Studying Synergies between SDEs and PDEs; Analysis of Kolmogorov and Feynman-Kac Results

Arnab Dey

***Abstract*—In this report, the synergy between stochastic differential equations (SDEs) and partial differential equations (PDEs) is studied. There are important results by Kolmogorov, Feynman and Kac that can be utilized to solve PDEs using SDEs and vice-versa. In this report, the results in the articles by Black-Scholes [1], and Harrison [2],[3] are analysed and their importance critiqued. These synergies between PDEs and SDEs are poised to be utilized in the energy markets, demand-response programs and reserve planning for the grid. The report is presented by instantiating the analysis to European options; the general theory is applicable widely.**

I. INTRODUCTION

Traditional financial markets, involving assets such as stocks, has shown unprecedented instability for a long time. The effects of various exogenous and endogenous factors on the price of a stock are very difficult to predict in real time and thus increases the risk of financial investments. Not only financial market, similar instabilities have intruded into other sectors also due to various governmental decisions and restructuring of trading mechanisms. For example, in the beginning of 1990s, the energy markets were deregulated in many states in the United States to promote the use of clean renewable energies and to increase the energy efficiency and lower cost. The deregulation allowed retailers to invest in generators and power lines, purchase electricity from the utilities and sell them to the consumers at competitive price. These restructures increased the volatility of energy price in the wholesale energy market. Therefore, in the context of decision analysis and risky investment valuation problem, the need of a mechanism to mitigate risk is becoming indispensable. Inherently, the uncertainties associated with the assets lead to stochastic differential equations (SDEs) that need to be solved in order to hedge risk. In general, solving an SDE is much more difficult compared to solving a partial differential equation (PDE), for which the theories are much older and more developed. Therefore, the synergy between SDEs and PDEs is of particular importance in energy markets and traditional financial markets to address asset price uncertainties. One of the most popular financial tools that take advantage of such synergies, to hedge risk of price volatility, is the *European Options*. It is a contract specifically designed to mitigate the future uncertainties of the underlying asset value. The buyer of a European option has the right, but not the obligation, to buy or sell one share of the underlying asset (stock) at a pre-specified *fixed price* on a given *future date*. The *fixed price* and the *future date* are decided when the contract is signed. Therefore, even if the underlying asset increases or decreases in value on that *future date*, the option holder gets to decide whether or not to buy or sell that asset at the *fixed price* based on rational decision. However, for the seller of the option, the fundamental problem is to decide how much the option is worth, when the contract is signed, to provide the buyer with such right.

The pioneering work by Fischer Black and Myron Scholes [1] along with Robert C. Merton, on the valuation of European options is one of the most influential works applied to financial markets. For this work, Scholes and Merton won the 1997 Nobel Prize in Economics while Black was mentioned as a contributor being ineligible for the prize due to his death in 1995. Black and Scholes assumed a probabilistic model of stock price dynamics and proposed to mitigate the uncertainty due to stock price by dynamically balancing a portfolio consisting of risky stocks and risk-less assets such as bonds, cash etc. The derivation of the price of a European option is built upon stochastic calculus and such valuation of the option provides the option seller *almost sure* guarantees, that means with probability one, of incurring no loss and no profit, irrespective of whether the buyer exercises the right to buy or sell the stock on the given date at the fixed price. They presented how the SDE, involving stock price uncertainties, can be converted to a PDE for which a closed-form solution is available. After their seminal work, various extensions to Black-Scholes model have been proposed in literature along with insights into evolution of probability densities of the underlying assets with time [2]–[4]. These extensions have formed a new paradigm in studying the link between SDEs and PDEs. Risk mitigation and evaluation of a European option using the models developed under this paradigm, known as *equivalent martingale measure*, is widely used in financial and energy markets. It has been shown that these models converge to the original Black-Scholes model [1] using Kolmogorov Backward Equation and Feynman-Kac theorem [5]. Based on *equivalent martingale measure*, a simplification of the option pricing model, to make it suitable to implement in discrete time, has been proposed in the seminal paper by Cox, Ross, and Rubinstein [6]. Option pricing using Black-Scholes model and *equivalent martingale measure* technique has been applied to various different domains [7]–[12].

In this report, we aim to provide a cohesive overview of the synergies between SDEs and PDEs, in the context of European option pricing theory, along with its applications in energy markets and energy resource planning. This includes the derivation of the Black-Scholes option pricing model following [1] and a detailed approach to *equivalent martingale measure* technique using Kolmogorov Backward Equation and Feynman-Kac theorem. We also report how these techniques converge to the original Black-Scholes model for completion of our analysis. In particular, we focus on the utilization of these approaches to solve SDEs by suitably taking feedback from the solution to PDEs and vice-versa.

The rest of the report is organized as follows: In Section II, we introduce the key definitions that are used throughout the report. Some basic theorems, built upon the definitions, to analyze the synergy between SDEs and PDEs are presented in Section III. We introduce the European option pricing problem in Section IV. Section V describes the analysis of European option pricing following the work by Black and Scholes. We discuss the interaction of SDEs and PDEs, in the context of option pricing, using Kolmogorov Backward Equation and Equivalent Martingale Measure in Section VI and Section VII respectively. Section VIII presents the concluding remarks and the future scope of works.

II. Definitions and Notations

Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers. First we state some definitions [5],[13],[14] that will be used throughout the report.

Definition 1 (Weiner Process). *A Weiner Process $W(t), t \in [0, \infty)$, is a stochastic process with the following properties:*

1) $W(0) = 0$.
2) $\forall\ 0 \leq t < s$, $W(s) - W(t) \sim N(0, s - t)$.
3) $W(t)$ *is continuous in* $t \in [0, \infty)$.
4) $W(t_2) - W(t_1)$ *is independent of* $W(s_2) - W(s_1)$ *for all* $0 \leq t_1 < t_2 \leq s_1 < s_2$ *where* $t_1, t_2, s_1, s_2 \in [0, \infty)$.

Definition 2 (Filtration). *Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a σ-algebra $\mathcal{F}(t)$ of the subsets of Ω. We then call the collection of σ-algebras $\mathcal{F}(t), 0 \leq t \leq T$, if for all $s \leq t$, $\mathcal{F}(s) \subseteq \mathcal{F}(t)$.*

Definition 3. *Let X be a random variable defined on a nonempty sample space Ω. The σ-algebra generated by X, denoted by $\sigma(X)$, is the collection of all subsets of Ω such that $\{w \in \Omega : X(w) \in B\}$, where B ranges over the Borel subsets of $\mathbb{R}$.*

Definition 4. *Let X be a random variable defined on a nonempty sample space Ω. Let $\mathcal{G}$ be a σ-algebra of subsets of Ω. If every set in $\sigma(X)$ is also in $\mathcal{G}$, we say that X is $\mathcal{G}$-measurable.*

Definition 5. *Let Ω be a nonempty sample space equipped with a filtration $\mathcal{F}(t), 0 \leq t \leq T$. Let $X(t)$ be a collection of random variables indexed by $t \in [0, T]$. We say this collection of random variables is a stochastic process adapted to $\mathcal{F}(t)$ if, for each t, the random variable $X(t)$ is $\mathcal{F}(t)$-measurable.*

Definition 6 (Martingale). *A stochastic process $X(t), t \in [0, \infty)$ is a martingale if*

$$\mathbb{E}[X(t)] < \infty, \quad \forall t,$$
$$\mathbb{E}[X(s)|\mathcal{F}_t] = X(t), \quad \forall s \geq t,$$

where $X(t)$ is adapted to the filtration $\mathcal{F}_t, t \in [0, \infty)$.

Definition 7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let X be a random variable that is integrable. The conditional expectation of X given $\mathcal{G}$, denoted by $\mathbb{E}[X|\mathcal{G}]$, is any random variable that satisfies:*
(i) $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable.
(ii) $\int_A \mathbb{E}[X|\mathcal{G}] d\mathbb{P} = \int_A X d\mathbb{P}$ for all $A \in \mathcal{G}$.
Note that, taking $A = \Omega$, we obtain $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$.

Definition 8 (Ito integral). *Let $\mathcal{F}(t)$ be a filtration of a Weiner process $W(t), t \in [0, T]$ where T is a positive number. Let $X(t)$ be a stochastic process adapted to $\mathcal{F}(t)$. We define the Ito integral of $X(t)$ as follows:*

$$I(t) := \int_0^t X(u) dW(u), \tag{1}$$

where $dW(t)$ is defined as $dW(t) := W(t + dt) - W(t)$.

Definition 9 (Equity portfolio). *An equity portfolio is a collection of financial assets such as stocks, bonds, cash etc. In this report, we will consider portfolios consisting of stocks and bonds only. Thus, the value of a portfolio is given by,*

$$V(S(t), t) = a(t)S(t) + b(t)B(t), \tag{2}$$

where $a(t)$ is the amount of stocks, $S(t)$ is the stock price, $b(t)$ is the amount of bond, $B(t)$ is the bond value at time $t \in [0, T]$ where T is a positive constant. The pair, $\{a(t), b(t)\}$, is called the strategy at time t.

