MODELS FOR FLOW AND TRANSPORT THROUGH POROUS MEDIA DERIVED BY HOMOGENIZATION

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Models for Flow and Transport through Porous Media Derived by Homogenization

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Abstract

Models for diffusion are considered such as two-continua, first-order kinetic, and double permeability models. It is shown that all these different models can be derived from a single micro model using homogenization and two-scale convergence.

1 Introduction

The purpose of this paper is to show that a large family of problems for flow and transport in porous media can be formulated and derived in a unified way using the concept of *two-scale convergence*. This concept was first implicitely used in the paper [3], then explicitely formulated in [13], and generalized in [1] and [2]. The notion of two-scale convergence combines the method of asymptotic expansions, i.e., the formal method of expanding into a series in terms of a small scale parameter ε, with the energy method of L. Tartar. This notion is a generalization of weak convergences, but at the same time it captures rapid oscillations of functions; for details see section 11.

In the context of diffusion type processes there are essentially five different macro models (dropping convection, chemical reactions, and distributed sources; for details of notations see section 2):

1. Diffusion through anisotropic media (section 3) is described by

$$C\partial_t u = \nabla \cdot (D\nabla u). \tag{1}$$

2. The two-continua model (section 4) is

$$\begin{cases} C_i \partial_t u_i + B_i = \nabla \cdot (D_i \nabla u_i), & x \in \Omega, \ i = 1,2 \\ B_1 = -B_2 = S_1 u_1 + S_2 u_2, & x \in \Omega \end{cases} \tag{2}$$

3. The first-order kinetic model (section 5) is

$$\begin{cases} C_1 \partial_t u_1 + |\Gamma| \partial_t u_0 = \nabla \cdot (D \nabla u_1), & x \in \Omega \\ c_0 \partial_t u_0 = s_1 u_1 + s_0 u_0, & x \in \Omega \end{cases} \tag{3}$$

4. The double permeability model (section 6) is

$$\begin{cases} C \partial_t u_1 + B = \nabla \cdot (D \nabla u_1), & x \in \Omega \\ B = -\int_\Gamma \nu \cdot d_2 \nabla_y u_2 \, d\Gamma(y), & x \in \Omega \\ c_2 \partial_t u_2 = \nabla_y \cdot (d_2 \nabla_y u_2), & x \in \Omega, \ y \in Y_2 \\ u_1 = u_2, & x \in \Omega, \ y \in \Gamma \end{cases} \tag{4}$$

5. The double permeability model with Robin type transmission conditions (section 7) is

$$\begin{cases} C \partial_t u_1 + B = \nabla \cdot (D \nabla u_1), & x \in \Omega \\ B = -\int_\Gamma b \, d\Gamma(y), & x \in \Omega \\ c_2 \partial_t u_2 = \nabla_y \cdot (d_2 \nabla_y u_2), & x \in \Omega, \ y \in Y_2 \\ \vec{\nu} \cdot d_2 \nabla_y u_2 = b = s_1 u_1 + s_2 u_2, & x \in \Omega, \ y \in \Gamma \end{cases} \tag{5}$$

We also consider certain combinations of the models described above (sections 8 through 10). It will be shown that these models can all be derived from a single micro model, namely system (6), using homogenization. Comparisons of some of the macro models have been carried out in [8] and [15], see also [10].

2 The General Diffusion Problem

Notations: Let Y be the unit cube in $\mathbf{R}^3$. For $i = 1,2$ we consider two disjoint open subsets $Y_i \subset Y$ with common smooth boundary Γ such that $Y = Y_1 \cup Y_2 \cup \Gamma$. The vector ν denotes the normal on Γ outward w.r.t. Y_1, i.e., inward w.r.t. Y_2. A two-dimensional section of Y is shown in figure 1. The tangential gradient on Γ is denoted by ∇^Γ, and $\nabla^\Gamma \cdot$ denotes the divergence operator with respect to Γ. Furthermore, the surface measure on

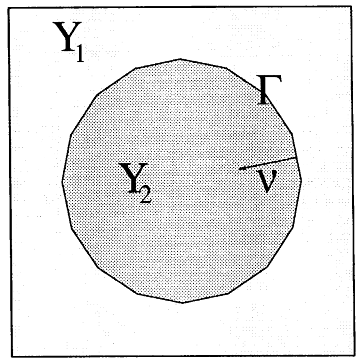

Figure 1: The unit cell Y

Γ is denoted by $d\Gamma$. Now we repeat Y_i and Γ periodically in all $\mathbf{R}^3$ (whenever we use the word *periodic* we mean Y-periodic) and use the same symbols Y_i and Γ. In all cases except the last two (sections 9 and 10) we assume that Y_1 is connected in $\mathbf{R}^3$; in sections sections 9 and 10 we assume that Γ is connected. For the multi-continua model (section 4) we assume that the sets Y_2 and Γ are also connected. Obviously, the boundary $\partial\Gamma$ is void in any case. The spaces $L^2_\#(Y_i)$, $H^1_\#(Y_i)$, $L^2_\#(\Gamma)$, $H^1_\#(\Gamma)$ are the spaces of periodic functions with scalar products

$$(u,v)_{L^2_\#(Y_i)} = \int_{Y_i} uv\ dy \text{ and } (u,v)_{H^1_\#(Y_i)} = \int_{Y_i} (uv + \nabla u \cdot \nabla v)\ dy,$$

$$(u,v)_{L^2_\#(\Gamma)} = \int_\Gamma \varepsilon uv\ d\Gamma(y) \text{ and } (u,v)_{H^1_\#(\Gamma)} = \int_\Gamma \varepsilon(uv + \nabla^\Gamma u \cdot \nabla^\Gamma v)\ d\Gamma(y),$$

where the integration takes place over one Y-period.

We assume that functions $c_i, d_i \in L^\infty_\#(Y_i)$ (the spaces of periodic essentially bounded functions), $i = 1,2$ and $c_0, d_0, s_{i,j} \in L^\infty_\#(\Gamma)$, $i = 1,2$, $j = 0,1,2$ and numbers $e_i \in \{0,1\}$ for $i = 0,1,2$ are given. We assume $c_i, d_i \geq 0$, the numbers $s_{i,j}$ may be negative. We use the shorthand notation

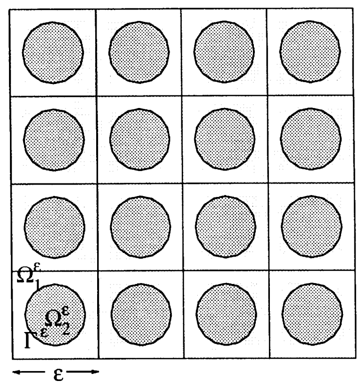

Figure 2: The periodic structure

$b_i = \sum_{j=0,1,2} s_{ij} u_j$ for $i = 1, 2$, where u_j are the unknowns of the problems described later; i.e., b_i denotes a linear combination of the variables u_j.

From now on we consider an open set $\Omega \subset \mathbf{R}^3$ and a small scale parameter $\varepsilon > 0$. We define the subsets of Ω (see figure 2)

$$\Omega_i^\varepsilon = \{x \in \Omega : \ x \in \varepsilon Y_i\}, \ i = 1, 2, \text{ and } \Gamma^\varepsilon = \{x \in \Omega : \ x \in \varepsilon\Gamma\}.$$

As before, the vector ν denotes the normal on Γ^ε outward w.r.t. Ω_1^ε, i.e., inward w.r.t. Ω_2^ε. Instead of $\nabla^{\Gamma^\varepsilon}$ we use ∇^ε for short. The surface measure on Γ^ε is denoted by $d\Gamma^\varepsilon$. The spaces $L^2_\#(\Omega_i^\varepsilon)$, $H^1_\#(\Omega_i^\varepsilon)$, $L^2_\#(\Gamma^\varepsilon)$, $H^1_\#(\Gamma^\varepsilon)$ are defined in the usual way.

We use the following functions with εY-periodicity: $c_i^\varepsilon(x) = c_i(\frac{x}{\varepsilon})$, $d_i^\varepsilon(x) = d_i(\frac{x}{\varepsilon})$, $s_{i,j}^\varepsilon(x) = s_{i,j}(\frac{x}{\varepsilon})$. In this way we get $b_i^\varepsilon = \sum_{j=0,1,2} s_{ij}^\varepsilon u_j^\varepsilon$.

It should be mentioned that the condition of strict periodicity of the functions may be relaxed by considering functions of the type $c^\varepsilon(x) = c(x, \frac{x}{\varepsilon})$ and $d^\varepsilon(x) = d(x, \frac{x}{\varepsilon})$ where $c(x, y)$ and $d(x, y)$ are periodic w.r.t. y for $x \in \Omega$. This is possible if the functions $c(x, y)$ and $d(x, y)$ satisfy certain "admissibility" conditions, see [2].

