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QUASI-INVARIANCE OF ANALYTIC MEASURES ON COMPACT GROUPS*

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1. Introduction

The study of analytic measures on compact groups with ordered duals has been the subject of several papers on Fourier Analysis in recent years (see W. Rudin [12] for references). In their remarkable papers [5], [6], H. Helson and D. Lowdenslager have used a new method to study the properties of analytic functions on the Bohr group. In his subsequent works [3], [4], Helson has emphasized the connection of this problem to the Weyl-Von Neumann operator equations ([8], [10]). In the meantime, K. de Leeuw and I.Glicksberg [2] have given an extension of the classical theorem of F and M. Riesz to compact groups. They obtain as its consequence refinements of some theorems of Helson-Lowdenslager [5] and S. Bochner [1].

Our purpose here is to use Helson's method in [4] to obtain a simple proof of the de Leeuw-Gliskberg theorem basing ourselves entirely on the Hilbert space geometry. We think that the interest of this proof, aside from its simplicity and clarity, lies in unifying the ideas of the above two approaches. This unity may eventually lead to a deeper knowledge of analytic measures on groups with ordered duals. Such a study has been made in the special case of the Bohr group by M. G. Nadkarni [9]. A complete study may also give an extension of the work of G. Kallianpur and V. Mandrekar [7] to the situation considered by Helson-Lowdenslager [6].

2. Quasi-invariance of Spectral Measures

Let G be a locally compact abelian group. Let Γ be the character group of G. Let H be a Hilbert space. Let $\{U_\gamma : \gamma \in \Gamma\}$ be a strongly continuous group of unitary operators on H. It is known ([11]p. 392) that there exists a hermitian projection valued measure β on the Borel subsets of G such that

$$(2.1) \qquad U_\gamma = \int_G \chi_\gamma(g)\beta(dg)$$

where χ_γ denotes the character on G corresponding to $\gamma \in \Gamma$.

Let ψ be a continuous homomorphism of Γ into R, the group of real numbers with the usual topology. ψ induces a homomorphism $\varphi: R \to G$ of the associated dual groups. In fact, φ is the unique mapping defined by $\chi_\gamma(\varphi(t)) = e^{i\psi(\gamma)t}$. With the above notation we obtain the following result of purely geometric nature which will be used in Section 3 to prove the de Leeuw-Glicksberg theorem.

Theorem 2.1. Let $\{V_t;\ t \in R\}$ be a group of unitary operators satisfying

(2.2) $$U_\gamma V_t = \chi_\gamma(\varphi(t)) V_t U_\gamma.$$

Then $V_t \beta(\Delta) V_{-t} = \beta(\Delta + \varphi(t))$ where β is the spectral measure on G corresponding to $\{U_\gamma,\ \gamma \in \Gamma\}$ and Δ is any Borel subset of G.

Proof: By 2.1 we have

(2.3) $$U_\gamma V_t = \int_G \chi_\gamma(g) \beta(dg) V_t.$$

But

(2.4) $$\chi_\gamma(\varphi(t)) V_t U_\gamma = \chi_\gamma(\varphi(t)) V_t \int_G \chi_\gamma(g)\beta(dg) = \chi_\gamma(\varphi(t)) \int_G \chi_\gamma(g) V_t \beta(dg)$$

$$= \int_G \chi_\gamma(g + \varphi(t))\, V_t\, \beta(dg) = \int_G \chi_\gamma(g) V_t \beta(dg - \varphi(t)).$$

Since $U_\gamma V_t = \chi_\gamma(\varphi(t)) V_t U_\gamma$, we can equate (2.3) and (2.4) to obtain

(2.5) $$\int_G \chi_\gamma(g)\beta(dg) V_t = \int_G \chi_\gamma(g) V_t \beta(dg - \varphi(t));$$

i.e., for all $x, y \in H$,

(2.6) $$\int_G \chi_\gamma(g)(\beta(dg)V_t x, y) = \int_G \chi_\gamma(g)(V_t \beta(dg - \varphi(t))x, y).$$

By the uniqueness of the Fourier transform it follows that for all $x, y \in H$, $(\beta(\Delta)V_t x, y) = (V_t \beta(\Delta - \varphi(t))x, y)$. In other words, $\beta(\Delta)V_t = V_t \beta(\Delta - \varphi(t))$. Hence $V_t \beta(\Delta) V_{-t} = \beta(\Delta + \varphi(t))$ q.e.d.

3. Quasi-invariance of Analytic Measures

Let G be a compact abelian group and Γ its discrete dual. An "ordering" of Γ is given by a fixed non-trivial homomorphism ψ of Γ into the group of real numbers. Since Γ is discrete the mapping ψ is a continuous homomorphism and thus induces a continuous homomorphism $\varphi: R \to G$ of the associated dual groups; φ is the unique mapping defined by $\gamma(\varphi(t)) = e^{i\psi(\gamma)t}$, $t \in R$, $\gamma \in \Gamma$.

Let μ be a finite complex regular measure on the Borel subsets of G. μ is said to be φ-analytic if $\int_G \chi_\gamma(g)\mu(dg) = 0$ whenever $\psi(\gamma) \leq 0$. Let $|\mu|$ denote the total variation measure associated with μ. It is easy to see that $\mu(dg) = e(g)|\mu|(dg)$ where $e(\cdot)$ is a complex valued measurable function on G of absolute value 1 almost everywhere $[|\mu|]$. μ is called quasi-invariant under φ if $|\mu|(\Delta) = 0$ implies $|\mu|(\Delta + \varphi(t)) = 0$ for all $t \in R$. We shall denote by $L_2(G,|\mu|)$ the space of complex-valued functions square integrable with respect to $|\mu|$, where functions equal a.e. $[|\mu|]$ are identified. We consider the subspaces $\mathcal{M}_s$ of $L_2(G|\mu|)$ generated $\{\chi_\gamma(\cdot)e(\cdot): \psi(\gamma) \leq s\}$ for each s. They have the following properties

(3.1) $\quad$ (i) $\mathcal{M}_s \leq \mathcal{M}_u$ if $s \leq u$, (ii) $\bigvee_s \mathcal{M}_s = L_2(G,|\mu|)$,

(iii) $\mathcal{M}_{s+0} = \mathcal{M}_s$, $\quad$ (iv) $\bigcap_s \mathcal{M}_s = \{0\}$.

(i), (ii) and (iii) are obvious. Only property (iv) needs a proof. Consider $\int_G \chi_\gamma(g)\chi_\tau(g)e(g)|\mu|(dg)$. If $\psi(\tau) \leq -t$, then for γ satisfying $\psi(\gamma) < t$ we have $\int_G \chi_\gamma(g)\chi_\tau(g)e(g)|\mu|(dg) = 0$ by φ-analyticity of μ. Since $\mathcal{M}_{-t}$ is spanned by $\{\chi_\tau(\cdot)e(\cdot): \psi(\tau) \leq -t\}$, we have for any $f \in \mathcal{M}_{-t}$ $\int_G \chi_\gamma(g)f(g)e(g)|\mu|(dg) = 0$ for γ with $\psi(\gamma) < t$. Let $f \in \mathcal{M}_{-\infty}$. Then $f \in \mathcal{M}_{-t}$ for each t. Hence we get $\int_G \chi_\gamma(g)f(g)e(g)|\mu|(dg) = 0$ for all γ. This implies $f = 0$ a.e. $[|\mu|]$ proving (iv).

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