Definition 10 (Self-financing portfolio). *An equity portfolio, $V(S(t), t)$, consisting of $a(t)$ amount of stocks and $b(t)$ amount of bonds at time $t \in [0, T]$, is self-financing if the change in value of the portfolio is only due to the change in the prices of the stock and the bond. This implies*

$$dV(S(t), t) = a(t)dS(t) + b(t)dB(t), \tag{3}$$

for all $t \in [0, T]$ where $S(t), B(t)$ are the stock price and bond value respectively, at time t.

Next we derive some fundamental results associated with the above definitions which will be used later.

III. Basic Theorems

Theorem 1. *For a Weiner process $W(t), t \in [0, \infty)$, the following equalities hold:*

$$(dW(t))^2 = dt, \ (dt)^2 = 0, \ dt dW(t) = 0.$$

Proof Sketch. Please see Appendix A. □

Lemma 1 (Ito's Lemma). *Let $X(t)$ be a stochastic process that satisfies the dynamic equation $dX(t) = \overline{\mu}(X(t), t)dt + \overline{\sigma}(X(t), t)dW(t)$, where $W(t)$ is a Weiner process and $\overline{\mu}(\cdot), \overline{\sigma}(\cdot)$ are known functions in $\mathbb{R}$. Suppose $f(X(t))$ is a twice continuously differentiable function. Then for all $t \geq 0$,*

$$df(X(t)) = \frac{\partial f(X(t))}{\partial X(t)} dX(t) + \frac{1}{2} \frac{\partial^2 f(X(t))}{\partial X^2(t)} (dX(t))^2.$$

Proof. Please see Appendix B. □

Theorem 2. *The Ito integral defined by (1) is a martingale.*

Proof. Please see Appendix C. □

Now, we formulate the European Option Pricing problem and present the option pricing theory introduced in [1].

IV. European Option Pricing Problem

A European option is a contract issued by a company (option seller) that gives the buyer the right (not obligation) to buy, then termed CALL option, or sell, then termed PUT option, a share of the stock at a given fixed price K (strike price) on a given future time T (maturity date). However, the option seller is *obliged* to sell (buy) the stock to (from) the option buyer for the strike price on the maturity date if it is a European call (put) option. Let $S(t)$ denote the stock price at time $t \in [0,T]$. Let us assume that the buyer needs to pay C amount to purchase the option at time $t=0$. In case of a call option, if at maturity, the stock price $S(T)$ is greater than K, the option buyer can buy the stock from the option seller for K and sell it at the market for a price $S(T)$ obtaining an instant profit of $S(T)-K$. On the contrary, if $S(T) \leq K$, the option buyer will not exercise her right to buy it form the option seller, since she could buy it at a lower price from the market. In such a case, the instant profit of the buyer would be zero. Therefore, the *payoff* of the call option contract is:

$$(S(T)-K)^+ := \max(S(T)-K, 0).$$

Similarly for a put option the payoff is $(K-S(T))^+ := \max(K-S(T),0)$. Note that, from the perspective of the seller, a net profit of the buyer at maturity would result into net loss of the same amount. Let us assume that the seller invests C amount, which the buyer pays to purchase the option, in a portfolio with $a(0)$ amount of stocks and $b(0)$ amount of bonds with a total valuation of

$$C = V(S(0),0) = a(0)S(0) + b(0)B(0), \tag{4}$$

where $S(0), B(0)$ denote the price of stock and bond at time $t=0$ (when the contract is signed) respectively. If the portfolio is self-financing, and at the maturity, the valuation of the portfolio is $V(S(T),T) = (S(T)-K)^+$ in case of a call option, then the seller can ensure that there would be zero profit and zero loss at the time of maturity, irrespective of the actual price of the stock at the maturity. Such a financial market is known as *arbitrage free* market. Therefore, the fundamental problems in European option pricing are:
(1) how much the buyer should pay for the option at $t=0$, in other words, what should be the value of $C = V(S(0),0)$ and,
(2) how the seller can guarantee the payoff $(S(T)-K)^+$, in case of a call option, from the price charged (C) for the option, in other words, how to guarantee that $V(S(T),T) = (S(T)-K)^+$ by maintaining a self-financing portfolio with an initial investment of $V(S(0),0) = C$ amount. Also, the initial amount of stock, $a(0)$, and bond, $b(0)$ need to be computed.

Next, we present the solutions to the above pricing problems following the Black-Scholes model. The solution approach explains the link between SDEs and PDEs, associated with stock price uncertainties, and show how the feedback from one to another helps to completely mitigate such uncertainties.

V. Black-Scholes Model

In this section, we derive the celebrated Black-Scholes model to calculate the European option price. We present the detailed derivation for a call option, and note that a similar approach can be taken to calculate a put option price, the only difference being the payoff at maturity as described in the previous section. We first state the assumptions taken in the derivation, and then develop the theory of option pricing using stochastic calculus.

A. Assumptions

In deriving the formula for the value of an option, the following assumptions are taken:

- Interest rate of bond (risk-free asset), r, is deterministic and constant. The bond value $B(t)$ at any time $t \in [0,T]$ follows the dynamic equation given by
$$\mathrm{d}B(t) = rB(t)\mathrm{d}t \implies B(t) = B(0)e^{rt}, \tag{5}$$
where $B(0)$ is the initial price of the bond at time $t=0$.
- Stock price $S(t), t \in [0,T]$ follows a Geometric Brownian motion governed by the SDE,
$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), \tag{6}$$
where μ, σ are known constants in $\mathbb{R}$ and $W(t)$ is a Weiner process.
- Stocks pay no dividend.
- There are no transaction cost in buying or selling the stocks or the bonds.
- There is no arbitrage opportunities in the market.
- It is possible to borrow any fraction of the price of a stock to buy it or to hold it, at the interest rate r.

Under these assumptions, a self-financing strategy, as described next, leads to the Black-Scholes option pricing model.

B. Self-financing Strategy

Since there is no arbitrage opportunity in the market, we need a self-financing equity portfolio, $V(S(t),t)$, consisting of $a(t)$ amount of stocks and $b(t)$ amount of bonds as given in (2). Now, applying Ito's Lemma to (2), we obtain

$$\begin{aligned}\mathrm{d}V(S(t),t) &= a(t)\mathrm{d}S(t) + b(t)\mathrm{d}B(t) + \mathrm{d}a(t)S(t) + \mathrm{d}b(t)B(t)\\ &\quad + \mathrm{d}a(t)\mathrm{d}S(t) + \mathrm{d}b(t)\mathrm{d}B(t)\\ &= a(t)\mathrm{d}S(t) + b(t)\mathrm{d}B(t)\\ &\quad + (S(t)+\mathrm{d}S(t))\mathrm{d}a(t) + (B(t)+\mathrm{d}B(t))\mathrm{d}b(t).\end{aligned}$$

Therefore, by the Definition 10, if the portfolio $V(S(t),t)$ is self-financing,

$$\underbrace{(S(t)+\mathrm{d}S(t))}_{\text{Stock price at } t+\mathrm{d}t}\mathrm{d}a(t) + \underbrace{(B(t)+\mathrm{d}B(t))}_{\text{Bond value at } t+\mathrm{d}t}\mathrm{d}b(t) = 0.$$

Hence, if there is any change in the amount of stocks and bonds at time t, the amount of funds available at time $t+\mathrm{d}t$ due to stock amount change must be balanced by amount of funds available due to bond amount change. Suppose, we choose to change the amount of bond in such a way that

$$\mathrm{d}b(t) = -\frac{(S(t)+\mathrm{d}S(t))\mathrm{d}a(t)}{B(t)}. \tag{7}$$

Then, $\mathrm{d}b(t)B(t) + (S(t)+\mathrm{d}S(t))\mathrm{d}a(t) = 0$. Note that, from Theorem 1, $\mathrm{d}B(t)\mathrm{d}b(t) = (rB(t)\mathrm{d}b(t))(\mathrm{d}t) = 0$. Therefore, $(B(t)+\mathrm{d}B(t))\mathrm{d}b(t) + (S(t)+\mathrm{d}S(t))\mathrm{d}a(t) = 0$. Thus, if we choose to change the amount of bonds using (7), the self-financing condition is satisfied. Equity portfolio given by (2) is assumed to be such a self-financing portfolio. Next we present how such self-financing strategy along with the assumptions leads to the solution to the European option pricing problem.

C. Black-Scholes Partial Differential Equation

The pricing problem is to compute how much the option buyer needs to pay at time $t = 0$ and how that amount can be invested in a self-financing portfolio such that at the maturity T, the portfolio value would be $(S(T) - K)^+$. At the initial time, we do not know the future stock price $S(t), 0 < t < T$, and thus our goal is to find a formula for the future value of the portfolio in terms of future stock price which guarantees $V(S(T),T) = (S(T) - K)^+$. The first step in deriving this formula is given in the following theorem which illustrates how the SDE of the equity portfolio can be converted to a PDE of the same with a suitable trading strategy.

Theorem 3 (Black-Scholes PDE)**.** *Suppose, in a financial market, following the assumptions given in Section V-A, we have a self-financing portfolio given in (2). Then $V(S(t),t)$ satisfies*

$$\frac{\partial V(S(t),t)}{\partial t} + rS(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V(S(t),t)}{\partial S^2(t)} - rV(S(t),t) = 0, \tag{8}$$

and the number of stocks that needs to be maintained in the portfolio is given by,

$$a(t) = \frac{\partial V(S(t),t)}{\partial S(t)}, \tag{9}$$

for all $t \in [0,T)$.