Now we are ready to formulate the *general micro problem* in strong form.

$$\begin{cases} c_i^\varepsilon \partial_t u_i^\varepsilon = \varepsilon^{2e_i} \nabla \cdot (d_i^\varepsilon \nabla u_i^\varepsilon), & x \in \Omega_i^\varepsilon,\ i = 1,2 \\ \varepsilon c_0^\varepsilon \partial_t u_0^\varepsilon + \vec{\nu} \cdot (\varepsilon^{2e_1} d_1^\varepsilon \nabla u_1^\varepsilon - \varepsilon^{2e_2} d_2^\varepsilon \nabla u_2^\varepsilon) = \varepsilon^{1+2e_0} \nabla^\varepsilon \cdot (d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon), & x \in \Gamma^\varepsilon \\ u_0^\varepsilon = u_1^\varepsilon = u_2^\varepsilon \text{ or } -\vec{\nu} \cdot \varepsilon^{2e_1} d_1^\varepsilon \nabla u_1^\varepsilon = \varepsilon b_1^\varepsilon,\ \vec{\nu} \cdot \varepsilon^{2e_2} d_2^\varepsilon \nabla u_2^\varepsilon = \varepsilon b_2^\varepsilon, & x \in \Gamma^\varepsilon \end{cases} \tag{6}$$

This system models the following sitation: In both domains Ω_1^ε and Ω_2^ε a standard diffusion process takes place. On the interface Γ^ε between these two domains there is another diffusion process behaving surface-like, which is similar to flow in a thin Hele-Shaw cell. The coupling between the three diffusion processes is such that the boundary fluxes from Ω_1^ε and Ω_2^ε appear as sources or sinks on the interface Γ^ε; in other words, there is continuity of mass or energy, resp. There are two different types of transmission conditions: If the *trace conditions* are "on" we have continuity $u_0^\varepsilon = u_1^\varepsilon = u_2^\varepsilon$ of the variables (which is a natural choice if one deals with concentrations, pressures, or temperatures); if the trace conditions are "off" we prescribe Robin type conditions, i.e., the boundary fluxes $-\vec{\nu} \cdot \varepsilon^{e_1} d_1^\varepsilon \nabla u_1^\varepsilon$ and $\vec{\nu} \cdot \varepsilon^{e_2} d_2^\varepsilon \nabla u_2^\varepsilon$ from Ω_1^ε and Ω_2^ε onto the interface Γ^ε depend as ε-multiples of functions $b_i^\varepsilon = \sum_{j=0,1,2} s_{ij}^\varepsilon u_j^\varepsilon$, $i = 1,2$ on the three variables u_j, $j = 0,1,2$. The factors ε^{e_i} govern the relative orders of magnitudes of the diffusivities: whenever $e_i = 1$ the diffusivity $\varepsilon^{e_i} d_i^\varepsilon$ is small compared to those with $e_i = 0$. In this sense, we allow combinations of fast and slow diffusion processes.

The objective of the paper is to show that one can derive all models mentioned in the introduction from this micro problem by using homogenization. Which specific model arises as the homogenized limit depends on the choice of the numbers e_i and on which type of transmission conditions is chosen on Γ^ε (trace condition on or off). In principle, there are 16 possible combinations. Not all of them are relevant or interesting. It turns out that whenever the trace condition is on, a reasonable choice is $e_0 = 0$. The case $e_0 = e_1 = e_2 = 1$ is of no special interest. At this point we summarize the important cases in the following table.

Model	Eqn	e_0	e_1	e_2	trace
Diffusion	1	0	0	0	on
Two-Continua	2	0	0	0	off
First Order	3	1	0	0	off
Double Perm.	4	0	0	1	on
Double Perm. Robin	5	0	0	1	off
Slow Membranes	29	1	0	1	off
Fissure/Blocks	33	0	1	1	on
Fast Membranes	35	0	1	1	off

The first step on the way to derive the homogenized equations is to formulate the micro problem in weak form. We say that a triple of functions $(\varphi_0^\varepsilon, \varphi_1^\varepsilon, \varphi_2^\varepsilon)$ is *admissible* iff $\varphi_0^\varepsilon = \varphi_1^\varepsilon = \varphi_2^\varepsilon$ provided that the trace conditions are "on"; if the trace conditions are "off" any triple of functions is admissible. Obviously, an equivalent form of problem (6) is to find admissible functions u_i such that

$$\sum_{i=1,2} \int_0^T \int_{\Omega_i^\varepsilon} \left(c_i^\varepsilon \partial_t u_i^\varepsilon - \varepsilon^{2e_i} \nabla \cdot (d_i^\varepsilon \nabla u_i^\varepsilon) \right) \varphi_i dx dt$$

$$+ \int_0^T \int_{\Gamma^\varepsilon} \Big(\varepsilon \Big((c_0^\varepsilon \partial_t u_0^\varepsilon - \varepsilon^{2e_0} \nabla^\varepsilon \cdot (d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon)) \varphi_0$$

$$+ b_1^\varepsilon(\varphi_1 - \varphi_0) + b_2^\varepsilon(\varphi_2 - \varphi_0) \Big)$$

$$+ \vec{\nu} \cdot (\varepsilon^{2e_1} d_1^\varepsilon \nabla u_1^\varepsilon \varphi_1 - \varepsilon^{2e_2} d_2^\varepsilon \nabla u_2^\varepsilon \varphi_2) \Big)\ d\Gamma^\varepsilon(x) dt = 0$$

for all admissible test functions. Using the spaces

$$\begin{aligned} V^\varepsilon &= \{\vec{u} = (u_0, u_1, u_2) :\ u_0 \in H^1(\Gamma^\varepsilon),\ u_i \in H^1(\Omega_i^\varepsilon),\ i = 1,2 \text{ admissible}\} \\ \mathcal{V}^\varepsilon &= \{\vec{u} \in L^2(0,T;V^\varepsilon) :\ \partial_t \vec{u} \in L^2(0,T;(V^\varepsilon)')\} \end{aligned}$$

and integrating by parts we get the following.

Problem 1 *The* weak micro-problem *is to find a vector $\vec{u}^\varepsilon \in \mathcal{V}^\varepsilon$ of functions u_i^ε such that*

$$\sum_{i=1,2} \int_0^T \int_{\Omega_i^\varepsilon} \left(c_i^\varepsilon \partial_t u_i^\varepsilon \varphi_i + \varepsilon^{2e_i} d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla \varphi_i \right)\ dx dt$$

$$+ \int_0^T \int_{\Gamma^\varepsilon} \varepsilon \Big(c_0^\varepsilon \partial_t u_0^\varepsilon \varphi_0 + \varepsilon^{2e_0} d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon \cdot \nabla^\varepsilon \varphi_0$$

$$+ b_1^\varepsilon(\varphi_1 - \varphi_0) + b_2^\varepsilon(\varphi_2 - \varphi_0) \Big)\ d\Gamma^\varepsilon(x) dt = 0 \qquad (7)$$

for all admissible $\varphi_i \in C_0^\infty([0,T] \times \Omega)$.

Obviously, if the trace conditions are on, the terms with b_i^ε vanish due to the admissibility of the test functions φ_i; they could be omitted in that situation.

This is where the concept of two-scale convergence comes into play, see section 11. The essential point is that for $\varepsilon \to 0$ the limit functions may depend not only on time t and the global variable x but also on the local variable y. We say that a triple $(\varphi_0, \varphi_1, \varphi_2)$ of functions that depends on t, x, y, periodic w.r.t y, is *admissible* iff $\varphi_0(t,x,y) = \varphi_1(t,x,y) = \varphi_2(t,x,y)$ holds for all $y \in \Gamma$, provided that the trace conditions are on; otherwise any triple is admissible. Using the spaces

$$\begin{aligned} V &= \{\vec{u} = (u_0, u_1, u_2) : \; u_0 \in H^1(\Omega; H^1_\#(\Gamma)), \\ &\quad u_i \in H^1(\Omega; H^1_\#(Y_i)), \; i = 1,2 \text{ admissible}\} \\ \mathcal{V} &= \{\vec{u} \in L^2(0,T;V) : \; \partial_t \vec{u} \in L^2(0,T;V')\} \end{aligned}$$

we get the following.

Problem 2 *The* weak macro-problem *is to find triples* $\vec{u}, \vec{v} \in \mathcal{V}$ *of functions* u_i *and* v_i *such that*

$$\begin{aligned} &\sum_{i=1,2} \int_0^T \int_\Omega \int_{Y_i} \Big(c_i \partial_t u_i \varphi_i \\ &\quad + (1-e_i) d_i (\nabla_x u_i + \nabla_y v_i) \cdot (\nabla_x \varphi_i + \nabla_y \psi_i) + e_i d_i \nabla_y u_i \cdot \nabla_y \varphi_i \Big) \, dy dx dt \\ &\quad + \int_0^T \int_\Omega \int_\Gamma \Big(c_0 \partial_t u_0 \varphi_0 + \\ &\quad + (1-e_0) d_0 (\nabla_x u_0 + \nabla_y^\Gamma v_0) \cdot (\nabla_x \varphi_0 + \nabla_y^\Gamma \psi_0) + e_0 d_0 \nabla_y^\Gamma u_0 \cdot \nabla_y^\Gamma \varphi_0 \\ &\quad + b_1(\varphi_1 - \varphi_0) + b_2(\varphi_2 - \varphi_0) \Big) \, d\Gamma(y) dx dt = 0 \end{aligned} \tag{8}$$

for all admissible $\varphi_i, \psi_i \in C_0^\infty([0,T] \times \Omega; C_\#^\infty(Y))$.

