{S}} F_{\mathbb{S}}(M,N)) \xrightarrow{\mathrm{id}\wedge\epsilon} R \wedge_{\mathbb{S}} N \xrightarrow{\beta} N$$

in which $\epsilon\colon M \wedge_{\mathbb{S}} F_{\mathbb{S}}(M,N) \to N$ is the topological analogue of the evaluation map

$$M \otimes_{\mathbb{Z}} \mathbf{Mod}_{\mathbb{Z}}(M,N) \to N \qquad m \otimes f \mapsto f(m).$$

Both $\wedge_R$ and F_R have properties that are analogous to those of $\otimes_R$ and Hom_R in ordinary algebra. For more on this, see [16, Ch. III].

Finally, we can define R-algebras for any commutative $\mathbb{S}$-algebra R just as in Definition 4.2.3 by replacing the category $\mathcal{M}_{\mathbb{S}}$ with the category $\mathcal{M}_R$.

Definition 4.2.9. Let R be a commutative $\mathbb{S}$-algebra. An *R-algebra* is a monoid in $\mathcal{M}_R$. Similarly, a *commutative R-algebra* is a commutative monoid in $\mathcal{M}_R$. We will write $\mathcal{C}_R$ for the category of commutative R-algebras.

4.3 Highly Structured Ring Spectra

It is shown in [16, pg. 41-43] that $\mathbb{S}$-algebras and commutative $\mathbb{S}$-algebras are, up to weak equivalence, the same as *A_∞-ring spectra* and *E_∞-ring spectra*, respectively. We refer to these collectively as *highly structured ring spectra*.

Definition 4.3.1. An *A_∞-ring spectrum* is an associative monoid in the category of $\mathbb{L}$-spectra. An E_∞-ring spectrum is an associative and commutative monoid in the category of $\mathbb{L}$-spectra.

Remark 4.3.2. If one works in another category of spectra with a well-behaved point-set level smash product other than the category of $\mathbb{S}$-modules, such as the category of symmetric spectra, then A_∞ and E_∞-ring spectra may be defined as spectra with an action of an A_∞ or E_∞-operad, just as for spaces. For example, we have the operad of Stasheff associahedra, $\mathcal{A}$, which is an operad in $\mathbf{Top}_*$. It makes sense to take the smash product of a space and a ring spectrum R, and so we have action maps $\mathcal{A}(k) \wedge R^{\wedge k} \to R$, defining an A_∞-structure on R.

There is a subtlety to comparing highly structured ring spectra with their $\mathbb{S}$-algebra analogues. As we've defined them, an A_∞-ring spectrum, for example, need not be an $\mathbb{S}$-algebra.

Proposition 4.3.3. *[16, pg. 43] An $\mathbb{S}$-algebra or commutative $\mathbb{S}$-algebra is an A_∞ or E_∞-ring spectrum which is also an $\mathbb{S}$-module. A module over an $\mathbb{S}$-algebra or commutative $\mathbb{S}$-algebra R is a module over R, regarded as an A_∞ or E_∞-ring spectrum, which is also an $\mathbb{S}$-module.*

Recall from §3 that the homology of any E_∞-space has an action of the Dyer-Lashof algebra. If R is an E_∞-ring spectrum, then $H_*(R;\mathbb{F}_p) = H_*(R)$ will also have an action of the "big" Dyer-Lashof algebra, $\mathcal{R}_p(-\infty)$.

Example 4.3.4. Because $\mathbb{F}_p$ is a commutative ring, the Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$ is an E_∞-ring spectrum. Its homology $H_*(H\mathbb{F}_p)$ is the *dual Steenrod algebra.* Its structure as a Hopf algebra over $\mathbb{F}_p$ was computed by Milnor [37]:

- $H_*(H\mathbb{F}_2) \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots]$. Here, ξ_i is dual to $\mathrm{Sq}^{2^{i-1}}\mathrm{Sq}^{2^{i-2}}\cdots\mathrm{Sq}^2\mathrm{Sq}^1$ and has degree $|\xi_i| = 2^i - 1$. (Also note that ξ_1^n is dual to Sq^n.)

- $H_*(H\mathbb{F}_p) \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_p}[\tau_0, \tau_1, \ldots]$. Here, ξ_i is dual to $P^{p^{i-1}}P^{p^{i-2}}\cdots P^pP^1$ and has degree $|\xi_i| = 2p^i - 2$; the element τ_i is dual to $P^{p^{i-1}}P^{p^{i-2}}\cdots P^p\beta$ and has degree $|\tau_i| = 2p^i - 1$.

As part of the Hopf algebra structure, there is also a coproduct,

$$\Delta\colon H_*(H\mathbb{F}_p) \to H_*(H\mathbb{F}_p) \otimes H_*(H\mathbb{F}_p),$$

and a conjugation map

$$\chi\colon H_*(H\mathbb{F}_p) \to H_*(H\mathbb{F}_p).$$

Formulas describing these are in Appendix A. As an algebra over the Dyer-Lashof algebra, $H_*(H\mathbb{F}_2)$ is generated by ξ_1 and $H_*(H\mathbb{F}_p)$ is generated by τ_0. Formulas for the Dyer-Lashof action, computed by Steinberger [11, Ch. III], are also in Appendix A.

Example 4.3.5. If X is a finite complex, then the spectrum $F(X,\mathbb{S})$ of maps $X \to \mathbb{S}$ is an E_∞-ring spectrum. The Dyer-Lashof action on $H_*(F(X,\mathbb{S}))$ coincides with the action of the Steenrod algebra on $H^*(X)$ [11, Ch. III]; specifically, $Q^{-i} = \mathrm{Sq}^i$.

4.4 The Spectrum of Units

In classical algebra, the elements of a commutative ring R which have a multiplicative inverse are the *units* of R. These elements form an abelian group under multiplication, called the *group of units of* R, written $\mathrm{GL}_1 R = R^\times$. Taking the group of units defines a functor $\mathrm{GL}_1(-)\colon \mathbf{CRing} \to \mathbf{AbGrp}$, from the category of commutative rings to the category of abelian groups. This functor is adjoint to the *group ring* functor $\mathbb{Z}[-]\colon \mathbf{AbGrp} \to \mathbf{CRing}$, sending an abelian group A to its group ring over the integers, $\mathbb{Z}[A]$. Continuing our analogy of spectra as analogues of abelian groups and E_∞-ring spectra as analogues of commutative rings, we would expect an E_∞-ring spectrum to have a *spectrum of units*. Moreover, the functor associating an E_∞-ring spectrum to its spectrum of units should be adjoint to the *spherical group ring* functor

$$\mathbb{S}[-] = \Sigma^\infty_+ \Omega^\infty(-)\colon \mathbf{Sp}_{\geq 0} \to E_\infty - \mathbf{Ring}$$

where $\mathbf{Sp}_{\geq 0}$ is the category of connective spectra.[2] As spelled out in [5, 31], there is such a spectrum of units satisfying these properties. To construct it, we begin by constructing the *space* of units associated to any A_∞-ring spectrum:

Definition 4.4.1. Let R be an A_∞-ring spectrum and let $\Omega^\infty R$ be its infinite loop space. The *space of units of* R, written $\mathrm{GL}_1 R$, is the union of the components of $\Omega^\infty R$ that correspond to units in the ring $\pi_0(\Omega^\infty R) \cong \pi_0 R$. Alternatively, if $(\pi_0 R)^\times \subseteq \pi_0 R$ is the group of units in $\pi_0 R$, then $\mathrm{GL}_1 R$ is the pullback in the following diagram of based spaces:

$$\begin{array}{ccc} \mathrm{GL}_1 R & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ (\pi_0 R)^\times & \longrightarrow & \pi_0 R \end{array}$$

(Here, both $\pi_0 R$ and $(\pi_0 R)^\times$ have the discrete topology.)

If the A_∞-ring structure on R extends to an E_∞-ring structure, then $\mathrm{GL}_1 R$ is a grouplike E_∞-space, being a union of components of the grouplike E_∞-space $\Omega^\infty R$. By the Recognition Principle, there is then a *spectrum* of units, $\mathrm{gl}_1 R$, with $\Omega^\infty \mathrm{gl}_1 R =$

[2] Σ^∞_+ is the composite functor $\Sigma^\infty \circ (-)_+$, where $(-)_+\colon \mathbf{Top} \to \mathbf{Top}_*$ adds a disjoint basepoint. This is needed so that $\Sigma^\infty_+ \Omega^\infty X$ is a *unital* ring spectrum. We view the basepoint of X as the multiplicative identity, "1", and the additional basepoint as the additive identity, "0".

$\mathrm{GL}_1 R$. It is proved in [5, Theorem 2.1] that the functor $\mathrm{gl}_1(-)$ is adjoint to $\Sigma^\infty_+\Omega^\infty$. This spectrum defines a generalized cohomology theory such that if X is a space, then

$$(\mathrm{gl}_1 R)^0(X) \cong R^0(X)^\times.$$

Remark 4.4.2. As mentioned in [44], there are difficulties related to model structures and comparison of operads in verifying that $\mathrm{gl}_1 R$ does participate in the adjunction

$$(\mathbf{E}_\infty - \mathbf{Ring})(\Sigma^\infty_+\Omega^\infty X, R) \cong \mathbf{Sp}(X, \mathrm{gl}_1 R)$$

Definition 4.4.3. If R is an A_∞-ring spectrum, then the *space of strict units of* R, written $\mathrm{SL}_1 R$, is the component of $\Omega^\infty R$ corresponding to the multiplicative identity $1 \in \pi_0 R = \pi_0(\Omega^\infty R)$. If R is an E_∞-ring spectrum, then $\mathrm{SL}_1 R$ deloops to give the *spectrum of strict units*, $\mathrm{sl}_1 R$.

Example 4.4.4. If $R = \mathbb{S}$, then $\mathrm{GL}_1\mathbb{S}$ (written "F" in [14]) is the monoid of stable self-equivalences of the sphere. The space of strict units, $\mathrm{SL}_1\mathbb{S}$ (written "SF" in [14]), is the submonoid of self-equivalences of degree 1.

Example 4.4.5. Recall that if A is an ordinary commutative ring, then HA is an E_∞-ring spectrum. In this case, $\mathrm{gl}_1 HA \simeq HA^\times$, the Eilenberg-Mac Lane spectrum of the group of units of A.

The spectra $\mathrm{gl}_1 R$ and $\mathrm{sl}_1 R$ are, in many ways, like R itself. For example, for all $n \geq 1$, we have [41]

$$\pi_n(\mathrm{sl}_1 R) = \pi_n(\mathrm{gl}_1 R) = (\mathrm{gl}_1 R)^0(S^n) \cong (1 + \tilde{R}^0(S^n))^\times \subseteq \tilde{R}^0(S^n)^\times.$$

Therefore, for all $n \geq 1$,

$$\pi_n(\mathrm{sl}_1 R) = \pi_n(\mathrm{gl}_1 R) \cong \pi_n(R).$$

The isomorphism is given by the map[3]

$$\pi_n(\mathrm{gl}_1 R) \to \pi_n(R) \qquad \text{“}1 + x \mapsto x\text{”}.$$

[3] This is induced by the inclusion $\mathrm{GL}_1 R \hookrightarrow \Omega^\infty R$ of spaces, and *not* in general by a map of spectra $\mathrm{gl}_1 R \to R$ [41].

In dimension $n = 0$, the group $\pi_0(\mathrm{sl}_1 R)$ is trivial and $\pi_0(\mathrm{gl}_1 R) = (\pi_0 R)^\times$. There are also isomorphisms of graded abelian groups

$$H_*(\mathrm{GL}_1 R) \cong \bigoplus_{u \in (\pi_0 R)^\times} [u] \# H_*(\Omega^\infty R_0) \qquad H_*(\mathrm{SL}_1 R) \cong [1] \# H_*(\Omega^\infty R_0)$$

where $\Omega^\infty R_0 \subseteq \Omega^\infty R$ is the component corresponding to $0 \in \pi_0(R)$. In addition, because the E_∞-structure on $\mathrm{GL}_1 R$ and $\mathrm{SL}_1 R$ is inherited from the multiplicative E_∞-structure on R, both $H_*(\mathrm{GL}_1 R)$ and $H_*(\mathrm{SL}_1 R)$ inherit the multiplicative Pontryagin product, $\circ$, and its associated Dyer-Lashof operations, $\widetilde{Q}^r$. The above isomorphisms may be used to compute these operations on $H_*(\mathrm{GL}_1 R)$ or $H_*(\mathrm{SL}_1 R)$ using the distributivity and mixed Cartan formulas (Theorem 3.2.5) on $H_*(\Omega^\infty R)$. However, the unit spectra $\mathrm{gl}_1 R$ and $\mathrm{sl}_1 R$ can have k-invariants that are, in general, very different from the k-invariants of R. We will explore the relationship between these k-invariants in more detail in §6, with the aid of the Miller spectral sequence.

Example 4.4.6. ([14, II; Theorem 5.1, 5.2]) Because the sphere spectrum, $\mathbb{S}$, is an E_∞-ring spectrum, $\Omega^\infty\mathbb{S}$ is an E_∞-ring space. Its mod p homology, $H_*(\Omega^\infty\mathbb{S}; \mathbb{F}_p)$, is generated under the additive product $\#$ by $[1], [-1] \in H_0(\Omega^\infty\mathbb{S})$ and the elements $Q^I[1]$, where I is an admissible sequence with $e(I) + b(I) \geq 1$. Note that if $\ell(I) = k$, then the element $Q^I[1]\#[1-p^k]$ lies in the homology of the 1-component of Ω^∞; in other words, $Q^I[1]\#[1-p^k] \in H_*(\mathrm{SL}_1\mathbb{S})$. If $p \geq 3$, then $H_*(\mathrm{SL}_1\mathbb{S})$ is the free graded commutative algebra generated by these elements under the multiplicative product $\circ$. If $p = 2$, then $H_*(\mathrm{SL}_1\mathbb{S})$ is isomorphic to the tensor product of the exterior algebra on the elements $Q^I[1]\#[1-2^k]$ (with $\ell(I) \geq 2$) and the polynomial algebra on the elements $Q^rQ^r[1]\#[-3]$. (Note that although $Q^rQ^r[1] = Q^r[1]\#Q^r[1]$, the elements $Q^rQ^r[1]\#[-3]$ are indecomposable under $\circ$).

The main motivation for studying the spectrum $\mathrm{gl}_1 R$ is its connection to *orientation theory* [31], [5]. Recall that for a based space X and a vector bundle ξ over X, the associated *Thom space*, $\mathrm{Th}(\xi)$, is the quotient $S(\xi)/X$. Here, $S(\xi)$ is the fiberwise one-point compactification of ξ; this is a sphere bundle over X and has a section $X \to S(\xi)$ given by sending $x \in X$ to the point at infinity in the fiber over x. The *Thom spectrum* of X is the suspension spectrum $\Sigma^\infty \mathrm{Th}(\xi)$.

Definition 4.4.7. Given a based space X, a rank n vector bundle ξ over X, and a ring spectrum R, an *R-orientation* of ξ is a choice of cohomology class $\omega \in R^n(\mathrm{Th}(\xi))$ such that for every point $x \in X$, the restriction of ω along the map

$$\mathrm{Th}(\mathbb{R}^n) \simeq S^n \to \mathrm{Th}(\xi)$$

is a generator.

If an R-orientation of ξ exists, then we have a Thom isomorphism

$$R^*(X) \xrightarrow{\sim} R^{*+n}(\mathrm{Th}(\xi))$$

The obstruction to the existence of such an orientation turns out to be the *R-theory Stiefel-Whitney class*, $w(\xi; R) \in [X, B\mathrm{GL}_1 R] = (\mathrm{gl}_1 R)^1(X)$. For more details, see [31, IV, §3].

Chapter 5

The Miller Spectral Sequence

Let X be a connective spectrum and let $\Omega^\infty X$ be its associated infinite loop space. The purpose of this section is to examine how $H_*(\Omega^\infty X) = H_*(\Omega^\infty X; \mathbb{F}_p)$, as a Hopf algebra over $\mathbb{F}_p$ with an action of the Dyer-Lashof algebra $\mathcal{R}_p$, determines $H_*(X)$.

As a starting point, consider the relationship between $H_*(\Omega^n X)$ and $H_*(\Omega^{n-1} X)$, where X is a space and $n \geq 1$. Since the classifying space $B\Omega^n X$ is homotopy equivalent to $\Omega^{n-1}X$, there is the bar spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^{H_*(\Omega^n X)}_{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow H_{s+t}(B\Omega^n X) \cong H_{s+t}(\Omega^{n-1} X)$$

that computes $H_*(\Omega^{n-1}X)$ from knowledge of $H_*(\Omega^n X)$ as a Hopf algebra over $\mathbb{F}_p$ with Dyer-Lashof operations. It follows that if we begin with knowledge of $H_*(\Omega^n X)$ for some n, then we may iterate the bar spectral sequence n times to compute $H_*(X)$. In particular, if X is a spectrum, then we may iterate the bar spectral sequence infinitely many times to compute $H_*(X)$ from knowledge of $H_*(\Omega^\infty X)$. To streamline this process, Miller [35] constructed a spectral sequence that does this computation all at once [35]. This spectral sequence takes the following form:

Theorem 5.0.1. *There is a first-quadrant spectral sequence*

$$E^2_{s,t} = \mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)(H_t(\Omega^\infty X)) \Longrightarrow H_{s+t}(X),$$

called the Miller spectral sequence*, that computes the homology of a spectrum X from knowledge of the homology of its associated infinite loop space, $\Omega^\infty X$, as a Hopf algebra* *over the Dyer-Lashof algebra $\mathcal{R}_p$. Here, $\mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)$ is the s'th left derived functor of $\mathbb{F}_p \otimes_{\mathcal{R}_p} Q(-)$, which is the functor on the category $\mathbf{AR} - \mathbf{Hopf}_{\mathbb{F}_p}$ of Hopf algebras over $\mathbb{F}_p$ with an allowable action of $\mathcal{R}_p$ that takes indecomposables with respect to the Pontryagin product and then with respect to the action of $\mathcal{R}_p$.*

To construct this spectral sequence, we need to know which objects in $\mathbf{AR} - \mathbf{Hopf}_{\mathbb{F}_p}$ are projective objects.

Definition 5.0.2. An object $H_*(\Omega^\infty P) \in \mathbf{AR} - \mathbf{Hopf}_{\mathbb{F}_p}$ is *projective* if P is a retract of a suspension spectrum.

Proof of Theorem 5.0.1. The idea behind the construction of this spectral sequence is that if X is a suspension spectrum, so that $X = \Sigma^\infty Y$ for some space Y, then

$$H_*(X) = H_*(Y) \cong \mathbb{F}_p \otimes_{\mathcal{R}_p} QH_*(\Omega^\infty \Sigma^\infty Y).$$

For a general spectrum X, we may resolve X by suspension spectra and use the above result to construct a spectral sequence converging to $H_*(X)$. This type of resolution of $X = X(0)$ begins with the map $\Sigma^\infty\Omega^\infty X(0) \to X(0)$ that is adjoint to the identity $\Omega^\infty X(0) \to \Omega^\infty X(0)$. To construct the next stage in the resolution, let $X(1)$ be the fiber of $\Sigma^\infty\Omega^\infty X(0) \to X(0)$ and consider the map $\Sigma^\infty\Omega^\infty X(1) \to X(1)$. Let $X(2)$ be the fiber of this map, etc. Repeating this process indefinitely, we obtain a diagram

$$\begin{array}{ccccccccc}
X = X(0) & & & & X(1) & & & & X(2) \\
& \nwarrow & & \swarrow & & \nwarrow & & \swarrow & & \nwarrow \\
& & \Sigma^\infty\Omega^\infty X(0) & \xleftarrow{d^1} & & \Sigma^\infty\Omega^\infty X(1) & \xleftarrow{d^1} & & \Sigma^\infty\Omega^\infty X(2) & \cdots
\end{array}$$

in which the "V"-shaped segments are fiber sequences. This gives a spectral sequence converging to $H_*(X)$ with $E^1_{s,t} = H_t(\Sigma^\infty\Omega^\infty X(s))$ and differential d^1 given by the dashed maps along the bottom of the above diagram.

To identify the E^2 page, apply the functor $\Omega^\infty(-)$ to everything in the diagram, giving

$$\begin{array}{ccccccccc}
\Omega^\infty X = \Omega^\infty X(0) & & & & \Omega^\infty X(1) & & & & \Omega^\infty X(2) \\
& \nwarrow & & \swarrow & & \nwarrow & & \swarrow & & \nwarrow \\
& & \Omega^\infty\Sigma^\infty\Omega^\infty X(0) & \xleftarrow{d^1} & & \Omega^\infty\Sigma^\infty\Omega^\infty X(1) & \xleftarrow{d^1} & & \Omega^\infty\Sigma^\infty\Omega^\infty X(2) & \cdots
\end{array}$$

For each n, the fibration

$$\Omega^\infty X(n) \leftarrow \Omega^\infty \Sigma^\infty \Omega^\infty X(n) \leftarrow \Omega^\infty X(n+1)$$

is principal and has a section $\Omega^\infty X(n) \to \Omega^\infty \Sigma^\infty \Omega^\infty X(n)$. Therefore, $\Omega^\infty \Sigma^\infty \Omega^\infty X(n)$ and the product $\Omega^\infty X(n) \times \Omega^\infty X(n+1)$ are weakly equivalent. From this fact, we get short exact sequences

$$\mathbb{F}_p \to H_*(\Omega^\infty X(n+1)) \to H_*(\Omega^\infty \Sigma^\infty \Omega^\infty X(n)) \to H_*(\Omega^\infty X(n)) \to \mathbb{F}_p$$

of Hopf algebras over $\mathbb{F}_p$ with an action of $\mathcal{R}_p$. Splicing these short exact sequences together, we have a long exact sequence

$$\mathbb{F}_p \leftarrow H_*(\Omega^\infty X(0)) \leftarrow H_*(\Omega^\infty \Sigma^\infty \Omega^\infty X(0)) \xleftarrow{d^1} H_*(\Omega^\infty \Sigma^\infty \Omega^\infty X(1)) \xleftarrow{d^1} \cdots .$$

Since each term $H_*(\Omega^\infty \Sigma^\infty \Omega^\infty X(n))$ in this long exact sequence is a projective object, the above gives a projective resolution of $H_*(\Omega^\infty X(0))$. Recalling that

$$E^1_{s,*} = H_*(\Sigma^\infty \Omega^\infty X(s)) \cong \mathbb{F}_p \otimes_{\mathcal{R}_p} QH_*(\Omega^\infty \Sigma^\infty \Omega^\infty X(s)),$$

we see that taking homology of the E^1 page computes the value of the left derived functor of $\mathbb{F}_p \otimes_{\mathcal{R}_p} Q(-)$ on $H_*(\Omega^\infty X(0)) = H_*(\Omega^\infty X)$. □

Remark 5.0.3. There is an alternative construction of the Miller spectral sequence using the two-sided monadic bar construction (Appendix B.1). If we are given an infinite loop space Y, then the bar construction $B_\bullet(\Sigma^\infty, \mathbf{E}_\infty, Y)$, where $\mathbf{E}_\infty$ is the monad associated to the little ∞-cubes operad, is a simplicial spectrum whose geometric realization is a spectrum, X, satisfying $\Omega^\infty X \simeq Y$. We may therefore compute $H_*(X)$ from knowledge of $H_*(Y)$ using the bar spectral sequence.

