APPROXIMATE NEWTON METHODS AND HOMOTOPY FOR STATIONARY OPERATOR EQUATIONS

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APPROXIMATE NEWTON METHODS AND HOMOTOPY FOR STATIONARY OPERATOR EQUATIONS

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Abstract

A quadratically convergent algorithm, based upon a Newton-type iteration, is defined to approximate roots of opeartor equations in Banach spaces. Fréchet derivative operator invertibility is not required; approximate right inverses are used in a neighborhood of the root. This result, which requires an initially small residual, is sufficiently robust to yield existence; it may be viewed as a generalized version of the Kantorovich theorem. A second algorithm, based on continuation via single, Euler-predictor/Newton-corrector iterates, is also presented. It has the merit of controlling the residual until the homotopy terminates, at which point the first algorithm applies. This method is capable of yielding existence of a solution curve as well. An application is given for operators described by compact perturbations of the identity.

1. Introduction

In this paper we derive approximation and existence theorems related to the roots of operator equations in Banach spaces, and to solution curves associated with operator equations depending on a parameter. In Section 2 we make use of approximate right inverses to define inexact or approximate Newton iterations. These are quadratically convergent, in a sense to be made precise below. In order to guarantee the sufficiently small residual required to begin the iteration, we present a continuation method in Section 3. This is based recursively upon an explicit Euler-predictor iteration, followed by a single approximate Newton-corrector iteration. The residual is controlled, rather than reduced, during the homotopy. We presume that the residual can be made small at the commencement of the process; at termination, the residual is sufficiently small to permit the approximate Newton iteration to proceed.

These results were motivated by an attempt to understand and define linearization procedures for systems of partial differential equations. For such systems, such as the system modeling the flow of electrons and holes in a semiconductor (cf. Bank, Rose and Fichtner [3] and Jerome [9]), one has a block structure with diagonal of uniformly elliptic type; under simplifying assumptions, this holds for the linearized model as well. One would naturally seek to construct a homotopy, leading from the diagonal structure to the full perturbation. In this paper, we consider only those applications in which the corresponding operator equation can be written as a compact perturbation of the identity. The approximate right inverses are defined in terms of "de facto" spectral expansions, an admittedly ineffective computational procedure. Obviously preferred is the replacement of such spectral methods by equivalent, and computationally efficient, procedures. Conjugate gradient and multigrid methods suggest themselves for future study.

The study of Newton-type iteration schemes in Banach spaces was systematically carried out by Kantorovich in his fundamental paper [11] (see also Kantorovich and Akilov [12]). Refinements of this classical result, and variations in the arguments leading to the result, have been extensively investigated. Among the contributors have been Dennis [7], Gragg and Tapia [8], Ortega [14], Ortega and Rheinboldt [15], Ostrowski [16], Rall and Tapia [17], Rheinboldt [18], and Tapia [20]. The present paper is not strictly in the vein of these previous studies, though the results of Section 2, in particular, depend upon the recurrence relations of Kantorovich. Rather, this approach was motivated by an attempt to understand the success of the Nash-Moser implicit function theorem (cf. Nirenberg [13]), and the approximate Newton method upon which it is based. For those familiar with this theorem, let us make the disclaimer right at the outset, that we do not exploit the complete generality of that setting; in particular, the "loss of derivative" phenomenon is not included. The basic idea of using approximate right inverses of the Fréchet derivative, whose approximation properties are governed by the residual, is taken from the Nash-Moser theory, though the quadratic estimates derived here are our own. Other investigators, particularly Dennis [6], have considered approximate right inverses which, in the terminology of Tapia [19], are "almost right inverses." This type of hypothesis is actually weaker near the root than our own; however, rather than the R-linear convergence derived in [6], in terms of a geometric sequence ρ^{-n}, we obtain a quadratic convergence in terms of a sequence ρ^{-2^n} (cf. (2.18)).

The existence of an "almost right inverse" or, in our own work, a sufficiently small residual, implies surjectivity of the Fréchet derivative maps near the root. However, what distinguishes the procedure of this paper, that of [6], that of Dembo, Eisenstat, and Steihaug [5], and that of Bank and Rose [2], from some of the Kantorovich type studies cited above, is that the iteration is based upon approximate, rather than exact, inverses, whether these be one-sided or two-sided. As Tapia [19] has noted, however, significant parts of the Newton-Kantorovich theorem remain valid with only left or right inverse assumptions, and these can be directed merely toward the inverse associated with the starting iterate.