As for the weak micro problem, the terms with b_i could be omitted if the trace conditions are on.

It is clear that one has to specify initial conditions for $t = 0$ and boundary conditions on the boundary $\partial\Omega$ for $t > 0$ in order to obtain a well-posed initial boundary value problem. Furthermore, it is necessary to make sure that not all of the functions c_i and d_i vanish simultaneously. Under reasonable assumptions of this kind one gets in the usual way existence, uniqueness and continuous dependence on the data. One can prove the following result (the symbol "$\Rightarrow$" means two-scale convergence, which is explained in section 11).

Theorem 1 *Let $\vec{u}^\varepsilon = (u_0^\varepsilon, u_1^\varepsilon, u_2^\varepsilon)$ be the solution of the micro problem and $\vec{u} = (u_0, u_1, u_2)$ the solution of the macro problem. Then one has $u_i^\varepsilon \Rightarrow u_i$, i=0,1,2.*

The way to prove this result is using test functions of the form $\varphi_i(x) + \varepsilon\psi_i(x, \frac{x}{\varepsilon})$ instead of $\varphi_i(x)$ in the micro problem and then passing to the two-scale limit by taking advantage of the tools mentioned in section 11. It turns out that for $e_i = 0$ the functions v_i can be eliminated by expressing them explicitely in terms of u_i, a procedure that is well known in homogenization theory for elliptic or parabolic problems (see section 3). In this case u_i is a function of t and x only, i.e. it is independent of y. It should be mentioned that strong convergence $u_i \to u$ in $L^2(0,T;\Omega)$ can be proved after appropriate extension of u_i^ε from Ω_i^ε to all Ω.

Integration by parts gives for the macro problem

$$
\begin{aligned}
&\sum_{i=1,2} \int_0^T \int_\Omega \int_{Y_i} \Big(c_i \partial_t u_i \\
&\quad -(1-e_i)\nabla_x \cdot (d_i(\nabla_x u_i + \nabla_y v_i)) - e_i \nabla_y \cdot (d_i \nabla_y u_i)\Big)\varphi_i \; dydxdt \\
&\quad + \int_0^T \int_\Omega \int_\Gamma \Big(\Big(c_0 \partial_t u_0 \\
&\quad -(1-e_0)\nabla_x \cdot (d_0(\nabla_x u_0 + \nabla_y^\Gamma v_0)) - e_0 \nabla_y^\Gamma \cdot (d_0 \nabla_y^\Gamma u_0)\Big)\varphi_0 \\
&\quad + e_1 \vec{\nu} \cdot d_1 \nabla_y u_1 \varphi_1 - e_2 \vec{\nu} \cdot d_2 \nabla_y u_2 \varphi_2 \\
&\quad + b_1(\varphi_1 - \varphi_0) + b_2(\varphi_2 - \varphi_0)\Big) \; d\Gamma(y)dxdt \\
&\quad - \sum_{i=1,2} \int_0^T \int_\Omega \int_{Y_i} (1-e_i)\nabla_y \cdot (d_i(\nabla_x u_i + \nabla_y v_i))\psi_i \; dydxdt \\
&\quad + \int_0^T \int_\Omega \int_\Gamma \Big(-(1-e_0)\nabla_y^\Gamma \cdot (d_0(\nabla_x u_0 + \nabla_y^\Gamma v_0))\psi_0 \\
&\quad + \vec{\nu} \cdot \Big((1-e_1)d_1(\nabla_x u_1 + \nabla_y v_1)\psi_1 \\
&\quad -(1-e_2)d_2(\nabla_x u_2 + \nabla_y v_2)\psi_2\Big)\Big) \; d\Gamma(y)dxdt = 0.
\end{aligned}
$$

In the following sections we are going to identify the strong problems that correspond to each of the weak problems for the different situations.

3 Diffusion in Anisotropic Media

Here we have $e_0 = e_1 = e_2 = 0$, and the trace conditions are on. The weak micro problem is

$$\sum_{i=1,2} \int_0^T \int_{\Omega_i^\varepsilon} \left(c_i^\varepsilon \partial_t u_i^\varepsilon \varphi_i + d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla \varphi_i \right) dx dt$$
$$+ \int_0^T \int_{\Gamma^\varepsilon} \varepsilon \left(c_0^\varepsilon \partial_t u_0^\varepsilon \varphi_0 + d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon \cdot \nabla^\varepsilon \varphi_0 \right) d\Gamma^\varepsilon(x) dt = 0.$$

The weak macro problem is

$$\sum_{i=1,2} \int_0^T \int_\Omega \int_{Y_i} \left(c_i \partial_t u_i \varphi_i + d_i (\nabla_x u_i + \nabla_y v_i) \cdot (\nabla_x \varphi_i + \nabla_y \psi_i) \right) dy dx dt$$
$$+ \int_0^T \int_\Omega \int_\Gamma \left(c_0 \partial_t u_0 \varphi_0 + d_0 (\nabla_x u_0 + \nabla_y^\Gamma v_0) \cdot (\nabla_x \varphi_0 + \nabla_y^\Gamma \psi_0) \right) d\Gamma(y) dx dt = 0 \quad (9)$$

The strong micro problem is

$$\begin{cases} c_i^\varepsilon \partial_t u_i^\varepsilon = \nabla \cdot (d_i^\varepsilon \nabla u_i^\varepsilon), & x \in \Omega_i^\varepsilon, \ i = 1,2 \\ \varepsilon c_0^\varepsilon \partial_t u_0^\varepsilon + \vec{\nu} \cdot (d_1^\varepsilon \nabla u_1^\varepsilon - d_2^\varepsilon \nabla u_2^\varepsilon) = \varepsilon \nabla^\varepsilon \cdot (d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon), & x \in \Gamma^\varepsilon \\ u_0^\varepsilon = u_1^\varepsilon = u_2^\varepsilon, & x \in \Gamma^\varepsilon \end{cases} \quad (10)$$

This model describes diffusion in a medium that is a periodic mixture of three different materials one of which is very thin.

The strong macro problem is to find a function $u(t,x)$ such that

$$\boxed{C \partial_t u = \nabla \cdot (D \nabla u), \ x \in \Omega} \quad (11)$$

which is the standard diffusion equation (1). Here C is the average

$$C = \sum_{i=1,2} \int_{Y_i} c_i \, dy + \int_\Gamma c_0 \, d\Gamma(y)$$

and the tensor D has the coefficients

$$D_{kl} = \sum_{i=1,2} \int_{Y_i} d_i (\nabla_y \omega_k + \vec{e}_k) \cdot \vec{e}_l \, dy + \int_\Gamma d_0 (\nabla_y^\Gamma \omega_k + \vec{e}_k) \cdot \vec{e}_l \, d\Gamma(y),$$

and $\omega_k(y)$ is the periodic solution of the cell problem

$$\begin{cases} \nabla_y \cdot \left(d_i (\nabla_y \omega_k + \vec{e}_k) \right) = 0, & y \in Y_i, \ i = 1,2 \\ \vec{\nu} \cdot \left(d_1 (\nabla_y \omega_k + \vec{e}_k) - d_2 (\nabla_y \omega_k + \vec{e}_k) \right) = \nabla_y^\Gamma \cdot \left(d_0 (\nabla_y^\Gamma \omega_k + \vec{e}_k) \right), & y \in \Gamma \end{cases}$$

The way to derive equation (11) from (9) goes along the following lines (see, e.g., [2]): Since the trace condition is on, we can identify the functions u_0, u_1, and u_2, with one function u. The same is true for the functions v_i and the test functions φ_i and ψ_i. Since φ and ψ can be chosen independently, we get from equation (9)

$$\sum_{i=1,2} \int_0^T \int_\Omega \int_{Y_i} d_i(\nabla_x u + \nabla_y v) \cdot \nabla_y \psi \; dydxdt + \int_0^T \int_\Omega \int_\Gamma d_0(\nabla_x u + \nabla_y^\Gamma v) \cdot \nabla_y^\Gamma \psi \; d\Gamma(y)dxdt = 0 \tag{12}$$

and

$$\sum_{i=1,2} \int_0^T \int_\Omega \int_{Y_i} \Big(c_i \partial_t u \varphi + d_i(\nabla_x u + \nabla_y v) \cdot \nabla_x \varphi\Big) \; dydxdt + \int_0^T \int_\Omega \int_\Gamma \Big(c_0 \partial_t u \varphi + d_0(\nabla_x u + \nabla_y^\Gamma v) \cdot \nabla_x \varphi\Big) \; d\Gamma(y)dxdt = 0. \tag{13}$$