5.1 Computing E^2

The E^2 page of the Miller spectral sequence,

$$E^2_{s,t} = \mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)(H_t(\Omega^\infty X)),$$

may be viewed as the value of the left derived functor of a composite of functors,

1. the indecomposables functor, $Q\colon \mathbf{AR}-\mathbf{Hopf}_{\mathbb{F}_p} \to \mathbf{A_1R}-\mathbf{Mod}_{\mathbb{F}_p}$, and

2. the tensor product, $\mathbb{F}_p \otimes_{\mathcal{R}_p} (-)\colon \mathbf{A_1R}-\mathbf{Mod}_{\mathbb{F}_p} \to \mathbf{GrMod}_{\mathbb{F}_p}$,

on $H_*(\Omega^\infty X)$. Here, the category $\mathbf{A_1R}-\mathbf{Mod}_{\mathbb{F}_p}$ is the category of 1-allowable graded $\mathbb{F}_p$-modules with an action of the Dyer-Lashof algebra. Note that all of the categories $\mathbf{AR}-\mathbf{Hopf}_{\mathbb{F}_p}$, $\mathbf{A_1R}-\mathbf{Mod}_{\mathbb{F}_p}$, and $\mathbf{GrMod}_{\mathbb{F}_p}$ are abelian categories and that the functor Q takes projective objects to projective objects. As a result, there is a Grothendieck spectral sequence that computes the left derived functor $\mathbf{L}_*(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)$. If we let $\mathbf{L}_*Q$ denote the left derived functor of Q and $\mathrm{Untor}_*^{\mathcal{R}_p}(\mathbb{F}_p, -)$ denote the left derived functor of $\mathbb{F}_p \otimes_{\mathcal{R}_p} (-)$ [1] , then the Grothendieck spectral sequence takes the form

$$E^2_{s,t} = \mathrm{Untor}_s^{\mathcal{R}_p}(\mathbb{F}_p, \mathbf{L}_tQ(H_*(\Omega^\infty X))) \Longrightarrow \mathbf{L}_{s+t}(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)(H_*(\Omega^\infty X)).$$

Proposition 5.1.1. *[35, §2.3] The left derived functors of*

$$Q\colon \mathbf{AR}-\mathbf{Hopf}_{\mathbb{F}_p} \to \mathbf{A_1R}-\mathbf{Mod}_{\mathbb{F}_p}$$

satisfy

$$\mathbf{L}_tQ(A) = \begin{cases} Q(A) & \text{if } t = 0 \\ Q\,\mathrm{Tor}_2^A(\mathbb{F}_p, \mathbb{F}_p) & \text{if } t = 1 \\ 0 & \text{if } t \geq 2 \end{cases}$$

for any $A \in \mathbf{AR}-\mathbf{Hopf}_{\mathbb{F}_p}$.

5.2 Koszul Resolutions

To compute $\mathrm{Untor}_*^{\mathcal{R}_p}(\mathbb{F}_p, M)$ for a module $M \in \mathbf{A_1R}-\mathbf{Mod}_{\mathbb{F}_p}$, one option is to use the bar complex $B_*(\mathbb{F}_p, \mathcal{R}_p(1), M)$, where $\mathcal{R}_p(1)$ is the quotient of the Dyer-Lashof algebra by the operations Q^I with excess less than 1, as described in Definition 3.1.4. However, the bar complex is typically an intractable method for computing $\mathrm{Untor}_*^{\mathcal{R}_p}$ due to its size. Miller's original approach to computing this (as used in [35] and also in [21]) is to instead take advantage of the fact that the category $\mathbf{A_1R}-\mathbf{Mod}_{\mathbb{F}_p}$ admits a theory

[1] As in Miller's original paper [35], we use the notation $\mathrm{Untor}_*^{\mathcal{R}_p}$ here isntead of $\mathrm{Tor}_*^{\mathcal{R}_p}$ to emphasize that $\mathbb{F}_p \otimes_{\mathcal{R}_p} (-)$ is a functor defined on a category of graded modules with an *unstable* action of $\mathcal{R}_p$, meaning that $Q^r(x) = 0$ for $r < |x|$.

of *Koszul resolutions*. To begin this section, we recall some definitions from Priddy's foundational paper on Koszul resolutions, [38]. In §5.3, we will apply these ideas to compute $\mathrm{Untor}_*^{\mathcal{R}_p}(\mathbb{F}_p, M)$ for some specific modules $M \in \mathbf{A_1R} - \mathbf{Mod}_{\mathbb{F}_p}$.

Definition 5.2.1. Let k be a field and let A be an associative graded k-algebra with a presentation $A \cong T\{x_i\}/R$, where $T\{x_i\}$ is the tensor algebra over k on the generators $\{x_i\}$ and $R \subseteq T\{x_i\}$ is the two-sided ideal of relations of A. We say that A is a

- *pre-Koszul algebra* if R is generated by elements of the form

$$\sum_i c_i x_i + \sum_{j,k} c_{j,k} x_j x_k$$

  where c_i and $c_{j,k}$ are elements of k, and a

- *homogeneous pre-Koszul algebra* if R is generated by elements of the form

$$\sum_{j,k} c_{j,k} x_j x_k.$$

If we write a_i for the image of x_i in A, then we call $\{a_i\}$ a *pre-Koszul set of generators of* A if the a_i are linearly independent. In this situation, the presentation $A \cong T\{x_i\}/R$ is a *pre-Koszul presentation of* A.

Note that the tensor algebra $T\{x_i\}$ has an increasing filtration (the *length filtration*)

$$F_0T\{x_i\} \subseteq F_1T\{x_i\} \subseteq F_2T\{x_i\} \subseteq \cdots$$

where $F_\ell T\{x_i\}$ is spanned by all monomials in $T\{x_i\}$ with length at most ℓ. (The element $1 \in T\{x_i\}$ is viewed as a monomial with length 0.) This clearly induces a corresponding filtration $\{F_\ell A\}$ of A.

Definition 5.2.2. Suppose that A has a pre-Koszul presentation and let $\{F_\ell A\}$ be the corresponding length filtration on A. The bigraded k-module E^0A with

$$(E^0A)_{\ell,t} = (F_\ell A/F_{\ell-1}A)_{\ell+t}$$

is called the *associated homogeneous pre-Koszul algebra of* A.

In the above definition, A is assumed to be a pre-Koszul algebra, but not assumed to be a *homogeneous* pre-Koszul algebra. The idea behind the definition of E^0A is that it is isomorphic to A if A is a homogeneous pre-Koszul algebra, but E^0A is a suitable replacement for A by a homogeneous pre-Koszul algebra if A is pre-Koszul but not homogeneous itself. If $\{a_i\}$ is a pre-Koszul set of generators of A, then E^0A is generated by the images, b_i, of the a_i in $(E^0A)_{1,*} = (F_1A/F_0A)_{1+*}$. If the relations of A are of the form

$$\sum_i c_i a_i + \sum_{j,k} c_{j,k} a_j a_k = 0,$$

then the relations of E^0A are of the form

$$\sum_{j,k} c_{j,k} b_j b_k = 0.$$

Notice that since E^0A is bigraded, $\mathrm{Tor}^{E^0A}(k,k) = \mathrm{Tor}^{E^0A}_{*,*,*}(k,k)$ and $\mathrm{Ext}_{E^0A}(k,k) = \mathrm{Ext}^{*,*,*}_{E^0A}(k,k)$ are trigraded. The first degree is the (co)homological degree, the second is the length grading, and the sum of the last two gradings gives the (usual) internal degree. The set of generators $\{b_i\}$ of E^0A determines a basis $\{\beta_i\}$ for $\mathrm{Ext}^{1,*,*}_{E^0A}(k,k) = \mathrm{Ext}^{1,1,*}_{E^0A}(k,k)$, where $\beta_i \in \mathrm{Ext}^{1,1,|b_i|-1}_{E^0A}(k,k)$ corresponds to $b_i \in (E^0A)_{1,|b_i|-1}$.

Remark 5.2.3. We use the notation $|\beta_i|$ to refer to the *total* degree of the element $\beta_i \in \mathrm{Ext}^{1,1,|b_i|-1}_{E^0A}(k,k)$, which is $|b_i| + 1$.

Definition 5.2.4. We say that pre-Koszul algebra A is a *Koszul algebra* if $\mathrm{Ext}^{*,*,*}_{E^0A}(k,k)$ is generated by $\{\beta_i\}$.

Remark 5.2.5. If A is a *homogeneous* pre-Koszul algebra, then $E^0A \cong A$. As a result, we write α_i instead of β_i for the basis elements of $\mathrm{Ext}^{*,*,*}_A(k,k)$ determined by the pre-Koszul generators $\{a_i\}$ of A. The above definition then says that A is Koszul if $\mathrm{Ext}^{*,*,*}_A(k,k)$ is generated by $\{\alpha_i\}$. In this case, we say that $\{a_i\}$ is a set of *Koszul generators* of A (instead of *pre*-Koszul generators).

Proposition 5.2.6. *A homogeneous algebra A is Koszul if and only if $\mathrm{Ext}^{s,\ell,*}_A(k,k) = 0$ unless $s = \ell$ (equivalently, if and only if $\mathrm{Tor}^A_{s,\ell,*}(k,k) = 0$ unless $s = \ell$). In other words, A is Koszul if and only if A has no cohomology "off the main diagonal".*

Proof. If A is Koszul, then $\mathrm{Ext}_A^{*,*,*}(k,k)$ is generated by classes $\alpha_i \in \mathrm{Ext}_A^{1,1,*}(k,k)$. In the cobar complex, let $[\tilde{\alpha}_i] \in C^{1,1,*}(k,A,k)$ be a representative of the cohomology class α_i. Note that any product of the classes $[\tilde{\alpha}_i]$ of length s must lie in $C^{s,s,*}(k,A,k)$. It follows that $\mathrm{Ext}_A^{s,\ell,*}(k,k) = 0$ unless $s = \ell$. Conversely, if $\mathrm{Ext}_A^{s,\ell,*}(k,k) = 0$ unless $s = \ell$, then $C^{s,\ell,*}(k,A,k) = 0$ unless $s = \ell$. Therefore the elements of $C^{1,1,*}(k,A,k)$ generate $\mathrm{Ext}_A(k,k)$, so that A is Koszul. $\square$

Example 5.2.7. The tensor algebra $T\{x_i\}$ has a pre-Koszul presentation given by $T\{x_i\} \cong T\{x_i\}/0$. Since $\mathrm{Ext}_{T\{x_i\}}^{*,*,*}(k,k)$ is the trivial algebra on generators σx_i (of tridegree $(1,1,|x_i|)$), the tensor algebra $T\{x_i\}$ is a homogeneous Koszul algebra.

Example 5.2.8. The free graded-commutative A-algebra on the generators $\{x_i\}$, $A\{x_i\}$, has a pre-Koszul presentation

$$A\{x_i\} \cong T\{x_i\}/(x_i x_j - (-1)^{|x_i||x_j|} x_j x_i)$$

If $\mathrm{char}(k) = 2$, then $A\{x_i\} = k[x_i]$ and if $\mathrm{char}(k) \neq 2$, then $A\{x_i\} = k[x_i] \otimes \Lambda_k\{x_j\}$, where the x_i have even degree and the x_j have odd degree. Since $\mathrm{Ext}_{k[x_i]}^{*,*,*}(k,k) \cong \Lambda_k\{\sigma x_i\}$ and $\mathrm{Ext}_{\Lambda_k\{x_j\}}^{*,*,*}(k,k) \cong k[\sigma x_j]$ ([12, Prop 2.1-2.3]), both $k[x_i]$ and $\Lambda_k\{x_j\}$ are homogeneous Koszul algebras. It follows that $A\{x_i\}$ is a homogeneous Koszul algebra.

Example 5.2.9. The Steenrod algebra $\mathcal{A}_2$ has a pre-Koszul set of generators $\{\mathrm{Sq}^i\}$ $(i \geq 1)$. If p is odd, then $\mathcal{A}_p$ has a pre-Koszul set of generators $\{P^i\} \cup \{\beta P^j\}$, where $i \geq 1$ and $j \geq 0$. In either case, note that the relations in $\mathcal{A}_p$ are *not* homogeneous because $\mathrm{Sq}^0 = P^0 = 1$. The associated homogenous pre-Koszul algebra of $\mathcal{A}_p$ is $E^0\mathcal{A}_p = \mathcal{A}_p^L$, the mod-$p$ Steenrod algebra for simplicial restricted Lie algebras [38, 2.2(6)]. This has the same generators and relations as $\mathcal{A}_p$, except that $\mathrm{Sq}^0 = P^0 = 0$. It turns out that $\mathcal{A}_p^L$ is Koszul, so that $\mathcal{A}_p$ is Koszul as well.

5.2.1 The Koszul Complex

Let R and L be graded right and left modules, respectively, over the graded k-algebra A. Let $B_*(R,A,L)$ be the usual bar complex. The upshot of knowing that A is Koszul is that $B_*(R,A,L)$ has a relatively small subcomplex $K_*(R,A,L)$, called the *Koszul complex*, that is homotopy equivalent to $B_*(R,A,L)$ itself. Therefore, $K_*(R,A,L)$ can be used to more easily compute $\mathrm{Tor}_*^A(R,L)$.

Theorem 5.2.10. *If A is a Koszul algebra, R is a right A-module, and L is a left A-module, then* $\mathrm{Tor}_*^A(R,L)$ *is the homology of the Koszul complex* $K_*(R,A,L)$.

We defer the proof of Theorem 5.2.10 for a moment to define the Koszul complex and the injection $K_*(R,A,L) \hookrightarrow B_*(R,A,L)$. Let A be a Koszul algebra over k with a fixed set $\{a_i\}$ of pre-Koszul generators and consider the inclusion $i\colon (E^0A)_{1,*} \hookrightarrow A$ given by sending the generators $a_i \in A$, viewed as lying in $(E^0A)_{1,|a_i|-1} = (F_1A/F_0A)_{|a_i|}$, to themselves. This induces an injective map

$$i\colon B_{s,s,*}(k, E^0A, k) \hookrightarrow B_s(k, A, k).$$

Here, $B_{s,s,*}(k, E^0A, k)$ is generated by the length s elements $[b_{i_1}|b_{i_2}|\cdots|b_{i_s}]$, where $i(b_{i_j})$ is a Koszul generator of A for all $1 \leq j \leq s$. Each element of $\mathrm{Tor}_{s,s,*}^{E^0A}(k,k)$ may be represented by a cycle in $B_{s,s,*}(k, E^0A, k)$ of the form

$$\sum_i c_i[b_{i_1}|b_{i_2}|\cdots|b_{i_s}]$$

where $c_i \in k$ for all i. Moreover, this representation is unique since $B_{s+1,s,*}(k, E^0A, k) = 0$. (This is due to the fact that any $(s+1)$-chain must have length at least $s+1$.) Therefore, this determines an injection

$$j\colon \mathrm{Tor}_{s,s,*}^{E^0A}(k,k) \hookrightarrow B_{s,s,*}(k, E^0A, k).$$

Tensoring the composite $i \circ j$ over k with R on the left and L on the right gives an injection

$$\iota\colon R \otimes \mathrm{Tor}_{s,s,*}^{E^0A}(k,k) \otimes L \hookrightarrow B_s(R, A, L)$$

Explicitly,

$$\iota\left(r \otimes \left(\sum_i c_i[b_{i_1}|b_{i_2}|\cdots|b_{i_s}]\right) \otimes \ell\right) = r \otimes \left(\sum_i c_i[a_{i_1}|a_{i_2}|\cdots|a_{i_s}]\right) \otimes \ell$$

Definition 5.2.11. Given a Koszul algebra A, a right A-module R, and a left A-module L, the *Koszul complex* is the complex $K_*(R,A,L)$ with

$$K_s(R,A,L) = R \otimes \mathrm{Tor}_{s,s,*}^{E^0A}(k,k) \otimes L,$$

The differential $K_s(R,A,L) \to K_{s-1}(R,A,L)$ (which preserves the internal degree) is defined by formula (3.5) in [38].

Remark 5.2.12. $\iota\colon K_*(R,A,L) \hookrightarrow B_*(R,A,L)$ is an injection of differential coalgebras [38, Prop 3.10].

In lieu of giving the general formula for the differential on $K_*(R,A,L)$, we illustrate what it does via the following example, taken from [38, Example 3.6].

Example 5.2.13. Consider the Koszul complex $K_*(\mathbb{F}_2, \mathcal{A}_2, \mathbb{F}_2)$, where $\mathcal{A}_2$ is the mod-2 Steenrod algebra. There is an element

$$[\mathrm{Sq}^2 \,|\, \mathrm{Sq}^2 \,|\, \mathrm{Sq}^3] + [\mathrm{Sq}^2 \,|\, \mathrm{Sq}^4 \,|\, \mathrm{Sq}^1] + [\mathrm{Sq}^5 \,|\, \mathrm{Sq}^1 \,|\, \mathrm{Sq}^1]$$

in the bar complex $B_{3,3,4}(\mathbb{F}_2, E^0\mathcal{A}_2, \mathbb{F}_2)$, which represents an element of $K_{3,7}(\mathbb{F}_2, \mathcal{A}_2, \mathbb{F}_2)$. The differential on $K_{3,7}(\mathbb{F}_2, \mathcal{A}_2, \mathbb{F}_2)$ sends this element to

$$\begin{aligned}&[\mathrm{Sq}^2\,\mathrm{Sq}^2 \,|\, \mathrm{Sq}^3] + [\mathrm{Sq}^2 \,|\, \mathrm{Sq}^2\,\mathrm{Sq}^3] + [\mathrm{Sq}^2\,\mathrm{Sq}^4 \,|\, \mathrm{Sq}^1]\\ &\quad + [\mathrm{Sq}^2 \,|\, \mathrm{Sq}^4\,\mathrm{Sq}^1] + [\mathrm{Sq}^5\,\mathrm{Sq}^1 \,|\, \mathrm{Sq}^1] + [\mathrm{Sq}^5 \,|\, \mathrm{Sq}^1\,\mathrm{Sq}^1].\end{aligned}$$

Using the Adem relations

$$\mathrm{Sq}^1\,\mathrm{Sq}^1 = 0, \quad \mathrm{Sq}^2\,\mathrm{Sq}^2 = 0, \quad \mathrm{Sq}^2\,\mathrm{Sq}^3 = \mathrm{Sq}^5 + \mathrm{Sq}^4\,\mathrm{Sq}^1, \quad \text{and} \quad \mathrm{Sq}^2\,\mathrm{Sq}^4 = \mathrm{Sq}^6 + \mathrm{Sq}^5\,\mathrm{Sq}^1,$$

this reduces to $[\mathrm{Sq}^2 \,|\, \mathrm{Sq}^5] + [\mathrm{Sq}^6 \,|\, \mathrm{Sq}^1]$, which is an element of $B_{2,2,5}(\mathbb{F}_2, E^0\mathcal{A}_2, \mathbb{F}_2)$ that determines an element of $K_{2,7}(\mathbb{F}_2, \mathcal{A}_2, \mathbb{F}_2)$.

Proof of Theorem 5.2.10. Suppose that A is Koszul. The length filtration of A induces a filtration on the bar complex $B(R,A,L)$. The associated spectral sequence (the bar spectral sequence) takes the form

$$E^1_{s,\ell,*} = R \otimes \mathrm{Tor}^A_{s,\ell,*}(k,k) \otimes L \Longrightarrow \mathrm{Tor}^A_{s+\ell,*}(R,L)$$

Since A is a Koszul algebra, $\mathrm{Tor}^A_{s,\ell,*}(k,k) = 0$ unless $s = \ell$; it follows that E^1 is exactly the Koszul complex. This restriction also forces the spectral sequence to collapse at E^1, which gives the result. □

5.2.2 The co-Koszul Complex

Definition 5.2.14. The *co-Koszul complex* is the complex $K^*(R,A,L) = K_*(R,A,L)^\vee$. Explicitly,

$$K^s(R,A,L) = R^\vee \otimes \mathrm{Ext}^{s,s,*}_{E^0A}(k,k) \otimes L^\vee$$

The differential $K^s(R,A,L) \to K^{s+1}(R,A,L)$ is described in [38, pg. 49].

Remark 5.2.15. Dual to Theorem 5.2.10 is the fact that if A is Koszul, then the co-Koszul complex computes $\mathrm{Ext}_A(L, R^\vee) = \mathrm{Tor}^A(R, L)^\vee$, where $(-)^\vee$ denotes the k-linear dual.

Remark 5.2.16. Note that the injection

$$\iota\colon K_s(R, A, L) \hookrightarrow B_s(R, A, L)$$

of the previous section dualizes to give a surjection

$$\iota^\vee\colon C^s(R, A, L) \twoheadrightarrow K^s(R, A, L)$$

of the cobar complex onto the co-Koszul complex.

There is an alternative way to describe the co-Koszul complex in terms of the *Koszul dual* of a Koszul algebra.

Definition 5.2.17. The *Koszul dual* of a Koszul algebra A is the algebra $A^! = \mathrm{Ext}_A(k, k)^{\mathrm{op}}$. This is a graded k-algebra with $A^!_s = \mathrm{Ext}_A^{s,s}(k, k)^{\mathrm{op}}$.

Remark 5.2.18. If A is a Koszul algebra, then A is a quadratic algebra. If we write $A = T(V)/R$, where V is the vector space of generators and R is the subspace of homogeneous quadratic relations, then $A^! \cong T(V^\vee)/R^\perp$.

From this perspective, the co-Koszul complex $K^*(R, A, L)$ takes the form

$$R^\vee \otimes L^\vee \to R^\vee \otimes (A^!_1) \otimes L^\vee \to \cdots \to R^\vee \otimes (A^!_s) \otimes L^\vee \to R^\vee \otimes (A^!_{s+1}) \otimes L^\vee \to \cdots,$$

where the coboundary map is to be described in Theorem 5.2.22.

Example 5.2.19. Let $A = T(V)/R$, where V is a finite-dimensional vector space and $R = \{v \otimes w - w \otimes v \mid v, w \in V\}$. Then A is the symmetric algebra on V. Note that for any $\alpha \in V^\vee$, we have

$$(\alpha \otimes \alpha)(v \otimes w - w \otimes v) = \alpha(v)\alpha(w) - \alpha(w)\alpha(v) = 0$$

and so $A^! = T(V^\vee)/\{\alpha \otimes \alpha \mid \alpha \in V^\vee\}$. This is isomorphic to the exterior algebra on $V^\vee$. If we let $\{x_1, \ldots, x_n\}$ be a basis for V, then this says that the Koszul dual of $k[x_1, \ldots, x_n]$ is $\Lambda_k[x_1, \ldots, x_n]$. The co-Koszul complex $K^*(k, A, k)$ takes the form

$$k \to \Lambda^1_k[x_1, \ldots, x_n] \to \cdots \to \Lambda^s_k[x_1, \ldots, x_n] \to \Lambda^{s+1}_k[x_1, \ldots, x_n] \to \cdots.$$

Note that this is the dual of the classical Koszul complex, originally introduced by J.L. Koszul [20] in 1950.

One major advantage of working with the co-Koszul complex instead of the Koszul complex is that its generators, relations, and differential are more easily described. We give this general description in Theorem 5.2.22 below, which makes use of the following definition:

Definition 5.2.20. If $\{a_i\}_{i \in I}$ is a set of pre-Koszul generators of A, then the set B_A consisting of the monomials 1, a_i, and $a_{i_1} \cdots a_{i_n}$, where $n \geq 1$ and $i_j \in I$ for $1 \leq j \leq n$, is a basis of A as a k-module. We say that a subset

$$S \subseteq U = I \cup (I \times I) \cup (I \times I \times I) \cup \cdots$$

is a *labeling set* for B_A if for all $a \in B_A$ with $a \neq 1$, there is a unique sequence (called a *label*) $(i_1, \ldots, i_n) \in S$ such that $a = a_{i_1} \cdots a_{i_n}$.