The use of uniformly bounded, approximate inverses appears in the work of Bank and Rose [2], where strict residual reduction is achieved at each outer iteration step. These authors require invertibility of the Fréchet derivative at the root, in the sense of an exact inverse, however, in order to guarantee both the beginning of the iterative process, and quadratic convergence. However, the damping of the iterations by Bank and Rose may be thought of broadly as local reduction of the approximation error for the approximate right inverse. In fact, the constants $\{\alpha_k\}$ of those authors and the constants $\{\eta_k\}$ in [5] provide lower bounds for the term $M\|z\|_Z$ in inequality (2.2a), when z is the residual $F(v)$, and v is the computed iterate u_k. Inequality (2.2a) is one of our fundamental hypotheses governing the approximate right inverse. The condition of Bank and Rose, and the comparable hypothesis of Dembo, Eisenstat and Steilhaug, that $\alpha_0 < 1$ to begin the iteration, is comparable to the residual condition $\|F(u_0)\|_Z \leq \rho^{-1}$ of Lemma 2.2; the latter guarantees that $M\|F(u_0)\|_Z < 1$ in (2.2a). An interesting observation is that the quadratic convergence obtained in [2] and [5], under the assumption of a two-sided inverse at the root, follows in our work without any left inverse assumptions.

The continuation method discussed herein is not an arc length continuation method. In the language of Chan and Keller [4], the method can handle singular points of bifurcation type, but not limit points. The continuation method we present here stresses residual control rather than reduction; in particular, the right inverse required for the predictor steps satisfies relatively crude bounds, with much weaker hypotheses than required for the approximate right inverses used in defining the Newton iterations. Finally, as the reader will no doubt have noted by now, the point of this paper is not "still another Newton method" but rather the logical join of this method, with continuation, to produce a globally convergent method for root determination. Weaker versions of the results of Section 2 appeared in [10], where superlinear convergence was derived. Alternate versions are also possible in which approximation of the right inverse is governed not by the residual, but by the order of the current residual. This also leads to quadratic convergence. Moreover, compact perturbations of the identity require the approximate right inverse only on the closed range of the derivative maps. These extensions will be discussed in a subsequent paper.

2. An Approximate Newton Method for Operator Equations

We consider a mapping F, with domain containing a closed ball $B = B_r = \{v \in X : \|v - v_0\| \leq r\}$ in a Banach space X, and range in a Banach space Z. We assume the following properties concerning F.

(i) F is continuously Lipschitz differentiable on an open set U containing B:

$$(2.1) \qquad \|F'(v) - F'(w)\|_{X,Z} \leq 2M\|v - w\|_X \quad , \qquad v, w \in U \quad .$$

(ii) For every $v \in B$ there is a linear map $G(v) \in L(Z,X)$ such that, for all $z \in Z$,

$$(2.2a) \qquad \|[F'(v)G(v) - I]z\|_Z \leq M\|F(v)\|_Z \|z\|_Z \quad ,$$

and

$$(2.2b) \qquad \|G(v)z\|_X \leq M\|z\|_Z \quad .$$

We establish first the Kantorovich recurrence relations.

Lemma 2.1. Let $0 < h \leq 1/2$ and $\rho > 0$ be given. Define a sequence of nonnegative real numbers by

$$(2.3) \qquad t_0 := 0 \ , \quad t_1 := \rho^{-1} \ , \quad t_{k+1} - t_k := \frac{h}{2} \frac{(t_k - t_{k-1})^2}{(\rho^{-1} - ht_k)} \ , \qquad k \geq 1 \ .$$

Then $\{t_k\}$ is a strictly increasing sequence, converging to t^*, where

$$(2.4) \qquad t^* := (h\rho)^{-1}(1 - \sqrt{1 - 2h}) \quad .$$

The convergence of $\{t_k\}$ is described by the relations

$$(2.5a) \qquad t^* - t_k = \frac{\theta_k (h\rho)^{-1}(1 - \sqrt{1-2h})^{2^k}}{2^k} \ , \qquad k \geq 0 \ ,$$

where $0 < \theta_{k+1} \leqslant \theta_k \leqslant 1$ and $\{\theta_k\}$ are given recursively by