The strong form of the subproblem (12) is

$$\begin{cases} \nabla_y \cdot \Big(d_i(\nabla_x u + \nabla_y v)\Big) = 0, & y \in Y_i, \; i = 1,2 \\ \vec{\nu} \cdot \Big(d_1(\nabla_x u + \nabla_y v) - d_2(\nabla_x u + \nabla_y v)\Big) = \nabla_y^\Gamma \cdot \Big(d_0(\nabla_x u + \nabla_y^\Gamma v)\Big), & y \in \Gamma \end{cases} \tag{14}$$

and the strong form of the subproblem (13) is

$$\Big(\sum_{i=1,2} \int_{Y_i} c_i \; dy + \int_\Gamma c_0 \; d\Gamma(y)\Big) \partial_t u = \nabla_x \cdot \Big(\sum_{i=1,2} \int_{Y_i} d_i(\nabla_x u + \nabla_y v) \; dy + \int_\Gamma d_0(\nabla_x u + \nabla_y^\Gamma v) \; d\Gamma(y)\Big) \tag{15}$$

From equation (14) one deduces the representation

$$v(t,x,y) = \sum_{k=1}^3 \omega_k(y) \partial_{x_k} u(t,x).$$

Plugging this formula into equation (15) directly gives equation (11). This type of argument applies also to the examples in the following sections.

Special Cases:

$c_0 = d_0 = 0$: One gets standard homogenization for anisotropic media. This case is studied in textbooks of homogenization theory, see [5] [7].

$c_0 = d_0 = c_2 = d_2 = 0$: One has homogenization of diffusion in media with holes.

$c_2 = d_2 = 0$: This is a system of blocks and fissures.

$c_1 = d_1 = c_2 = d_2 = 0$: This is a system of fissures only.

In the same way one could also treat a system consisting of small tubes or pipes, i.e., use curves instead of surfaces Γ. It is important to notice that in all these cases the homogenized limit is that of diffusion in an anisotropic medium.

4 Multi-Continua

Here we have $e_0 = e_1 = e_2 = 0$, and the trace conditions are off. The weak micro problem is

$$\sum_{i=1,2} \int_0^T \int_{\Omega_i^\varepsilon} \left(c_i^\varepsilon \partial_t u_i^\varepsilon \varphi_i + d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla \varphi_i \right) dx dt$$
$$+ \int_0^T \int_{\Gamma^\varepsilon} \varepsilon \Big(c_0^\varepsilon \partial_t u_0^\varepsilon \varphi_0 + d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon \cdot \nabla^\varepsilon \varphi_0$$
$$+ b_1^\varepsilon (\varphi_1 - \varphi_0) + b_2^\varepsilon (\varphi_2 - \varphi_0) \Big) \, d\Gamma^\varepsilon(x) dt = 0.$$

The weak macro problem is

$$\sum_{i=1,2} \int_0^T \int_\Omega \int_{Y_i} \Big(c_i \partial_t u_i \varphi_i + d_i (\nabla_x u_i + \nabla_y v_i) \cdot (\nabla_x \varphi_i + \nabla_y \psi_i) \Big) \, dy dx dt$$
$$+ \int_0^T \int_\Omega \int_\Gamma \Big(c_0 \partial_t u_0 \varphi_0 + d_0 (\nabla_x u_0 + \nabla_y^\Gamma v_0) \cdot (\nabla_x \varphi_0 + \nabla_y^\Gamma \psi_0)$$
$$+ b_1 (\varphi_1 - \varphi_0) + b_2 (\varphi_2 - \varphi_0) \Big) \, d\Gamma(y) dx dt = 0.$$

The strong form of the micro problem is

$$\begin{cases} c_i^\varepsilon \partial_t u_i^\varepsilon = \nabla \cdot (d_i^\varepsilon \nabla u_i^\varepsilon), & x \in \Omega_i^\varepsilon, \ i = 1,2 \\ -\vec{\nu} \cdot d_1^\varepsilon \nabla u_1^\varepsilon = \varepsilon b_1^\varepsilon, & x \in \Gamma^\varepsilon \\ \vec{\nu} \cdot d_2^\varepsilon \nabla u_2^\varepsilon = \varepsilon b_2^\varepsilon, & x \in \Gamma^\varepsilon \\ c_0^\varepsilon \partial_t u_0^\varepsilon - b_1^\varepsilon - b_2^\varepsilon = \nabla^\varepsilon \cdot (d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon), & x \in \Gamma^\varepsilon \end{cases} \tag{16}$$

This model describes simultaneous diffusion in three continua that are intertwined and between which exchange takes place depending on the variables on the common boundaries.

The strong macro problem is to find functions $u_i(t,x)$ such that

$$\left\{\begin{array}{ll} C_i\partial_t u_i + B_i = \nabla\cdot(D_i\nabla u_i), & x\in\Omega,\ i=0,1,2 \\ B_i = \sum_{j=0,1,2} S_{ij}u_j, & x\in\Omega,\ i=1,2 \\ B_0 = -B_1 - B_2, & x\in\Omega \end{array}\right. \tag{17}$$

Here

$$C_i = \int_{Y_i} c_i\ dy,\ i=1,2,\ C_0 = \int_\Gamma c_0\ d\Gamma(y),$$

$$S_{ij} = \int_\Gamma s_{ij}\ d\Gamma(y),\ i=1,2,\ j=0,1,2.$$

The tensors D_i, $i=1,2$ have the coefficients

$$D_{ikl} = \int_{Y_i} d_i(\nabla_y\omega_{ik} + \vec{e}_k)\cdot\vec{e}_l\ dy$$

where ω_{ik} are the periodic solutions of the cell problem

$$\left\{\begin{array}{ll} \nabla_y\cdot\Big(d_i(\nabla_y\omega_{ik}+\vec{e}_k)\Big) = 0, & y\in Y_i \\ \vec{\nu}\cdot\Big(d_i(\nabla_y\omega_{ik}+\vec{e}_k)\Big) = 0, & y\in\Gamma \end{array}\right.$$

and the tensor D_0 has the coefficients

$$D_{0kl} = \int_\Gamma (\nabla_y^\Gamma\omega_{0k} + \vec{e}_k)\cdot\vec{e}_l\ d\Gamma(y)$$

where ω_{0k} is the periodic solution of the cell problem

$$\nabla_y^\Gamma\cdot\Big(d_0(\nabla_y^\Gamma\omega_{0k}+\vec{e}_k)\Big) = 0,\ y\in\Gamma.$$

Special Cases:

$c_0 = d_0$: one gets $b_1^\varepsilon + b_2^\varepsilon = 0$ and $B_0 = B_1 + B_2 = 0$. Therefore, a reasonable choice is $b_1 = -b_2 = s_1u_1 + s_2u_2$ with $s_1>0, s_2<0$. The macro model is

$$\boxed{\begin{array}{ll} C_i\partial_t u_i + B_i = \nabla\cdot(D_i\nabla u_i), & x\in\Omega,\ i=1,2 \\ B_i = \sum_{j=1,2} S_{ij}u_j, & x\in\Omega,\ i=1,2 \\ B_1 + B_2 = 0, & x\in\Omega \end{array}} \tag{18}$$

This is the two-continua model (2) which was proposed in [6]. In [15] this macro model is called the "double porosity parallel model".

$c_2 = d_2 = 0$: one gets $b_2 = 0$ and thus $B_2 = 0$. Therefore, a reasonable choice is $b_1 = -b_0 = s_1 u_1 + s_0 u_0$ with $s_1 > 0, s_0 < 0$. The macro model is

$$\begin{cases} C_i \partial_t u_i + B_i = \nabla \cdot (D_i \nabla u_i), & x \in \Omega,\ i = 0,1 \\ B_i = \sum_{j=0,1} S_{ij} u_j, & x \in \Omega,\ i = 0,1 \\ B_0 + B_1 = 0, & x \in \Omega \end{cases} \tag{19}$$

This is again the two-continua model.