Remark 5.2.21. In [38, §5], it is shown that if A is a homogeneous pre-Koszul algebra and (B_A, S) satisfies

- $(i_1, \ldots, i_k) \in S$ and $(j_1, \ldots, j_\ell) \in S$ implies that $(i_1, \ldots, i_k, j_1, \ldots, j_\ell) \in S$, or else the label of each monomial appearing in an expression for $a_{i_1} \ldots a_{i_k} a_{j_1} \ldots a_{j_\ell}$ (after applying the admissible relations of A) has a label that is strictly greater than $(i_1, \ldots, i_k, j_1, \ldots, j_\ell)$ in the lexicographic order, and

- for $k > 2$, we have $(i_1, \ldots, i_k) \in S$ if and only if for all j, we have $(i_1, \ldots, i_j) \in S$ and $(i_{j+1}, \ldots, i_k) \in S$,

then A is Koszul. In this case, we call (B_A, S) a *Poincaré-Birkhoff-Witt (PBW) basis* and A a *PBW algebra*. This provides a purely algebraic criterion for determining whether or not a homogeneous pre-Koszul algebra is Koszul. It is satisfied by many common examples, including the algebras $k[x_i]$, $\Lambda\{x_i\}$, and $\mathcal{A}_p^L$ for all primes p.

As before, we (formally) obtain a basis $B_{E^0 A}$ by replacing the letter a_i with b_i. We also let $\alpha_i \in \mathrm{Ext}_A^{1,1,*}(k, k)$ and $\beta_i \in \mathrm{Ext}_{E^0 A}^{1,1,*}(k, k)$ be "dual" to a_i and b_i, respectively. (In other words, α_i is represented by $[a_i^\vee]$ in the cobar complex $C^*(k, A, k)$ and similarly for β_i.)

Theorem 5.2.22. ([38, Theorem 4.6]) *Suppose that A has a set of pre-Koszul generators $\{a_i\}_{i\in I}$, a labeled basis (B_A, S), and "admissible" relations*

$$a_r a_s = \sum_{(i,j)\in S} c_{r,s,i,j} a_i a_j,$$

where $c_{r,s,i,j} \in k$ is a constant depending on r, s, i, and j. Then

- *$K^*(k, A, k)$ is generated by the set $\{\beta_i\}_{i\in I}$, where $\beta_i \in K^1(k, A, k) = \mathrm{Ext}^{1,1,*}_{E^0 A}(k, k)$.*
- *For each pair $(i, j) \in S$, there is a relation in $K^*(k, A, k)$ of the form*

  $$(-1)^{n_{i,j}} \beta_i \beta_j = - \sum_{(r,s)\in U-S} (-1)^{n_{r,s}} c_{r,s,i,j} \beta_r \beta_s,$$

  where $n_{i,j} = |\beta_i| + (|\beta_i| - 1)(|\beta_j| - 1)$.
- *The differential $\delta\colon K^*(k, A, k) \to K^{*+1}(k, A, k)$ is determined by the formula*

  $$\delta(\beta_i) = \sum_j (-1)^{|\beta'_j|} \beta'_j \otimes \beta''_j,$$

  where $\sum_j \beta'_j \otimes \beta''_j$ is the coproduct of β_i in $A^\vee$. This is equivalent to

  $$\delta(\beta_i) = \sum_{(r,s)\in U-S} (-1)^{n_{r,s}} c_{r,s,i} \beta_r \beta_s.$$

Example 5.2.23. [38, §7] As an application of the results of Theorem 5.2.22, we describe the co-Koszul complexes $K^*(\mathbb{F}_p, \mathcal{A}_p, \mathbb{F}_p)$ for all primes p. It is generated by

- (if $p = 2$): $\sigma_r \in \mathrm{Ext}^{1,1,r-1}_{E^0\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$, represented by $[(\mathrm{Sq}^r)^\vee]$, for $r \geq 1$.
- (if $p \geq 3$): $\pi_r \in \mathrm{Ext}^{1,1,2r(p-1)-1}_{E^0\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$ and $\rho_s \in \mathrm{Ext}^{1,1,2s(p-1)}_{E^0\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$, represented by $[(P^r)^\vee]$ and $[(\beta P^s)^\vee]$, respectively, for $r \geq 1$ and $s \geq 0$.

The relations of $K^*(\mathbb{F}_p, \mathcal{A}_p, \mathbb{F}_p)$ are

- (if $p = 2$):

  $$\sigma_i \sigma_j = \sum_{r=2j}^{\lfloor 2(i+j)/3 \rfloor} \binom{i-r-1}{r-2j} \sigma_r \sigma_{i+j-r} \qquad (i \geq 2j)$$

- (if $p \geq 3$):

$$-\pi_i\pi_j = \sum_{r=pj}^{\lfloor p(i+j)/(p+1)\rfloor} (-1)^{r+j}\binom{(p-1)(i-r)-1}{r-pj}\pi_r\pi_{i+j-r} \qquad (i \geq pj)$$

$$-\pi_i\rho_j = \sum_{r=pj+1}^{\lfloor p(i+j)/(p+1)\rfloor} (-1)^{r+j-1}\binom{(p-1)(i-r)-1}{r-pj-1}\pi_r\rho_{i+j-r} \qquad (i > pj)$$

$$\rho_i\pi_j = \sum_{r=pj}^{\lfloor p(i+j)/(p+1)\rfloor} (-1)^{r+j}\left[\binom{(p-1)(i-r)}{r-pj}\pi_r\rho_{i+j-r} - \binom{(p-1)(i-r)-1}{r-pj}\rho_r\pi_{i+j-r}\right] \qquad (i > pj)$$

$$-\rho_i\rho_j = \sum_{r=pj+1}^{\lfloor p(i+j)/(p+1)\rfloor} (-1)^{r+j-1}\binom{(p-1)(i-r)-1}{r-pj-1}\rho_r\rho_{i+j-r} \qquad (i > pj)$$

The differential on $K^*(\mathbb{F}_p, \mathcal{A}_p, \mathbb{F}_p)$ is determined by the following formula(s):

- (if $p = 2$):

$$\delta(\sigma_i) = \sum_{r=1}^{i-1} \sigma_r\sigma_{i-r}$$

- (if $p \geq 3$):

$$\delta(\pi_i) = \sum_{r=1}^{i-1}\binom{i}{r}\pi_r\pi_{i-r}, \qquad \delta(\rho_j) = \sum_{r=1}^{j-1}\binom{j}{r}(\pi_r\rho_{j-r} - \rho_r\pi_{j-r})$$

Notation 5.2.24. If $I = (i_1, \ldots, i_n)$ is any sequence of positive integers, we will abbreviate the product $\sigma_{i_1}\cdots\sigma_{i_n}$ as σ_I. Similarly, if $J = (\epsilon_1, i_1, \ldots, \epsilon_n, i_n)$, where each ϵ_k is 0 or 1 and $i_k \geq \epsilon_k$ for all $1 \leq k \leq n$, then π_J is a product whose k'th term is π_{i_k} if $\epsilon_k = 0$ and is ρ_{i_k} if $\epsilon_k = 1$.

Remark 5.2.25. As noted in [38], for any prime p, the co-Koszul complex $K^*(\mathbb{F}_p, \mathcal{A}_p, \mathbb{F}_p)$ is isomorphic to the opposite of the (mod-p) *lambda algebra*, Λ. The lambda algebra was first defined at the prime $p = 2$ in [13]; a reference for the odd primary case is in [18]. It is most often used to compute the E_2-term of the Adams spectral sequence

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_*(\mathbb{S})^\wedge_p,$$

converging to the p-completion of the ring of stable homotopy groups of spheres. It is a bigraded algebra with generators

- (at $p = 2$): λ_i (for $i \geq 0$) of bidegree $(1, i)$
- (at $p \geq 3$): λ_i (for $i \geq 1$) of bidegree $(1, 2i(p-1)-1)$ and μ_i (for $i \geq 0$) of bidegree $(1, 2i(p-1))$.

The isomorphism from $K^*(\mathbb{F}_2, \mathcal{A}_2, \mathbb{F}_2)$ to the opposite of the lambda algebra sends the generator σ_{i+1} to the generator λ_i ($i \geq 0$). If p is odd, then the isomorphism from $K^*(\mathbb{F}_p, \mathcal{A}_p, \mathbb{F}_p)$ to the opposite of the lambda algebra sends the generator π_{i+1} to λ_i ($i \geq 0$) and the generator ρ_j to $-\mu_{j-1}$ ($j \geq 0$).

We will also need a description of the differential of $K^*(k, A, M) = K^*(k, A, k) \otimes M^\vee$ for an A-module M. To this end, let $\lambda\colon A \otimes M \to M$ be the A-module structure map of M. Dualizing, we have a coaction $\lambda^\vee\colon M^\vee \to A^\vee \otimes M^\vee$ of $A^\vee$ on $M^\vee$. Consider the composite

$$\bar{\lambda}^\vee\colon M^\vee \xrightarrow{\lambda^\vee} A^\vee \otimes M^\vee \xrightarrow{i^\vee \otimes \mathrm{id}} (E^0 A)^\vee_1 \otimes M^\vee$$

For $\mu \in M^\vee$, write

$$\bar{\lambda}^\vee(\mu) = \sum_i \beta_i \otimes \mu_i$$

where $\{\mu_i\}$ is a basis for $M^\vee$ and the β_i are dual to the pre-Koszul generators of A. Given $\bar{\lambda}^\vee(\mu)$, we let

$$\Delta_M(\mu) = \sum_i (-1)^{|\beta_i|} \beta_i \otimes \mu_i.$$

Theorem 5.2.26. *[38, Theorem 4.3] If M is an A-module, then*

$$K^*(k, A, M) = K^*(k, A, k) \otimes M^\vee$$

is a differential $K^*(k, A, k)$-module. The differential, δ, is determined by the formula $\delta(1 \otimes \mu) = \Delta_M(\mu)$.

Example 5.2.27. If M is an $\mathcal{A}_p$-module, then the differential of $K^*(\mathbb{F}_p, \mathcal{A}_p, M)$ is determined by

$$\delta(1 \otimes \mu) = \begin{cases} \sum_{i \geq 1} \sigma_i \otimes \mathrm{Sq}^i_*(\mu) & \text{if } p = 2 \\ \sum_{i \geq 1} -\pi_i \otimes P^i_*(\mu) + (-1)^{|\mu|} \sum_{j \geq 0} \rho_j \otimes \beta P^j_*(\mu) & \text{if } p \geq 3 \end{cases}$$

5.3 Computing $\mathrm{Untor}^{\mathcal{R}_p}_*(\mathbb{F}_p, M)$

Recall from Definition 3.1.4 that $\mathcal{R}_p(-\infty)$ is the unital, associative, graded $\mathbb{F}_p$-algebra generated by

- the symbols Q^r (for $r \geq 0$) of degree r if $p = 2$, and
- the symbols Q^r (for $r \geq 0$) and βQ^s (for $s \geq 1$) of degree $2r(p-1)$ and $2s(p-1)-1$, respectively, if $p \geq 3$.

The relations of $\mathcal{R}_p(-\infty)$ are the Adem relations. Because $Q^0 \neq 1$, the relations of $R(-\infty)$ are homogeneous, in contrast to the case of the Steenrod algebra. Moreover, the set $\{Q^I \mid I \text{ admissible}\}$ is a PBW basis for $\mathcal{R}_p(-\infty)$. Therefore, $\mathcal{R}_p(-\infty)$ is a Koszul algebra with Koszul generators $\{Q^r\}_{r \geq 0}$ if $p = 2$, and $\{Q^r\}_{r \geq 0} \cup \{\beta Q^s\}_{s \geq 1}$ if $p \geq 3$.

Recall that for an integer $n \geq 1$, we write $\mathcal{R}_p(n)$ for the quotient $\mathcal{R}_p(-\infty)/B(n)$, where $B(n)$ is the ideal generated by the set $\{Q^I \mid e(I) < n\}$. For any $n \geq 1$, the algebra $\mathcal{R}_p(n)$ is a homogeneous Koszul algebra with relations given by the image of the Adem relations. Since any operation Q^I in $\mathcal{R}_p(n)$ has excess at least n by definition, the sets of Koszul generators for $\mathcal{R}_p(n)$ are

- $\{Q^r\}_{r \geq n}$ if $p = 2$, and
- $\{Q^r\}_{r \geq n} \cup \{\beta Q^s\}_{s \geq n+1}$ if $p \geq 3$.

Our goal in this section is to compute $\mathrm{Untor}^{\mathcal{R}_p}_*(\mathbb{F}_p, M)$, where M is a 1-allowable module over $\mathcal{R}_p$ (i.e., for an $\mathcal{R}_p(1)$-module M). Since $\mathcal{R}_p(1)$ is Koszul, there is a Koszul complex $K_*(\mathbb{F}_p, \mathcal{R}_p(1), M)$ whose homology is $\mathrm{Untor}^{\mathcal{R}_p}_*(\mathbb{F}_p, M)$. As mentioned in the previous section, computing the dual, $\mathrm{Untor}^{\mathcal{R}_p}_*(\mathbb{F}_p, M)^\vee$, via the co-Koszul complex $K^*(\mathbb{F}_p, \mathcal{R}_p(1), M)$ will be easier than working with $K_*(\mathbb{F}_p, \mathcal{R}_p(1), M)$ directly.

Theorem 5.3.1. *[35, 22] The co-Koszul complex $K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbb{F}_p)$ is generated by*

- *(if $p = 2$): $\sigma^r \in \mathrm{Ext}^{1,1,r-1}_{\mathcal{R}_2(1)}(\mathbb{F}_2, \mathbb{F}_2)$, represented by $[(Q^r)^\vee]$, for $r \geq 1$*
- *(if $p \geq 3$): $\pi^r \in \mathrm{Ext}^{1,1,2r(p-1)-1}_{\mathcal{R}_p(1)}(\mathbb{F}_p, \mathbb{F}_p)$ and $\rho^s \in \mathrm{Ext}^{1,1,2s(p-1)-2}_{\mathcal{R}_p(1)}(\mathbb{F}_p, \mathbb{F}_p)$, represented by $[(Q^r)^\vee]$ and $[(\beta Q^s)^\vee]$, respectively, for $r \geq 1$ and $s \geq 2$.*

The relations of $K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbb{F}_p)$ are

- *(if $p = 2$):*
$$\sigma^i \sigma^j = \sum_{s=1}^{\lfloor (i-1)/2 \rfloor} \binom{j-s-1}{i-2s-1} \sigma^{i+j-s}\sigma^s \qquad (i \leq 2j)$$
- *(if $p \geq 3$):*
$$\pi^i \pi^j = \sum_{s=1}^{\lfloor (i-1)/p \rfloor} (-1)^{i+s+1} \binom{(p-1)(j-s)-1}{i-ps-1} \pi^{i+j-s}\pi^s \qquad (i \leq pj)$$
$$\pi^i \rho^j = \sum_{s=2}^{\lfloor i/p \rfloor} (-1)^{i+s} \binom{(p-1)(j-s)-1}{i-ps} \pi^{i+j-s}\rho^s \qquad (i < pj)$$
$$\rho^i \pi^j = \sum_{s=2}^{\lfloor i/p \rfloor} (-1)^{i+s} \binom{(p-1)(j-s)}{i-ps} \pi^{i+j-s}\rho^s + \sum_{s=1}^{\lfloor (i-1)/p \rfloor} (-1)^{i+s+1} \binom{(p-1)(j-s)-1}{i-ps-1} \rho^{i+j-s}\pi^s \qquad (i \leq pj)$$
$$\rho^i \rho^j = \sum_{s=2}^{\lfloor i/p \rfloor} (-1)^{i+s} \binom{(p-1)(j-s)}{i-ps} \rho^{i+j-s}\rho^s \qquad (i < pj)$$

The differential of $K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbb{F}_p)$ is determined by the following formulas:

- *(if $p = 2$):*
$$\delta(\sigma^r) = \sum_{k=1}^{r} \binom{r}{k} \sigma^k \sigma^{r-k}$$
- *(if $p \geq 3$):*
$$\delta(\pi^r) = \sum_{k=1}^{r} -\binom{r}{k} \pi^k \pi^{r-k} \qquad \delta(\rho^s) = \sum_{k=2}^{s} \binom{s}{k} (-\pi^k \rho^{s-k} + \rho^k \pi^{s-k})$$

To prove Theorem 5.3.1, we will need to know some facts about the structure of the dual Dyer-Lashof algebra, $\mathcal{R}_p^\vee$. As a commutative $\mathbb{F}_p$-algebra,

$$\mathcal{R}_p^\vee = \prod_{k\geq 0} \mathcal{R}_p^\vee[k]$$

where $\mathcal{R}_p^\vee[0] = \mathbb{F}_p$ [14, I, §3]. With the aim of describing the factors $\mathcal{R}_p^\vee[k]$, we define the following elements of $\mathcal{R}_p^\vee[k]$:

- $\xi_{0k} = (Q^{I_{0k}})^\vee$, if $k \geq 0$,
- $\xi_{jk} = (Q^{I_{jk}})^\vee$, if $1 \leq j \leq k$,
- $\tau_{jk} = (Q^{J_{jk}})^\vee$, if $1 \leq j \leq k$,
- $\sigma_{ijk} = (Q^{K_{ijk}})^\vee$, if $1 \leq i < j \leq k$,

(if $p = 2$, then only ξ_{0k} and ξ_{jk} are defined). The element ξ_{0k} is the identity element of $\mathcal{R}_p^\vee[k]$ and $\prod_k \xi_{0k}$ is the identity of $\mathcal{R}_p^\vee$. The sequences I_{jk}, J_{jk}, and K_{ijk} are defined below:

- $I_{0k} = (0, \ldots, 0)$
- $I_{jk} = \begin{cases} (2^{k-1} - 2^{k-1-j},\ 2^{k-2} - 2^{k-2-j},\ \ldots,\ 2^j - 1,\ 2^{j-1},\ 2^{j-2},\ \ldots,\ 1) & \text{if } p = 2 \\ (0,\ p^{k-1} - p^{k-1-j},\ \ldots,\ 0,\ p^j - 1,\ 0,\ p^{j-1},\ 0,\ p^{j-2},\ \ldots,\ 0,\ 1) & \text{if } p \geq 3 \end{cases}$
- $J_{jk} = (0,\ p^{k-1} - p^{k-1-j},\ \ldots,\ 0,\ p^j - 1,\ 1,\ p^{j-1},\ 0,\ p^{j-2},\ \ldots,\ 0,\ 1)$
- $K_{ijk} = (0,\ p^{k-1} - p^{k-1-j},\ \ldots,\ 0,\ p^j - p^{j-i} - 1,\ 1,\ p^{j-1} - p^{j-1-i},\ J_{i,j-1})$

Remark 5.3.2. Recall that the operations Q^r are defined in terms of lower-indexed operations Q_i (see the construction end of §3.1). If we let Q_i act on a class in dimension 0, then there are correspondences

- $Q^{I_{jk}} \longleftrightarrow \begin{cases} Q_0 \cdots Q_0 Q_1 \cdots Q_1 & (\text{if } p = 2) \\ Q_0 \cdots Q_0 Q_{2(p-1)} \cdots Q_{2(p-1)} & (\text{if } p \geq 3) \end{cases}$

- $Q^{J_{jk}} \longleftrightarrow Q_{p-1} \cdots Q_{p-1} \beta Q_{2(p-1)} \cdots Q_{2(p-1)}$

- $Q^{K_{ijk}} \longleftrightarrow Q_0 \cdots Q_0 \beta Q_{p-1} \cdots Q_{p-1} \beta Q_{2(p-1)} \cdots Q_{2(p-1)}$

If $p \geq 3$, let $M[k] \subseteq \mathcal{R}_p^\vee[k]$ be the submodule spanned by ξ_{0k} and the monomials

$$\sigma_{e_1 e_2 k} \cdots \sigma_{e_{j-1} e_j k} \quad (j \text{ even}), \qquad \sigma_{e_1 e_2 k} \cdots \sigma_{e_{j-2} e_{j-1} k} \tau_{e_j k} \quad (j \text{ odd})$$

where $1 \leq e_1 < e_2 < \cdots < e_j \leq k$.

Proposition 5.3.3. *If $p = 2$, then*

$$\mathcal{R}_p^\vee[k] = \mathbb{F}_2[\xi_{1k}, \ldots, \xi_{kk}]$$

as an algebra. If $p \geq 3$, then

$$\mathcal{R}_p^\vee[k] \cong \mathbb{F}_p[\xi_{1k}, \ldots, \xi_{kk}] \otimes M[k]$$

as $\mathbb{F}_p$-modules. As an algebra, $\mathcal{R}_p^\vee[k]$ is determined by requiring the product to be commutative and to satisfy the following relations:

1. $\tau_{ik}\tau_{jk} = \begin{cases} \xi_{kk}\sigma_{ijk} & \text{if } i < j \\ 0 & \text{if } i = j \end{cases}$
2. $\sigma_{ijk}\tau_{nk} = (\tau_{ik}\tau_{jk}\tau_{nk})/\xi_{kk}$
3. $\sigma_{ijk}\sigma_{mnk} = (\tau_{ik}\tau_{jk}\tau_{mk}\tau_{nk})/\xi_{kk}^2$

(Note that the numerators of the right-hand sides of relations 2 and 3 are divisible by ξ_{kk} and ξ_{kk}^2, respectively, after applying relation 1.)

Remark 5.3.4. The component $\mathcal{R}_p^\vee[1] \subseteq \mathcal{R}_p^\vee$ contains the duals of the operations Q^r and βQ^s. In particular, ξ_{11}^r is dual to Q^r and $\tau_{11}\xi_{11}^s$ is dual to βQ^s.

Although $\mathcal{R}_p^\vee$ is *not* a coalgebra[2] , there is a well-defined coproduct on the positive degree elements of $\mathcal{R}_p^\vee$ [14, I, Theorem 3.13].

[2] This is due to the fact that $(\mathcal{R}_p)_0 = \mathbb{F}_p[Q^0]$ is not a finitely generated $\mathbb{F}_p$-module.

