Analyze Interface Stability of a Gel Material Surrounded by Fluid

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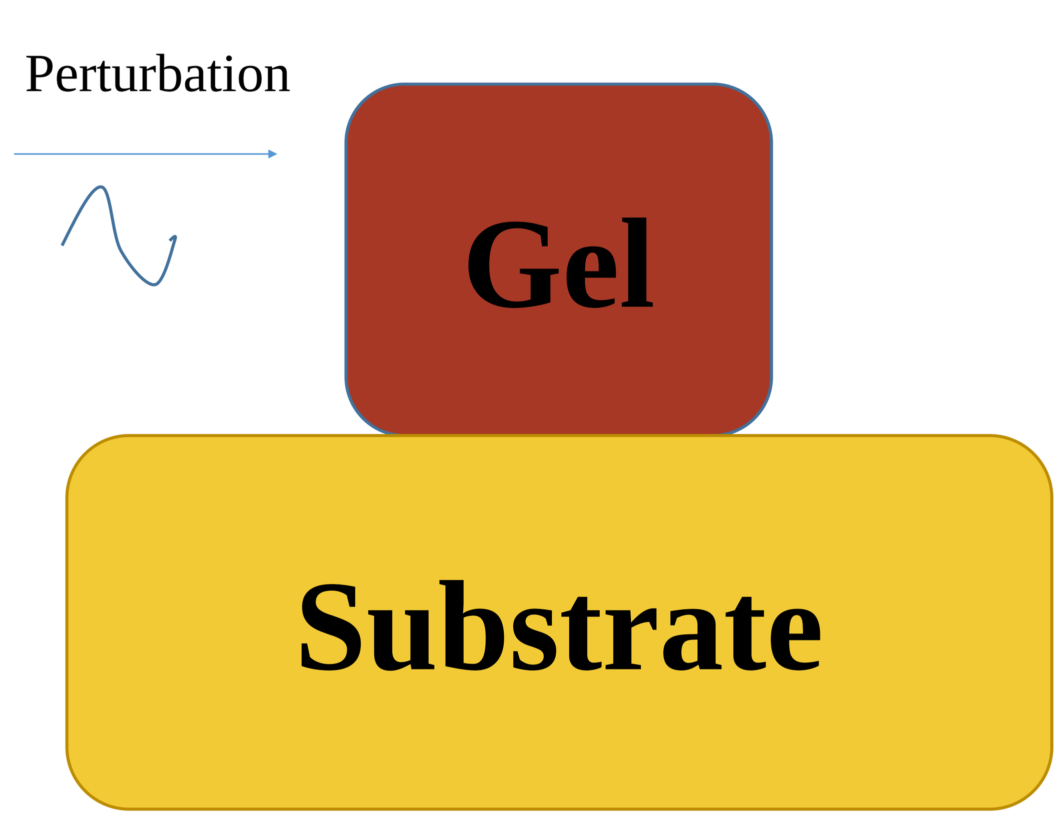
Abstract

This study aims to analyze whether the interface of gel material surrounded by fluid is stable. Its mathematical model is proposed by my mentor Dr. Carme Calderer professor at the School of Mathematics at the University of Minnesota and graduate assistant Carlos Garavito in reference [1]. In their study, they focus on characterizing the interface stability of gel materials surrounded by air through solving a nonlinear system of partial differential equations (PDE) with boundary conditions. My study is an extension of their work, which replaces boundary conditions with new ones in order to specify gel materials surrounded by fluid. Since solving the original systems of equations is difficult, I tried to solve a similar but simpler example. By applying the normal perturbation equations and substitute them into PDE system, the PDE system was reduced into an equivalent system of ordinary differential equations (ODE). After that, I concluded a series of procedures to solve the ODE system and obtain some useful results in fluid physics. This study has a far-reaching significance in many applications of gel materials. For instance: applying a gel as a carrier to deliver the drugs and implanting devices to human body.

Model



Partial Differential Equations

Governing equations are

$$\begin{aligned} \phi_0 \nabla \cdot \mathbf{v}_1 + (1-\phi_0)\nabla \cdot \mathbf{v}_2 &= 0, \\ (1-\phi_0)\Big(-\nabla p + \mathcal{G}_1 \vec{\triangle} \mathbf{v}_2 \Big) - (\mathbf{v}_2 - \mathbf{v}_1) &= 0, \\ -\nabla p + \phi_0 \Big(\mathcal{G}_2 \vec{\triangle} \mathbf{u} + \mathcal{G}_3 \vec{\triangle} \mathbf{v}_1 \Big) + (1-\phi_0)\mathcal{G}_1 \vec{\triangle} \mathbf{v}_2 &= 0, \\ \frac{\partial \mathbf{u}}{\partial t} &= \mathbf{v}_1, \end{aligned}$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{u}$, and p are the polymer and fluid velocities, the polymer displacement and the pressure, respectively. The notation $\vec{\triangle}\mathbf{v} = (\triangle \mathbf{v}_x, \triangle \mathbf{v}_z)$, represents the so called vector Laplacian in a Cartesian coordinate systems.