(2.5b) $$\theta_0 := 1 , \qquad \theta_{k+1} := \frac{\theta_k^2}{2^k \sqrt{1-2h} + \theta_k (1 - \sqrt{1-2h})^{2^k}} , \qquad k \geqslant 1 .$$

Proof: We sketch the proof. It is a slight variant of that in Rall and Tapia [17]. First we observe that an alternate characterization of $\{t_k\}$ is given by $t_0 = 0$ and

(2.6a) $$t_{k+1} = t_k - \frac{p(t_k)}{p'(t_k)} , \qquad k = 0, 1, \ldots ,$$

where

(2.6b) $$p(t) = \frac{1}{2} h\rho t^2 - t + \rho^{-1} .$$

Relations (2.4) and (2.6) can be used to derive the relation

(2.7) $$t^* - t_k = \frac{h}{2} \frac{(t^* - t_{k-1})^2}{(\rho^{-1} - ht_{k-1})} , \qquad k \geqslant 1 .$$

In turn, relations (2.6) and (2.7) can be used, inductively, to show that $\{t_k\}$ is strictly increasing and bounded above by t^*; here we use the fact that $t^* \leqslant (h\rho)^{-1}$. Now define $\{\theta_k\}$ implicitly by

$$t^* - t_k = \frac{\theta_k (h\rho)^{-1} (1 - \sqrt{1-2h})^{2^k}}{2^k} , \qquad k \geqslant 0 ,$$

i.e., by (2.5a). Equating (2.7) with the latter relation gives (2.5b). To see that $\theta_{k+1} \leqslant \theta_k$, assume that $\theta_k \leqslant 1$ and use the fact that the function

$$g_k(t) = 2^k t + \theta_k (1-t)^{2^k} , \qquad 0 \leqslant t \leqslant 1 ,$$

satisfies $g_k(0) = \theta_k$ and $g_k'(t) \geqslant 0$, $t \geqslant 0$, and then compare with the denominator of (2.5b). The convergence of $\{t_k\}$ to t^* is immediate from (2.5a). ∎

Remark 2.1. An easy calculation, based upon (2.4), shows that, if $h \leqslant 1/2$, then $t^* \leqslant 2\rho^{-1}$.

The following lemma isolates the fundamental inequalities required of an approximate Newton iteration procedure in order to achieve what we shall characterize as quadratic convergence. Although F and G retain their roles via (2.9) in this lemma, (2.1) and (2.2) are not used.

Lemma 2.2. Suppose $u_0 \in B_{\alpha r}$, $0 \leqslant \alpha < 1$, with

$$\|F(u_0)\|_Z \leqslant \rho^{-1} \tag{2.8}$$

for some $\rho > 0$. Suppose also that the iterates $\{u_k\}$ are formally defined by the procedure

$$u_k - u_{k-1} = -G(u_{k-1})F(u_{k-1}), \quad k \geqslant 1. \tag{2.9}$$

The procedure is consistent, i.e., $\{u_k\} \subset B_r$, if the following inequalities are satisfied:

$$\|u_k - u_{k-1}\|_X \leqslant c\|F(u_{k-1})\|_Z, \quad k \geqslant 1, \tag{2.10a}$$

$$\|F(u_k)\|_Z \leqslant \frac{h\rho}{2}\|F(u_{k-1})\|_Z^2, \quad k \geqslant 1, \tag{2.10b}$$

for some positive constants $c \leqslant (1-\alpha)r/t^*$, and $h \leqslant 1/2$. The bootstrapping ensures $u_\ell \in B_r$ if (2.10a) holds for $k \leqslant \ell$, and (2.10b) for $k < \ell$, each $\ell \geqslant 1$. The residual sequence $\{F(u_k)\}$ satisfies

$$\|F(u_{k-1})\|_Z \leqslant \left(\prod_{j=0}^{k-1} \tau_j^{2^{k-j-1}}\right)(t_k - t_{k-1}), \quad k \geqslant 1, \tag{2.11a}$$

and the sequence $\{u_k\}$ is a Cauchy sequence satisfying the estimate

$$\|u_k - u_{k-1}\|_X \leqslant c\left(\prod_{j=0}^{k-1} \tau_j^{2^{k-j-1}}\right)(t_k - t_{k-1}), \quad k \geqslant 1, \tag{2.11b}$$