Proposition 5.3.5. *[14, I, Theorem 3.13] The coproduct, Δ, is given on the generators of $\mathcal{R}_p^\vee$ by*

$$\begin{aligned}
\Delta(\xi_{0k}) &= \sum_i \xi_{0,k-i} \otimes \xi_{0i},\\
\Delta(\xi_{jk}) &= \sum_{(h,i)} \xi_{k-i,k-i}^{p^i-p^{i-h}} \xi_{j-h,k-i}^{p^{i-h}} \otimes \xi_{hi},\\
\Delta(\tau_{jk}) &= \sum_{(h,i)} \xi_{k-i,k-i}^{p^i-p^{i-h}} \xi_{j-h,k-i}^{p^{i-h}} \otimes \tau_{hi} + \sum_i \xi_{k-i,k-i}^{p^i-1} \tau_{j-i,k-i} \otimes \xi_{ii},\\
\Delta(\sigma_{ijk}) &= \sum_{(f,g,h)} \xi_{k-h,k-h}^{p^h-p^{h-f}-p^{h-g}} \left(\xi_{j-g,k-h}^{p^{h-g}} \xi_{i-f,k-h}^{p^{h-f}} - \xi_{j-f,k-h}^{p^{h-f}} \xi_{i-g,k-h}^{p^{h-g}}\right) \otimes \sigma_{fgh}\\
&\qquad + \sum_{(g,h)} \xi_{k-h,k-h}^{p^h-p^{h-g}-1} \left(\xi_{j-g,k-h}^{p^{h-g}} \tau_{i-h,k-h} - \xi_{i-g,k-h}^{p^{h-g}} \tau_{j-h,k-h}\right) \otimes \tau_{gh}\\
&\qquad\qquad + \sum_h \xi_{k-h,k-h}^{p^h-1} \sigma_{i-h,j-h,k-h} \otimes \xi_{hh}.
\end{aligned}$$

In particular, we have

$$\Delta(\xi_{11}) = 1 \otimes \xi_{11} + \xi_{11} \otimes 1, \qquad \Delta(\tau_{11}) = 1 \otimes \tau_{11} + \tau_{11} \otimes 1,$$

and so

$$\Delta(\xi_{11}^r) = \sum_{k=0}^{r} \binom{r}{k} \xi_{11}^k \otimes \xi_{11}^{r-k}, \qquad \Delta(\tau_{11}\xi_{11}^s) = \sum_{k=0}^{s} \binom{s}{k} (\xi_{11}^k \otimes \tau_{11}\xi_{11}^{s-k} + \tau_{11}\xi_{11}^k \otimes \xi_{11}^{s-k}).$$

Proof of Theorem 5.3.1. First, we consider the case $p = 2$. Because the algebra $\mathcal{R}_2(1)$ has Koszul generators $\{Q^r\}_{r\geq 1}$, the co-Koszul complex $K^*(\mathbb{F}_2, \mathcal{R}_2(1), \mathbb{F}_2)$ is generated by the elements $\sigma^r \in \mathrm{Ext}_{\mathcal{R}_2(1)}^{1,1,r-1}(\mathbb{F}_2, \mathbb{F}_2)$, where $r \geq 1$ and σ^r is dual to Q^r (i.e., σ^r is represented by ξ_{11}^r in the cobar complex). If $r > 2s$, then we have a relation

$$Q^r Q^s = \sum_{j=\lceil r/2 \rceil}^{r-s-1} \binom{j-s-1}{2j-r} Q^{r+s-j} Q^j = \sum_{(i,j)} \binom{j-s-1}{2j-r} Q^i Q^j,$$

where (i, j) satisfies $i + j = r + s$ and $i \leq 2j$. By Theorem 5.2.22, if $i \leq 2j$, then there is a relation in the co-Koszul complex of the form

$$\sigma^i \sigma^j = \sum_{(r,s)} \binom{j-s-1}{2j-r} \sigma^r \sigma^s,$$

where (r, s) satisfies $r > 2s$. If we use the equation $r = i + j - s$ to eliminate r from this sum, then we have

$$\sigma^i \sigma^j = \sum_{s=1}^{\lfloor (i-1)/2 \rfloor} \binom{j-s-1}{2j-(i+j-s)} \sigma^{i+j-s} \sigma^s = \sum_{s=1}^{\lfloor (i-1)/2 \rfloor} \binom{j-s-1}{j-i+s} \sigma^{i+j-s} \sigma^s$$

Using the formula $\binom{m+n}{m} = \binom{m+n}{n}$, we can rewrite this binomial coefficient as

$$\binom{j-s-1}{j-i+s} = \binom{(j-i+s)+(i-2s-1)}{j-i+s} = \binom{j-s-1}{i-2s-1}$$

and hence

$$\sigma^i \sigma^j = \sum_{s=1}^{\lfloor (i-1)/2 \rfloor} \binom{j-s-1}{i-2s-1} \sigma^{i+j-s} \sigma^s.$$

To determine the differential in the co-Koszul complex, recall that σ^r is represented by $[\xi_{11}^r]$ in the cobar complex. By Proposition 5.3.5, the coproduct of ξ_{11}^r in $\mathcal{R}_p^\vee$ is

$$\Delta(\xi_{11}^r) = \sum_{k=0}^{r} \binom{r}{k} \xi_{11}^k \otimes \xi_{11}^{r-k},$$

and so

$$\delta(\sigma^r) = \sum_{k=0}^{r} \binom{r}{k} \sigma^k \sigma^{r-k}.$$

Now let $p \geq 3$. The duals of the Koszul generators $\{Q^r\}_{r \geq 1} \cup \{\beta Q^s\}_{s \geq 2}$ of $\mathcal{R}_p(1)$ determine classes $\pi^r \in \mathrm{Ext}_{\mathcal{R}_p(1)}^{1,1,2r(p-1)-1}(\mathbb{F}_p, \mathbb{F}_p)$ and $\rho^s \in \mathrm{Ext}_{\mathcal{R}_p(1)}^{1,1,2r(p-1)-2}(\mathbb{F}_p, \mathbb{F}_p)$, respectively, that generate $K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbb{F}_p)$. Note that π^r is represented by ξ_{11}^r in the cobar complex and that ρ^s is represented by $\tau_{11}\xi_{11}^s$. Recall that in this case the relations of $\mathcal{R}_p(1)$ are:

- If $r > ps$, then

$$Q^r Q^s = \sum_{j=\lceil r/p \rceil}^{r-(p-1)s-1} (-1)^{r+j} \binom{(p-1)(j-s)-1}{pj-r} Q^{r+s-j} Q^j.$$

- If $r \geq ps$, then

$$\begin{aligned} Q^r \beta Q^s = &\sum_{j=\lceil r/p \rceil}^{r-(p-1)s} (-1)^{r+j} \binom{(p-1)(j-s)}{pj-r} \beta Q^{r+s-j} Q^j \\ &+ \sum_{j=\lceil (r+1)/p \rceil}^{r-(p-1)s} (-1)^{r+j-1} \binom{(p-1)(j-s)-1}{pj-r-1} Q^{r+s-j} \beta Q^j. \end{aligned}$$

formally applying β and the identity $\beta^2 = 0$ to the above gives

- If $r > ps$, then

$$\beta Q^r Q^s = \sum_{j=\lceil r/p \rceil}^{r-(p-1)s-1} (-1)^{r+j} \binom{(p-1)(j-s)-1}{pj-r} \beta Q^{r+s-j} Q^j.$$

- If $r \geq ps$, then

$$\beta Q^r \beta Q^s = \sum_{j=\lceil (r+1)/p \rceil}^{r-(p-1)s} (-1)^{r+j-1} \binom{(p-1)(j-s)-1}{pj-r-1} \beta Q^{r+s-j} \beta Q^j.$$

To illustrate how Theorem 5.2.22 leads to the claimed relations in the co-Koszul complex, we derive the formula for $\rho^i \pi^j$ if $i \leq pj$. This product corresponds to the operation $\beta Q^i Q^j$, which appears in the relation for $Q^r \beta Q^s$, with coefficient $(-1)^{r+j}\binom{(p-1)(j-s)}{pj-r}$, and in the relation for $\beta Q^r Q^s$, with coefficient $(-1)^{r+j}\binom{(p-1)(j-s)-1}{pj-r}$. Therefore,

$$\begin{aligned}(-1)^{n_{i,j}} \rho^i \pi^j = &-\sum_{(r,s)} (-1)^{n_{r,s}} (-1)^{r+j} \binom{(p-1)(j-s)}{pj-r} \pi^r \rho^s \\ &- \sum_{(r,s)} (-1)^{n'_{r,s}} (-1)^{r+j} \binom{(p-1)(j-s)-1}{pj-r} \rho^r \pi^s,\end{aligned}$$

where (r, s) satisfies $i + j = r + s$ and $r > ps$, and

$$\begin{aligned} n_{i,j} &= |\rho^i| + (|\rho^i| - 1)(|\pi^j| - 1) = 2i(p-1) + [2i(p-1)-1][2j(p-1)] \\ n_{r,s} &= |\pi^r| + (|\pi^r| - 1)(|\rho^s| - 1) = 2r(p-1) + 1 + [2r(p-1)][2s(p-1)-1] \\ n'_{r,s} &= |\rho^r| + (|\rho^r| - 1)(\pi^s - 1) = 2r(p-1) + [2r(p-1)-1][2s(p-1)]. \end{aligned}$$

This implies that

$$(-1)^{n_{i,j}} = 1, \qquad (-1)^{n_{r,s}} = -1, \qquad (-1)^{n'_{r,s}} = 1.$$

Hence,

$$\rho^i \pi^j = \sum_{(r,s)} (-1)^{r+j} \binom{(p-1)(j-s)}{pj-r} \pi^r \rho^s + \sum_{(r,s)} (-1)^{r+j+1} \binom{(p-1)(j-s)-1}{pj-r} \rho^r \pi^s.$$

Using the equation $r = i + j - s$ to eliminate r in the sums above gives

$$\rho^i \pi^j = \sum_{s=2}^{\lfloor i/p \rfloor} (-1)^{i+s} \binom{(p-1)(j-s)}{(p-1)j - i + s} \pi^{i+j-s} \rho^s + \sum_{s=1}^{\lfloor (i-1)/p \rfloor} (-1)^{i+s+1} \binom{(p-1)(j-s)-1}{(p-1)j - i + s} \rho^{i+j-s} \pi^s$$

Since

$$\binom{(p-1)(j-s)}{(p-1)j-i+s} = \binom{[(p-1)j-i+s]+[i-ps]}{(p-1)j-i+s} = \binom{(p-1)(j-s)}{i-ps}$$
$$\binom{(p-1)(j-s)-1}{(p-1)j-i+s} = \binom{[(p-1)j-i+s]+[i-ps-1]}{(p-1)j-i+s} = \binom{(p-1)(j-s)-1}{i-ps-1}$$

we have

$$\rho^i \pi^j = \sum_{s=2}^{\lfloor i/p \rfloor} (-1)^{i+s} \binom{(p-1)(j-s)}{i-ps} \pi^{i+j-s} \rho^s + \sum_{s=1}^{\lfloor (i-1)/p \rfloor} (-1)^{i+s+1} \binom{(p-1)(j-s)-1}{i-ps-1} \rho^{i+j-s} \pi^s$$

if $i \leq pj$, as desired. The other three relations are similar.

Finally, recall that π^r is represented by $[\xi_{11}^r]$ in the cobar complex and ρ^s is represented by $[\tau_{11}\xi_{11}^r]$. Since

$$\Delta(\xi_{11}^r) = \sum_{k=0}^{r} \binom{r}{k} \xi_{11}^k \otimes \xi_{11}^{r-k} \qquad \Delta(\tau_{11}\xi_{11}^s) = \sum_{k=0}^{s} \binom{s}{k} (\xi_{11}^k \otimes \tau_{11}\xi_{11}^{s-k} + \tau_{11}\xi_{11}^k \otimes \xi_{11}^{s-k}),$$

we have

$$\delta(\pi^r) = \sum_{k=1}^{r-1} -\binom{r}{k} \pi^k \pi^{r-k} \qquad \delta(\rho^s) = \sum_{k=2}^{s-2} \binom{s}{k} (-\pi^k \rho^{s-k} + \rho^k \pi^{s-k})$$

where the signs are introduced according to Theorem 5.2.22. □

The relations of $K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbb{F}_p)$ bear an obvious resemblance to the Adem relations of the Steenrod algebra, particularly in the odd primary case (Appendix A). To be more precise about the relationship between the co-Koszul complex of $\mathcal{R}_p(1)$ and the Steenrod algebra, we have the following proposition:

Proposition 5.3.6. *If $p = 2$, then the map of $\mathbb{F}_2$-algebras*

$$K^*(\mathbb{F}_2, \mathcal{R}_2(1), \mathbb{F}_2) \to \mathcal{A}_2 \qquad \sigma^r \mapsto \mathrm{Sq}^{r+1}\ (r \geq 1)$$

is injective. Similarly, if $p \geq 3$, then the map of $\mathbb{F}_p$-algebras

$$K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbb{F}_p) \to \mathcal{A}_p \qquad \pi^r \mapsto \beta P^r\ (r \geq 1), \quad \rho^s \mapsto P^s\ (s \geq 2)$$

is injective.

Remark 5.3.7. One interpretation of the above proposition is that the Koszul dual of $\mathcal{R}_p(1)$ is a subalgebra of the Steenrod algebra, $\mathcal{A}_p$.

Proof of Proposition 5.3.6. The result follows in the case $p \geq 3$ from the fact that the relations of $K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbb{F}_p)$ are sent to the Adem relations of $\mathcal{A}_p$ under the correspondence $\pi^r \mapsto \beta P^r$ and $\rho^s \mapsto P^s$. In the case $p = 2$, recall that the relations of $K^*(\mathbb{F}_2, \mathcal{R}_2(1), \mathbb{F}_2)$ are of the form

$$\sigma^i \sigma^j = \sum_{s=1}^{\lfloor (i-1)/2 \rfloor} \binom{j-s-1}{i-2s-1} \sigma^{i+j-s} \sigma^s$$

where $i \leq 2j$. The image of this under the map $\sigma^r \mapsto \mathrm{Sq}^{r+1}$ is

$$\begin{aligned} \mathrm{Sq}^{i+1} \mathrm{Sq}^{j+1} &= \sum_{s=1}^{\lfloor i/2 \rfloor} \binom{j-s-1}{i-2s-1} \mathrm{Sq}^{i+j-s+1} \mathrm{Sq}^{s+1} \\ &= \sum_{s=1}^{\lfloor i/2 \rfloor} \binom{(j+1)-(s+1)-1}{(i+1)-2(s+1)} \mathrm{Sq}^{(i+1)+(j+1)-(s+1)} \mathrm{Sq}^{s+1} \end{aligned}$$

Changing the index of summation to $k = s+1$ gives

$$\mathrm{Sq}^{i+1} \mathrm{Sq}^{j+1} = \sum_{k=2}^{\lfloor (i+1)/2 \rfloor} \binom{(j+1)-k-1}{(i+1)-2k} \mathrm{Sq}^{(i+1)+(j+1)-k} \mathrm{Sq}^k,$$

which agrees with the Adem relation for $\mathrm{Sq}^{i+1} \mathrm{Sq}^{j+1}$. (This relation is applicable because $i \leq 2j$ implies that $i + 1 \leq 2(j+1)$.) □

Theorem 5.3.8. *If M is a 1-allowable $\mathcal{R}_p$-module, then the differential of*

$$K^*(\mathbb{F}_p, \mathcal{R}_p(1), M) = K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbb{F}_p) \otimes M^\vee$$

is determined by the formula

- *(if $p=2$):* $\delta(1\otimes\mu)=\displaystyle\sum_{i=\lceil|\mu|/2\rceil}^{|\mu|-1}\sigma^i\otimes\mu Q^i$

- *(if $p\geq 3$):* $\delta(1\otimes\mu)=\displaystyle\sum_{i=\lceil|\mu|/2p\rceil}^{\lfloor(|\mu|-1)/2(p-1)\rfloor}-\pi^i\otimes\mu Q^i+(-1)^{|\mu|}\sum_{j=\lceil|\mu|/2p\rceil}^{\lfloor(|\mu|/2(p-1)\rfloor}\rho^j\otimes\mu\beta Q^j$

Proof. Because M is a $\mathcal{R}_p(1)$-module, there is a coaction $\lambda^\vee\colon M^\vee\to\mathcal{R}_p(1)^\vee\otimes M^\vee$ of $\mathcal{R}_p(1)^\vee$ on $M^\vee$. For $\mu\in M^\vee$, we write

$$\lambda^\vee(\mu)=\sum_i\alpha_i\otimes\mu_i.$$

In addition, there is an adjoint, right action of $\mathcal{R}_p(1)$ on $M^\vee$. The latter has the following explicit formulas [6]:

$$\mu Q^r=(-1)^{|Q^r|\cdot|\mu|}\sum_i\langle Q^r,\alpha_i\rangle\,\mu_i=\sum_i\langle Q^r,\alpha_i\rangle\,\mu_i$$
$$\mu\beta Q^s=(-1)^{|\beta Q^s|\cdot|\mu|}\sum_i\langle\beta Q^s,\alpha_i\rangle\,\mu_i=(-1)^{|\mu|}\sum_i\langle\beta Q^s,\alpha_i\rangle\,\mu_i$$

where $\langle -,-\rangle:\mathcal{R}_p(1)\otimes\mathcal{R}_p(1)^\vee\to\mathbb{F}_p$ is the pairing of $\mathcal{R}_p(1)$ with its dual. By Theorem 5.2.26,

$$\begin{aligned}\delta(1\otimes\mu)&=\sum_i(-1)^{|\pi^i|}\pi^i\otimes\mu_i+\sum_j(-1)^{|\rho^j|}\rho^j\otimes\mu_j\\&=\sum_i-\pi^i\otimes\mu_i+\sum_j\rho^j\otimes\mu_j\\&=\sum_i-\pi^i\otimes\left(\sum_k\left\langle Q^i,\pi^k\right\rangle\mu_k\right)+(-1)^{|\mu|}\sum_j\rho^j\otimes\left(\sum_\ell(-1)^{|\mu|}\left\langle\beta Q^j,\rho^\ell\right\rangle\mu_\ell\right)\\&=\sum_i-\pi^i\otimes\mu Q^i+(-1)^{|\mu|}\sum_j\rho^j\otimes\mu\beta Q^j\end{aligned}$$

since $\left\langle Q^i,\pi^k\right\rangle=1$ if $k=i$ and is zero otherwise. To determine the bounds of summation, note that because $Q^i(x)=0$ if $2i<|x|$, we have $\mu Q^i=0$ if

$$2i<|\mu Q^i|=|\mu|-2i(p-1),$$

or, in other words, if $2ip<|\mu|$. Similarly, $\mu\beta Q^j=0$ if $2ip<|\mu|$. The upper bounds follow from the fact that μQ^i and $\mu\beta Q^j$ must have positive degree.

If $p = 2$, then a similar argument shows that

$$\delta(1 \otimes \mu) = \sum_{i=\lceil |\mu|/2 \rceil}^{|\mu|-1} \sigma^i \otimes \mu Q^i.$$

□

Proposition 5.3.9. *If M is a 1-allowable $\mathcal{R}_p$-module, then a basis for $K^*(\mathbb{F}_p, \mathcal{R}_p(1), M)$ is*

- *($p = 2$):* $\left\{ \sigma^I \otimes \mu \;\middle|\; \begin{array}{c} I = (i_1, \dots, i_s) \text{ is admissible, } \mu \in M^\vee \text{, and} \\ i_k \geq i_{k+1} + \cdots + i_s + |\mu| + 1 \; (1 \leq k \leq s) \end{array} \right\}$
- *($p \geq 3$):* $\left\{ \pi^J \otimes \mu \;\middle|\; \begin{array}{c} J = (\epsilon_1, i_1, \dots, \epsilon_s, i_s) \text{ is admissible, } \mu \in M^\vee \text{, and} \\ i_k - \epsilon_k \geq (i_{k+1} - \epsilon_{k+1}) + \cdots + (i_s - \epsilon_s) + |\mu| + 1 \; (1 \leq k \leq s) \end{array} \right\}$

(The monomials σ^I and π^J are defined as in Notation 5.2.24.)

Proof. First suppose that $p = 2$. The bar complex $B_*(\mathbb{F}_2, \mathcal{R}_2(1), M)$ has (in homological degree s) basis elements of the form $[Q^{I_1}|\cdots|Q^{I_s}]m$, where $m \in M$, each sequence I_k is admissible, and

$$e(I_k) \geq |I_{k+1}| + \cdots + |I_s| + |m| + 1$$

for all $1 \leq k \leq s$. This implies that in the Koszul complex $K_*(\mathbb{F}_2, \mathcal{R}_2(1), M)$, we have basis elements $Q^{i_1} \otimes \cdots \otimes Q^{i_s} \otimes m$, where $(i_1, \dots, i_s)$ is admissible and

$$i_k \geq i_{k+1} + \cdots + i_s + |m| + 1.$$

for all $1 \leq k \leq s$. The corresponding dual basis (where we write μ for $m^\vee \in M^\vee$) is the claimed basis of $K^*(\mathbb{F}_2, \mathcal{R}_2(1), M)$.

In the case $p \geq 3$, the bar complex $B_*(\mathbb{F}_p, \mathcal{R}_p(1), M)$ has basis elements of the form $[Q^{J_1}|\cdots|Q^{J_s}]m$, where each sequence J_k is admissible and

$$e(J_k) \geq |J_{k+1}| + \cdots + |J_s| + |m| + 1$$

If $\ell(J_k) = n$, then $J_k = (\epsilon_1, i_1, \dots, \epsilon_n, i_n)$ and

$$Q^{J_k} = \beta^{\epsilon_1} Q^{i_1} \cdots \beta^{\epsilon_n} Q^{i_n}$$

with $e(J_k) = i_1 - \epsilon_1 - 2(p-1)[i_2 + i_3 + \cdots i_n]$. This implies that in the Koszul complex $K_*(\mathbb{F}_p, \mathcal{R}_p(1), M)$, we have basis elements $\beta^{\epsilon_1} Q^{i_1} \otimes \cdots \otimes \beta^{\epsilon_s} Q^{i_s} \otimes m$, where $(\epsilon_1, i_1, \ldots, \epsilon_n, i_n)$ is admissible and

$$i_k - \epsilon_k \geq (i_{k+1} - \epsilon_{k+1}) + \cdots + (i_s - \epsilon_s) + |m| + 1$$

for all $1 \leq k \leq s$. The corresponding dual basis is the claimed basis of $K^*(\mathbb{F}_p, \mathcal{R}_p(1), M)$.
□

Remark 5.3.10. More generally, if M is an n-allowable $\mathcal{R}_p$-module, then a basis for the co-Koszul complex $K^*(\mathbb{F}_p, \mathcal{R}_p(n), M)$ is

- $(p=2)$: $\left\{ \sigma^I \otimes \mu \;\middle|\; \begin{array}{c} I = (i_1, \ldots, i_s) \text{ is admissible, } \mu \in M^\vee \text{, and} \\ i_k \geq i_{k+1} + \cdots + i_s + |\mu| + n \ (1 \leq k \leq s) \end{array} \right\}$
- $(p \geq 3)$: $\left\{ \pi^J \otimes \mu \;\middle|\; \begin{array}{c} J = (\epsilon_1, i_1, \ldots, \epsilon_s, i_s) \text{ is admissible, } \mu \in M^\vee \text{, and} \\ i_k - \epsilon_k \geq (i_{k+1} - \epsilon_{k+1}) + \cdots + (i_s - \epsilon_s) + |\mu| + n \ (1 \leq k \leq s) \end{array} \right\}$

Example 5.3.11. Let p be any prime and let $X = H\mathbb{Z}$, so that

$$H_*(\Omega^\infty H\mathbb{Z}; \mathbb{F}_p) = H_*(K(\mathbb{Z}, 0); \mathbb{F}_p) \cong \mathbb{F}_p[\mathbb{Z}] \cong \mathbb{F}_p[x, x^{-1}]$$

where $x = [1] - [0]$ and has degree 0. In this case, $QH_*(\Omega^\infty H\mathbb{Z})$ is spanned by x. Moreover, the homology of the unit component, [1], is polynomial and has no p-torsion. Therefore,

$$\mathbf{L}_1 QH_*(\Omega^\infty H\mathbb{Z}) = Q\,\mathrm{Tor}_2^{H_*(\Omega^\infty H\mathbb{Z})}(\mathbb{F}_p, \mathbb{F}_p) = 0$$

and we have

$$\mathbf{L}_k QH_*(\Omega^\infty H\mathbb{Z}) = \begin{cases} \mathbb{F}_p\{x\} & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}$$

In this case, the Grothendieck spectral sequence

$$E^2_{s,t} = \mathrm{Untor}_s^{\mathcal{R}_p}(\mathbb{F}_p, \mathbf{L}_t Q(H_*(\Omega^\infty H\mathbb{Z}))) \Longrightarrow \mathbf{L}_{s+t}(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)(H_*(\Omega^\infty H\mathbb{Z}))$$

for the E^2 page of the Miller spectral sequence collapses at E^2 and shows that

$$\mathbf{L}_k(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)(H_*(\Omega^\infty H\mathbb{Z})) \cong \mathrm{Untor}_k^{\mathcal{R}_p}(\mathbb{F}_p, \mathbb{F}_p\{x\})$$

for all $k \geq 0$.