Coefficient	Meaning
$\mathcal{G}_1 = \frac{\eta_2}{2d^2\beta}$	$\frac{\text{Fluid Viscosity}}{\text{Mixture Friction}}$
$\mathcal{G}_2 = \frac{G}{dW_0\beta}$	$\frac{\text{Polymer Shear}}{\text{Mixture Shear}}$
$\mathcal{G}_3 = \frac{\eta_1}{2d^2\beta}$	$\frac{\text{Polymer Viscosity}}{\text{Mixture Friction}}$

Table 1: Dimensionless numbers describing the mechanical response of the gel. Here, $\eta_{1,2}$ represents the fluid and polymer viscosities, G is the polymer shear modulus, β is the friction coefficient between polymer and solvent and d and W_0 are the characteristic distance and velocity of the problem, respectively.

Boundary Conditions

$$\begin{aligned} \mathbf{u}(x,-d) = \mathbf{0}, \quad \mathbf{v}_2(x,-d) &= \mathbf{0}, \text{ (no-slip)}, \\ (\mathbf{v}_1 - \mathbf{v}_2)(x,0) &= \mathbf{0}, \text{ (impermeability)}, \\ (\mathbf{T}_1(x,0,t) + \mathbf{T}_2(x,0,t) + \frac{\gamma}{R}\mathbf{I})\,\boldsymbol{\nu} &= -P_{\text{Ext}}\,\boldsymbol{\nu}, \text{ (force balance at the top)}, \\ \frac{\partial u_z}{\partial t}(x,0,t) &= \frac{\partial S(x,0,t)}{\partial t}, \text{ (kinematic)}. \end{aligned}$$

Perturbed System of Equations

We apply the normal perturbation,

$$\begin{aligned} p \approx p_0 + \epsilon \tilde{p}(z)\exp(ikx + \alpha t), \ \mathbf{v}_2 \approx \epsilon \tilde{\mathbf{v}}_2(z)\exp(ikx+\alpha t), \\ \mathbf{u} \approx \epsilon \tilde{\mathbf{u}}(z)\exp(ikx+\alpha t), \ S \approx \epsilon \exp(ikx + \alpha t), \end{aligned}$$

The perturbed system of equations is,

$$\begin{aligned} & f_{\tilde{\mathbf{v}}_2} \equiv \nabla \cdot \tilde{\mathbf{v}}_2 = ik\tilde{\mathbf{v}}_{2,x} + \frac{d}{dz}\tilde{\mathbf{v}}_{2,z}, \\ & \frac{d^2 f_{\tilde{\mathbf{v}}_2}}{dz^2} - \Gamma_1(\alpha,k) f_{\tilde{\mathbf{v}}_2} = 0, \\ & \frac{d^2 p}{dz^2} - k^2 p = h(\alpha, k, f_{\tilde{\mathbf{v}}_2}), \\ & \frac{d^4 \tilde{\mathbf{v}}_{2,z}}{dz^4} - \Gamma_2(k,\alpha)\frac{d^2 \tilde{\mathbf{v}}_{2,z}}{dz^2} + k^4 \tilde{\mathbf{v}}_{2,z} = g\Big(\alpha, k, f_{\tilde{\mathbf{v}}_2}, \frac{d\tilde{p}}{dz}\Big), \\ & \Gamma_1(\alpha,k) \equiv k^2 + \frac{1}{2\phi_0(1-\phi_0)\left((1-\phi_0)\left(\frac{\mathcal{G}_3}{\alpha} + \mathcal{G}_2\right) + \phi_0\mathcal{G}_1\right)}, \\ & \Gamma_2(\alpha,k) \equiv 2k^2 + \frac{1}{(1-\phi_0)K_1}\left[1 - \frac{(1-\phi_0)\mathcal{G}_1\alpha}{\phi_0(\mathcal{G}_3 + \alpha\mathcal{G}_2)}\right], \end{aligned}$$

where f and g are suitable functions. The notation $\tilde{\mathbf{v}}_{2,x}$, $\tilde{\mathbf{v}}_{2,z}$ represents the x- and z-components of the fluid velocity $\mathbf{v}_2$.