Here $\{t_k\}$ are given by (2.3), and $\{\tau_k\}$ are decreasing positive constants, not exceeding one, given by

$$(2.12) \qquad \tau_k := \sqrt{1-2h} + \frac{\theta_k(1-\sqrt{1-2h})^{2^k}}{2^k} \quad , \quad k \geqslant 0 \; ,$$

where θ_k is defined in (2.5b). The limit u of $\{u_k\}$ is a root of F, $F(u) = 0$, if F is continuous. The quadratic convergence estimate,

$$(2.13) \qquad \|u - u_k\|_X \leqslant c\Big(\prod_{j=1}^{k} \tau_j^{2^{k-j}}\Big)(t^* - t_k) \; ,$$

holds, where $t^* - t_k$ is given explicitly in (2.5a).

Proof: Inequality (2.11a) holds for $k = 1$ via (2.8); (2.11b) then follows from (2.10a) in this case. Note that $\tau_0 = 1$ and $t_1 - t_0 = \rho^{-1}$. Assume, inductively, that $u_0, u_1, \dots, u_{k-1}$ are in B_r and that (2.11a,b) hold (for k). Then, by (2.11b), we have

$$\|u_k - u_0\|_X \leqslant \sum_{j=1}^{k} \|u_j - u_{j-1}\|_X \leqslant c \sum_{j=1}^{k} (t_j - t_{j-1}) = ct_k \leqslant ct^* \; .$$

It follows that

$$\|u_k - v_0\|_X \leqslant \|u_k - u_0\|_X + \|u_0 - v_0\|_X \leqslant (1-\alpha)r + \alpha r = r \; ,$$

so that $u_k \in B_r$. Moreover, by (2.10b) and (2.11a) we have

$$(2.14a) \qquad \|F(u_k)\|_Z \leqslant \Big(\prod_{j=0}^{k} \tau_j^{2^{k-j}}\Big) \frac{h(t_k - t_{k-1})^2}{2(\rho^{-1} - ht_k)} \; ,$$

provided it can be shown that

$$(2.14b) \qquad \tau_k = (\rho^{-1} - ht_k)/\rho^{-1} \; .$$

The verification that (2.12) and (2.14b) agree proceeds from the identity

$$\rho\tau_k = (\rho^{-1} - ht^*) + h(t^* - t_k) \quad ,$$

established by conjunction of (2.4) and (2.5a). The inductive step in (2.11a) for $k \leftarrow k+1$ is completed by applying (2.3) to (2.14a). Inequality (2.11b) for $k+1$ now follows by applying (2.10a) and (2.11a).

To see that $\{u_k\}$ is Cauchy, write

$$\begin{aligned}
\|u_{k+m} - u_k\|_X &\leqslant \sum_{\ell=k}^{k+m-1} \|u_{\ell+1} - u_\ell\|_X \\
&\leqslant \sum_{\ell=k}^{k+m-1} \gamma_\ell (t_{\ell+1} - t_\ell) \\
&\leqslant \gamma_k \sum_{\ell=k}^{k+m-1} (t_{\ell+1} - t_\ell) \\
&= \gamma_k (t_{k+m} - t_k) \\
&\leqslant \gamma_k (t^* - t_k)
\end{aligned}$$

for $k \geqslant 0$ and $m > k$. Here we have written

$$\text{(2.15)} \qquad \gamma_\ell = c \prod_{j=0}^{\ell} \tau_j^{2^{\ell-j}} \quad ,$$

and have used the decrease of $\{\gamma_\ell\}$ and the monotone convergence of $\{t_k\}$. Once the limit u has been selected, let $m \to \infty$ to obtain (2.13). $F(u) = 0$ follows from (2.11a) and the continuity of F. ∎

We are now prepared to state and prove the approximate Newton iteration theorem.