To compute $\mathrm{Untor}_*^{\mathcal{R}_p}(\mathbb{F}_p, \mathbb{F}_p\{x\})$, we pass to the dual and consider the co-Koszul complex

$$K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbb{F}_p\{x\}) \cong K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbb{F}_p) \otimes \mathbb{F}_p\{\alpha\},$$

where $x^\vee$ has degree 0 and is dual to x. A basis for the co-Koszul complex has the form

- $(p = 2)$: $\left\{ \sigma^I \otimes x^\vee \;\middle|\; \begin{array}{c} I = (i_1, \ldots, i_s) \text{ is admissible and} \\ i_k \geq i_{k+1} + \cdots + i_s + 1 \ (1 \leq k \leq s) \end{array} \right\}$

- $(p \geq 3)$: $\left\{ \pi^J \otimes x^\vee \;\middle|\; \begin{array}{c} J = (\epsilon_1, i_1, \ldots, \epsilon_s, i_s) \text{ is admissible and} \\ i_k - \epsilon_k \geq (i_{k+1} - \epsilon_{k+1}) + \cdots + (i_s - \epsilon_s) + 1 \ (1 \leq k \leq s) \end{array} \right\}$

The $\mathbb{F}_p$-algebra map

$$\mathrm{Untor}_*^{\mathcal{R}_p}(\mathbb{F}_p, \mathbb{F}_p\{x\})^\vee \to \mathcal{A}_p$$

defined by sending $x^\vee \mapsto 1$ and

- $\sigma^I \mapsto \mathrm{Sq}^{I+1}$ (if $p = 2$), where $I + 1 = (i_1 + 1, \ldots, i_s + 1)$, and
- $\pi^J \mapsto P^J$ (if $p \geq 3$)

is injective and has image $\mathcal{A}_p/\mathcal{A}_p\beta$ (where we write Sq^1 for β in the case $p = 2$). It is a standard result that

$$H_*(H\mathbb{Z})^\vee \cong H^*(H\mathbb{Z}) \cong \mathcal{A}_p/\mathcal{A}_p\beta,$$

and so the Miller spectral sequence,

$$E^2_{s,t} = \mathbf{L}_s(\mathbb{F}_2 \otimes_{\mathcal{R}_2} Q)(H_t(\Omega^\infty H\mathbb{Z})) \cong \mathrm{Untor}_{s,t}^{\mathcal{R}_2}(\mathbb{F}_2, \mathbb{F}_2\{x\}) \Longrightarrow H_*(H\mathbb{Z}),$$

must collapse at E^2.

Example 5.3.12. Let $X = H\mathbb{F}_p$, so that

$$H_*(\Omega^\infty X; \mathbb{F}_p) = H_*(K(\mathbb{F}_p, 0); \mathbb{F}_p) \cong \mathbb{F}_p[C_p] \cong \mathbb{F}_p[x]/(x^p)$$

where $x = [1] - [0]$ and has degree 0. Since the homology of the identity component is not polynomial, $\mathbf{L}_*Q(H_*\Omega^\infty H\mathbb{F}_p)$ is not quite as simple as $\mathbf{L}_*Q(H_*\Omega^\infty H\mathbb{Z})$. In this case, we have $QH_*\Omega^\infty H\mathbb{F}_p = \mathbb{F}_p\{x\}$, and so

$$\mathbf{L}_k Q(H_*\Omega^\infty H\mathbb{F}_p) = \begin{cases} \mathbb{F}_p\{x\} & \text{if } k = 0 \\ Q\operatorname{Tor}_2^{\mathbb{F}_p[x]/(x^p)}(\mathbb{F}_p, \mathbb{F}_p) & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases}$$

Since $\operatorname{Tor}_*^{\mathbb{F}_p[x]/(x^p)}(\mathbb{F}_p, \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\sigma x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$, where $|\sigma x| = 1$ and $|\varphi^0 x| = 2$, we have

$$\mathbf{L}_k Q(H_*(\Omega^\infty H\mathbb{F}_p)) = \begin{cases} \mathbb{F}_p\{x\} & \text{if } k = 0 \\ \mathbb{F}_p\{y\} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $|x| = 0$ and $|y| = 0$, so that x has total degree 0 and y has total degree 1.

Let M be the module $\mathbf{L}_* Q(H_*\Omega^\infty H\mathbb{F}_p)$, graded by total degree. Let $x^\vee$ and $y^\vee$ be dual to x and y, respectively. A basis for the co-Koszul complex $K^*(\mathbb{F}_p, \mathcal{R}_p(1), M)$ is

- (if $p = 2$): $\{\sigma^I \otimes x^\vee\} \cup \{\sigma^I \otimes y^\vee\}$
- (if $p \geq 3$): $\{\pi^J \otimes x^\vee\} \cup \{\pi^J \otimes y^\vee\}$

where the sequences $I = (i_1, \ldots, i_s)$ and $J = (\epsilon_1, i_1, \ldots, \epsilon_s, i_s)$ satisfy the same conditions as in the previous example. The differentials in the co-Koszul complex are all 0 (for degree reasons) and therefore we conclude that

$$\operatorname{Untor}_*^{\mathcal{R}_p(1)}(\mathbb{F}_p, M)^\vee \cong K^*(\mathbb{F}_p, \mathcal{R}_p(1), M).$$

The map of $\mathbb{F}_p$-algebras

$$K^*(\mathbb{F}_p, \mathcal{R}_p(1), M) \to \mathcal{A}_p$$

defined by sending $x^\vee \mapsto 1$, $y^\vee \mapsto \beta$, and $\sigma^I \mapsto \mathrm{Sq}^{I+1}$ if $p = 2$, and $\pi^J \mapsto P^J$ if $p \geq 3$ is an isomorphism. Since $H^*(H\mathbb{F}_p) \cong \mathcal{A}_p$, both the Grothendieck spectral sequence

$$\operatorname{Untor}_*^{\mathcal{R}_p}(\mathbb{F}_p, M)^\vee \cong K^*(\mathbb{F}_p, \mathcal{R}_p(1), M) \Longrightarrow \mathbf{L}_*(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)(H_*(\Omega^\infty H\mathbb{F}_p))^\vee$$

and the Miller spectral sequence

$$\mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)(H_t(\Omega^\infty H\mathbb{F}_p))^\vee \Longrightarrow H^*(H\mathbb{F}_p)$$

must collapse at E_2.

The result of Example 5.3.12 can be generalized as follows, which will be used in the proof of Theorem 6.0.5, given in the next section.

Lemma 5.3.13. *The Miller spectral sequence*

$$\mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)(H_*(K(\mathbb{F}_p, n))) \Longrightarrow H_*(H\mathbb{F}_p)$$

collapses at E^2, where $n \geq 0$ and p is any prime.

Proof. First consider the case $n \geq 2$. Recalling the structure of $H_*(K(\mathbb{F}_p, n))$ from Example A.0.1, we see that

$$\mathbf{L}_0 Q(H_*(K(\mathbb{F}_p, n))) = \begin{cases} \mathbb{F}_2\{x_S\} & \text{if } p = 2 \\ \mathbb{F}_p\{x_S\} \otimes \mathbb{F}_p\{y_S\} & \text{if } p \geq 3 \end{cases}$$

where

- if $p = 2$, then x_S denotes the dual of the class $\mathrm{Sq}^S \iota_n$, where S ranges over all admissible sequences with $e(S) < n$, and
- if $p \geq 3$, then x_S denotes the dual of the class $P^S \iota_n$ when it has even degree and y_S denotes its dual if it has odd degree. Here, S ranges over all admissible sequences with either $e(S) < n$ or, if $e(S) = n$, then $P^S = \beta P^{S'}$ for some admissible sequence S'.

Since $n \geq 2$, the bottommost class in $H_*(K(\mathbb{F}_p, n))$ has degree at least 2. It follows that $\mathrm{Tor}_*^{H_*(K(\mathbb{F}_p,n))}(\mathbb{F}_p, \mathbb{F}_p)$ is concentrated in degree at least 3, and so

$$\mathbf{L}_1 Q(H_*(K(\mathbb{F}_p, n))) = Q\, \mathrm{Tor}_2^{H_*(K(\mathbb{F}_p,n))}(\mathbb{F}_p, \mathbb{F}_p) = 0.$$

If M denotes the module $\mathbf{L}_* Q(H_*(K(\mathbb{F}_p, n)))$, then the basis elements of the co-Koszul complex $K^*(\mathbb{F}_p, \mathcal{R}_p(1), M)$ have the form $\sigma^I \otimes x_S^\vee$ if $p = 2$, and $\pi^J \otimes x_S^\vee$, $\pi^J \otimes y_S^\vee$ if $p \geq 3$, where the sequences I and J satisfy the same conditions as in Example 5.3.11. Also note that the co-Koszul complex does not have any nontrivial differentials, so that

$$K^*(\mathbb{F}_p, \mathcal{R}_p(1), M) \cong \mathrm{Untor}_*^{\mathcal{R}_p(1)}(\mathbb{F}_p, M)^\vee.$$

Define a map of $\mathbb{F}_p$-algebras

$$K^*(\mathbb{F}_p, \mathcal{R}_p(1), M) \to \Sigma^n \mathcal{A}_p$$

by

- if $p = 2$: $x_S^\vee \mapsto \mathrm{Sq}^S$ (if S is empty, then $\mathrm{Sq}^S = 1$), and $\sigma^I \mapsto \mathrm{Sq}^{I+1}$.
- if $p \geq 3$: $x_S^\vee \mapsto P^S$, $y_S^\vee \mapsto P^S$ (if S is empty, then $P^S = 1$), and $\pi^J \mapsto P^J$.

This map is an isomorphism, and therefore the Grothendeick spectral sequence

$$\mathrm{Untor}_*^{\mathcal{R}_p(1)}(\mathbb{F}_p, M)^\vee \Longrightarrow \mathbf{L}_*(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)(H_*(K(\mathbb{F}_p, n)))^\vee$$

and the Miller spectral sequence

$$\mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q)(H_t(K(\mathbb{F}_p, n)))^\vee \Longrightarrow H^*(\Sigma^n H\mathbb{F}_p)$$

both collapse at E^2.

Since the case $n = 0$ is covered in Example 5.3.12, the only remaining case to consider is $n = 1$. In this case, the only difference is that the $\mathbb{F}_p$-module

$$\mathbf{L}_1 Q(H_*(K(\mathbb{F}_p, 1))) = Q\,\mathrm{Tor}_2^{H_*(K(\mathbb{F}_p,1))}(\mathbb{F}_p, \mathbb{F}_p)$$

is nonzero and generated by the suspension of the class $\iota_1^\vee \in H_*(K(\mathbb{F}_p, 1))$. The isomorphism

$$K^*(\mathbb{F}_p, \mathcal{R}_p(1), M) \to \Sigma\mathcal{A}_p$$

is defined in the same way as it is for the case $n \geq 2$, except that the suspension of $\iota_1^\vee$ is sent to the Bockstein, β. □

Chapter 6

Application to Postnikov Towers of $\mathrm{sl}_1(R)$

Recall from §4 that many constructions in the category of topological spaces carry over to the category of spectra. This includes the notion of a *Postnikov tower*, which is a particular type of decomposition of a space into pieces that are more easily understood, at least from the standpoint of homotopy theory.

Proposition 6.0.1. *A connective spectrum X admits a Postnikov tower. In other words, there is a tower of spectra*

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & \tau_{\leq 2}X & \xrightarrow{k_2} & \Sigma^4 H\pi_3(X) \\
 & \nearrow & \downarrow & & \\
 & & \tau_{\leq 1}X & \xrightarrow{k_1} & \Sigma^3 H\pi_2(X) \\
 & \nearrow & \downarrow & & \\
X & \longrightarrow & \tau_{\leq 0}X & \xrightarrow{k_0} & \Sigma^2 H\pi_1(X)
\end{array}$$

in which:

- *the map $X \to \tau_{\leq n}X$ induces an isomorphism on π_* in the range $[0, n]$;*

- *$\pi_*(\tau_{\leq n}X) = 0$ for $* \geq n+1$; and*
- *the spectrum $\tau_{\leq n+1}X$ is constructed as the pullback (in the category of $\mathbb{S}$-modules) of the map $* \to \Sigma^{n+2}H\pi_{n+1}(X)$ along the map k_n for all n.*

Proof. Choose a map $f_0 \colon X \to H\pi_0(X) =: \tau_{\leq 0}X$ inducing an isomorphism on π_0. The cofiber of this map, Cf_0, has homotopy groups

$$\pi_k(Cf_0) \cong \begin{cases} 0 & \text{if } k = 0, 1 \\ \pi_{k-1}(X) & \text{if } k \geq 2 \end{cases}$$

By attaching cells to kill all homotopy groups of Cf_0 above dimension two, we obtain a map $c_0 \colon Cf_0 \to \Sigma^2 H\pi_1(X)$.

The fiber of the composite

$$k_0 \colon \tau_{\leq 0}X \to Cf_0 \xrightarrow{c_0} \Sigma^2 H\pi_1(X)$$

which we will denote by $\tau_{\leq 1}X$, has homotopy groups

$$\pi_k(\tau_{\leq 1}X) \cong \begin{cases} \pi_k(X) & \text{if } k = 0, 1 \\ 0 & \text{if } k \geq 2 \end{cases}$$

and fits into a pullback square of the form

$$\begin{array}{ccc} \tau_{\leq 1}X & \longrightarrow & * \\ \downarrow & & \downarrow \\ \tau_{\leq 0}X & \xrightarrow[k_0]{} & \Sigma^2 H\pi_1(X) \end{array}$$

The maps $f_0 \colon X \to \tau_{\leq 0}X$ and $X \to *$ induce a map $f_1 \colon X \to \tau_{\leq 1}X$ that is a lift of f_0 along $\tau_{\leq 1}X \to \tau_{\leq 0}X$. Moreover, f_1 induces an isomorphism on π_* in degrees 0 and 1. To construct the next stage in the tower, $\tau_{\leq 2}X$, proceed as above with f_1 playing the role of f_0. □

Notice that the construction of the Postnikov tower of a spectrum is essentially the same as the construction for spaces, with the spectra $\Sigma^{n+2}H\pi_{n+1}(X)$ playing the role of the Eilenberg-Mac Lane spaces $K(\pi_{n+1}(X), n+2)$. Each map

$$k_n \colon \tau_{\leq n}X \to \Sigma^{n+2}H\pi_{n+1}(X)$$

can be viewed as a cohomology class $[k_n] \in H^{n+2}(\tau_{\leq n}X; \pi_{n+1}(X))$.

Definition 6.0.2. The class $[k_n]$ is the n'th *k-invariant* of the spectrum X.

Recall that if R is an E_∞-ring spectrum, then there are associated spectra of units, $\mathrm{gl}_1 R$ and $\mathrm{sl}_1 R$. These spectra are formed by delooping the components $\bigcup_{u \in (\pi_0 R)^\times} \Omega^\infty R_u$ and $\Omega^\infty R_1$, respectively, of the E_∞-ring space $\Omega^\infty R$. As a result, there is a functorial equivalence between $\mathrm{gl}_1 R$ and the underlying spectrum of R in the stable range; specifically, if $n \geq 1$, then [29, Cor. 5.2.3]

$$\tau_{[n,2n-1]} R \simeq \tau_{[n,2n-1]} \mathrm{gl}_1 R.$$

The same holds if we replace $\mathrm{gl}_1 R$ with $\mathrm{sl}_1 R$. Although this result implies that certain k-invariants of $\mathrm{gl}_1 R$ (or $\mathrm{sl}_1 R$) and R must be identical, this is not always the case.

Example 6.0.3. Let R be the commutative $H\mathbb{F}_2$-algebra associated[1] to the commutative differential graded algebra (cdga) $\mathbb{F}_2[x]/(x^3)$, where $|x| = n \geq 1$ and the differential is identically 0. The only nonzero homotopy groups of R are

$$\pi_0(R) = \mathbb{F}_2\{1\}, \qquad \pi_n(R) = \mathbb{F}_2\{x\}, \qquad \pi_{2n}(R) = \mathbb{F}_2\{x^2\}.$$

Moreover, because R is an $H\mathbb{F}_2$-algebra, the underlying spectrum of R splits as a wedge of Eilenberg-Mac Lane spectra:

$$R \simeq H\mathbb{F}_2 \vee \Sigma^n H\mathbb{F}_2 \vee \Sigma^{2n} H\mathbb{F}_2.$$

Although the only nonzero homotopy groups of $\mathrm{gl}_1 R$ (which is the same as $\mathrm{sl}_1 R$ in this case) are

$$\pi_n(\mathrm{gl}_1 R) = \mathbb{F}_2\{y\}, \qquad \pi_{2n}(\mathrm{gl}_1 R) = \mathbb{F}_2\{y^2\},$$

where y corresponds to $1 + x$, we will show in this example that $\mathrm{gl}_1 R$ is *not* equivalent to $\Sigma^n H\mathbb{F}_2 \vee \Sigma^{2n} H\mathbb{F}_2$, and therefore the k-invariant linking $\pi_n(\mathrm{gl}_1 R)$ and $\pi_{2n}(\mathrm{gl}_1 R)$ is not 0.

To do this, consider the infinite loop space $\mathrm{GL}_1 R = K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, 2n)$. Recall that the Pontryagin algebra structure and the Dyer-Lashof operations on

$$H_*(\mathrm{GL}_1 R) \cong [1] \# H_*(\Omega^\infty R_0)$$

[1] Starting with any cdga, we may produce an E_∞-ring spectrum via the Dold-Kan Correspondence. See Theorem B.0.7 and Remark B.0.8.

are inherited from the multiplicative product $\circ$ and operations $\widetilde{Q}^n$ on $H_*(\Omega^\infty R)$. Let $\iota_n^\vee \in H_n(\Omega^\infty R_0)$ and $\iota_{2n}^\vee \in H_{2n}(\Omega^\infty R_0)$ be dual to the fundamental classes

$$\iota_n \in H^n(K(\mathbb{F}_2, n)) \hookrightarrow H^n(\Omega^\infty R_0), \qquad \iota_{2n} \in H^{2n}(K(\mathbb{F}_2, 2n)) \hookrightarrow H^{2n}(\Omega^\infty R_0).$$

We claim that in the ring $H_*(\mathrm{GL}_1 R)$, there is a relation

$$([1]\#\iota_n^\vee) \circ ([1]\#\iota_n^\vee) = [1]\#\iota_{2n}^\vee,$$

and therefore $\mathrm{gl}_1 R$ has a nonzero k-invariant.

The relation can be seen from the distributivity law: since $\iota_n^\vee$ is primitive, we have

$$([1]\#\iota_n^\vee) \circ ([1]\#\iota_n^\vee) = [1]\#(\iota_n^\vee\#\iota_n^\vee) + [1]\#(\iota_n^\vee \circ \iota_n^\vee).$$

As rings with the additive product $\#$, there is an isomorphism

$$\begin{aligned} H_*(\Omega^\infty R_0) &\cong H_*(K(\mathbb{F}_2, n)) \otimes H_*(K(\mathbb{F}_2, 2n)) \\ &\cong \Lambda_{\mathbb{F}_2}[(\mathrm{Sq}^I\, \iota_n)^\vee,\ (\mathrm{Sq}^J\, \iota_{2n})^\vee \mid e(I) < n,\ e(J) < 2n]. \end{aligned}$$

In particular, $\iota_n^\vee\#\iota_n^\vee = 0$. To calculate $\iota_n^\vee \circ \iota_n^\vee$, note that the Hurewicz map sends the generator $x \in \pi_n(\Omega^\infty R_0)$ to the class $\iota_n^\vee$, and sends the product in the ring $\pi_*(\Omega^\infty R_0)$ to the $\circ$ product in $H_*(\Omega^\infty R_0)$. Since $x^2 \neq 0$, we have $\iota_n^\vee \circ \iota_n^\vee \neq 0$. In fact, this element must be dual to ι_{2n}; this is because the remaining elements of $H^{2n}(\Omega^\infty R_0)$, which are ι_n^2 and $\mathrm{Sq}^I\, \iota_n$ $(e(I) < n)$, are both primitive and therefore must have indecomposable duals.

Example 6.0.4. One also sees a similar phenomenon at odd primes. For example, let R be the commutative $H\mathbb{F}_p$-algebra associated to the cdga $\mathbb{F}_p[x]/(x^{p+1})$, where $|x| = n \geq 2$ is even (so that x^{p+1} is the first power of x that equals 0) and the differential is identically 0. In this case, there is an equivalence of spectra

$$R \simeq H\mathbb{F}_p \vee \Sigma^n H\mathbb{F}_p \vee \cdots \vee \Sigma^{np} H\mathbb{F}_p.$$

Consider the space of strict units, $\mathrm{SL}_1 R$. Since

$$\Omega^\infty R \cong K(\mathbb{F}_p, 0) \times K(\mathbb{F}_p, n) \cdots \times K(\mathbb{F}_p, np),$$

we have

$$\mathrm{SL}_1 R \cong \{1\} \times K(\mathbb{F}_p, n) \times \cdots \times K(\mathbb{F}_p, np).$$

For $1 \le i \le p$, let $\iota_{in}^\vee \in H_{in}(\Omega^\infty R_0)$ denote the dual of the fundamental class $\iota_{in} \in H^{in}(K(\mathbb{F}_p, in)) \hookrightarrow H^*(\Omega^\infty R_0)$. Let $[1]\#\iota_{in}^\vee \in H_{in}(\mathrm{SL}_1 R)$ denote the corresponding classes in the ring $H_*(\mathrm{SL}_1 R)$. We claim that there is a relation

$$([1]\#\iota_n^\vee)^{\circ p} = \iota_{pn}^\vee$$

in this ring, and therefore the spectrum $\mathrm{sl}_1 R$ has a nonzero k-invariant linking π_n and π_{pn}, unlike the spectrum R.

To show this, note that

$$([1]\#\iota_n^\vee)^{\circ p} = \widetilde{Q}^{n/2}([1]\#\iota_n^\vee)$$

and that the element $\iota_n^\vee$ is primitive. By the mixed Cartan formula (Theorem 3.2.5),

$$\widetilde{Q}^{n/2}([1]\#\iota_n^\vee) = \sum_{n_0+\cdots+n_p=\frac{n}{2}} \sum \widetilde{Q}_0^{n_0}([1]^{(0)} \otimes (\iota_n^\vee)^{(0)})\# \cdots \# \widetilde{Q}_p^{n_p}([1]^{(p)} \otimes (\iota_n^\vee)^{(p)})$$

where

$$[1] \otimes \iota_n^\vee \mapsto \sum [1]^{(0)} \otimes (\iota_n^\vee)^{(0)} \otimes \cdots \otimes [1]^{(p)} \otimes (\iota_n^\vee)^{(p)}$$

under the iterated coproduct of $H_*(\Omega^\infty R) \otimes H_*(\Omega^\infty R)$ and

$$\widetilde{Q}_0^n(x \otimes y) = [\epsilon(y)]\widetilde{Q}^n(x)$$

$$\widetilde{Q}_i^n(x \otimes y) = \tfrac{1}{p}\tbinom{p}{i} Q^n\left(\sum\sum x^{(1)} \circ \cdots \circ x^{(p-i)} \circ y^{(1)} \circ \cdots \circ y^{(i)}\right) \text{ (for } 1 \le i \le p-1)$$

$$\widetilde{Q}_p^n(x \otimes y) = [\epsilon(x)]\widetilde{Q}^n(y).$$

Because $\Delta([1]) = [1] \otimes [1]$ and $\Delta(\iota_n^\vee) = [0] \otimes \iota_n^\vee + \iota_n^\vee \otimes [0]$, the terms in the iterated coproduct of $[1] \otimes \iota_n^\vee$ are of the form

$$\begin{gathered}
[1] \otimes [1] \otimes \iota_n^\vee \otimes [0] \otimes [0] \otimes \cdots \otimes [0], \\
[1] \otimes [1] \otimes [0] \otimes \iota_n^\vee \otimes [0] \otimes \cdots \otimes [0], \\
\vdots \\
[1] \otimes [1] \otimes [0] \otimes [0] \otimes [0] \otimes \cdots \otimes \iota_n^\vee.
\end{gathered}$$

Recall that because $[1]$ is the unit for the $\circ$ product, the only nonzero multiplicative Dyer-Lashof operation on $[1]$ is $\widetilde{Q}^0[1] = [1^2] = [1]$. On the element $[0]$, all of the additive and multiplicative Dyer-Lashof operations vanish except $Q^0[0] = \widetilde{Q}^0[0] = [0]$. With this in mind, almost all terms in the mixed Cartan formula vanish, and we are left with

$$\widetilde{Q}^{n/2}([1]\#\iota_n^\vee) = [1]\#\widetilde{Q}^{n/2}(\iota_n^\vee) = [1]\#(\iota_n^\vee)^{\circ p}.$$

Since $x^p \neq 0$ in $\pi_*(R)$, we have $(\iota_n^\vee)^{\circ p} \neq 0$. By reasoning that is similar to that in Example 6.0.3, the only possibility is that $(\iota_n^\vee)^{\circ p} = \iota_{pn}^\vee$.