Proof. From self-financing condition (3) and SDE (6) we get,

$$\begin{aligned} &\mathrm{d}V(S(t),t) = a(t)\mathrm{d}S(t) + b(t)\mathrm{d}B(t) \\ &= a(t)[\mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t)] + rb(t)B(t)\mathrm{d}t \\ &= a(t)[\mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t)] + r[V(S(t),t) - a(t)S(t)]\mathrm{d}t \\ &= [\mu a(t)S(t) + r(V(S(t),t) - a(t)S(t))]\mathrm{d}t \\ &+ [\sigma a(t)S(t)]\mathrm{d}W(t). \end{aligned} \tag{10}$$

On the other hand, applying Ito's Lemma to $V(S(t),t)$,

$$\mathrm{d}V(S(t),t) = \frac{\partial V(S(t),t)}{\partial t}\mathrm{d}t + \frac{\partial V(S(t),t)}{\partial S(t)}\mathrm{d}S(t) + \frac{1}{2}\frac{\partial^2 V(S(t),t)}{\partial S^2(t)}(\mathrm{d}S(t))^2.$$

Substituting the expression from (6), and since $(\mathrm{d}t)^2 = (\mathrm{d}t)(\mathrm{d}W(t)) = 0$, $(\mathrm{d}W(t))^2 = \mathrm{d}t$ (Theorem 1), we obtain,

$$\mathrm{d}V(S(t),t) = \Big[\frac{\partial V(S(t),t)}{\partial t} + \mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V(S(t),t)}{\partial S^2(t)}\Big]\mathrm{d}t + \Big[\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\Big]\mathrm{d}W(t). \tag{11}$$

Equating the terms associated with $\mathrm{d}W(t)$ in (10) and (11), we get,

$$a(t) = \frac{\partial V(S(t),t)}{\partial S(t)}, \tag{12}$$

and equating the terms associated with $\mathrm{d}t$, we have

$$\frac{\partial V(S(t),t)}{\partial t} + \mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V(S(t),t)}{\partial S^2(t)} = \mu a(t)S(t) + r(V(S(t),t) - a(t)S(t)).$$

Substituting the expression of $a(t)$ from (12), we obtain

$$\frac{\partial V(S(t),t)}{\partial t} + rS(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V(S(t),t)}{\partial S^2(t)} - rV(S(t),t) = 0.$$

This completes the proof. □

Now, our goal is to find a solution to the Black-Scholes PDE with the terminal condition $V(S(T),T) = (S(T)-K)^+$. Suppose, we have found the solution. If an investor starts with initial capital of $V(S(0),0)$ and balances the number of stocks $a(t)$ as given in (9) and $\frac{V(S(t),t)-a(t)S(t)}{B(t)}$ number of bonds for all time $[0,T)$, then (8) would be satisfied and it would be guaranteed that as $t \to T$, the portfolio value would be $(S(T)-K)^+$ *almost surely*. Thus the risk associated with option pricing can be successfully hedged. Next, we present the solution methodology to calculate the option price, $V(S(0),0)$, and find the strategy $\{a(t), b(t)\}$ for all $t \in [0,T)$.

D. Black-Scholes Option Pricing Formula

Note that, (8) is a PDE and $t, S(t)$ are dummy variables. Therefore, the solution to (8) can be obtained by standard approach to solve PDEs as presented in [1]. Here, we present the solution and the detailed proof is given in Appendix D.

Theorem 4. *The solution to the PDE (8) is given by,*

$$V(S(t),t) = S(t)\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \tag{13}$$

where

$$\begin{aligned} d_+ &= \frac{\ln(S(t)/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \\ d_- &= \frac{\ln(S(t)/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \end{aligned} \tag{14}$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}du$ is the standard normal cumulative distribution function. The corresponding strategy at time $t \in [0,T)$ is given by

$$a(t) = \Phi(d_+), \; b(t) = -\frac{Ke^{-r(T-t)}}{B(t)}\Phi(d_-). \tag{15}$$

Proof. Please see Appendix D. □

Thus, the option seller can calculate the option price at $t=0$ (when the contract is signed) using (13) and (14) by simply plugging in the value of the stock price at $t=0$. The option seller charges this amount to the buyer and invests the same in a self-financing portfolio starting with $a(0)$ amount of stocks and $b(0)$ amount of bonds per (15). Upon investing the initial amount, the seller has to balance the number of stocks and bonds using (15) for all time $0 < t < T$ to ensure a portfolio valuation of $(S(T)-K)^+$ at the maturity. Note that, (13) does not define $V(S(t),t)$ when $t = T$ (because then $T - t = 0$ and it appears in the denominator of (14)). However, it can be easily verified that $\lim_{t\to T} V(S(t),t) = (S(T) - K)^+$. Thus, in conclusion, we have solved the option pricing problem.

Remark 4.1. *The most important contributions by Black and Scholes is two-fold; (1) to provide a link between the SDE (10) and the PDE (8) utilizing the self-financing strategy (7), as presented in Theorem 3, and (2) to further convert the PDE (8) to a heat equation (refer to Appendix D), by suitable change of variables, for which there is a closed-form solution. The PDE solution leads to the expressions for the number of stocks and bonds, as given in (15), which is fed back to the portfolio $V(S(t),t)$ to satisfy its associated SDE. Continuously balancing the assets as per (15) helps to completely mitigate the uncertainties of future stock price.*

To complete the analysis of European option pricing using Black-Scholes model, we analyze (13) in simulation and present the effects of initial stock price, strike price and time to maturity on the option value. For the simulation study, we assume a stock price SDE as given in (6) with parameters $\mu = 0.01, \sigma = 0.3$. We assume that the bond has a monthly interest rate, $r = 0.05$. Fig. 1 illustrates the effect of initial

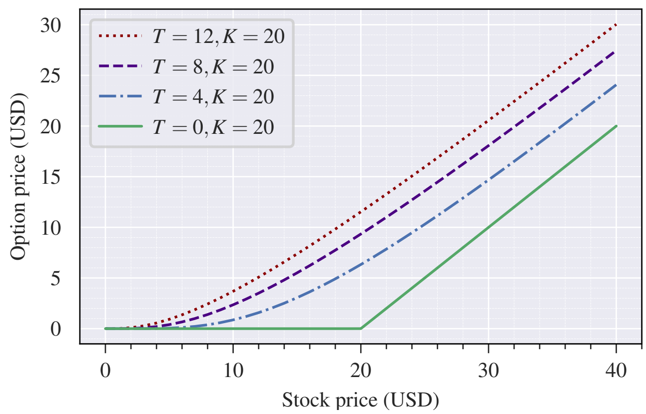

Fig. 1. For a fixed K, option price is lesser for shorter maturity for a certain initial stock price. Option price always stay above the payoff curve (in green).

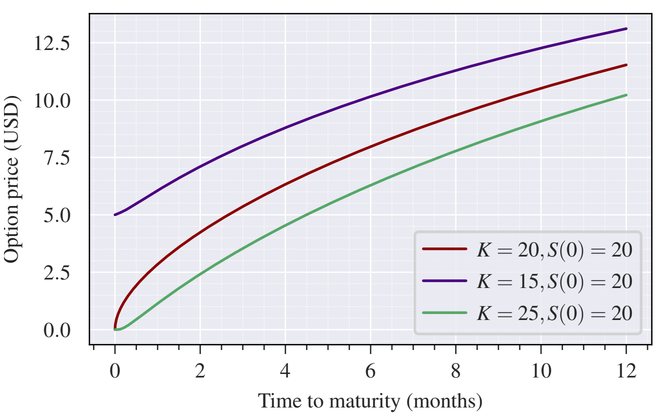

Fig. 2. For a fixed initial stock price, option value increases with time to maturity. Higher strike price leads to lower option value. Lower strike price compared to the initial stock price increases the option valuation.

stock price on the call option value for a fixed strike price of 20 dollars. It is observed that option price increases with initial stock price while shorter maturity reduces the option valuation. Fig. 2 illustrates the effect of time to maturity on the option price for a fixed initial stock price of 20 dollars. As the time to maturity decreases, option price reduces while higher value of strike price also reduces the option valuation. Such effects are carefully utilized in financial markets to hedge risk. Along with that, with the evolution of modern wholesale energy market, the entire Black-Scholes model is utilized for system reserve planning [15]–[19], energy trading, demand-response and renewable projects [20]–[22].