5 First-Order Kinetic

Here we have $e_0 = 1$, $e_1 = e_2 = 0$, and the trace conditions are off. The weak micro problem is

$$\begin{aligned} &\int_0^T \int_{\Omega_1^\varepsilon} \left(c_1^\varepsilon \partial_t u_1^\varepsilon \varphi_1 + d_1^\varepsilon \nabla u_1^\varepsilon \cdot \nabla \varphi_1 \right) dx dt \\ &\quad + \int_0^T \int_{\Omega_2^\varepsilon} \left(c_2^\varepsilon \partial_t u_2^\varepsilon \varphi_2 + d_2^\varepsilon \nabla u_2^\varepsilon \cdot \nabla \varphi_2 \right) dx dt \\ &\quad + \int_0^T \int_{\Gamma^\varepsilon} \varepsilon \Big(c_0^\varepsilon \partial_t u_0^\varepsilon \varphi_0 + \varepsilon^2 d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon \cdot \nabla^\varepsilon \varphi_0 \\ &\quad + b_1^\varepsilon (\varphi_1 - \varphi_0) + b_2^\varepsilon (\varphi_2 - \varphi_0) \Big) \, d\Gamma^\varepsilon(x) dt = 0. \end{aligned}$$

The weak macro problem is

$$\begin{aligned} &\int_0^T \int_\Omega \int_{Y_1} \Big(c_1 \partial_t u_1 \varphi_1 + d_1 (\nabla_x u_1 + \nabla_y v_1) \cdot (\nabla_x \varphi_1 + \nabla_y \psi_1) \Big) \, dy dx dt \\ &\quad + \int_0^T \int_\Omega \int_{Y_2} \Big(c_2 \partial_t u_2 \varphi_2 + d_2 (\nabla_x u_2 + \nabla_y v_2) \cdot (\nabla_x \varphi_2 + \nabla_y \psi_2) \Big) \, dy dx dt \\ &\quad + \int_0^T \int_\Omega \int_\Gamma \Big(c_0 \partial_t u_0 \varphi_0 + d_0 \nabla_y^\Gamma u_0 \cdot \nabla_y^\Gamma \varphi_0 \\ &\quad + b_1 (\varphi_1 - \varphi_0) + b_2 (\varphi_2 - \varphi_0) \Big) \, d\Gamma(y) dx dt = 0. \end{aligned}$$

The strong form of the micro problem is

$$\begin{cases} c_i^\varepsilon \partial_t u_i^\varepsilon = \nabla \cdot (d_i^\varepsilon \nabla u_i^\varepsilon), & x \in \Omega_i^\varepsilon,\ i = 1,2 \\ -\vec{\nu} \cdot d_1^\varepsilon \nabla u_1^\varepsilon = \varepsilon b_1^\varepsilon,\ \vec{\nu} \cdot d_2^\varepsilon \nabla u_2^\varepsilon = \varepsilon b_2^\varepsilon, & x \in \Gamma^\varepsilon \\ c_0^\varepsilon \partial_t u_0^\varepsilon - b_1^\varepsilon - b_2^\varepsilon = \varepsilon^2 \nabla^\varepsilon \cdot (d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon), & x \in \Gamma^\varepsilon \end{cases} \tag{20}$$

This model describes simultaneous diffusion in three intertwined continua similar to the previous section, where now the thin continuum - the interface between the other two - is slow.

The strong macro problem is to find functions $u_1(t,x)$, $u_2(t,x)$ and $u_0(t,x,y)$ such that

$$\begin{cases} C_i \partial_t u_i + B_i = \nabla \cdot (D_i \nabla u_i), & x \in \Omega,\ i = 1,2 \\ c_0 \partial_t u_0 - b_1 - b_2 = \nabla^\Gamma \cdot (d_0 \nabla^\Gamma u_0), & x \in \Omega,\ y \in \Gamma \\ B_i = \int_\Gamma b_i \, d\Gamma(y), & x \in \Omega,\ i = 1,2 \end{cases} \tag{21}$$

Here C_i is the average

$$C_i = \int_{Y_i} c_i \, dy,$$

and the tensors D_i, $i = 1,2$ have the coefficients

$$D_{ikl} = \int_{Y_i} d_i (\nabla_y \omega_{ik} + \vec{e}_k) \cdot \vec{e}_l \, dy$$

where ω_{ik} are the periodic solutions of the cell problem

$$\begin{cases} \nabla_y \cdot \Big(d_i (\nabla_y \omega_{ik} + \vec{e}_k) \Big) = 0, & y \in Y_i \\ \vec{\nu} \cdot \Big(d_i (\nabla_y \omega_{ik} + \vec{e}_k) \Big) = 0, & y \in \Gamma \end{cases}$$

Special Case:

$c_2 = d_2$: One gets $b_2 = 0$ and therefore $B_2 = C_2 = D_2 = 0$. Therefore, the macro model is

$$\begin{cases} C \partial_t u_1 + B = \nabla \cdot (D \nabla u_1), & x \in \Omega \\ c_0 \partial_t u_0 - b_1 = \nabla \cdot (d_0 \nabla u_0), & x \in \Omega,\ y \in \Gamma \\ B = \int_\Gamma b_1 \, d\Gamma(y), & x \in \Omega \end{cases}$$

This model was studied in [9].

Specializing further by assuming that $b_1 = s_1 u_1 + s_0 u_0$ (with $s_1 > 0, s_0 < 0$) is independent of y and $d_0 = 0$, one gets that u_0 is also y-independent and thus

$$\boxed{\begin{array}{ll} C \partial_t u_1 + |\Gamma| b_1 = \nabla \cdot (D \nabla u_1), & x \in \Omega \\ c_0 \partial_t u_0 = b_1, & x \in \Omega,\ y \in \Gamma \end{array}} \tag{22}$$

In this situation the micro model describes adsorption of chemical species on the surfaces Γ^ε. The macro model is the well known first-order kinetic model (3). The degenerate case $C = 0$ is called the "fissured medium system" in [15].

6 Double Permeability

Here we have $e_0 = e_1 = 0$, $e_2 = 1$, and the trace conditions are on. This case is equivalent to $e_0 = e_2 = 0$, $e_1 = 1$. The weak micro problem is

$$\int_0^T \int_{\Omega_1^\varepsilon} \left(c_1^\varepsilon \partial_t u_1^\varepsilon \varphi_1 + d_1^\varepsilon \nabla u_1^\varepsilon \cdot \nabla \varphi_1 \right) dxdt$$
$$+ \int_0^T \int_{\Omega_2^\varepsilon} \left(c_2^\varepsilon \partial_t u_2^\varepsilon \varphi_2 + \varepsilon^2 d_2^\varepsilon \nabla u_2^\varepsilon \cdot \nabla \varphi_2 \right) dxdt$$
$$+ \int_0^T \int_{\Gamma^\varepsilon} \varepsilon \left(c_0^\varepsilon \partial_t u_0^\varepsilon \varphi_0 + d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon \cdot \nabla^\varepsilon \varphi_0 \right) d\Gamma^\varepsilon(x) dt = 0.$$

The weak macro problem is

$$\int_0^T \int_\Omega \int_{Y_1} \Big(c_1 \partial_t u_1 \varphi_1 + d_1 (\nabla_x u_1 + \nabla_y v_1) \cdot (\nabla_x \varphi_1 + \nabla_y \psi_1) \Big) dydxdt$$
$$+ \int_0^T \int_\Omega \int_{Y_2} \left(c_2 \partial_t u_2 \varphi_2 + d_2 \nabla_y u_2 \cdot \nabla_y \varphi_2 \right) dydxdt$$
$$+ \int_0^T \int_\Omega \int_\Gamma \Big(c_0 \partial_t u_0 \varphi_0 + d_0 (\nabla_x u_0 + \nabla_y^\Gamma v_0) \cdot (\nabla_x \varphi_0 + \nabla_y^\Gamma \psi_0) \Big) d\Gamma(y) dxdt = 0.$$

The strong form of the micro problem is

$$\begin{cases} c_1^\varepsilon \partial_t u_1^\varepsilon = \nabla \cdot (d_1^\varepsilon \nabla u_1^\varepsilon), & x \in \Omega_1^\varepsilon \\ c_2^\varepsilon \partial_t u_2^\varepsilon = \varepsilon^2 \nabla \cdot (d_2^\varepsilon \nabla u_2^\varepsilon), & x \in \Omega_2^\varepsilon \\ \varepsilon c_0^\varepsilon \partial_t u_0^\varepsilon + \vec{\nu} \cdot (d_1^\varepsilon \nabla u_1^\varepsilon - \varepsilon^2 d_2^\varepsilon \nabla u_2^\varepsilon) = \varepsilon \nabla^\varepsilon \cdot (d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon), & x \in \Gamma^\varepsilon \\ u_0^\varepsilon = u_1^\varepsilon = u_2^\varepsilon, & x \in \Gamma^\varepsilon \end{cases} \tag{23}$$

This model describes diffusion in a mixture of three materials one of which is slow.