Based on Examples 6.0.3 and 6.0.4, one might suspect that if R is any commutative $H\mathbb{F}_p$-algebra, then nontrivial k-invariants of $\mathrm{sl}_1 R$ arise from nonzero p'th powers in the ring $\pi_*(R)$. Our main result in this section is that if we work in a range that is small enough to force p'th powers to be zero, then R and $\mathrm{sl}_1 R$ are equivalent as spectra.

Theorem 6.0.5. *If R is a commutative $H\mathbb{F}_p$-algebra and $n \geq 1$ is any integer, then there is a functorial equivalence of spectra*

$$\tau_{[n,pn-1]}R \simeq \tau_{[n,pn-1]}\mathrm{sl}_1 R.$$

Remark 6.0.6. This theorem is motivated by the following observations [29, Remark 5.2.6]. First, recall that there is an equivalence between R and $\mathrm{sl}_1 R$ in the stable range [29, Theorem 5.1.2]:

$$\tau_{[n,2n-1]}R \simeq \tau_{[n,2n-1]}\mathrm{sl}_1 R.$$

This is also true if we replace $\mathrm{sl}_1 R$ with $\mathrm{gl}_1 R$. This is analogous to the fact that if A is an ordinary commutative ring and $I \subseteq A$ is an ideal satisfying $I^2 = 0$, then $1 + I \subseteq A^\times$ and there is an isomorphism of abelian groups

$$I \to 1 + I \subseteq A^\times, \qquad x \mapsto 1 + x,$$

which is similar to the isomorphism

$$\pi_*(\tau_{[n,2n-1]}R) \to \pi_*(\tau_{[n,2n-1]}\mathrm{sl}_1 R), \qquad x \mapsto 1 + x.$$

If we instead assume that A is a p-local commutative ring and that $J \subseteq A$ is an ideal satisfying $J^p = 0$, then $1 + J \subseteq A^\times$ and there is an isomorphism of abelian groups

$$J \to 1 + J \subseteq A^\times, \qquad x \mapsto 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{p-1}}{(p-1)!}$$

Based on this, A. Mathew and V. Stojanoska conjectured a result similar to that of Theorem 6.0.5 in [29], namely that there is a functorial equivalence of spectra

$$\tau_{[n,pn-1]}R \simeq \tau_{[n,pn-1]}\mathrm{gl}_1 R.$$

Since $\tau_{[n,pn-1]}\mathrm{gl}_1 R \simeq \tau_{[n,pn-1]}\mathrm{sl}_1 R$, we would also expect an equivalence

$$\tau_{[n,pn-1]}R \simeq \tau_{[n,pn-1]}\mathrm{sl}_1 R.$$

Our proof of Theorem 6.0.5 will rely on the results of the following three lemmas: Lemma 6.0.7, Lemma 6.0.8, and Lemma 6.0.9. Collectively, they are intended to show that a certain map resembling a truncated exponential map ultimately determines the equivalence $\tau_{[n,pn-1]}R \simeq \tau_{[n,pn-1]}\mathrm{sl}_1 R$; this is analogous to the situation in ordinary algebra, as mentioned in the preceding remark. The reason why we work with $\mathrm{sl}_1 R$ instead of $\mathrm{gl}_1 R$ throughout can be most clearly seen in the proof of Lemma 6.0.9, which states that the module of #-indecomposable elements of the ring $H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)$ is isomorphic to the module of $\circ$-indecomposable elements of the ring $H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$. We will use $Q^\#$ to denote the first type of indecomposables and Q° for the second. This result is shown by directly computing the structure the ring $H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$. This is considerably more difficult to do for the ring $H_*(\tau_{[n,pn-1]}\mathrm{GL}_1 R)$ due to the fact that $\pi_0(\mathrm{GL}_1 R) = \pi_0(R)^\times$ can be many possible abelian groups; as a result, one can say much less about the $\circ$ product in $H_*(\tau_{[n,pn-1]}\mathrm{GL}_1 R)$ than one can about the $\circ$ product in $H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$.

Lemma 6.0.7. *Let R be an E_∞-ring spectrum such that $H_*(\Omega^\infty R)^{\circ p} = 0$ and such that $(p-1)!$ is invertible in the ring $\pi_0(R)$. Then there is a map*

$$\exp\colon (H_*(\Omega^\infty R), \#) \to (H_*(\Omega^\infty R), \circ)$$

of rings defined by sending a class $x \in H_*(\Omega^\infty R)$ to

$$\sum [1]\# x^{(1)} \# ([\tfrac{1}{2!}] \circ x^{(2)} \circ x^{(3)}) \# \cdots \# ([\tfrac{1}{(p-1)!}] \circ x^{([(p-2)(p-1)/2]+1)} \circ \cdots \circ x^{(p(p-1)/2)}),$$

where the sum is over the $(\frac{p(p-1)}{2} - 1)$-fold iterated coproduct of x.

Proof. Since $(p-1)!$ is invertible in $\pi_0(R)$, for each integer k satisfying $1 \le k \le p-1$, there exists a map

$$\frac{1}{k!}\colon \mathbb{F}_p \to H_0(\Omega^\infty R_0) \subseteq H_*(\Omega^\infty R)$$

sending $1 \in \mathbb{F}_p$ to the class $[\frac{1}{k!}] \in H_0(\Omega^\infty R)$. Using these maps, we may write the expression for

$$\exp\colon H_*(\Omega^\infty R) \to H_*(\Omega^\infty R)$$

as the following composite (note that all tensor products are taken over $\mathbb{F}_p$):

$$\begin{array}{c}
H_*(\Omega^\infty R) \\
\Big\downarrow {\scriptstyle \Delta^{\frac{p(p-1)}{2}-1}} \\
H_*(\Omega^\infty R)^{\otimes p(p-1)/2} \\
\| \\
H_*(\Omega^\infty R) \otimes H_*(\Omega^\infty R)^{\otimes 2} \otimes H_*(\Omega^\infty R)^{\otimes 3} \otimes \cdots \otimes H_*(\Omega^\infty R)^{\otimes (p-1)} \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes \circ \otimes \circ^2 \otimes \cdots \otimes \circ^{p-2}} \\
H_*(\Omega^\infty R)^{\otimes (p-1)} \\
\| \\
(\mathbb{F}_p \otimes H_*(\Omega^\infty R)) \otimes (\mathbb{F}_p \otimes H_*(\Omega^\infty R)) \otimes \cdots \otimes (\mathbb{F}_p \otimes H_*(\Omega^\infty R)) \\
\Big\downarrow {\scriptstyle (1 \otimes \mathrm{id}) \otimes (\frac{1}{2!} \otimes \mathrm{id}) \otimes \cdots \otimes (\frac{1}{(p-1)!} \otimes \mathrm{id})} \\
(H_*(\Omega^\infty R) \otimes H_*(\Omega^\infty R))^{\otimes (p-1)} \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes \mathrm{id} \otimes \circ^{\otimes (p-2)}} \\
H_*(\Omega^\infty R)^{\otimes p} \\
\Big\downarrow {\scriptstyle \#^{p-1}} \\
H_*(\Omega^\infty R)
\end{array}
\qquad (\text{composite: } \exp)$$

To verify that exp satisfies $\exp(x \# y) = \exp(x) \circ \exp(y)$ for all $x, y \in H_*(\Omega^\infty R)$, we first note that the maps Δ, $\#$, and $\circ$ induce natural transformations

$$\begin{aligned}
\Delta\colon & \operatorname{Hom}(-, H_*(\Omega^\infty R)) \to \operatorname{Hom}(-, H_*(\Omega^\infty R)) \otimes \operatorname{Hom}(-, H_*(\Omega^\infty R)), \\
\#\colon & \operatorname{Hom}(-, H_*(\Omega^\infty R)) \otimes \operatorname{Hom}(-, H_*(\Omega^\infty R)) \to \operatorname{Hom}(-, H_*(\Omega^\infty R)), \\
\circ\colon & \operatorname{Hom}(-, H_*(\Omega^\infty R)) \otimes \operatorname{Hom}(-, H_*(\Omega^\infty R)) \to \operatorname{Hom}(-, H_*(\Omega^\infty R)),
\end{aligned}$$

where Hom is taken in the category of cocommutative graded coalgebras, **CCGrCoalg**.

Because $H_*(\Omega^\infty R)$ is a Hopf ring, by Remark 3.2.4, each of these is a natural transformation between ring-valued functors. Moreover, the diagram

$$\begin{array}{ccc} H_*(\Omega^\infty R) \otimes H_*(\Omega^\infty R) & \xrightarrow{\#} & H_*(\Omega^\infty R) \\ \Big\downarrow{\scriptstyle \exp\otimes\exp} & & \Big\downarrow{\scriptstyle \exp} \\ H_*(\Omega^\infty R) \otimes H_*(\Omega^\infty R) & \xrightarrow[\circ]{} & H_*(\Omega^\infty R) \end{array}$$

will commute; i.e., $\exp(x\#y) = \exp(x) \circ \exp(y)$, if and only if the diagram $(*)$ of rings

$$\begin{array}{ccc} \mathrm{Hom}(H_*(\Omega^\infty R), H_*(\Omega^\infty R)) \otimes \mathrm{Hom}(H_*(\Omega^\infty R), H_*(\Omega^\infty R)) & \xrightarrow{\#} & \mathrm{Hom}(H_*(\Omega^\infty R), H_*(\Omega^\infty R)) \\ \Big\downarrow{\scriptstyle \exp\otimes\exp} & & \Big\downarrow{\scriptstyle \exp} \\ \mathrm{Hom}(H_*(\Omega^\infty R), H_*(\Omega^\infty R)) \otimes \mathrm{Hom}(H_*(\Omega^\infty R), H_*(\Omega^\infty R)) & \xrightarrow[\circ]{} & \mathrm{Hom}(H_*(\Omega^\infty R), H_*(\Omega^\infty R)) \end{array}$$

commutes.

To show this, we first introduce some notation. In the ring $\mathrm{Hom}(H_*(\Omega^\infty R), H_*(\Omega^\infty R))$, we write $+$ for its addition and $\cdot$ for its multiplication. We also write X for the element of $\mathrm{Hom}(H_*(\Omega^\infty R), H_*(\Omega^\infty R))$ corresponding to $x \in H_*(\Omega^\infty R)$; i.e., X is the map $H_*(\Omega^\infty R) \to H_*(\Omega^\infty R)$ that sends 1 to x. In particular, note that the element of $\mathrm{Hom}(H_*(\Omega^\infty R), H_*(\Omega^\infty R))$ corresponding to the class $[i] \in H_0(\Omega^\infty R)$ is the integer i. We have that

- Δ sends X to $X \otimes X$,
- $\#$ sends (X, Y) to $X + Y$, and
- $\circ$ sends (X, Y) to $X \cdot Y$,

so that exp sends X to

$$\exp(X) = 1 + X + \frac{1}{2!}X^2 + \cdots + \frac{1}{(p-1)!}X^{p-1}.$$

Commutativity of $(*)$ is then equivalent to showing that $\exp(X+Y) = \exp(X)\cdot\exp(Y)$. For each $1 \le d \le p-1$, the terms of the product

$$\begin{aligned} &\exp(X) \cdot \exp(Y) \\ &\quad = \left(1 + X + \frac{1}{2!}X^2 + \cdots + \frac{1}{(p-1)!}X^{p-1}\right)\left(1 + Y + \frac{1}{2!}Y^2 + \cdots + \frac{1}{(p-1)!}Y^{p-1}\right) \end{aligned}$$

with homogeneous degree d are

$$\frac{1}{d!}X^d,\ \frac{1}{(d-1)!}X^{d-1}Y,\ \frac{1}{(d-2)!2!}X^{d-2}Y^2\cdots,\ \frac{1}{(d-1)!}XY^{d-1},\ \frac{1}{d!}Y^d.$$

Since

$$\begin{aligned}
&\frac{1}{d!}X^d+\frac{1}{(d-1)!}X^{d-1}Y+\frac{1}{(d-2)!2!}X^{d-2}Y^2+\cdots+\frac{1}{(d-1)!}XY^{d-1}+\frac{1}{d!}Y^d\\
&\quad=\frac{1}{d!}\left(X^d+\frac{d!}{(d-1)!}X^{d-1}Y+\frac{d!}{(d-2)!2!}X^{d-2}Y^2+\cdots+\frac{d!}{(d-1)!}XY^{d-1}+Y^d\right)\\
&\quad=\frac{1}{d!}\left(X^d+\binom{d}{1}X^{d-1}Y+\binom{d}{2}X^{d-2}Y^2+\cdots+\binom{d}{d-1}XY^{d-1}+Y^d\right)\\
&\quad=\frac{1}{d!}(X+Y)^d,
\end{aligned}$$

we have

$$\begin{aligned}
&\exp(X)\cdot\exp(Y)\\
&\quad=1+(X+Y)+\frac{1}{2!}(X+Y)^2+\cdots+\frac{1}{(p-1)!}(X+Y)^{p-1}+(\text{terms of deg}\geq p).
\end{aligned}$$

Since $H_*(\Omega^\infty R)^{\circ p}=0$ by assumption, all products in the ring $\mathrm{Hom}(H_*(\Omega^\infty R),H_*(\Omega^\infty R))$ of length at least p will vanish. Therefore,

$$\begin{aligned}
\exp(X)\cdot\exp(Y)&=1+(X+Y)+\frac{1}{2!}(X+Y)^2+\cdots+\frac{1}{(p-1)!}(X+Y)^{p-1}\\
&=\exp(X+Y),
\end{aligned}$$

as desired. □

Lemma 6.0.8. *Suppose that R is a commutative $H\mathbb{F}_p$-algebra. If the nonzero homotopy groups of R are concentrated in the range $[n,pn-1]$ for some $n\geq 1$, then $\Omega^\infty R$ is an E_∞-ring space that satisfies $H_*(\Omega^\infty R)^{\circ p}=0$.*

Proof. Suppose that the nontrivial homotopy groups of R occur in the range $[n,pn-1]$ for some $n\geq 1$. Let $d_1,d_2,\dots,d_s$ be the degrees in which $\pi_*(R)$ is nontrivial. Since R is an $H\mathbb{F}_p$-algebra, it is equivalent—as a spectrum—to a wedge of Eilenberg-Mac Lane spectra; specifically,

$$R\simeq\Sigma^{d_1}H\mathbb{F}_p\vee\cdots\vee\Sigma^{d_s}H\mathbb{F}_p.$$

It follows that

$$\Omega^\infty R \simeq K(\mathbb{F}_p, d_1) \times \cdots \times K(\mathbb{F}_p, d_s),$$

For $1 \le i \le s$, let $x_i \in \pi_{d_i}(\Omega^\infty R)$ be a generator and let $\iota_{d_i}^\vee \in H_{d_i}(\Omega^\infty R)$ be dual to the fundamental class $\iota_{d_i} \in H^{d_i}(K(\mathbb{F}_p, d_i)) \hookrightarrow H^{d_i}(\Omega^\infty R)$.

The Hurewicz map

$$h\colon \pi_*(\Omega^\infty R) \to H_*(\Omega^\infty R)$$

sends x_i to $\iota_{d_i}^\vee$ and sends the product in the ring $\pi_*(\Omega^\infty R)$ to the $\circ$ product in $H_*(\Omega^\infty R)$. Specifically, the diagram

$$\begin{array}{ccc} \pi_{d_i}(\Omega^\infty R) \otimes \pi_{d_j}(\Omega^\infty R) & \xrightarrow{\mu_*} & \pi_{d_i+d_j}(\Omega^\infty R) \\ \downarrow h & & \downarrow h \\ H_{d_i}(\Omega^\infty R) \otimes H_{d_j}(\Omega^\infty R) & \xrightarrow[\circ]{} & H_{d_i+d_j}(\Omega^\infty R) \end{array} \qquad \begin{array}{ccc} x_i \otimes x_j & \longmapsto & x_i x_j \\ \downarrow & & \downarrow \\ \iota_{d_i}^\vee \otimes \iota_{d_j}^\vee & \longmapsto & \iota_{d_i}^\vee \circ \iota_{d_j}^\vee \end{array}$$

commutes. Moreover, the $\circ$ product on $H_*(\Omega^\infty R)$ is completely determined by the product in the ring $\pi_*(\Omega^\infty R) = \pi_*(R)$. To see this, note that because R is an $H\mathbb{F}_p$-algebra, there is a map of E_∞-ring spectra

$$\alpha\colon H\mathbb{F}_p \wedge_{\mathbb{S}} R \to R$$

expressing the action of $H\mathbb{F}_p$ on R. Using the unit map $\mathbb{S} \to H\mathbb{F}_p$, we have a composite

$$R = \mathbb{S} \wedge_{\mathbb{S}} R \to H\mathbb{F}_p \wedge_{\mathbb{S}} R \xrightarrow{\alpha} R.$$

After taking $\pi_*(-)$, we have

$$\pi_*(R) \xrightarrow{h} \pi_*(H\mathbb{F}_p \wedge_{\mathbb{S}} R) = H_*(R) \xrightarrow{\alpha} \pi_*(R), \qquad \alpha \circ h = \mathrm{id}.$$

As a result, the Hurewicz map $\pi_*(\Omega^\infty R) \to H_*(\Omega^\infty R)$ has a section $H_*(\Omega^\infty R) \to \pi_*(\Omega^\infty R)$. Therefore, if $x, y \in H_*(\Omega^\infty R)$ and either x or y is *not* in the image of the Hurewicz map, then $x \circ y = 0$.

It follows from the above that the only nonzero $\circ$ products in the ring $H_*(\Omega^\infty R)$ are those involving the classes $\iota_{d_i}^\vee$. Moreover, because the ring $\pi_*(\Omega^\infty R)$ has no nonzero products of length p or larger for degree reasons, it follows that $H_*(\Omega^\infty R)^{\circ p} = 0$. $\square$

Lemma 6.0.9. *Suppose that R is a commutative $H\mathbb{F}_p$-algebra and that $n \geq 1$. Then the map*

$$\exp\colon H_*(\tau_{[n,pn-1]}\Omega^\infty R) \to H_*(\tau_{[n,pn-1]}\Omega^\infty R)$$

of Lemma 6.0.7 determines an isomorphism of $\mathbb{F}_p$-modules

$$Q^{\#}H_*(\tau_{[n,pn-1]}\Omega^\infty R_0) \xrightarrow{\sim} Q^{\circ}H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R).$$

Proof. First, note that $\tau_{[n,pn-1]}R$ is a commutative $H\mathbb{F}_p$-algebra that satisfies the hypotheses of Lemma 6.0.8, so that

$$\exp\colon (H_*(\tau_{[n,pn-1]}\Omega^\infty R), \#) \to (H_*(\tau_{[n,pn-1]}\Omega^\infty R), \circ)$$

is defined and is a map of rings. It is a map of $\mathbb{F}_p$-modules because each map in the diagram defining exp (see the proof of Lemma 6.0.7) is a map of $\mathbb{F}_p$-modules. To show that exp gives the desired isomorphism, we will explicitly compute the structures of $Q^{\#}H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)$ and $Q^{\circ}H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$, as well as the value of exp on each element of $Q^{\#}H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)$, and observe that exp sets up a bijection between them.

As in the proof of Lemma 6.0.8, let $d_1, d_2, \ldots, d_s$ denote the degrees in which $\pi_*(\tau_{[n,pn-1]}\Omega^\infty R)$ is nonzero. We let $x_i \in \pi_{d_i}(\tau_{[n,pn-1]}\Omega^\infty R)$ denote a generator and we let $\iota_{d_i}^\vee$ be dual to the fundamental class $\iota_{d_i} \in H^{d_i}(K(\mathbb{F}_p, d_i)) \hookrightarrow H^{d_i}(\tau_{[n,pn-1]}\Omega^\infty R)$, so that the Hurewicz map sends x_i to $\iota_{d_i}^\vee$.

As rings with the *additive* product, we have

$$H_*(\tau_{[n,pn-1]}\Omega^\infty R_0) \cong H_*(K(\mathbb{F}_p, d_1)) \otimes \cdots \otimes H_*(K(\mathbb{F}_p, d_s)),$$

where, for each $1 \leq i \leq s$,

$$H_*(K(\mathbb{F}_p, d_i)) \cong \Gamma_{\mathbb{F}_p}[(P^I\iota_{d_i})^\vee \mid |P^I\iota_{d_i}| \text{ even}] \otimes \Lambda_{\mathbb{F}_p}[(P^I\iota_{d_i})^\vee \mid |P^I\iota_{d_i}| \text{ odd}].$$

Here, I ranges over all admissible sequences (including the empty sequence) such that either $e(I) < d_i$ or, if $e(I) = d_i$, then P^I is of the form $P^I = \beta P^{I'}$ for some sequence I'. For the sake of brevity, we will denote the set of all such sequences by $S(d_i)$. It follows that

$$Q^{\#}H_*(\tau_{[n,pn-1]}\Omega^\infty R_0) \cong \mathbb{F}_p\{(P^I\iota_{d_1})^\vee \mid I \in S(d_1)\} \oplus \cdots \oplus \mathbb{F}_p\{(P^I\iota_{d_s})^\vee \mid I \in S(d_s)\}.$$

Now consider the space $\mathrm{SL}_1 R = \Omega^\infty R_1$. Since $n \geq 1$, we have

$$\tau_{[n,pn-1]}\mathrm{SL}_1 R \simeq \tau_{[n,pn-1]}\Omega^\infty R \simeq K(\mathbb{F}_p, d_1) \times \cdots \times K(\mathbb{F}_p, d_s);$$

however, we view

$$H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R) \cong [1]\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)$$

as a ring with product given by the *multiplicative* product, $\circ$, which is inherited from $H_*(\tau_{[n,pn-1]}\Omega^\infty R)$.