VI. European Option Pricing using KBE

In many applications, such as valuation of investment in a risky project, determination of return of investment, analysis of macroeconomic factors etc., it is required to compute the expected payoff of an investment at a future date [23]–[27]. The evolution of transition probabilities of financial assets and portfolios with time are also of interest to determine the effects of various macroeconomic factors that drive the market. In an actual stock market, the option issuing companies usually incur operational costs to maintain financial portfolios. Therefore, calculation of expected payoff of any risky investment is of utmost importance to these companies. In the previous section, we derived the option pricing formula to evaluate the European option price and a formula for the portfolio. However, with the previous approach, it is not clear how to derive the future expected payoff of the portfolio or how the transition probabilities of the stock price affects the portfolio and evolves with time. Notice that, the option pricing problem involves finding the current value of a portfolio given a valuation goal ($(S(T) - K)^+$) to reach at a future time. Therefore, one can utilize the Kolmogorov Backward Equation (KBE) [5] to find out future expected payoff along with the time evolution of stock price probability densities. In this section, we present a comprehensive analysis of European option pricing using KBE and show that under certain assumptions it leads to the same solution as given in Theorem 4.

Theorem 5. *Suppose the mean rate of return of the stock, μ, is equal to the interest rate of the bond, r. Hence, the stock price, denoted by $S'(t)$, follows a Geometric Brownian Motion given by*

$$dS'(t) = rS'(t)dt + \sigma S'(t)dW(t), \tag{16}$$

where σ is a known positive constant and $W(t)$ is a Weiner process. Then, the transition probabilities of the stock price satisfies

$$\frac{\partial p(z,T|x,t)}{\partial t} = -rx\frac{\partial}{\partial x}p(z,T|x,t) - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}p(z,T|x,t), \tag{17}$$

for all $0 \leq t \leq T$, where $p(z,T|x,t) = p(S'(T) = z|S'(t) = x)$ and $p(z,s|x,s) = \delta(z-x)$ for all $s \in [0,T]$, and the solution to the Black-Scholes PDE (8) with the terminal condition of $V(S'(T),T) = (S'(T) - K)^+$, is given by

$$V(x,t) = e^{-r(T-t)} \int_{-\infty}^{\infty} p(z,T|x,t)V(z,T)dz, \tag{18}$$

Proof. We will apply Kolmogorov Backward Equation (KBE) for our proof. We first state the KBE and then show how it can be utilized to prove the intended results.

Given a random process $X(t)$ in $\mathbb{R}$ following the SDE,

$$\mathrm{d}X(t) = \overline{\mu}(X(t),t)\mathrm{d}t + \overline{\sigma}(X(t),t)\mathrm{d}\overline{W}(t), \tag{19}$$

where $\overline{W}(t)$ is a Weiner Process, and $\overline{\mu}(\cdot)$, $\overline{\sigma}(\cdot)$ are known functions in $\mathbb{R}$, KBE satisfies,

$$\frac{\partial p(z,s|x,t)}{\partial t} = -\overline{\mu}(x,t)\frac{\partial}{\partial x}p(z,s|x,t) - \frac{1}{2}\overline{\sigma}^2(x,t)\frac{\partial^2}{\partial x^2}p(z,s|x,t),$$

for all $s \geq t$, subject to a given final condition $p(z,s|x,t) = u(z,s)$, where, $p(z,s|x,t) = p(X(s) = z|X(t) = x)$ and $p(z,s|x,s) = \delta(z-x)$.

Now, let us present how KBE can be applied to prove Theorem 5. Consider $\overline{\mu}(X(t),t) = rX(t)$ and $\overline{\sigma}(X(t),t) = \sigma X(t)$. Then from (19), we obtain

$$\mathrm{d}X(t) = rX(t)\mathrm{d}t + \sigma X(t)\mathrm{d}\overline{W}(t),$$

which is same as the stock price dynamics given in (16) with $X(t), \overline{W}(t)$ replaced by $S'(t)$ and $W(t)$ respectively. Therefore applying KBE for the process satisfying (16) for $T \geq t$, $p(z,T|x,t) = p(S'(T) = z|S'(t) = x)$ satisfies (17).

Let us assume that the option seller maintains a self-financing portfolio $V(S'(t),t)$ consisting of stocks, satisfying price dynamics given in (16), and bonds, satisfying (5). Now define, for all $t \in [0,T]$,

$$V(x,t) := e^{-r(T-t)} \int_{-\infty}^{\infty} p(z,T|x,t)V(z,T)\mathrm{d}z. \qquad (20)$$

Therefore, for $T \geq t$,

$$\frac{\partial V(x,t)}{\partial t} = re^{-r(T-t)} \int_{-\infty}^{\infty} p(z,T|x,t)V(z,T)\mathrm{d}z + e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{\partial p(z,T|x,t)}{\partial t} V(z,T)\mathrm{d}z.$$

From (17), we obtain,

$$\begin{aligned}\frac{\partial V(x,t)}{\partial t} &= rV(x,t) + e^{-r(T-t)} \int_{-\infty}^{\infty} \Big[-rx\frac{\partial}{\partial x}p(z,T|x,t) \\ &\quad -\tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} p(z,T|x,t)\Big] V(z,T)\mathrm{d}z \\ &= rV(x,t) - rx\frac{\partial}{\partial x}\left[e^{-r(T-t)} \int_{-\infty}^{\infty} p(z,T|x,t)V(z,T)\mathrm{d}z\right] \\ &\quad - \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\left[e^{-r(T-t)} \int_{-\infty}^{\infty} p(z,T|x,t)V(z,T)\mathrm{d}z\right]\end{aligned}$$

From our definition of $V(x,t)$ as given in (20), we obtain

$$\frac{\partial V(x,t)}{\partial t} = rV(x,t) - rx\frac{\partial V(x,t)}{\partial x} - \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 V(x,t)}{\partial x^2}. \qquad (21)$$

Note that, both (21) and the Black-Scholes PDE (8) are same. Therefore, the solution to the Black-Scholes PDE (8) with the terminal condition of $V(S'(T),T) = (S'(T)-K)^+$ must be of the form given in (20). This completes the proof. □

To complete the analysis of European option pricing, next we show that a solution given by (20) leads to the same option pricing formula (13) derived in the previous section.

Theorem 6. *Suppose, the stock price $S'(t)$ follows the stochastic differential equation (16) and the self-financing portfolio $V(S'(t),t)$, consisting of $a(t)$ number of stocks and $b(t)$ number of bonds, is given by (18). Then $V(S'(t),t), a(t),$ and $b(t)$ satisfy*

$$V(S'(t),t) = S'(t)\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-),$$
$$a(t) = \Phi(d_+),\ b(t) = \frac{Ke^{-r(T-t)}}{B(t)}\Phi(d_-),$$

where

$$d_+ = \frac{\ln(S'(t)/K) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$d_- = \frac{\ln(S'(t)/K) + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

Proof. Please see Appendix E. □

Notice that the pricing formula given in Theorem 6 is same as Black-Scholes pricing formula given in Theorem 4.

In summary, under the assumption that the mean rate of return of the stock is equal to the interest rate of the bond, (17) and (18) can be utilized to characterize the evolution of stock price transition probabilities with time and to calculate the expected future payoff. However, the question that arises immediately is what happens if the mean rate of return of the stock is not equal to the bond interest rate? Next we delve into this problem and show that, by a suitable transformation of probability measures, known as *equivalent martingale measure* [2],[3], it is possible to characterize the stock price transition probabilities, future expected payoffs, and the option pricing formula even when $\mu \neq r$. In the subsequent sections, we illustrate this concept and discuss how the change of probability measure leads to some other useful properties of European option pricing problem.

VII. Equivalent Martingale Measure

Let the Weiner process $W(t)$, associated with our original stock price $S(t)$, be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. From the stock price dynamics given by (6),

$$\begin{aligned}\mathrm{d}S(t) &= \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t) \\ &= rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\left(W(t) + \tfrac{\mu-r}{\sigma}t\right) \\ &= rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\left(W(t) + \eta t\right),\end{aligned}$$

where $\eta = \frac{\mu-r}{\sigma}$. Let us define $W^Q(t) := W(t) + \eta t$. Then,

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^Q(t). \qquad (22)$$

Note that, if $\mu \neq r$, then $\eta \neq 0$, and the process $W^Q(t)$ will no longer be a Weiner process under the probability measure $\mathbb{P}$ because of non-zero drift. Thus Theorem 5 will not hold true. In this section, we aim to find out a probability measure under which $W^Q(t)$ will be a Weiner process. We start with a theorem illustrating how a transformation of probability measure is performed, and then present the concept of equivalent martingale measure based on the theorem.

Theorem 7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely non-negative variable with $\mathbb{E}[Z] = 1$. For, $A \in \mathcal{F}$, define,*

$$\mathbb{Q}(A) := \int_A Z d\mathbb{P}.$$

Then, $\mathbb{Q}$ is a probability measure.

Proof. Please see Appendix F. □

Let us denote the expected value of a random variable X under the probability measure $\mathbb{Q}$ as $\mathbb{E}^Q[X] = \int_\Omega X\mathrm{d}\mathbb{Q}$. Then we can derive the following corollary.

Corollary 7.1. *If X is a non-negative random variable, then,*

$$\mathbb{E}^Q[X] = \mathbb{E}[XZ].$$

Proof. By definition, $\mathbb{E}^Q[X] = \int_\Omega X\mathrm{d}\mathbb{Q} = \int_\Omega XZ\mathrm{d}\mathbb{P} = \mathbb{E}[XZ]$. This completes the proof. □

Our objective is to define a process $Z(t)$ with $\mathbb{E}[Z(t)]=1$ for all $t \in [0,T]$ such that it can be utilized to define a probability measure $\mathbb{Q}$ under which $W^Q(t)$ will be a Weiner process.