The strong macro problem is to find functions $u_1(t, x)$ and $u_2(t, x, y)$ such that

$$\boxed{\begin{array}{ll} C \partial_t u_1 + B = \nabla \cdot (D \nabla u_1), & x \in \Omega \\ B = - \int_\Gamma \nu \cdot d_2 \nabla_y u_2 \, d\Gamma(y), & x \in \Omega \\ c_2 \partial_t u_2 = \nabla_y \cdot (d_2 \nabla_y u_2), & x \in \Omega, \ y \in Y_2 \\ u_1 = u_2, & x \in \Omega, \ y \in \Gamma \end{array}} \tag{24}$$

which is equation (4). Here C is the average

$$C = \int_{Y_1} c_1 \, dy + \int_\Gamma c_0 \, d\Gamma(y),$$

and the tensor D has the coefficients

$$D_{kl} = \int_{Y_1} d_1(\nabla_y \omega_k + \vec{e}_k) \cdot \vec{e}_l \; dy + \int_\Gamma d_0(\nabla_y^\Gamma \omega_k + \vec{e}_k) \cdot \vec{e}_l \; d\Gamma(y)$$

where ω_k is the periodic solution of the cell problem

$$\begin{cases} \nabla_y \cdot \Big(d_1(\nabla_y \omega_k + \vec{e}_k) \Big) = 0, & y \in Y_1 \\ \vec{\nu} \cdot \Big(d_1(\nabla_y \omega_k + \vec{e}_k) \Big) = \nabla_y^\Gamma \cdot \Big(d_0(\nabla_y^\Gamma \omega_k + \vec{e}_k) \Big), & y \in \Gamma \end{cases}$$

Special Case: $c_0 = d_0 = 0$: This is the well known double permeability model which is often called the "double porosity" model, see [3] and [12]. In [15] the macro problem of this section is called the "matched micro-structure model". An early reference to this idea in the soil sciences is [14]. Nonlinear problems of this type were studied in [4].

7 Double Permeability Model with Robin Type Transmission Conditions

Here we have $e_0 = e_1 = 0$, $e_2 = 1$, and the trace conditions are off. This case is equivalent to $e_0 = e_2 = 0$, $e_1 = 1$. The weak micro problem is

$$\begin{aligned} &\int_0^T \int_{\Omega_1^\varepsilon} \Big(c_1^\varepsilon \partial_t u_1^\varepsilon \varphi_1 + d_1^\varepsilon \nabla u_1^\varepsilon \cdot \nabla \varphi_1 \Big) \; dxdt \\ &\quad + \int_0^T \int_{\Omega_2^\varepsilon} \Big(c_2^\varepsilon \partial_t u_2^\varepsilon \varphi_2 + \varepsilon^2 d_2^\varepsilon \nabla u_2^\varepsilon \cdot \nabla \varphi_2 \Big) dxdt \\ &\quad + \int_0^T \int_{\Gamma^\varepsilon} \varepsilon \Big(c_0^\varepsilon \partial_t u_0^\varepsilon \varphi_0 + d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon \cdot \nabla^\varepsilon \varphi_0 \\ &\quad + b_1^\varepsilon (\varphi_1 - \varphi_0) + b_2^\varepsilon (\varphi_2 - \varphi_0) \Big) \; d\Gamma^\varepsilon(x) dt = 0. \end{aligned}$$

The weak macro problem is

$$\begin{aligned} &\int_0^T \int_\Omega \int_{Y_1} \Big(c_1 \partial_t u_1 \varphi_1 + d_1 (\nabla_x u_1 + \nabla_y v_1) \cdot (\nabla_x \varphi_1 + \nabla_y \psi_1) \Big) \; dydxdt \\ &\quad + \int_0^T \int_\Omega \int_{Y_2} \Big(c_2 \partial_t u_2 \varphi_2 + d_2 \nabla_y u_2 \cdot \nabla_y \varphi_2 \Big) \; dydxdt \\ &\quad + \int_0^T \int_\Omega \int_\Gamma \Big(c_0 \partial_t u_0 \varphi_0 + d_0 (\nabla_x u_0 + \nabla_y v_0) \cdot (\nabla_x \varphi_0 + \nabla_y \psi_0) \\ &\quad + b_1 (\varphi_1 - \varphi_0) + b_2 (\varphi_2 - \varphi_0) \Big) \; d\Gamma(y) dxdt = 0. \end{aligned}$$

The strong form of the micro problem is

$$
\begin{cases}
c_1^\varepsilon \partial_t u_1^\varepsilon = \nabla \cdot (d_1^\varepsilon \nabla u_1^\varepsilon), & x \in \Omega_1^\varepsilon \\
c_2^\varepsilon \partial_t u_2^\varepsilon = \varepsilon^2 \nabla \cdot (d_2^\varepsilon \nabla u_2^\varepsilon), & x \in \Omega_2^\varepsilon \\
-\vec{\nu} \cdot d_1^\varepsilon \nabla u_1^\varepsilon = \varepsilon b_1^\varepsilon, \ \varepsilon \vec{\nu} \cdot d_2^\varepsilon \nabla u_2^\varepsilon = b_2^\varepsilon, & x \in \Gamma^\varepsilon \\
c_0^\varepsilon \partial_t u_0^\varepsilon - b_1^\varepsilon - b_2^\varepsilon = \nabla^\varepsilon \cdot (d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon), & x \in \Gamma^\varepsilon
\end{cases}
\tag{25}
$$

This model describes simultaneous diffusion in three intertwined continua one of which is slow.

The strong macro problem is to find functions $u_0(t,x), u_1(t,x)$ and $u_2(t,x,y)$ such that

$$
\begin{cases}
C_i \partial_t u_i + B_i = \nabla \cdot (D_i \nabla u_i), & x \in \Omega, \ i = 0, 1 \\
B_1 = \int_\Gamma b_1 \ d\Gamma(y), & x \in \Omega \\
B_0 = -B_1 - \int_\Gamma b_2 \ d\Gamma(y), & x \in \Omega \\
c_2 \partial_t u_2 = \nabla_y \cdot (d_2 \nabla_y u_2), & x \in \Omega, \ y \in Y_2 \\
\vec{\nu} \cdot d_2 \nabla_y u_2 = b_2, & x \in \Omega, \ y \in \Gamma
\end{cases}
\tag{26}
$$

Here we have

$$
C_1 = \int_{Y_1} c_1 \ dy, \ C_0 = \int_\Gamma c_0 \ d\Gamma(y).
$$

The tensor D_1 has the coefficients

$$
D_{1kl} = \int_{Y_1} d_1 (\nabla_y \omega_{1k} + \vec{e}_k) \cdot \vec{e}_l \ dy
$$

where ω_{1k} is the periodic solution of the cell problem

$$
\begin{cases}
\nabla_y \cdot \Big(d_i (\nabla_y \omega_{1k} + \vec{e}_k) \Big) = 0, & y \in Y_i \\
\vec{\nu} \cdot \Big(d_i (\nabla_y \omega_{1k} + \vec{e}_k) \Big) = 0, & y \in \Gamma
\end{cases}
$$

and the tensor D_0 has the coefficients

$$
D_{0kl} = \int_\Gamma (\nabla_y^\Gamma \omega_{0k} + \vec{e}_k) \cdot \vec{e}_l \ d\Gamma(y)
$$

where ω_{0k} is the periodic solution of the cell problem

$$
\nabla_y^\Gamma \cdot \Big(d_0 (\nabla_y^\Gamma \omega_{0k} + \vec{e}_k) \Big) = 0, \ y \in \Gamma.
$$

Special Case: $c_0 = d_0 = 0$. Then the macro model is

$$
\boxed{
\begin{array}{ll}
C \partial_t u_1 + B = \nabla \cdot (D \nabla u_1), & x \in \Omega \\
B = -\int_\Gamma b_2 \ d\Gamma(y), & x \in \Omega \\
c_2 \partial_t u_2 = \nabla_y \cdot (d_2 \nabla_y u_2), & x \in \Omega, \ y \in Y_2 \\
\vec{\nu} \cdot d_2 \nabla_y u_2 = b_2, & x \in \Omega, \ y \in \Gamma
\end{array}
}
\tag{27}
$$

Then one gets a double permeability model with Robin type transmission conditions, see equation (5). In [15] this macro model is called the "regularized micro-structure model". Models of this kind with nonlinear transmission conditions were studied in [11].

8 Slow Membranes

Here we have $e_1 = 0$, $e_0 = e_2 = 1$, and the trace conditions are off. This case is equivalent to $e_0 = e_1 = 1$, $e_2 = 0$. The weak micro problem is

$$\begin{aligned}
&\int_0^T \int_{\Omega_1^\varepsilon} \Big(c_1^\varepsilon \partial_t u_1^\varepsilon \varphi_1 + d_1^\varepsilon \nabla u_1^\varepsilon \cdot \nabla \varphi_1 \Big) \, dx dt \\
&\quad + \int_0^T \int_{\Omega_2^\varepsilon} \Big(c_2^\varepsilon \partial_t u_2^\varepsilon \varphi_2 + \varepsilon^2 d_2^\varepsilon \nabla u_2^\varepsilon \cdot \nabla \varphi_2 \Big) dx dt \\
&\quad + \int_0^T \int_{\Gamma^\varepsilon} \varepsilon \Big(c_0^\varepsilon \partial_t u_0^\varepsilon \varphi_0 + \varepsilon^2 d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon \cdot \nabla^\varepsilon \varphi_0 \\
&\quad + b_1^\varepsilon (\varphi_1 - \varphi_0) + b_2^\varepsilon (\varphi_2 - \varphi_0) \Big) \, d\Gamma^\varepsilon(x) dt = 0.
\end{aligned}$$