To determine $Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$, we will explicitly compute the $\circ$ product on the (additive) generators

$$[1]\#(P^I \iota_{d_i})^\vee \in H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R) = [1]\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0).$$

To do this, we will make use of the distributivity formula. First, note that the classes $\iota_{d_i}^\vee$ and $(P^I \iota_{d_i})^\vee$ in $H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)$ have coproducts given by

$$\Delta(\iota_{d_i}^\vee) = [0] \otimes \iota_{d_i}^\vee + \iota_{d_i}^\vee \otimes [0], \qquad \Delta((P^I \iota_{d_i})^\vee) = [0] \otimes (P^I \iota_{d_i})^\vee + (P^I \iota_{d_i})^\vee \otimes [0];$$

in other words, these elements are primitive. This is because the cohomology classes ι_{d_i} and $P^I \iota_{d_i}$ are indecomposable; if not, then $\Omega^\infty R_0$ would not be a product of Eilenberg-Mac Lane spaces. Therefore, by the distributivity law,

$$([1]\#\iota_{d_i}^\vee) \circ ([1]\#\iota_{d_j}^\vee) = [1]\#(\iota_{d_i}^\vee \# \iota_{d_j}^\vee) + [1]\#(\iota_{d_i}^\vee \circ \iota_{d_j}^\vee),$$

$$([1]\#\iota_{d_i}^\vee) \circ ([1]\#(P^I \iota_{d_j})^\vee) = [1]\#(\iota_{d_i}^\vee \# (P^I \iota_{d_j})^\vee),$$

$$([1]\#(P^I \iota_{d_i})^\vee) \circ ([1]\#(P^I \iota_{d_j})^\vee) = [1]\#((P^I \iota_{d_i})^\vee \# (P^I \iota_{d_j})^\vee).$$

It is evident from the latter two equations that elements of the form $[1]\#(\iota_{d_i}^\vee \# (P^I \iota_{d_j})^\vee)$ and $[1]\#((P^I \iota_{d_i})^\vee \# (P^I \iota_{d_j})^\vee)$ are $\circ$-decomposable. To determine whether or not the remaining elements of $H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$ are $\circ$-decomposable, we compute the dual of the product $([1]\#\iota_{d_i}^\vee) \circ ([1]\#\iota_{d_j}^\vee)$. This is done using the usual pairing

$$\langle -, - \rangle : H^*(\mathrm{SL}_1 R) \otimes H_*(\mathrm{SL}_1 R) \to \mathbb{F}_p$$

of $H_*(\mathrm{SL}_1 R)$ with its dual. Because $([1]\#\iota_{d_i}^\vee) \circ ([1]\#\iota_{d_j}^\vee)$ is the image of the tensor product $([1]\#\iota_{d_i}^\vee) \otimes ([1]\#\iota_{d_j}^\vee)$ under the $\circ$ product map

$$\circ \colon H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R) \otimes H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R) \to H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R),$$

for any cohomology class $\alpha \in H^*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$, we have

$$\left\langle \alpha, ([1]\#\iota_{d_i}^\vee) \circ ([1]\#\iota_{d_j}^\vee) \right\rangle = \sum \left\langle \alpha', [1]\#\iota_{d_i}^\vee \right\rangle \left\langle \alpha'', [1]\#\iota_{d_j}^\vee \right\rangle$$

where $\sum \alpha' \otimes \alpha''$ is the coproduct of α and the multiplication in the terms on the right-hand side is the product in $\mathbb{F}_p$. For this pairing to be nonzero, the coproduct of α must have a nonzero component in $H^{d_i}(\tau_{[n,pn-1]}\mathrm{SL}_1 R) \otimes H^{d_j}(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$. The cohomology class $\iota_{d_i}\iota_{d_j}$ is such an element, and therefore it must be dual to $([1]\#\iota_{d_i}^\vee) \circ ([1]\#\iota_{d_j}^\vee)$. Similarly, any nonzero power $([1]\#\iota_{d_i}^\vee)^{\circ k}$ must be dual to $\iota_{d_i}^k$. This is because k must be strictly less than p for $([1]\#\iota_{d_i}^\vee)^{\circ k}$ to be nonzero, and so, by the binomial theorem, the coproduct of $\iota_{d_i}^k$ has a nonzero component in $H^{d_i}(\tau_{[n,pn-1]}\mathrm{SL}_1 R)^{\otimes k}$. Therefore, each element of the form $[1]\#\iota_{d_i}$ or $[1]\#P^I\iota_{d_i}$, where $1 \le i \le s$ and $I \in S(d_i)$ is $\circ$-indecomposable, and we have

$$Q^\circ H_*(\mathrm{SL}_1 R) \cong \mathbb{F}_p\{[1]\#(P^I\iota_{d_1})^\vee \mid I \in S(d_1)\} \oplus \cdots \oplus \mathbb{F}_p\{[1]\#(P^I\iota_{d_s})^\vee \mid I \in S(d_s)\}.$$

To check that the map

$$\exp \colon H_*(\tau_{[n,pn-1]}\Omega^\infty R) \to H_*(\tau_{[n,pn-1]}\Omega^\infty R)$$

induces an isomorphism

$$Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0) \to Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R),$$

notice that each element, x, of the domain is a primitive element. Therefore, its iterated coproduct $\Delta^{\frac{p(p-1)}{2}-1}(x)$ is given by

$$\Delta^{\frac{p(p-1)}{2}-1}(x) = x \otimes [0] \otimes [0] \otimes \cdots \otimes [0] + \otimes [0] \otimes x \otimes [0] \otimes \cdots \otimes [0] + \cdots + [0] \otimes [0] \otimes [0] \otimes \cdots \otimes x.$$

Then

$$\begin{aligned}\exp(x) &= [1]\#x\#([\tfrac{1}{2!}]\circ[0]\circ[0])\#\cdots\#([\tfrac{1}{(p-1)!}]\circ[0]\circ\cdots\circ[0])\\ &\quad+[1]\#[0]\#([\tfrac{1}{2!}]\circ x\circ[0])\#\cdots\#([\tfrac{1}{(p-1)!}]\circ[0]\circ\cdots\circ[0])\\ &\quad\quad\vdots\\ &\quad\quad+[1]\#[0]\#([\tfrac{1}{2!}]\circ[0]\circ[0])\#\cdots\#([\tfrac{1}{(p-1)!}]\circ[0]\circ\cdots\circ x)\\ &= [1]\#x\end{aligned}$$

The map exp therefore sends the generator $(P^I\iota_{d_i})^\vee \in Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)$ to the generator $[1]\#(P^I\iota_{d_i})^\vee \in Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$. It follows that

$$\exp\colon Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0) \to Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$$

is a map of $\mathbb{F}_p$-modules that is both injective and surjective, and hence an isomorphism. □

Proof of Theorem 6.0.5. Consider the Grothendieck spectral sequences that compute the E^2 pages of the Miller spectral sequences for $H_*(\tau_{[n,pn-1]}R)$ and $H_*(\tau_{[n,pn-1]}\mathrm{sl}_1 R)$:

$$\begin{aligned}\mathrm{Untor}_s^{\mathcal{R}_p}(\mathbb{F}_p, \mathbf{L}_t Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R)) &\Longrightarrow H_{s+t}(\tau_{[n,pn-1]}R)\\ \mathrm{Untor}_s^{\mathcal{R}_p}(\mathbb{F}_p, \mathbf{L}_t Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)) &\Longrightarrow H_{s+t}(\tau_{[n,pn-1]}\mathrm{sl}_1 R)\end{aligned}$$

Since $n \geq 1$, the spaces $\tau_{[n,pn-1]}\Omega^\infty R$ and $\tau_{[n,pn-1]}\Omega^\infty R_0$ are identical; both are the product

$$K(\mathbb{F}_p, d_1)\times\cdots\times K(\mathbb{F}_p, d_s)$$

where $n \leq d_1 \leq \cdots \leq d_s \leq pn-1$. It follows that we can replace the E^2 page of the first spectral sequence with $\mathrm{Untor}_s^{\mathcal{R}_p}(\mathbb{F}_p, \mathbf{L}_t Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0))$.

We first claim that the E^2 pages of these Grothendieck spectral sequences are isomorphic. To do this, we will show that there is an isomorphism of co-Koszul complexes

$$K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbf{L}_* Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)) \xrightarrow{\sim} K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbf{L}_* Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0))$$

induced by the map

$$\exp\colon H_*(\tau_{[n,pn-1]}\Omega^\infty R) \to H_*(\tau_{[n,pn-1]}\Omega^\infty R).$$

Since both co-Koszul complexes have trivial differentials (there are no nonzero p'th powers in the rings $H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)$ and $H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)$, as shown in the proofs of Lemmas 6.0.8 and 6.0.9, and therefore no nonzero Dyer-Lashof operations on either ring), this is equivalent to showing that exp induces an isomorphism on each term.

Since

$$\mathbf{L}_0 Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0) \cong Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)$$

and

$$\mathbf{L}_0 Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R) \cong Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R),$$

the result of Lemma 6.0.9 shows that exp gives an isomorphism

$$\mathbf{L}_0 Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0) \xrightarrow{\sim} \mathbf{L}_0 Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R).$$

To show that it also gives an isomorphism between the modules

$$\mathbf{L}_1 Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0) = Q\,\mathrm{Tor}_2^{H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)}(\mathbb{F}_p, \mathbb{F}_p)$$

and

$$\mathbf{L}_1 Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R) = Q\,\mathrm{Tor}_2^{H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)}(\mathbb{F}_p, \mathbb{F}_p),$$

consider the bar complexes

$$B_*(\mathbb{F}_p, H_*(\tau_{[n,pn-1]}\Omega^\infty R_0), \mathbb{F}_p) \quad \text{and} \quad B_*(\mathbb{F}_p, H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R), \mathbb{F}_p).$$

Note that the kernel of the boundary map

$$B_2(\mathbb{F}_p, H_*(\tau_{[n,pn-1]}\Omega^\infty R_0), \mathbb{F}_p) \to B_1(\mathbb{F}_p, H_*(\tau_{[n,pn-1]}\Omega^\infty R_0), \mathbb{F}_p)$$

is generated by two families of elements: one family consists of the elements of the form $[((P^I \iota_{d_i})^\vee)^{\#(p-1)} \mid (P^I \iota_{d_i})^\vee]$ and $[(P^I \iota_{d_i}^\vee) \mid (P^I \iota_{d_i}^\vee)^{\#(p-1)}]$ where $|P^I \iota_{d_i}|$ is even; the other consists of elements of the form $[(P^I \iota_{d_i})^\vee \mid (P^I \iota_{d_i})^\vee]$ where $|P^I \iota_{d_i}|$ is odd. In either case, we have $1 \le i \le s$ and $I \in S(d_i)$. Because none of these elements are in the image of the boundary map of the bar complex, or products of classes in degree Tor_1, these elements generate $Q\,\mathrm{Tor}_2^{H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)}(\mathbb{F}_p, \mathbb{F}_p)$. Also note that these generators are carried to the elements

$$\begin{aligned}
&[[1]\#((P^I \iota_{d_i})^\vee)^{\#(p-1)} \mid [1]\#(P^I \iota_{d_i})^\vee],\\
&[[1]\#(P^I \iota_{d_i})^\vee \mid [1]\#((P^I \iota_{d_i})^\vee)^{\#(p-1)}], \text{ and}\\
&[[1]\#(P^I \iota_{d_i})^\vee \mid [1]\#(P^I \iota_{d_i})^\vee],
\end{aligned}$$

in $B_2(\mathbb{F}_p, H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R), \mathbb{F}_p)$, respectively, by the map exp.

As shown in the proof of Lemma 6.0.9, the $\circ$ product on $H_*(\mathrm{SL}_1 R)$ agrees with the $\#$ product (shifted from the 0-component to the 1-component) in all cases except

$$([1] \circ \iota_{d_i}^\vee) \circ ([1] \circ \iota_{d_j}^\vee) = [1]\#(\iota_{d_i}^\vee \# \iota_{d_j}^\vee) + [1]\#(\iota_{d_i}^\vee \circ \iota_{d_j}^\vee).$$

However, $[1]\#(\iota_{d_i}^\vee \# \iota_{d_j}^\vee)$ is nonzero unless either $p = 2$ and $d_i = d_j$ or p is odd and both d_i and d_j are odd. Thus the kernel of the map

$$B_2(\mathbb{F}_p, H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R), \mathbb{F}_p) \to B_1(\mathbb{F}_p, H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R), \mathbb{F}_p)$$

is isomorphic to the kernel of the map

$$B_2(\mathbb{F}_p, H_*(\tau_{[n,pn-1]}\Omega^\infty R_0), \mathbb{F}_p) \to B_1(\mathbb{F}_p, H_*(\tau_{[n,pn-1]}\Omega^\infty R_0), \mathbb{F}_p),$$

where the isomorphism is once again induced by exp. Therefore, exp gives an isomorphism

$$Q\,\mathrm{Tor}_2^{H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)}(\mathbb{F}_p, \mathbb{F}_p) \xrightarrow{\sim} Q\,\mathrm{Tor}_2^{H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)}(\mathbb{F}_p, \mathbb{F}_p).$$

As a result, we have shown that exp gives an isomorphism

$$K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbf{L}_* Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)) \xrightarrow{\sim} K^*(\mathbb{F}_p, \mathcal{R}_p(1), \mathbf{L}_* Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)).$$

This implies that we also have an isomorphism between the E^2 pages of each Grothendieck spectral sequence:

$$\mathrm{Untor}_s^{\mathcal{R}_p}(\mathbb{F}_p, \mathbf{L}_t Q^\# H_*(\tau_{[n,pn-1]}\Omega^\infty R_0)) \xrightarrow{\sim} \mathrm{Untor}_s^{\mathcal{R}_p}(\mathbb{F}_p, \mathbf{L}_t Q^\circ H_*(\tau_{[n,pn-1]}\mathrm{SL}_1 R)).$$

Because each of these spectral sequences have only two nonzero rows, each one degenerates to a long exact sequence. It follows that by the Five Lemma, the map exp induces an isomorphism

$$\mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q^\#)(H_t(\tau_{[n,pn-1]}\Omega^\infty R_0)) \xrightarrow{\sim} \mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q^\circ)(H_t(\tau_{[n,pn-1]}\mathrm{SL}_1 R))$$

between the elements on the E^∞ pages of the Grothendieck spectral sequences.

The above shows that the elements on the Miller E^2 pages for $H_*(\tau_{[n,pn-1]}R)$ and $H_*(\tau_{[n,pn-1]}\mathrm{sl}_1 R)$ are isomorphic. To determine the differentials, note that the diagram in the proof of Lemma 6.0.7 may be lifted to a diagram of infinite loop spaces, ultimately giving a map $\Omega^\infty R_0 \to \mathrm{SL}_1 R$. This gives a map of spectra $\Sigma^\infty\Omega^\infty R_0 \to \Sigma^\infty \mathrm{SL}_1 R$, and therefore a map

$$\begin{array}{ccccc} \tau_{[n,pn-1]}R & \longleftarrow & \tau_{[n,pn-1]}\Sigma^\infty\Omega^\infty R_0 & \longleftarrow & \cdots \\ & & \downarrow & & \\ \tau_{[n,pn-1]}\mathrm{sl}_1 R & \longleftarrow & \tau_{[n,pn-1]}\Sigma^\infty \mathrm{SL}_1 R & \longleftarrow & \cdots \end{array}$$

between the resolutions of $\tau_{[n,pn-1]}R$ and $\tau_{[n,pn-1]}\mathrm{sl}_1 R$ that are used to construct the Miller spectral sequence. The map exp therefore defines a map between the two Miller spectral sequences.

Also note that because $\tau_{[n,pn-1]}R$ is an $H\mathbb{F}_p$-algebra, the Miller spectral sequence

$$\mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q^{\#})(H_t(\tau_{[n,pn-1]}\Omega^\infty R_0)) \Longrightarrow H_{s+t}(\tau_{[n,pn-1]}R)$$

must collapse at E^2 (see Proposition 5.3.13). Since

$$\mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q^{\#})(H_t(\tau_{[n,pn-1]}\Omega^\infty R_0)) \xrightarrow{\sim} \mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q^{\circ})(H_t(\tau_{[n,pn-1]}\mathrm{SL}_1 R))$$

is a map of spectral sequences, this forces the differentials in the Miller spectral sequence

$$\mathbf{L}_s(\mathbb{F}_p \otimes_{\mathcal{R}_p} Q^{\circ})(H_t(\tau_{[n,pn-1]}\mathrm{SL}_1 R)) \Longrightarrow H_{s+t}(\tau_{[n,pn-1]}\mathrm{sl}_1 R)$$

to be trivial as well. It follows that exp induces an isomorphism

$$H_*(\tau_{[n,pn-1]}R) \cong H_*(\tau_{[n,pn-1]}\mathrm{sl}_1 R),$$

and therefore there is an equivalence of spectra $\tau_{[n,pn-1]}R \simeq \tau_{[n,pn-1]}\mathrm{sl}_1 R$. □

6.1 Possible Connections to TAQ Cohomology

If R is a highly structured ring spectrum, then the Postnikov tower of the underlying spectrum of R lifts to a Postnikov tower of highly structured ring spectra. In the case that R is an E_∞-ring spectrum, this was studied by I. Kriz [23] and M. Basterra [8] in the 1990s. The k-invariants in this case, which we will refer to as E_∞ *k-invariants*, are classes in the *topological André-Quillen cohomology* (or, briefly, TAQ *cohomology*) of the truncations of R; specifically,

$$[\tilde{k}_n] \in \mathrm{TAQ}^{n+2}(\tau_{\leq n}R; H\pi_{n+1}(R)).$$

TAQ cohomology is a topological refinement of *André-Quillen cohomology*, which is a cohomology theory for commutative rings (or, more generally, commutative algebras over a fixed commutative ring) that was independently constructed by M. André [3] and D. Quillen [39] in the late 1960s/early 1970s. In analogy with André-Quillen cohomology, each element of the abelian group $\mathrm{TAQ}^{n+2}(\tau_{\leq n}R; H\pi_{n+1}(R))$ can be represented by a map

$$\tau_{\leq n}R \to \tau_{\leq n}R \vee \Sigma^{n+2}H\pi_{n+1}(R),$$

where $\tau_{\leq n}R \vee \Sigma^{n+2}H\pi_{n+1}(R)$ is the *square-zero extension* of $\tau_{\leq n}R$ by $\Sigma^{n+2}H\pi_{n+1}(R)$, which is an E_∞-ring spectrum (given the "zero multiplication").

Remark 6.1.1. Kriz and Basterra's work on Postnikov towers for E_∞-ring spectra was motivated by a problem posed by J.P. May in the mid 1970s, which asked whether or not the *Brown-Peterson spectrum*, BP, is an E_∞-ring spectrum. The problem remained open until 2017, when T. Lawson [26] proved that BP does not have an E_{12}-structure at the prime $p = 2$. In the odd primary case, BP does not have a $E_{2(p^2+2)}$-structure, as shown by A. Senger in [36].

For all $n \geq 0$, there is a "forgetful" map

$$\mathrm{TAQ}^{n+2}(\tau_{\leq n}R; H\pi_{n+1}(R)) \to H^{n+2}(\tau_{\leq n}R; H\pi_{n+1}(R))$$

induced by the projection

$$\tau_{\leq n}R \vee \Sigma^{n+2}H\pi_{n+1}(R) \to \Sigma^{n+2}H\pi_{n+1}(R).$$

This sends the E_∞ k-invariant $[\tilde{k}_n]$, which "remembers" both the additive and multiplicative structures of R, to the spectrum-level k-invariant $[k_n]$, which only "remembers" the additive structure of R. As seen in Examples 6.0.3 and 6.0.4, as well as in the proof of Theorem 6.0.5, the k-invariants of the spectra $\mathrm{gl}_1 R$ and $\mathrm{sl}_1 R$ are related to the multiplicative structure of R. As a result, we pose the following question:

Question 6.1.2. *How are the E_∞ k-invariants of an E_∞-ring spectrum R related to its spectrum level k-invariants and the k-invariants of $\mathrm{gl}_1 R$? Is it possible to compute the E_∞ k-invariants of R from knowledge of its spectrum level k-invariants and the k-invariants of either $\mathrm{gl}_1 R$ or $\mathrm{sl}_1 R$?*

It is known [25, Example 1.5.5] that there are power operations in TAQ cohomology and that there is a Browder bracket on the homotopy groups of E_n-ring spectra [25, Theorem 1.4.2]. Roughly, the k-invariants of $\mathrm{gl}_1 R$ may be related to these structures in particular.

Remark 6.1.3. If R is an A_∞-ring spectrum, then the A_∞ k-invariants of R are classes in *topological Hochschild cohomology*, $\mathrm{THH}^{n+2}(\tau_{\leq n}R; H\pi_{n+1}R)$. If this A_∞-structure happens to lift to an E_∞-structure, then there are forgetful maps

$$\mathrm{TAQ}^{n+2}(\tau_{\leq n}R; H\pi_{n+1}R) \to \mathrm{THH}^{n+2}(\tau_{\leq n}R; H\pi_{n+1}R) \to H^{n+2}(\tau_{\leq n}R; H\pi_{n+1}R)$$

sending one type of structured k-invariant of R to another. In this case, it is possible to construct unit *spaces* $\mathrm{GL}_1 R$ and $\mathrm{SL}_1 R$, and one might hope for a relationship between the A_∞ k-invariants of R, the spectrum level k-invariants of R, and the k-invariants of the spaces $\mathrm{GL}_1 R$ and $\mathrm{SL}_1 R$.

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Appendix A

The Steenrod Algebra & Its Dual

The *mod-p Steenrod algebra*, $\mathcal{A}_p$, is the $\mathbb{F}_p$-algebra of cohomology operations (natural transformations)

$$H^*(-;\mathbb{F}_p) \to H^{*+d}(-;\mathbb{F}_p).$$

Equivalently, it is $H^*(H\mathbb{F}_p)$, the cohomology of the Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$. For $p = 2$, this is generated by the operations Sq^r $(r \geq 0)$, which raise degree by r; if p is odd, then it is generated by the operations P^r, which raise degree by $2r(p-1)$, and by the mod-p Bockstein, β, which is the connecting homomorphism in cohomology associated to the short exact sequence

$$0 \to \mathbb{Z}/p \xrightarrow{\cdot p} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

and raises degree by 1. These operations act on both the cohomology of spaces and spectra and satisfy the following properties (modifications for the case $p = 2$ are in brackets):

- *(Unit):* $P^0 = \mathrm{id}$.

- *(Additivity):* $P^r(x+y) = P^r(x) + P^r(y)$.

- *(Cartan Formula):* $P^r(xy) = \sum_{i+j=r} P^i(x)P^j(y)$.

- *(p'th Power):* $P^r(x) = x^p$ if $r = \frac{|x|}{2}$ [$\mathrm{Sq}^r(x) = x^2$ if $r = |x|$].

- *(Instability):* If x is an element in the cohomology of a space, then $P^r(x) = 0$ if $r > \frac{|x|}{2}$ [$\mathrm{Sq}^r(x) = 0$ if $r > |x|$].

- *(Stability):* If $\sigma\colon H^k(X) \xrightarrow{\sim} H^{k+1}(\Sigma X)$ is the suspension isomorphism, then $\sigma P^r = P^r \sigma$.

- *(Adem Relations):*

  - ($p = 2$): If $r < 2s$, then

$$\mathrm{Sq}^r \mathrm{Sq}^s = \sum_{j=0}^{\lfloor r/2 \rfloor} \binom{s-j-1}{r-2j} \mathrm{Sq}^{r+s-j} \mathrm{Sq}^j$$

  - ($p \geq 3$): If $r < ps$, then

$$P^r P^s = \sum_{j=0}^{\lfloor r/p \rfloor} (-1)^{r+j} \binom{(p-1)(s-j)-1}{r-pj} P^{r+s-j} P^j$$

  If $r \leq ps$, then

$$P^r \beta P^s = \sum_{j=0}^{\lfloor r/p \rfloor} (-1)^{r+j} \binom{(p-1)(s-j)}{r-pj} \beta P^{r+s-j} P^j + \sum_{j=0}^{\lfloor (r-1)/p \rfloor} (-1)^{r+j+1} \binom{(p-1)(s-j)-1}{r-pj-1} P^{r+s-j} \beta P^j$$

Just as for Dyer-Lashof operations, given a sequence $I = (i_1, i_2, \ldots, i_n)$, we let $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_n}$ (similarly for P^I). We may also define the *length*, *degree*, and *excess* of Sq^I (or P^I) as in Definition 3.1.3.