Let $Z(t)$ be the process defined by

$$Z(t) := e^{-\eta W(t) - \frac{1}{2}\eta^2 t}, \qquad (23)$$

for all $t \in [0,T]$. Let $\gamma(x,t) := e^{-\eta x - \frac{1}{2}\eta^2 t}$. Applying Ito's Lemma on $Z(t)$, we obtain,

$$\mathrm{d}Z(t) = \mathrm{d}(\gamma(W(t),t)) = \frac{\partial \gamma(W(t),t)}{\partial t}\mathrm{d}t + \frac{\partial \gamma(W(t),t)}{\partial W(t)}\mathrm{d}W(t) + \frac{1}{2}\frac{\partial^2 \gamma(W(t),t)}{\partial W^2(t)}(\mathrm{d}W(t))^2$$

$$= (-\tfrac{1}{2}\eta^2)Z(t) - (\eta)Z(t)\mathrm{d}W(t) + \tfrac{1}{2}\eta^2 Z(t)$$
$$= -\eta Z(t)\mathrm{d}W(t). \quad (24)$$

Therefore, integrating both sides, we obtain,

$$Z(t) = Z(0) - \int_0^t \eta Z(u)\mathrm{d}W(u)$$

As Ito integrals are martingale (Theorem 2), $Z(t)$ is a martingale. In particular, $\mathbb{E}[Z(t)] = Z(0) = 1$. Hence, $Z(t)$ can be utilized to define the transformed probability measure $\mathbb{Q}$. Further, let, $\mathcal{F}(t), t \in [0,T]$ be a filtration generated by the Weiner process $W(t)$. Then we have, $Z(t) = \mathbb{E}[Z(T)|\mathcal{F}(t)]$ for all $t \in [0,T]$. Next, we present two lemmas which would be applied to prove that $W^Q(t)$ is a Weiner process under the probability measure $\mathbb{Q}$.

Lemma 2. *Let $t \in [0,T]$ be given and let X be an $\mathcal{F}(t)$-measurable random variable. Then,*

$$\mathbb{E}^Q[X] = \mathbb{E}[XZ(t)].$$

Proof. Please see Appendix G. □

Lemma 3. *Let $0 \leq s \leq t \leq T$. If X is a $\mathcal{F}(t)$-measurable random variable, then,*

$$\mathbb{E}^Q[X|\mathcal{F}_s] = \tfrac{1}{Z(s)}\mathbb{E}[Z(t)X|\mathcal{F}_s]$$

Proof. Please see Appendix H. □

Remember that, our objective is to prove that $W^Q(t)$ is a Weiner process under the measure $\mathbb{Q}$. Here, we will apply Levy's theorem [28] which states that if $M(t), t \geq 0$, is a process such that (1) it is a martingale to a filtration $\mathcal{F}(t), t \geq 0$, (2) $M(0) = 0$, (3) $M(t)$ has continuous paths and (4) $(\mathrm{d}M(t))^2 = \mathrm{d}t$, then $M(t)$ is a Weiner process. Note that, $W^Q(0) = W(0) = 0$ and has continuous paths by definition. Next we show that, $(\mathrm{d}W^Q(t))^2 = \mathrm{d}t$.

$$(\mathrm{d}W^Q(t))^2 = (\mathrm{d}W(t) + \eta\mathrm{d}t)^2$$
$$= (\mathrm{d}W(t))^2 + \eta^2(\mathrm{d}t)^2 + 2\eta\mathrm{d}W(t)\mathrm{d}t = \mathrm{d}t,$$

where the last equality follows from Theorem 1. It remains to show that $W^Q(t)$ is a martingale under $\mathbb{Q}$ to prove that $W^Q(t)$ is a Weiner process under $\mathbb{Q}$.

First, we show that $W^Q(t)Z(t)$ is a martingale under $\mathbb{P}$. Consider the following,

$$\mathrm{d}(W^Q(t)Z(t)) = Z(t)\mathrm{d}W^Q(t) + W^Q(t)\mathrm{d}Z(t) + \mathrm{d}W^Q(t)\mathrm{d}Z(t)$$
$$= Z(t)(\mathrm{d}W(t) + \eta\mathrm{d}t) + W^Q(t)(-\eta Z(t)\mathrm{d}W(t))$$
$$+ (\mathrm{d}W(t) + \eta\mathrm{d}t)(-\eta Z(t)\mathrm{d}W(t)),$$

where the last step follows from the definition of $W^Q(t)$ and (24). Using Theorem 1, and rearranging, we obtain

$$\mathrm{d}(W^Q(t)Z(t)) = Z(t)\mathrm{d}W(t) - \eta Z(t)W^Q(t)\mathrm{d}W(t).$$

Therefore, using the martingale property of Ito integral (Theorem 2), $W^Q(t)Z(t)$ is a martingale under $\mathbb{P}$.

Now, let $0 \leq s \leq t \leq T$ be given. To prove that $W^Q(t)$ is a martingale under $\mathbb{Q}$, we have to show that $\mathbb{E}^Q[W^Q(t)|\mathcal{F}(s)] = W^Q(s)$. Using the martingale property of $W^Q(t)Z(t)$ under $\mathbb{P}$, and using Lemma 3, we obtain

$$\mathbb{E}^Q[W^Q(t)|\mathcal{F}_s] = \tfrac{1}{Z(s)}\mathbb{E}[W^Q(t)Z(t)|\mathcal{F}_s]$$
$$= \tfrac{1}{Z(s)}W^Q(s)Z(s) = W^Q(s).$$

Therefore, $W^Q(t)$ is a martingale under $\mathbb{Q}$. Hence, by Levy's theorem, $W^Q(t)$ is a Weiner process under the probability measure $\mathbb{Q}$.

In summary, upon defining an adapted process $Z(t)$ as per (23) and transforming the probability measure from $\mathbb{P}$ to $\mathbb{Q}$ as given in Theorem 7, $W^Q(t)$ becomes a Weiner process. Thus, under the transformed probability measure $\mathbb{Q}$, the stock price dynamics can be written as (16) and Theorem 5 will hold true even in the case when $\mu \neq r$. Moreover, the transformation of probability measure along with the martingale properties, enables us to use Feynman-Kac theorem to calculate European option pricing, which is an useful technique to compute option price, when solving PDE becomes intractable [29]. We conclude the report with the theorem illustrating the option pricing technique based on equivalent martingale measure utilizing Feynman-Kac theorem.

Theorem 8. *Suppose the stock price $S(t)$ satisfies the stochastic differential equaltion given by*

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t),$$

where the constant r denotes the interest rate of bond, σ is a positive constant, and $W^Q(t)$ is a Weiner process. Then, Feynman-Kac representation for the solution to the Black-Scholes PDE (8) is given by:

$$V(S(t),t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S(T)-K)^+|S(t)\right]. \quad (25)$$

Proof. Please see Appendix I. □

Remark 8.1. *It can be easily shown that (25) is equal to the Black-Scholes pricing formula given in Theorem (4). Also, note that, under the probability measure $\mathbb{Q}$, the discounted equity portfolio, $e^{-rt}V(S(t),t)$, is a martingale, by definition. Also, the discounted stock price $e^{-rt}S(t)$ is a martingale, which can be shown as follows:*

$$d(e^{-rt}S(t)) = -re^{-rt}S(t)dt + e^{-rt}dS(t)$$
$$= -re^{-rt}S(t)dt + e^{-rt}(rS(t)dt + \sigma S(t)dW^Q(t))$$
$$= e^{-rt}\sigma S(t)dW^Q(t).$$

Therefore, the discounted stock price can be written as an Ito integral and thus a martingale by Theorem 2. Hence $\mathbb{Q}$ is called equivalent martingale measure. From the martingale properties, it can be seen that under the probability measure $\mathbb{Q}$, the expected payoff does not depend on the risk of the asset, hence the name risk neutral measure.

One of the most important feature of Feynman-Kac representation of Black-Scholes PDE solution is it's direct applicability to simulations. The solution to the stock price SDE (6) can be used to compute the expectation in the right hand side of (25). In this context, using Euler method to generate multiple sample paths from the SDE (6) to compute the expectation is widely deployed in many applications [30]. Feynman-Kac representation plays a major role to solve PDEs arising in scenarios when assumptions of Black-Scholes model do not hold true, for example when the stocks pays dividend or with stochastic interest rate [31]. Calculation of option price using (25) is also widely used when finding a closed-form solution to PDEs is intractable, for example, when there are multiple assets in the portfolio [29],[32]–[34]. Evaluation of other types of options, derived from European options, are also performed using risk neutral measure [32],[33],[35]–[39].