The weak macro problem is

$$\begin{aligned}
&\int_0^T \int_\Omega \int_{Y_1} \Big(c_1 \partial_t u_1 \varphi_1 + d_1 (\nabla_x u_1 + \nabla_y v_1) \cdot (\nabla_x \varphi_1 + \nabla_y \psi_1) \Big) \, dy dx dt \\
&\quad + \int_0^T \int_\Omega \int_{Y_2} \Big(c_2 \partial_t u_2 \varphi_2 + d_2 \nabla_y u_2 \cdot \nabla_y \varphi_2 \Big) \, dy dx dt \\
&\quad + \int_0^T \int_\Omega \int_\Gamma \Big(c_0 \partial_t u_0 \varphi_0 + d_0 \nabla_y^\Gamma u_0 \cdot \nabla_y^\Gamma \varphi_0 \\
&\quad + b_1 (\varphi_1 - \varphi_0) + b_2 (\varphi_2 - \varphi_0) \Big) \, d\Gamma(y) dx dt = 0.
\end{aligned}$$

The strong form of the micro problem is

$$\left\{ \begin{array}{ll}
c_1^\varepsilon \partial_t u_1^\varepsilon = \nabla \cdot (d_1^\varepsilon \nabla u_1^\varepsilon), & x \in \Omega_1^\varepsilon \\
c_2^\varepsilon \partial_t u_2^\varepsilon = \varepsilon^2 \nabla \cdot (d_2^\varepsilon \nabla u_2^\varepsilon), & x \in \Omega_2^\varepsilon \\
-\vec{\nu} \cdot d_1^\varepsilon \nabla u_1^\varepsilon = \varepsilon b_1^\varepsilon, \ \varepsilon \vec{\nu} \cdot d_2^\varepsilon \nabla u_2^\varepsilon = b_2^\varepsilon, & x \in \Gamma^\varepsilon \\
c_0^\varepsilon \partial_t u_0^\varepsilon - b_1^\varepsilon - b_2^\varepsilon = \varepsilon^2 \nabla^\varepsilon \cdot (d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon), & x \in \Gamma^\varepsilon
\end{array} \right. \tag{28}$$

This model describes simultaneous diffusion in three intertwined continua two of which - including the thin one - are slow.

The strong macro is to find functions $u_1(t,x)$ and $u_2(t,x,y), u_0(t,x,y)$ such that

$$\begin{cases} C\partial_t u_1 + B = \nabla \cdot (D\nabla u_1), & x \in \Omega \\ B = \int_\Gamma b_1 \, d\Gamma(y), & x \in \Omega \\ c_2 \partial_t u_2 = \nabla_y \cdot (d_2 \nabla_y u_2), & x \in \Omega, \; y \in Y_2 \\ \vec{\nu} \cdot d_2 \nabla_y u_2 = b_2, & x \in \Omega, \; y \in \Gamma \\ c_0 \partial_t u_0 - b_1 - b_2 = \nabla^\Gamma \cdot (d_0 \nabla^\Gamma u_0), & x \in \Omega, \; y \in \Gamma \end{cases} \tag{29}$$

Here C_1 is the average

$$C = \int_{Y_1} c_1 \, dy,$$

and the tensor D has the coefficients

$$D_{kl} = \int_{Y_1} d_1(\nabla_y \omega_k + \vec{e}_k) \cdot \vec{e}_l \, dy$$

where ω_k are the periodic solutions of the cell problem

$$\begin{cases} \nabla_y \cdot \Big(d_1(\nabla_y \omega_k + \vec{e}_k) \Big) = 0, & y \in Y_1 \\ \vec{\nu} \cdot \Big(d_1(\nabla_y \omega_k + \vec{e}_k) \Big) = 0, & y \in \Gamma \end{cases}$$

Special Cases: $c_0 = d_0$: Then one gets $b_1 + b_2 = 0$. Therefore, the macro model is

$$\begin{cases} C\partial_t u_1 + B = \nabla \cdot (D\nabla u_1), & x \in \Omega \\ B = -\int_\Gamma b_2 \, d\Gamma(y), & x \in \Omega \\ c_2 \partial_t u_2 = \nabla_y \cdot (d_2 \nabla_y u_2), & x \in \Omega, \; y \in Y_2 \\ \vec{\nu} \cdot d_2 \nabla_y u_2 = b_2, & x \in \Omega, \; y \in \Gamma \end{cases} \tag{30}$$

This is the same as the special case in section 7.

$c_2 = d_2 = 0$. Then one gets

$$\begin{cases} C\partial_t u_1 + B = \nabla \cdot (D\nabla u_1), & x \in \Omega \\ B = \int_\Gamma b_1 \, d\Gamma(y), & x \in \Omega \\ c_0 \partial_t u_0 - b_1 = \nabla \cdot (d_0 \nabla u_0), & x \in \Omega, \; y \in \Gamma \end{cases} \tag{31}$$

This is the same as the special case of section 5.

9 Systems of Fast Fissures and Slow Blocks

Here we have $e_0 = 0$, $e_1 = e_2 = 1$, and the trace conditions are on. The weak micro problem is

$$\sum_{i=1,2} \int_0^T \int_{\Omega_1^\varepsilon} \left(c_i^\varepsilon \partial_t u_i^\varepsilon \varphi_i + \varepsilon^2 d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla \varphi_i \right) \, dxdt$$
$$+ \int_0^T \int_{\Gamma^\varepsilon} \varepsilon \left(c_0^\varepsilon \partial_t u_0^\varepsilon \varphi_0 + d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon \cdot \nabla^\varepsilon \varphi_0 \right) \, d\Gamma^\varepsilon(x) dt = 0.$$

The weak macro problem is

$$\sum_{i=1,2} \int_0^T \int_\Omega \int_{Y_i} \left(c_i \partial_t u_i \varphi_i + d_i \nabla_y u_i \cdot \nabla_y \varphi_i \right) \, dydxdt$$
$$+ \int_0^T \int_\Omega \int_\Gamma \left(c_0 \partial_t u_0 \varphi_0 + d_0 (\nabla_x u_0 + \nabla_y^\Gamma v_0) \cdot (\nabla_x \varphi_0 + \nabla_y^\Gamma \psi_0) \right) \, d\Gamma(y) dxdt = 0.$$

The strong form of the micro problem is

$$\begin{cases} c_i^\varepsilon \partial_t u_i^\varepsilon = \varepsilon^2 \nabla \cdot (d_i^\varepsilon \nabla u_i^\varepsilon), & x \in \Omega_i^\varepsilon, \; i = 1,2 \\ c_0^\varepsilon \partial_t u_0^\varepsilon + \varepsilon \vec{\nu} \cdot (d_1^\varepsilon \nabla u_1^\varepsilon - d_2^\varepsilon \nabla u_2^\varepsilon) = \nabla^\varepsilon \cdot (d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon), & x \in \Gamma^\varepsilon \\ u_0^\varepsilon = u_1^\varepsilon = u_2^\varepsilon, & x \in \Gamma^\varepsilon \end{cases} \tag{32}$$

This model describes diffusion in a mixture of fast fissures that are surrounded by slow blocks.

The strong macro problem is to find functions $u_0(t,x)$ and $u_1(t,x,y), u_2(t,x,y)$ such that

$$\begin{cases} C \partial_t u_0 + B = \nabla \cdot (D \nabla u_0), & x \in \Omega \\ B = \int_\Gamma \nu \cdot (d_1 \nabla_y u_1 - d_2 \nabla_y u_2) \, d\Gamma(y), & x \in \Omega \\ c_i \partial_t u_i = \nabla_y \cdot (d_i \nabla_y u_i), & x \in \Omega, \; y \in Y_i, \; i = 1,2 \\ u_0 = u_1 = u_2, & x \in \Omega, \; y \in \Gamma \end{cases} \tag{33}$$

Here C is the average

$$C = \int_\Gamma c_0 \, d\Gamma(y),$$

and the tensor D has the coefficients

$$D_{kl} = \int_\Gamma d_0 (\nabla_y^\Gamma \omega_k + \vec{e}_k) \cdot \vec{e}_l \, d\Gamma(y)$$

where ω_k is the periodic solution of the cell problem

$$\nabla_y^\Gamma \cdot \Big(d_0(\nabla_y^\Gamma \omega_k + \vec{e}_k)\Big) = 0,\ y \in \Gamma.$$

Special Case: $c_1 = d_1 = 0$. Then the macro model is

$$\begin{cases} C\partial_t u_0 + B = \nabla \cdot (D\nabla u_0), & x \in \Omega \\ B = -\int_\Gamma \nu \cdot d_2 \nabla_y u_2 \, d\Gamma(y), & x \in \Omega \\ c_2 \partial_t u_2 = \nabla_y \cdot (d_2 \nabla_y u_2), & x \in \Omega,\ y \in Y_2 \\ u_0 = u_2, & x \in \Omega,\ y \in \Gamma \end{cases}$$

This is the same double permeability model as in section 6. The micro model of this section was used in [16]; it seems likely that using the homogenized model could save quite some computer time in such situations.