Example A.0.1. The mod-p cohomology of the Eilenberg-Mac Lane spaces $K(\mathbb{F}_p, n)$ can be described in terms of the *fundamental class* $\iota_n \in H^n(K(\mathbb{F}_p, n); \mathbb{F}_p)$ and the Steenrod operations:

- $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2) \cong \mathbb{F}_2[\mathrm{Sq}^I(\iota_n) \mid I \text{ admissible}, e(I) < n]$

- $H^*(K(\mathbb{F}_p, n); \mathbb{F}_p) \cong \mathbb{F}_p[P^I(\iota_n) \mid |P^I(\iota_n)| \text{ even}] \otimes \Lambda_{\mathbb{F}_p}[P^I(\iota_n) \mid |P^I(\iota_n)| \text{ odd}]$

  In both factors of this tensor product, I ranges over all admissible sequences, with either $e(I) < n$ or, if $e(I) = n$, then P^I must be of the form $P^I = \beta P^{I'}$ for some sequence I'.

Dualizing, we have

- $H_*(K(\mathbb{F}_2, n); \mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}[x_I]$, where x_I is dual to the polynomial generator $\mathrm{Sq}^I(\iota_n)$, and

- $H_*(K(\mathbb{F}_p, n); \mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}[x_I] \otimes \Lambda_{\mathbb{F}_p}[y_I]$, where the x_I are dual to the even degree generators of $H^*(K(\mathbb{F}_p, n))$ and the y_I are dual to the odd degree generators.

The Steenrod algebra has the structure of a graded Hopf algebra over $\mathbb{F}_p$, where the coproduct, $\Delta\colon \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$, is given by the Cartan Formula:

$$\Delta(P^r) = \sum_{i+j=r} P^i \otimes P^j.$$

The conjugation on $\mathcal{A}_p$, written $\chi\colon \mathcal{A}_p \to \mathcal{A}_p$, is determined by the formulas

$$\chi(P^r) = \sum_{i+j=r} P^i \chi(P^j) \qquad \chi(P^0) = 1.$$

Therefore, its linear dual, $\mathcal{A}_p^*$, is also a Hopf algebra with conjugation. As an algebra, its structure is very simple (due to the fact that the Steenrod algebra has a simple coproduct) and was computed by Milnor in 1958 [37]:

- $\mathcal{A}_2^* = H_*(H\mathbb{F}_2) \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots]$. Here, ξ_i is dual to $\mathrm{Sq}^{2^{i-1}} \mathrm{Sq}^{2^{i-2}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1$ and has degree $|\xi_i| = 2^i - 1$. (Also note that ξ_1^n is dual to Sq^n.)

- $\mathcal{A}_p^* = H_*(H\mathbb{F}_p) \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_p}[\tau_0, \tau_1, \ldots]$. Here, ξ_i is dual to $P^{p^{i-1}} P^{p^{i-2}} \cdots P^p P^1$ and has degree $|\xi_i| = 2p^i - 2$; the element τ_i is dual to $P^{p^{i-1}} P^{p^{i-2}} \cdots P^p \beta$ and has degree $|\tau_i| = 2p^i - 1$. (We also have that ξ_1^n is dual to P^n and $\tau_0 \xi_1^n$ is dual to βP^n.)

The rest of the Hopf algebra structure is determined by the coproduct, Δ, and the conjugation, χ. The coproduct formulas (for any prime p) are:

$$\Delta(\xi_n) = \sum_{i+j=n} \xi_i^{p^j} \otimes \xi_j \qquad\qquad \Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i+j=n} \xi_i^{p^j} \otimes \tau_j$$

Moreover, since $\mathcal{A}_p^* = H_*(H\mathbb{F}_p)$ is the homology of an E_∞-ring spectrum, it has an action of the Dyer-Lashof algebra. The following formulas for this action are due to Steinberger [11, Ch III]:

- (if $p = 2$): As an algebra over the Dyer-Lashof algebra, $\mathcal{A}_2^*$ is generated by ξ_1. We have the following formulas:

  $Q^{2^i-2}(\xi_1) = \chi\xi_i$ $(i \geq 2)$ and $Q^n(\xi_1) \neq 0$ $(n \geq 1)$

  $Q^{2^i}(\xi_i) = \xi_{i+1} + \xi_1\xi_i^2$ $(i \geq 1)$

  $$Q^{2^i}(\chi\xi_i) = \chi\xi_{i+1} \text{ and } Q^n(\chi\xi_i) = \begin{cases} Q^{n+2^i-2}(\xi_1) & \text{if } n \equiv 0, -1 \bmod 2^i \\ 0 & \text{otherwise} \end{cases} \quad (i \geq 1)$$

  The p'th Power Lemma: $Q^{2n-1}(\xi_1) = [Q^{n-1}(\xi_1)]^2$ $(n \geq 2)$

- (if $p \geq 3$): Let $\rho(i) = \frac{p^i-1}{p-1}$. As an algebra over the Dyer-Lashof algebra, $\mathcal{A}_p^*$ is generated by τ_0. We have the following formulas:

  $Q^{\rho(i)}(\tau_0) = (-1)^i\chi\tau_i$ $(i \geq 1)$

  $\beta Q^{\rho(i)}(\tau_0) = (-1)^i\chi\xi_i$ and $\beta Q^n(\tau_0) \neq 0$ $(n \geq 1)$

  $\beta Q^n(\tau_0) \neq 0$ $(n \geq 1)$

  $$Q^{p^i}(\chi\xi_i) = \chi\xi_{i+1} \text{ and } Q^n(\chi\xi_i) = \begin{cases} (-1)^i\beta Q^{n+\rho(i)}(\tau_0) & \text{if } n \equiv -1 \bmod p^i \\ (-1)^{i+1}\beta Q^{n+\rho(i)}(\tau_0) & \text{if } n \equiv 0 \bmod p^i \\ 0 & \text{otherwise} \end{cases} \quad (i \geq 1)$$

  $$Q^{p^i}(\chi\tau_i) = \chi\tau_{i+1} \text{ and } Q^n(\chi\tau_i) = \begin{cases} (-1)^{i+1}Q^{n+\rho(i)}(\tau_0) & \text{if } n \equiv 0 \bmod p^i \\ 0 & \text{otherwise} \end{cases} \quad (i \geq 0)$$

  The p'th Power Lemma: $\beta Q^{pn}(\tau_0) = [\beta Q^n(\tau_0)]^p$ $(n \geq 1)$

Appendix B

Simplicial Objects

Recall that a simplicial complex K is a topological space that is built up from n-simplices Δ^n in the following way:

- Any face of a simplex in K is also in K.
- Any two non-disjoint simplices in K intersect in a common face.

Simplicial objects in a category $\mathcal{C}$ are built up from objects of $\mathcal{C}$ in a similar way.

Definition B.0.1. The *simplex category*, denoted by $\boldsymbol{\Delta}$, is the category whose objects are nonempty totally ordered sets $[n] = \{0 < 1 < \cdots < n\}$ $(n \geq 0)$ and whose morphisms are order preserving maps $[n] \to [m]$. These maps are generated by the

- *face maps* $\delta_i \colon [n] \to [n+1]$, the unique order preserving map missing only the element $i \in [n+1]$;
- *degeneracy maps* $\sigma_j \colon [n+1] \to [n]$, the unique order preserving map hitting $j \in [n]$ exactly twice and every other element exactly once.

Moreover, these maps satisfy the following *simplicial identities*:

$$\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1} \qquad \text{if } i < j$$

$$\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1} \qquad \text{if } i \leq j$$

$$\sigma_j \circ \delta_i = \begin{cases} \delta_i \circ \sigma_{j-1} & \text{if } i < j \\ 1 & \text{if } i = j, j+1 \\ \delta_{i-1} \circ \sigma_j & \text{if } i > j+1 \end{cases}$$

Definition B.0.2. A *simplicial object* in a category $\mathcal{C}$ is a functor $F\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. This may be viewed a sequence $A_\bullet = \{A_n\}_{n\geq 0}$ of objects of $\mathcal{C}$ (where $A_n = F[n]$) with face maps $F(\delta_i) = d_i\colon A_{n+1} \to A_n$ and degeneracy maps $F(\sigma_j) = s_j\colon A_n \to A_{n+1}$. This is often written as a diagram of the form

$$A_0 \leftleftarrows A_1 \ \substack{\leftarrow\\\leftarrow\\\leftarrow} \cdots$$

where there is an arrow for each face map and the degeneracy maps are omitted. We will denote the category of simplicial objects in $\mathcal{C}$ by $s\mathcal{C}$.

Remark B.0.3. Dually, a *cosimplicial object* in $\mathcal{C}$ is a functor $\mathbf{\Delta} \to \mathcal{C}$.

Definition B.0.4. The *geometric realization* of a simplicial set $A_\bullet$ is the following topological space:

$$|A| = \coprod_n (A_n \times \Delta^n)/\sim$$

Here, Δ^n is the standard n-simplex, and $\sim$ is an equivalence relation defined by the requirement that $(d_i(a), x) \sim (a, \delta_i(x))$ and $(s_j(a), x) \simeq (a, \sigma_j(x))$, where (d_i, s_j) are the face and degeneracy maps on $A_\bullet$, and (δ_i, σ_j) are the face and degeneracy maps on Δ^n.

Example B.0.5. For any object C in a category $\mathcal{C}$, there is the *constant* simplicial object $\mathrm{const}_\bullet(C)$. This has terms $\mathrm{const}_n(C) = C$ for all $n \geq 0$ and all face and degeneracy maps given by the identity on C. If $\mathcal{C}$ is a small category, then the functor $\mathrm{const}_\bullet\colon \mathcal{C} \to \mathbf{s}\mathcal{C}$ is a fully faithful embedding.

If $\mathcal{C}$ is the category of commutative rings or the category of modules over a commutative ring R, then the category of simplicial objects in $\mathcal{C}$ form a model category. It therefore makes sense to consider the homotopy groups of a simplicial ring or a simplicial R-module. These groups have a simple description in terms of the homology groups of a chain complex we can associate to any such simplicial object.

Definition B.0.6. Given a simplicial R-module $A_\bullet$, its *associated chain complex*, $\mathrm{Ch}(A_\bullet)$, is the chain complex whose terms are the same as those of $A_\bullet$ and with boundary maps $\partial_n\colon A_n \to A_{n-1}$ given by the alternating sum of face maps $d_0 - d_1 + d_2 - d_3 + \cdots$.

Theorem B.0.7 (The Dold-Kan Correspondence)**.** *The categories of simplicial R-modules and nonnegatively graded chain complexes over R are equivalent. Under this equivalence, the homotopy groups of a simplicial R-module correspond to the homology groups of the associated chain complex. In addition, the categories of simplicial (commutative) R-algebras and (commutative) differential, nonnegatively graded algebras over R are equivalent.*

Remark B.0.8. There is also a "stable" version of the Dold-Kan Correspondence [42]: the category of $H\mathbb{Z}$-module spectra is equivalent to the category of differential graded algebras over $\mathbb{Z}$. This also holds if $\mathbb{Z}$ is replaced by $\mathbb{F}_p$ for any prime p.

Note that any simplicial ring $A_\bullet$ is also a simplicial abelian group, and Theorem B.0.7 applies in this case to identify $\pi_*(A_\bullet)$ with $H_*(\mathrm{Ch}(A_\bullet))$ as well.

B.1 The Bar Complex

Let R be a commutative ring and let A be a graded commutative R-algebra. Given two A-modules M and N, we may form the following simplicial A-module:

$$M \otimes_R N \leftleftarrows M \otimes_R A \otimes_R N \;\overleftarrow{\overleftarrow{\overleftarrow{}}}\; M \otimes_R A^{\otimes 2} \otimes_R N \;\cdots$$

This is an example of a *two-sided bar construction*, $B_\bullet(M, A, N)$. The terminology comes from the fact that the element

$$m \otimes a_1 \otimes \cdots \otimes a_n \otimes n \in B_n(M, A, N) = M \otimes A^{\otimes n} \otimes N$$

is often written more compactly as "$m[a_1|\cdots|a_n]n$". This simplicial object has face maps $d_i\colon B_s(M, A, N) \to B_{s-1}(M, A, N)$ given by

$$d_i(m[a_1|\cdots|a_s]n) = m[a_1|\cdots|a_i a_{i+1}|\cdots|a_s]n.$$

If $i = 0$, we use the action of A on M; if $i = n$, we use the action of A on N. The degeneracy maps $s_j\colon B_s(M, A, N) \to B_{s+1}(M, A, N)$ are given by

$$s_j(m[a_1|\cdots|a_s]n) = m[a_1|\cdots|a_j|1|a_{j+1}|\cdots|a_s]n$$

Note that the map $M \otimes_R N \to M \otimes_A N$ gives an augmentation $B_\bullet(M, A, N) \to M \otimes_A N$.

Definition B.1.1. The *bar complex*, $B_*(M, A, N)$, is the associated chain complex of the bar construction $B_\bullet(M, A, N)$.

If A, M, and N are all graded, then the bar complex is a bigraded complex. The bidegree of the element $m[a_1|\cdots|a_s]n$ is $(s, |m| + |a_1| + \cdots + |a_s| + |n|)$. The differential $\partial\colon B_{s,t}(M, A, N) \to B_{s-1,t}(M, A, N)$ is given by the formula

$$\begin{aligned}\partial(m[a_1|\cdots|a_s]n) = (-1)^{e_0}&(ma_1)[a_2|\cdots|a_s]n \\ &+ \sum_{i=1}^{s-1}(-1)^{e_i} m[a_1|\cdots|a_ia_{i+1}|\cdots|a_s]n \\ &- (-1)^{e_{s-1}} m[a_1|\cdots|a_{s-1}]a_sn\end{aligned}$$

where $e_0 = |m|$ and $e_i = i + |m| + |a_1| + \cdots + |a_i|$ for $1 \le i \le s-1$. Moreover, the bar complex $B_*(R, A, R)$ has the structure of a differential coalgebra; the coproduct is defined by

$$[a_1|\cdots|a_s] \mapsto \sum_{i=1}^{s-1}([a_1|\cdots|a_i]) \otimes ([a_{i+1}|\cdots|a_s]).$$

Remark B.1.2. If A is an augmented R-algebra, then the augmentation $\epsilon\colon A \to R$ gives R the structure of an A-module. Explicitly, for $r \in R$ and $a \in A$, $ar = \epsilon(a)r$, where the right-hand side is the product in R. In this case, the differential of $B_*(R, A, R)$ is given by

$$\begin{aligned}\partial(r[a_1|\cdots|a_s]r') = (-1)^{e_0}&r\epsilon(a_1)[a_2|\cdots|a_s]r' \\ &+ \sum_{i=1}^{s-1}(-1)^{e_i} r[a_1|\cdots|a_ia_{i+1}|\cdots|a_s]r' \\ &- (-1)^{e_{s-1}} r[a_1|\cdots|a_{s-1}]\epsilon(a_s)r'\end{aligned}$$

The bar complex is useful because it provides a straightforward way to construct resolutions of algebras/modules, as the following proposition shows:

Proposition B.1.3. *The bar complex $B_*(A, A, A)$ gives a resolution of an algebra A. In addition, $B_*(A, A, N) = B_*(A, A, A) \otimes_R N$ (the tensor product is taken levelwise) gives a resolution of an A-module N, with the augmentation given by the action of A on N.*

Corollary B.1.4. *The homology of $B_*(M, A, N)$ is isomorphic to $\mathrm{Tor}_*^A(M, N)$.*

There are similar constructions taking place in other categories as well:

Example B.1.5. Given a group G and two G-spaces X and Y, we may consider the bar construction $B_\bullet(X, G, Y)$, which is a simplicial space:

$$X \times Y \leftleftarrows X \times G \times Y \leftleftarrows\!\!\!\!\leftarrow X \times G^2 \times Y \;\cdots$$

If $X = Y = *$, a one-point space, then the geometric realization of $B_\bullet(*, G, *)$ is the *classifying space* of G, written BG. Also, note that the space $EG = |B(G, G, *)|$ is contractible and has a right action by G. The quotient map

$$|B_\bullet(G, G, *)| \to |B_\bullet(G, G, *)|/G = |B_\bullet(*, G, *)| = BG$$

has fibers G and is an explicit construction of the universal G-bundle $EG \to BG$.

The connection between the above examples is that they are both special cases of a more general *monadic* two-sided bar construction [33], defined below:

Definition B.1.6. Let $\mathcal{C}$ be a category, let $T\colon \mathcal{C} \to \mathcal{C}$ be a monad, let $X \in \mathcal{C}$ be a left T-algebra, and let $F\colon \mathcal{C} \to \mathcal{C}$ be a functor with a right action of T. The *two-sided monadic bar construction*, $B_\bullet(F, T, X)$, is the following simplicial object in $\mathcal{C}$:

$$FX \leftleftarrows FTX \leftleftarrows\!\!\!\!\leftarrow FT^2X \;\cdots$$

The face maps $B_{n+1}(F, T, X) \to B_n(F, T, X)$ are given by the multiplication maps $\mu\colon T^2 \to T$ of the monad T, the right action $FT \to F$ of T on F, and the left action $T(X) \to X$ of T on X. The degeneracy maps are given by the unit $\mathrm{id}_{\mathcal{C}} \to T$ of the monad T.

Example B.1.7. The bar construction $B_\bullet(T, T, X)$ is given by

$$TX \leftleftarrows TTX \leftleftarrows\!\!\!\!\leftarrow TT^2X \;\cdots$$

and is augmented over X, since X is assumed to be an algebra over the monad T. The augmentation $T(X) \to X$ has a natural splitting map $X \to T(X)$ given by the unit, η, of the monad, giving an extra degeneracy map. It follows that the simplicial object $B_\bullet(T, T, X)$ is acyclic, and so it is a cofibrant replacement of X. (See Example B.3.3.)

B.2 The Cobar Complex

Definition B.2.1. The *cobar complex*, $C^*(M, A, N) = B_*(M, A, N)^\vee$, is the R-linear dual of the bar complex.

Explicitly, we have

$$C^s(M, A, N) = M^\vee \otimes (A^\vee)^{\otimes s} \otimes N^\vee.$$

The differential $\partial^\vee \colon C^s(M, A, N) \to C^{s+1}(M, A, N)$ is dual to the differential of $B_*(M, A, N)$ and is given by the formula

$$\begin{aligned}\partial^\vee(\mu[\alpha_1|\cdots|\alpha_s]\nu) = -(-1)^{\epsilon_{0,r}} \sum_r \mu'_r[\alpha'_r|\alpha_1|\cdots|\alpha_s]\nu \\ - \sum_{i=1}^{s} \sum_r (-1)^{\epsilon_{i,r}} \mu[\alpha_1|\cdots|\alpha'_{i,r}|\alpha''_{i,r}|\cdots|\alpha_s]\nu \\ - (-1)^{\epsilon_{s+1,r}} \sum_r \mu[\alpha_1|\cdots|\alpha_s|\alpha'_r]\nu'_r\end{aligned}$$

Here, the coproduct of $\alpha \in A^\vee$ is $\sum_r \alpha'_r \otimes \alpha''_r$, the (right) coaction of $A^\vee$ on $M^\vee$ sends $\mu \in M^\vee$ to $\sum_r \mu'_r \otimes \alpha'_r \in M^\vee \otimes A^\vee$ (similarly for the left coaction of $A^\vee$ on $N^\vee$) and we have

$$\epsilon_{i,r} = \begin{cases} |\mu'_r| & \text{if } i = 0 \\ i + |\alpha_1| + \cdots + |\alpha_{i-1}| + |\alpha'_{i,r}| & \text{if } 1 \le i \le s \\ (s+1) + |\alpha_1| + \cdots + |\alpha_n| + |\alpha'_r| + |\nu'_r| & \text{if } i = s+1 \end{cases}$$

Moreover, because the bar complex $B_*(R, A, R)$ is a differential coalgebra, the cobar complex $C^*(R, A, R)$ is a differential algebra, with product $\smile$ defined by juxtaposition:

$$[\alpha_1|\cdots|\alpha_m] \smile [\beta_1|\cdots|\beta_n] = [\alpha_1|\cdots|\alpha_m|\beta_1|\cdots|\beta_n]$$

Proposition B.2.2. *The cohomology of $C^*(M, A, N)$ is isomorphic to $\mathrm{Ext}^*_A(M, N)$.*

B.3 Simplicial Resolutions

Definition B.3.1. Let A be an object of a model category $\mathcal{C}$. A *simplicial resolution* $P_\bullet$ of A is an augmented simplicial object

$$A \leftarrow P_0 \leftleftarrows P_1 \; \substack{\leftarrow\\\leftarrow\\\leftarrow} \cdots$$

in $\mathcal{C}$ such that $\pi_0(P_\bullet) \cong A$ while $\pi_k(P_\bullet) = 0$ for all $k \geq 1$. In [17, Prop. 4.21], it is shown that simplicial resolutions exist and are unique up to homotopy equivalence.

Example B.3.2. Let A be a commutative algebra over a fixed commutative ring R, and let $\mathbf{Alg}_R/A$ be the category of commutative algebras augmented over A. (Note that this is not an abelian category.) Let $P_\bullet$ be a simplicial resolution of A in the model category $\mathbf{sAlg_R}$. The *cotangent complex* of A is the simplicial A-module

$$\mathbf{L}_R(A) := A \otimes_{P_\bullet} \Omega_R(P_\bullet),$$

which we can view as the total left derived functor of the functor $A \otimes_{(-)} \Omega_R(-)$ on $\mathbf{Alg}_R/A$. Its homotopy groups are the *André-Quillen homology* groups of A relative to R, written $AQ_*(A, R)$.

Simplicial resolutions are very flexible resolutions, as they may be used to compute derived functors on both abelian categories and non-abelian categories (where it is not possible to construct a projective/injective resolution). In certain situations, there is a straightforward way to construct simplicial resolutions of objects, even if the resulting resolutions are typically very large.

Example B.3.3 (The Standard/Bar Resolution)**.** Let T be a monad and let X be a T-algebra. Given this, we may construct a coaugmented cosimplicial object

$$X \rightarrow T(X) \rightrightarrows T^2(X) \; \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

where the coface maps $T^n(X) \rightarrow T^{n+1}(X)$ and the coaugmentation $X \rightarrow T(X)$ are constructed from the unit of the monad, and the codegeneracies $T^{n+1}(X) \rightarrow T^n(X)$ are constructed from the multiplication map $T^2 \rightarrow T$ of the monad. Since X is a T-algebra, there is a map $T(X) \rightarrow X$ that serves as an extra codegeneracy. This gives us an augmented *simplicial* object

$$X \leftarrow T(X) \leftleftarrows T^2(X) \; \substack{\leftarrow\\\leftarrow\\\leftarrow} \cdots$$

where the face maps are the old codegeneracy maps plus the action $T(X) \to X$, and the degeneracies are the old coface maps. This is exactly the two-sided bar construction $B_\bullet(T, T, X)$, which, as mentioned in Example B.4, is a cofibrant replacement of X; in other words, it is a simplicial resolution of X. This resolution is called the *standard resolution* or the *bar resolution* of the T-algebra X.

Similarly, starting with a comonad S and an S-coalgebra X, we may construct an augmented simplicial object

$$X \leftarrow S(X) \leftleftarrows S^2(X) \Lleftarrow \cdots$$

where the face maps $S^{n+1}(X) \to S^n(X)$ and the augmentation $S(X) \to X$ are constructed from the counit of the comonad, and the degeneracies $S^n(X) \to S^{n+1}(X)$ are constructed from the comultiplication map $S \to S^2$ of the comonad. This also gives a simplicial resolution of X, and is also called the standard/bar resolution of X.