VIII. Conclusions and Scope of Future Work

In this report, we presented a detailed study on SDEs and PDEs and the interaction between them. Analysis of SDEs is particularly important in energy trading and financial markets where mitigating the uncertainties of asset price is crucial. In this context, we discussed European option pricing theory, a popular tool which utilizes the synergy between SDEs and PDEs to hedge the risk associated with underlying assets. We presented the seminal work by Black and Scholes [1] on option pricing, which provides a framework for transformation of SDEs to PDEs for which a closed-form solution can be obtained. Here, we discussed the problem of finding the initial cost of an option contract and investment strategy of that initial amount in a portfolio consisting of risky assets (stocks) and non-risky assets (bonds). Even under future price uncertainty of the risky asset, the investment strategy, originally derived in [1], meets a certain future valuation of the portfolio with almost sure guarantee. This initial framework is extended in [4] to analyze the time evolution of transition probabilities of the assets satisfying their respective SDEs. The solution is built upon the important results by Kolmogorov to solve PDEs using SDEs and vice-versa. We discussed how this approach, under certain assumptions, leads to same investment strategy devised by Black-Scholes. We presented a further extension to the approach, known as equivalent martingale measure approach [2],[3], to provide a more general probabilistic representation of the solution to the PDE derived in [1]. Option pricing based on this approach is widely used in energy trading and reserve planning for smart grid. We presented how the equivalent martingale measure is intertwined with Feynman-Kac theorem and can be utilised to derive the Feynman-Kac representation of the solution to the original Black-Scholes PDE. Such representation is of utmost importance in the context of simulating the behavior of the portfolio satisfying the SDEs involved. There are many interesting directions that one can pursue for future research work. In the context of energy trading, the assumptions of Black-Scholes is restrictive due to different exogenous factors driving the electricity price. Analysis of electricity price dynamics and the synergy between SDEs and PDEs tailored to energy market needs to be done. In particular, it is important to analyze the electricity price behavior when the price does not follow geometric Browninan motion, and when the parameters, defining the SDEs associated with electricity price, are also stochastic. Another unique aspect of energy market is the requirement of meeting the energy demand. Such requirement affects the rational choice of option holder at the maturity. Thus it is important to analyze how the requirement of demand-supply balance affects the option price and the associated SDEs. Another interesting direction is to explore the applicability of the option pricing theory to mitigate the uncertainties of renewable energy sources. This will certainly have a major implication in reserve planning for smart grids with significant renewable penetration.

Appendix A
Proof Sketch of Theorem 1

Let us consider any T in $\mathbb{R}^+$ and partition the interval $[0,T]$ into n periods of length $\frac{T}{n}$. Consider the following integral,

$$\begin{aligned}\int_0^T (\mathrm{d}W(t))^2 &= \lim_{n\to\infty} \sum_{i=1}^n \left[W\left(i\tfrac{T}{n}\right) - W\left((i-1)\tfrac{T}{n}\right)\right]^2 \\ &= \lim_{n\to\infty} \sum_{i=1}^n X_i^2 = \lim_{n\to\infty} n\left[\tfrac{1}{n}\sum_{i=1}^n X_i^2\right],\end{aligned}$$

where $X_i := W\left(i\frac{T}{n}\right) - W\left((i-1)\frac{T}{n}\right)$. By the definition of Weiner process, $X_i \sim \mathcal{N}\left(0, \frac{T}{n}\right)$ and is independent of $X_j, j \neq i$. Also, note that $\mathbb{E}[X_i^2] = \frac{T}{n} < \infty$. Therefore, by the Law of large numbers, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^2 = \mathbb{E}[X_i^2] = \frac{T}{n}$. Hence,

$$\int_0^T (\mathrm{d}W(t))^2 = T = \int_0^T \mathrm{d}t \implies (\mathrm{d}W(t))^2 = \mathrm{d}t.$$

Next we prove $(\mathrm{d}t)^2 = 0$. Consider the following integral,

$$\begin{aligned}\int_0^T (\mathrm{d}t)^2 &= \lim_{n\to\infty} \sum_{i=1}^n \left[\left(i\tfrac{T}{n}\right) - \left((i-1)\tfrac{T}{n}\right)\right]^2 \\ &= \lim_{n\to\infty} \tfrac{T^2}{n} = 0.\end{aligned}$$

Hence $(\mathrm{d}t)^2 = 0$. In a similar way,

$$\int_0^T \mathrm{d}t\mathrm{d}W(t) = \lim_{n\to\infty} \sum_{i=1}^n [\tfrac{T}{n}][X_i] = T\mathbb{E}[X_i]$$

where the last equality follows from the Law of large numbers. Since, $\mathbb{E}[X_i] = 0$, $\int_0^T \mathrm{d}t\mathrm{d}W(t) = 0$ and therefore $\mathrm{d}t\mathrm{d}W(t) = 0$. This completes the proof.

Appendix B
Proof of Ito's Lemma

From the SDE of $X(t)$, we obtain,

$$\begin{aligned}(\mathrm{d}X(t))^2 &= (\overline{\mu}(X(t),t))^2(\mathrm{d}t)^2 + (\overline{\sigma}(X(t),t))^2(\mathrm{d}W(t))^2 \\ &\quad + 2\overline{\mu}(X(t),t)\overline{\sigma}(X(t),t)\mathrm{d}t\mathrm{d}W(t) \\ &= (\overline{\sigma}(X(t),t))^2\mathrm{d}t,\end{aligned}$$

where the last step follows from Theorem 1. Note that, all other higher order terms are 0 since $\mathrm{d}t\mathrm{d}W(t) = (\mathrm{d}t)^2 = 0$. Therefore, using Taylor series expansion of $\mathrm{d}f(X(t))$, we get,

$$\mathrm{d}f(X(t)) = \frac{\partial f(X(t))}{\partial X(t)}\mathrm{d}X(t) + \frac{1}{2}\frac{\partial^2 f(X(t))}{\partial X^2(t)}(\mathrm{d}X(t))^2.$$

This completes the proof.

Appendix C
Proof of Theorem 2

Let s,t be given such that $0 \le s \le t \le T$. Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,T]$ such that $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_n = T$. Let us also assume that s and t are in different sub-intervals of Π. Thus, there are $t_l, t_k \in \Pi$ such that $t_l < t_k$, $s \in [t_l, t_{l+1})$ and $t \in [t_k, t_{k+1})$. Hence,

$$\begin{aligned}I(t) &= \textstyle\sum_{j=0}^{l-1} X(t_j)(W(t_{j+1}) - W(t_j)) \\ &\quad + X(t_l)(W(t_{l+1}) - W(t_l)) \\ &\quad + \textstyle\sum_{j=l+1}^{k-1} X(t_j)(W(t_{j+1}) - W(t_j)) \\ &\quad + X(t_k)(W(t) - W(t_k)).\end{aligned}$$

We have to show that $\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$. Note that,

$$\mathbb{E}[\textstyle\sum_{j=0}^{l-1} X(t_j)(W(t_{j+1}) - W(t_j))|\mathcal{F}_s]$$

$$= \sum_{j=0}^{l-1} X(t_j)(W(t_{j+1}) - W(t_j)), \quad (26)$$

as $s \geq t_l$. Further,

$$\begin{aligned}\mathbb{E}[X(t_l)(W(t_{l+1}) - W(t_l))|\mathcal{F}(s)] \\ = X(t_l)(\mathbb{E}[W(t_{l+1})|\mathcal{F}(s)] - W(t_l)) \\ = X(t_l)(W(s) - W(t_l)),\end{aligned} \quad (27)$$

where the last equality follows from martingale property of Weiner process. Now, note that, for $t_j \geq t_{l+1} > s$,

$$\begin{aligned}&\mathbb{E}[X(t_j)(W(t_{j+1}) - W(t_j))|\mathcal{F}(s)] \\ &= \mathbb{E}[\mathbb{E}[X(t_j)(W(t_{j+1}) - W(t_j))|\mathcal{F}(t_j)]|\mathcal{F}(s)] \\ &= \mathbb{E}[X(t_j)(\mathbb{E}[W(t_{j+1})|\mathcal{F}(t_j)] - W(t_j))|\mathcal{F}(s)] \\ &= \mathbb{E}[X(t_j)(W(t_j) - W(t_j))|\mathcal{F}(s)] = 0.\end{aligned}$$

Therefore,

$$\mathbb{E}[\textstyle\sum_{j=l+1}^{k-1} X(t_j)(W(t_{j+1}) - W(t_j))|\mathcal{F}(s)] = 0. \quad (28)$$

Now,

$$\begin{aligned}&\mathbb{E}[X(t_k)(W(t) - W(t_k))|\mathcal{F}(s)] \\ &= \mathbb{E}[\mathbb{E}[X(t_k)(W(t) - W(t_k))|\mathcal{F}(t_k)]|\mathcal{F}(s)] \\ &= \mathbb{E}[X(t_k)(\mathbb{E}[W(t)|\mathcal{F}(t_k)] - W(t_k))|\mathcal{F}(s)] \\ &= \mathbb{E}[X(t_k)(W(t_k) - W(t_k))|\mathcal{F}(s)] = 0.\end{aligned} \quad (29)$$

Adding (26), (27), (28), and (29) we get $\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$. This completes the proof.