10 Fast Membranes

Here we have $e_0 = 0$, $e_1 = e_2 = 1$, and the trace conditions are off. The weak micro problem is

$$\begin{aligned} \sum_{i=1,2} \int_0^T \int_{\Omega_i^\varepsilon} \Big(c_i^\varepsilon \partial_t u_i^\varepsilon \varphi_i + \varepsilon^2 d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla \varphi_i\Big)\, dxdt \\ + \int_0^T \int_{\Gamma^\varepsilon} \varepsilon \Big(c_0^\varepsilon \partial_t u_0^\varepsilon \varphi_0 + d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon \cdot \nabla^\varepsilon \varphi_0 \\ + b_1^\varepsilon(\varphi_1 - \varphi_0) + b_2^\varepsilon(\varphi_2 - \varphi_0)\Big)\, d\Gamma^\varepsilon(x) dt = 0. \end{aligned}$$

The weak macro problem is

$$\begin{aligned} \sum_{i=1,2} \int_0^T \int_\Omega \int_{Y_i} \Big(c_i \partial_t u_i \varphi_i + d_i \nabla_y u_i \cdot \nabla_y \varphi_i\Big)\, dydxdt \\ + \int_0^T \int_\Omega \int_\Gamma \Big(c_0 \partial_t u_0 \varphi_0 + d_0(\nabla_x u_0 + \nabla_y^\Gamma v_0) \cdot (\nabla_x \varphi_0 + \nabla_y^\Gamma \psi_0) \\ + b_1(\varphi_1 - \varphi_0) + b_2(\varphi_2 - \varphi_0)\Big)\, d\Gamma(y) dxdt = 0. \end{aligned}$$

The strong form of the micro problem is

$$\begin{cases} c_i^\varepsilon \partial_t u_i^\varepsilon = \varepsilon^2 \nabla \cdot (d_i^\varepsilon \nabla u_i^\varepsilon), & x \in \Omega_i^\varepsilon,\ i = 1,2 \\ -\varepsilon \vec{\nu} \cdot d_1^\varepsilon \nabla u_1^\varepsilon = b_1^\varepsilon,\ \varepsilon \vec{\nu} \cdot d_2^\varepsilon \nabla u_2^\varepsilon = b_2^\varepsilon, & x \in \Gamma^\varepsilon \\ c_0^\varepsilon \partial_t u_0^\varepsilon - b_1^\varepsilon - b_2^\varepsilon = \nabla^\varepsilon \cdot (d_0^\varepsilon \nabla^\varepsilon u_0^\varepsilon), & x \in \Gamma^\varepsilon \end{cases} \tag{34}$$

This system describes simultaneous diffusion in three intertwined continua where the thin one is fast and the other two are slow.

The strong macro problem is to find functions $u_0(t,x)$ and $u_1(t,x,y), u_2(t,x,y)$ such that

$$\begin{cases} C\partial_t u_0 + B = \nabla\cdot(D\nabla u_0), & x\in\Omega \\ B = -\sum_{i=1,2}\int_\Gamma b_i \; d\Gamma(y), & x\in\Omega \\ c_i\partial_t u_i = \nabla_y\cdot(d_i\nabla_y u_i), & x\in\Omega,\ y\in Y_i,\ i=1,2 \\ -\vec{\nu}\cdot d_1\nabla_y u_1 = b_1, & x\in\Omega,\ y\in\Gamma \\ \vec{\nu}\cdot d_2\nabla_y u_2 = b_2, & x\in\Omega,\ y\in\Gamma \end{cases} \tag{35}$$

Here

$$C = \int_\Gamma c_0 \; d\Gamma(y),$$

and the tensor D has the coefficients

$$D_{kl} = \int_\Gamma (\nabla_y^\Gamma \omega_k + \vec{e}_k)\cdot\vec{e}_l \; d\Gamma(y)$$

where ω_k is the periodic solution of the cell problem

$$\nabla_y^\Gamma\cdot\left(d_0(\nabla_y^\Gamma\omega_k + \vec{e}_k)\right) = 0,\ y\in\Gamma.$$

Special Case: $c_1 = d_1 = 0$. Then the macro model is

$$\begin{cases} C\partial_t u_0 + B = \nabla\cdot(D\nabla u_0), & x\in\Omega \\ B = -\int_\Gamma b_2 \; d\Gamma(y), & x\in\Omega \\ c_2\partial_t u_2 = \nabla_y\cdot(d_2\nabla_y u_2), & x\in\Omega,\ y\in Y_2 \\ \vec{\nu}\cdot d_2\nabla_y u_2 = b_2, & x\in\Omega,\ y\in\Gamma \end{cases}$$

This is the same model as in the special case of sections 7.

11 Two-Scale Convergence

Definition 1 *We say that a sequence of functions* $u^\varepsilon \in L^2(\Omega)$ two-scale converges *to a function* $u\in L^2(\Omega; L^2_\#(Y))$ *iff we have*

$$\int_\Omega u^\varepsilon(x)\varphi(x,\frac{x}{\varepsilon})\; dx \to \int_\Omega\int_Y u(x,y)\varphi(x,y)\; dydx$$

for any test-function $\varphi\in C_0^\infty(\Omega; C_\#^\infty(Y))$. *We use the symbolic notation* $u^\varepsilon \Rightarrow u$.

The following lemma is very useful; proofs for the time-independent case can be found in Allaire ([2], 1992, Theorem 1.2 and Proposition 1.14).

Lemma 1 *(a) For any bounded sequence $u^\varepsilon \in L^2(\Omega)$ there exists a function $u \in L^2(\Omega; L^2_\#(Y))$ such that for a subsequence $u^\varepsilon \Rightarrow u$.*

(b) Let $u^\varepsilon \in H^1(\Omega)$ be a bounded sequence which converges weakly to a limit $u \in H^1(\Omega)$. Then u^ε two-scale converges to u, and there exists a function $v \in L^2(\Omega; H^1_\#(Y)/\mathbf{R})$ such that for a subsequence $\nabla u^\varepsilon \Rightarrow \nabla_x u + \nabla_y v$.

(c) Let u^ε and $\varepsilon\nabla u^\varepsilon$ be bounded sequences in $L^2(\Omega)$. Then there exists a function $u \in L^2(\Omega; H^1_\#(Y))$ such that for a subsequence $u^\varepsilon \Rightarrow u$, and $\varepsilon\nabla u^\varepsilon \Rightarrow \nabla_y u$.

We generalize this notion by introducing time and allowing subsets $Y_i \subset Y$ or surfaces $\Gamma \subset Y$.

Definition 2 *(a) We say that a sequence of functions $u^\varepsilon \in L^2(0,T; L^2(\Omega_i^\varepsilon))$* two-scale converges *to a function $u \in L^2([0,T] \times \Omega; L^2(Y_i))$ iff we have*

$$\int_0^T \int_{\Omega_i^\varepsilon} u^\varepsilon(t,x)\varphi(t,x,\frac{x}{\varepsilon})\; dxdt \to \int_0^T \int_\Omega \int_{Y_i} u(x,y)\varphi(t,x,y)\; dydxdt$$

for any test-function $\varphi \in C_0^\infty([0,T] \times \Omega; C_\#^\infty(Y))$.

(b) We say that a sequence of functions $u^\varepsilon \in L^2(0,T; L^2(\Gamma^\varepsilon))$ two-scale converges *to a function $u \in L^2([0,T] \times \Omega; L^2(\Gamma))$ iff we have*

$$\int_0^T \int_{\Gamma^\varepsilon} \varepsilon u^\varepsilon(t,x)\varphi(t,x,\frac{x}{\varepsilon})\; d\Gamma^\varepsilon(x)dt \to \int_0^T \int_\Omega \int_\Gamma u(t,x,y)\varphi(t,x,y)\; d\Gamma(y)dxdt$$

for any test-function $\varphi \in C_0^\infty([0,T] \times \Omega; C_\#^\infty(Y))$.

Similar results as lemma 1 hold in these situations. It is clear that the trace conditions for triples of functions, i.e., the conditions for admissibility, carry over from $(u_0^\varepsilon, u_1^\varepsilon, u_2^\varepsilon)$ to (u_0, u_1, u_2).

12 Conclusions

The advantage of the approach presented in this paper is that not only a common framework is given for deriving a variety of different macro models; but it is also made clear which assumptions are being made on the micro scale in order to obtain each individual macro model. These assumptions are twofold: the type of macro model depends on the particular scaling and on the transmission conditions, i.e., the trace conditions. The scaling by the small parameter ε indicates the order of magnitude of the diffusivities in the different subdomains. On the other hand, the trace conditions specify wether or not jumps of the variables (concentrations, pressures, temperatures, etc.) are allowed between the different subdomains.

Of course, the final answer to the question which model is the "best" can only be given by testing it with adequate experiments. But in any case, it is helpful to know the assumptions on which the macro models are based.

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