Solving Similar Problems

The following is the PDE system that we solved:

$$\eta_1\left(\frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{\partial^2 \mathbf{u}}{\partial y^2}\right) = \frac{\partial \mathbf{u}}{\partial t} + \mu_1\left(\frac{\partial \mathbf{u}}{\partial x} + \frac{\partial \boldsymbol{v}}{\partial y}\right)$$

$$\eta_2\left(\frac{\partial^2 \boldsymbol{v}}{\partial x^2} + \frac{\partial^2 \boldsymbol{v}}{\partial y^2}\right) = \frac{\partial \boldsymbol{v}}{\partial t} + \mu_2\left(\frac{\partial \mathbf{u}}{\partial x} + \frac{\partial \boldsymbol{v}}{\partial y}\right)$$

with the following boundary conditions:

$$\tilde{\mathbf{u}}(x,0) = 0, \tilde{\boldsymbol{v}}(x,0) = 0, \frac{\partial \tilde{\mathbf{u}}(x,1)}{\partial y} = 0, \frac{\partial \tilde{\boldsymbol{v}}(x,1)}{\partial y} = 0$$

We propose the following solutions for the above PDE system: $\mathbf{u} = \tilde{\mathbf{u}}(y)e^{ikx+\alpha t}, \boldsymbol{v} = \tilde{\boldsymbol{v}}(y)e^{ikx+\alpha t}$. After substituting the proposed answer into the partial differential equations system, we reduce it into the following system of ordinary differential equations (ODE):

$$\begin{pmatrix} \frac{d\tilde{\mathbf{u}}(y)}{dy} \\ \frac{d\tilde{\boldsymbol{v}}(y)}{dy} \\ \frac{d^2\tilde{\mathbf{u}}(y)}{dy^2} \\ \frac{d^2\tilde{\boldsymbol{v}}(y)}{dy^2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\Gamma_1}{\eta_1} & 0 & 0 & \frac{\mu_1}{\eta_1} \\ \frac{i\mu_2 k}{\eta_2} & \frac{\Gamma_2}{\eta_2} & 0 & \frac{\mu_2}{\eta_2} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}}(y) \\ \tilde{\boldsymbol{v}}(y) \\ \frac{d\tilde{\mathbf{u}}(y)}{dy} \\ \frac{d\tilde{\boldsymbol{v}}(y)}{dy} \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\Gamma_1}{\eta_1} & 0 & 0 & \frac{\mu_1}{\eta_1} \\ \frac{i\mu_2 k}{\eta_2} & \frac{\Gamma_2}{\eta_2} & 0 & \frac{\mu_2}{\eta_2} \end{pmatrix}, \boldsymbol{x}(y) = \begin{pmatrix} \tilde{\mathbf{u}}(y) \\ \tilde{\boldsymbol{v}}(y) \\ \frac{d\tilde{\mathbf{u}}(y)}{dy} \\ \frac{d\tilde{\boldsymbol{v}}(y)}{dy} \end{pmatrix}$$

Methods

1. Determine whether matrix A is diagonalizable by checking the matrix rank of A.
2. Find 4 eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of A and 4 corresponding linearly independent eigenvectors $\{\boldsymbol{v}^1, \boldsymbol{v}^2, \boldsymbol{v}^3, \boldsymbol{v}^4\}$
3. By Eigenpairs expression theorem, a set of fundamental solutions $\{\boldsymbol{x}^1(y) = \boldsymbol{v}^1 e^{\lambda_1 y}, \boldsymbol{x}^2(y) = \boldsymbol{v}^2 e^{\lambda_2 y}, \boldsymbol{x}^3(y) = \boldsymbol{v}^3 e^{\lambda_3 y}, \boldsymbol{x}^4(y) = \boldsymbol{v}^4 e^{\lambda_4 y}\}$ of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$ could be obtained. Then, the general solution of the system $\boldsymbol{x}'(y) = A\,\boldsymbol{x}(y)$ is:

$$\boldsymbol{x}(y) = c_1\boldsymbol{v}^1 e^{\lambda_1 y} + c_2\boldsymbol{v}^2 e^{\lambda_2 y} + c_3\boldsymbol{v}^3 e^{\lambda_3 y} + c_4\boldsymbol{v}^4 e^{\lambda_4 y}$$

here c_1, c_2, c_3, c_4 are arbitrary constants.

4. According to the boundary conditions $\tilde{\mathbf{u}}(x,0) = \tilde{\boldsymbol{v}}(x,0) = \frac{\partial \tilde{\mathbf{u}}(x,1)}{\partial y} = \frac{\partial \tilde{\boldsymbol{v}}(x,1)}{\partial y} = 0$, c_1, c_2, c_3, c_4 could be determined by substituting the boundary conditions to the above general solution.

References

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