Appendix D
Proof of Theorem 4

Let, $\tau = T - t$ and $x(t) = \ln S(t)$. Then $\frac{\partial V(S(t),t)}{\partial t} = -\frac{\partial V(e^{x(t)},T-\tau)}{\partial \tau}$, $\frac{\partial V(S(t),t)}{\partial S(t)} = \frac{1}{S(t)} \frac{\partial V(e^{x(t)},T-\tau)}{\partial x(t)}$, and, $\frac{\partial^2 V(S(t),t)}{\partial S^2(t)} = \frac{1}{(S(t))^2} \left[\frac{\partial^2 V(e^{x(t)},T-\tau)}{\partial x^2(t)} - \frac{\partial V(e^{x(t)},T-\tau)}{\partial x(t)} \right]$. Substituting the corresponding terms in (8), we have,

$$\begin{aligned}\frac{\partial V(e^{x(t)},T-\tau)}{\partial \tau} &= A \frac{\partial^2 V(e^{x(t)},T-\tau)}{\partial x^2(t)} + B \frac{\partial V(e^{x(t)},T-\tau)}{\partial x(t)} \\ &+ CV(e^{x(t)}, T - \tau),\end{aligned} \quad (30)$$

where $A = \frac{1}{2}\sigma^2 (> 0), B = (r - \frac{1}{2}\sigma^2), C = -r$. The final condition becomes

$$V(e^{x(T)}, T) = \begin{cases} e^{x(T)} - K, & \text{if } x(T) > \ln K \\ 0, & \text{if } x(T) \leq \ln K. \end{cases} \quad (31)$$

Let, $V(e^{x(t)}, T - \tau) = e^{-(\alpha x + \beta \tau)} u(x, \tau)$, where constants $\alpha, \beta \in \mathbb{R}$. We omit the explicit dependency of the variables $S(t), x(t)$ on time t, and of the variables $V(\cdot), u(\cdot)$ on $x(t)$ and τ for brevity of notation. Then,

$\frac{\partial V}{\partial \tau} = e^{-(\alpha x + \beta \tau)} \left[\frac{\partial u}{\partial \tau} - \beta u \right]$, $\frac{\partial V}{\partial x} = e^{-(\alpha x + \beta \tau)} \left[\frac{\partial u}{\partial x} - \alpha u \right]$, and, $\frac{\partial^2 V}{\partial x^2} = e^{-(\alpha x + \beta \tau)} \left[\frac{\partial^2 u}{\partial x^2} - 2\alpha \frac{\partial u}{\partial x} + \alpha^2 u \right]$.

Substituting the corresponding terms in (30), we get,

$$\frac{\partial u}{\partial \tau} = A \frac{\partial^2 u}{\partial x^2} + (B - 2\alpha A) \frac{\partial u}{\partial x} + (\beta + \alpha^2 A - \alpha B + C) u.$$

Choosing, $\alpha = \frac{B}{2A}$ and $\beta = \frac{B^2}{4A} - C$ we obtain,

$$\frac{\partial u}{\partial \tau} = A \frac{\partial^2 u}{\partial x^2}. \quad (32)$$

This is the heat-transfer equation of physics. Since $u(x, \tau) = e^{\alpha x + \beta \tau} V(e^x, T - \tau)$, the initial condition of u becomes,

$$u(x(T) = y, 0) = e^{\frac{B}{2A} y} V(e^y, T).$$

The solution of (32) is given by [40],

$$u(x, \tau) = \frac{1}{\sqrt{4\pi A \tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4A\tau}} u(z, 0) \mathrm{d}z$$

Substituting the expression of $u(z, 0)$,

$$\begin{aligned}u(x, \tau) &= \tfrac{1}{\sqrt{4\pi A \tau}} \textstyle\int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4A\tau}} e^{\frac{Bz}{2A}} V(e^z, T) \mathrm{d}z \\ &= \tfrac{1}{\sqrt{4\pi A \tau}} \textstyle\int_{-\infty}^{\infty} e^{\left(\frac{z-x-B\tau}{2\sqrt{A\tau}}\right)^2} e^{(\alpha x + \beta \tau + C\tau)} V(e^z, T) \mathrm{d}z.\end{aligned}$$

Since, $V(e^x, T - \tau) = e^{-(\alpha x + \beta \tau)} u(x, \tau)$, we obtain,

$$\begin{aligned}V(e^x, T - \tau) &= \tfrac{e^{C\tau}}{\sqrt{4\pi A \tau}} \textstyle\int_{-\infty}^{\infty} e^{\left(\frac{z-x-B\tau}{2\sqrt{A\tau}}\right)^2} V(e^z, T) \mathrm{d}z \\ &= \tfrac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \textstyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{z-x-\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2} V(e^z, T) \mathrm{d}z.\end{aligned}$$

From the terminal condition given in (31), we get

$$\begin{aligned}V(e^x, T - \tau) &= \tfrac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \textstyle\int_{\ln K}^{\infty} e^{-\frac{1}{2}\left(\frac{z-x-\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2} (e^z - K) \mathrm{d}z \\ &= \underbrace{\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} e^{-\frac{1}{2}\left(\frac{z-x-\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2} e^z \mathrm{d}z}_{I_1} \\ &- \underbrace{\frac{Ke^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} e^{-\frac{1}{2}\left(\frac{z-x-\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2} \mathrm{d}z}_{I_2}.\end{aligned} \quad (33)$$

Let us calculate I_1 first. Let, $m = \frac{1}{\sigma\sqrt{\tau}}(z - x - (r - \frac{1}{2}\sigma^2)\tau)$. Then, $\mathrm{d}m = \frac{1}{\sigma\sqrt{\tau}} \mathrm{d}z$. Hence,

$$\begin{aligned}I_1 &= \tfrac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \textstyle\int_{\ln K}^{\infty} e^{-\frac{1}{2}\left(\frac{z-x-\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2} e^z \mathrm{d}z \\ &= \tfrac{e^x}{\sqrt{2\pi}} \textstyle\int_{\frac{\ln K - x - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}(m - \sigma\sqrt{\tau})^2} \mathrm{d}m.\end{aligned}$$

Let $w = m - \sigma\sqrt{\tau} \implies \mathrm{d}w = \mathrm{d}m$. Since $x = \ln S$, we get,

$$I_1 = \tfrac{S}{\sqrt{2\pi}} \textstyle\int_{\frac{\ln(K/S) - (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}w^2} \mathrm{d}w = S\Phi\left(\frac{\ln(S/K) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right),$$

where $\Phi(\cdot)$ is the Cumulative Distribution Function (CDF) of $\sim \mathcal{N}(0, 1)$.

Now, let us calculate I_2. Let, $m = \frac{1}{\sigma\sqrt{\tau}}(z - x - (r - \frac{1}{2}\sigma^2)\tau)$. Then, $\mathrm{d}m = \frac{1}{\sigma\sqrt{\tau}} \mathrm{d}z$. Therefore,

$$\begin{aligned}I_2 &= \tfrac{Ke^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \textstyle\int_{\ln K}^{\infty} e^{-\frac{1}{2}\left(\frac{z-x-\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2} \mathrm{d}z \\ &= \tfrac{Ke^{-r\tau}}{\sqrt{2\pi}} \textstyle\int_{\frac{\ln K - x - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}m^2} \mathrm{d}m.\end{aligned}$$

Since $x = \ln S$, we obtain,

$$I_2 = Ke^{-r\tau}\Phi\left(\frac{\ln(S/K) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right),$$

where $\Phi(\cdot)$ is the CDF of $\sim \mathcal{N}(0,1)$.
Since $x = \ln S$ and $\tau = T - t$, from (33), we obtain the solution to the partial differential equation (8) as,

$$V(S(t),t) = S(t)\Phi\left(d_+\right) - Ke^{-r(T-t)}\Phi\left(d_-\right),$$

where

$$d_+ = \frac{\ln(S(t)/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$d_- = \frac{\ln(S(t)/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

and $V(S(0),0)$ is the value (premium) of the option.

Now, let us find the expressions for the portfolio strategy $a(t)$ and $b(t)$ for all time $t \in (0,T)$. From (9),

$$a(t) = \frac{\partial V(S(t),t)}{\partial S(t)}$$
$$= \Phi(d_+) + S(t)\frac{\partial\Phi(d_+)}{\partial d_+}\frac{\partial d_+}{\partial S(t)} - Ke^{-r(T-t)}\frac{\partial\Phi(d_-)}{\partial d_-}\frac{\partial d_-}{\partial S(t)}. \quad (34)$$

Since, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\mathrm{d}u$, $\frac{\partial\Phi(x)}{\partial x} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$. Also, note that, $\frac{\partial d_+}{\partial S(t)} = \frac{\partial d_-}{\partial S(t)} = \frac{1}{S(t)\sigma\sqrt{T-t}}$. Therefore,

$$Ke^{-r(T-t)}\frac{\partial\Phi(d_-)}{\partial d_-} = \frac{K}{\sqrt{2\pi}}e^{-r(T-t)}e^{-\frac{1}{2}d_-^2}$$
$$= \frac{K}{\sqrt{2\pi}}e^{-r(T-t)}e^{-\frac{1}{2}\left(\frac{\ln(S(t)/K)+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)^2}$$
$$= \frac{S(t)}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln(S(t)/K)+(r+\frac{\sigma^2}{2})(T-t)^2}{\sigma\sqrt{T-t}}\right)^2} = S(t)\frac{\partial\Phi(d_+)}{\partial S(t)}.$$

Therefore, from (34), $a(t) = \Phi(d_+)$ and from (2), $b(t) = \frac{Ke^{-r(T-t)}}{B(t)}\Phi(d_-)$. This completes the proof.

Appendix E
Proof of Theorem 6

Let $S'(t) = x$. From SDE (16), $\mathrm{d}x = rx\mathrm{d}t + \sigma x\mathrm{d}W(t)$. We omit explicit dependency of the variables on t for notational convenience. Let $y = \ln x$. Then from Ito's Lemma,

$$\mathrm{d}y = \frac{\partial y}{\partial x}\mathrm{d}x + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}(\mathrm{d}x)^2$$
$$= \frac{1}{x}(rx\mathrm{d}t + \sigma x\mathrm{d}W) + \frac{1}{2}\left(-\frac{1}{x^2}\right)(\sigma^2x^2)\mathrm{d}t$$
$$= \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}W.$$

Therefore, for any $s \geq t$, where $s,t \in [0,T]$,

$$y(s) = y(t) + \left(r - \frac{1}{2}\sigma^2\right)(s-t) + \sigma(W(s) - W(t)).$$

By the definition, $W(s) - W(t) \sim \mathcal{N}(0,(s-t))$. Therefore, denoting $y(m)$ as y_m for all $m \in [0,T]$, we obtain,

$$p(y_s,s|y_t,t) = \frac{1}{\sqrt{2\pi\sigma^2(s-t)}}\exp\left[-\frac{1}{2}\left(\frac{y_s - y_t - (r-\frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}}\right)^2\right].$$

Hence, by the definition of the portfolio (18),

$$V(x,t) = e^{-r(T-t)}\int_{-\infty}^{\infty} p(z,T|x,t)V(z,T)\mathrm{d}z$$
$$= e^{-r(T-t)}\int_{-\infty}^{\infty} p(y_T,T|y_t,t)V(e^{y_T},T)\mathrm{d}y_T. \quad (35)$$

Note, that the terminal condition is given by

$$V(e^{y_T},T) = \begin{cases} e^{y_T} - K, & \text{if } y_T > \ln K \\ 0, & \text{if } y_T \leq \ln K, \end{cases}$$

Substituting the above terminal condition in (35),

$$V(x,t) = e^{-r(T-t)}\int_{\ln K}^{\infty} p(y_T,T|y_t,t)(e^{y_T} - K)\mathrm{d}y_T$$
$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int_{\ln K}^{\infty} e^{-\frac{1}{2}\left(\frac{y_T - y_t - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)^2}(e^{y_T} - K)\mathrm{d}y_T.$$

Note that, the above integral is same as (33) which we have already calculated with $\tau \coloneqq T - t$. Therefore the result follows from Theorem 4. This completes the proof.

Appendix F
Proof of Theorem 7

To check that $\mathbb{Q}$ is a probability measure, we have to verify that $\mathbb{Q}(\Omega) = 1$ and $\mathbb{Q}$ is countably additive.
By assumption of $\mathbb{E}[Z] = 1$, we have, $\mathbb{Q}(\Omega) = \int_\Omega Z\mathrm{d}\mathbb{P} = \mathbb{E}[Z] = 1$. For countable additivity, let $A_1, A_2, \ldots$ be a sequence of disjoint sets in $\mathcal{F}$, and define $B_n \coloneqq \bigcup_{k=1}^n A_k$. Then,

$$\mathbb{Q}(\textstyle\bigcup_{k=1}^\infty A_k) = \lim_{n\to\infty}\int_\Omega \mathbb{1}_{B_n}Z\mathrm{d}\mathbb{P}$$

Since, $\mathbb{1}_{B_n} = \sum_{k=1}^n \mathbb{1}_{A_k}$,

$$\mathbb{Q}(\textstyle\bigcup_{k=1}^\infty A_k) = \lim_{n\to\infty}\sum_{k=1}^n\int_\Omega \mathbb{1}_{A_k}Z\mathrm{d}\mathbb{P}$$
$$= \lim_{n\to\infty}\sum_{k=1}^n\int_{A_k} Z\mathrm{d}\mathbb{P}$$
$$= \lim_{n\to\infty}\sum_{k=1}^n\mathbb{Q}(A_k) = \sum_{k=1}^\infty\mathbb{Q}(A_k).$$

Hence, $\mathbb{Q}$ is a probability measure. This completes the proof.

Appendix G
Proof of Lemma 2

From Corollary 7.1, $\mathbb{E}^Q[X] = \mathbb{E}[XZ(T)] = \mathbb{E}[\mathbb{E}[XZ(T)|\mathcal{F}_t]]$. Since X is a $\mathcal{F}_t$-measurable random variable, $\mathbb{E}^Q[X] = \mathbb{E}[X\mathbb{E}[Z(T)|\mathcal{F}_t]] = \mathbb{E}[XZ(t)]$. where the last equality follows from the fact that $Z(t)$ is a martingale. This completes the proof.

Appendix H
Proof of Lemma 3

Let $A \in \mathcal{F}_s$. As X is a $\mathcal{F}_t$-measurable random variable and since $0 \leq s \leq t \leq T$,

$$\int_A \mathbb{E}^Q[X|\mathcal{F}_s]\mathrm{d}\mathbb{Q} = \int_A X\mathrm{d}\mathbb{Q}. \quad (36)$$

Since $\mathbb{E}[Z(s)] = Z(0) = 1$, from Theorem 7,

$$\int_A \frac{1}{Z(s)}\mathbb{E}[Z(t)X|\mathcal{F}_s]\mathrm{d}\mathbb{Q} = \mathbb{E}^Q[\mathbb{1}_A\frac{1}{Z(s)}\mathbb{E}[Z(t)X|\mathcal{F}_s]],$$

where $\mathbb{1}_A(w) = 1$ if $w = A$, and 0 otherwise. As $\mathbb{1}_A\frac{1}{Z(s)}\mathbb{E}[Z(t)X|\mathcal{F}_s]$ is an $\mathcal{F}_s$-measurable random variable, using Lemma 2,

$$\mathbb{E}^Q[\mathbb{1}_A\frac{1}{Z(s)}\mathbb{E}[Z(t)X|\mathcal{F}_s]] = \mathbb{E}[\mathbb{1}_A\mathbb{E}[XZ(t)|\mathcal{F}_s]]$$

Since $A \in \mathcal{F}_s$,

$$\mathbb{E}[\mathbb{1}_A\mathbb{E}[XZ(t)|\mathcal{F}_s]] = \mathbb{E}[\mathbb{E}[\mathbb{1}_AXZ(t)|\mathcal{F}_s]] = \mathbb{E}[\mathbb{1}_AXZ(t)].$$

Using Lemma 2, $\mathbb{E}[\mathbb{1}_AXZ(t)] = \mathbb{E}^Q[\mathbb{1}_AX] = \int_A X\mathrm{d}\mathbb{Q}$. Therefore, from (36), $\mathbb{E}^Q[X|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[Z(t)X|\mathcal{F}_s]$. This completes the proof.

Appendix I
Proof of Theorem 8

Consider the stochastic differential equation

$$\mathrm{d}X(t) = \overline{\mu}(X(t),t)\mathrm{d}t + \overline{\sigma}(X(t),t)\mathrm{d}W^Q(t), \qquad (37)$$

where $\overline{\mu}(\cdot), \overline{\sigma}(\cdot)$ are known functions in $\mathbb{R}$, and $W^Q(t)$ is a Weiner process under probability measure $\mathbb{Q}$. Let $f(x)$ and $u(x,t)$ be known functions in $\mathbb{R}$. Let $t \in [0,T]$ be given where T is a positive constant. Define the function

$$V(x,t) \coloneqq \mathbb{E}^Q\left[e^{-\int_t^T u(x_\tau,\tau)\mathrm{d}\tau} f(X(T)) | X(t) = x\right].$$

Then, Feynman-Kac theorem states that $V(x,t)$ satisfies the following PDE,

$$\frac{\partial V(x,t)}{\partial t} = u(x,t)V(x,t) - \overline{\mu}(x,t)\frac{\partial V(x,t)}{\partial x} - \frac{1}{2}\overline{\sigma}^2(x,t)\frac{\partial^2 V(x,t)}{\partial x^2}, \qquad (38)$$

with the terminal condition $V(x,T) = f(x)$, for all $x \in \mathbb{R}$. Let us define $\overline{\mu}(X(t),t) \coloneqq rX(t)$ and $\overline{\sigma}(X(t),t) \coloneqq \sigma X(t)$, where r is the interest rate of the bond and σ is a positive constant. Then (37) can be written as

$$\mathrm{d}X(t) = rX(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W^Q(t),$$

which is same as the stock price SDE under $\mathbb{Q}$ as given in (22) with $W^Q(t)$ being a Weiner process. Let us define $u(x,t) \coloneqq r$. Then, (38) is same as the Black-Scholes PDE (8). Therefore, a straightforward application of Feynman-Kac theorem with $u(x,t) = r, \overline{\mu}(X(t),t) = rX(t), \overline{\sigma}(X(t),t) = \sigma X(t)$ and $V(S(T),T) = (S(T)-K)^+$ concludes the proof.

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