Theorem 2.3. Assume hypotheses (2.1) and (2.2), and let $u_0 \in B_{\alpha r}$, $0 \leqslant \alpha < 1$, such that u_0 satisfies (2.8). Define

$$h := 2(M+M^3)\rho^{-1} \quad , \tag{2.16}$$

and suppose that

$$h \leqslant 1/2 \ , \quad Mt^* = \frac{(1-\sqrt{1-2h})}{2(1+M^2)} \leqslant (1-\alpha)r \ , \tag{2.17}$$

where t^* is defined by (2.4). Then the definition (2.9) for the approximate Newton iterates is consistent, and $\{u_k\} \subset B_r$; in fact, $\|u_k - u_0\|_X \leqslant Mt^*$. Specifically, inequalities (2.10a, b) hold with $c = M$. In particular, $\{u_k\}$ is quadratically convergent to a root u of F, satisfying the estimate

$$\|u - u_k\|_X \leqslant \frac{\theta_k}{2(1+M^2)} \left(\prod_{j=0}^{k} \tau_j^{2^{k-j}} \right) \frac{(1-\sqrt{1-2h})^{2^k}}{2^k} \ . \tag{2.18}$$

Here $\{\theta_k\}$ and $\{\tau_k\}$ are decreasing sequences, bounded by one, and given explicitly by (2.5b) and (2.12), respectively.

Proof: According to Lemma 2.2, it suffices to verify inequality (2.10a) for $k = \ell$, followed by (2.10b) for $k = \ell$, as part of an inductive procedure for $\ell = 1, 2, \ldots$, which assumes $u_0, u_1, \ldots, u_{\ell-1}$ are in B_r. Suppose, then, that (2.10a, b) hold for $1 \leqslant k < \ell$ with $c = M$. Then the Newton iteration

$$u_\ell - u_{\ell-1} = -G(u_{\ell-1})F(u_{\ell-1}) \ ,$$

is well-defined and, by (2.2b),

$$\|u_\ell - u_{\ell-1}\|_X \leqslant M\|F(u_{\ell-1})\|_X \ ,$$

so that (2.10a) holds for $k = \ell$, with $c = M$. By Lemma 2.2, $u_\ell \in B_r$. The verification of (2.10b) for $k = \ell$ proceeds from the representation

(2.19a) $$F(u_\ell) = -\left[F'(u_{\ell-1})G(u_{\ell-1}) - I\right]F(u_{\ell-1}) + R(u_{\ell-1}, u_\ell)$$

where

(2.19b) $$R(u_{\ell-1}, u_\ell) = F(u_\ell) - F(u_{\ell-1}) - F'(u_{\ell-1})(u_\ell - u_{\ell-1}) .$$

The first term in (2.19a) may be dominated, via (2.2a), by $M\|F(u_{\ell-1})\|_Z^2$ and the second term by

$$\|R(u_{\ell-1}, u_\ell)\|_Z = \left\|\int_0^1 \left[F'\left(u_{\ell-1} + s(u_\ell - u_{\ell-1})\right) - F'(u_{\ell-1})\right](u_\ell - u_{\ell-1})ds\right\|_Z$$

$$\leqslant M\|u_\ell - u_{\ell-1}\|_X^2$$

upon use of a standard Taylor expansion (cf. Wouk [22]) and (2.1). From these remarks we have the estimate

(2.20) $$\|F(u_\ell)\|_Z \leqslant M\left(\|F(u_{\ell-1})\|_Z^2 + \|u_\ell - u_{\ell-1}\|_X^2\right) .$$

An application to (2.20) of (2.10a), with $k = \ell$ and $c = M$, yields

$$\|F(u_\ell)\|_Z \leqslant (M + M^3)\|F(u_{\ell-1})\|_Z^2 = (h\rho/2)\|F(u_{\ell-1})\|_Z^2 ,$$

which is (2.10b) for $k = \ell$. This completes the induction. Inequality (2.18) follows from (2.5a) and (2.13). ∎

<u>Remark 2.2</u>. The limitation of the results presented thus far is the necessity of a condition such as (2.8) on u_0. In the next section, we present a method of continuation type to circumvent this difficulty. The mapping F is generalized to depend upon a parameter λ.

3. An Euler-Newton Continuation Procedure for Approximation

We shall show how the usual Euler-Newton predictor-corrector method can be modified in terms of an approximate such method. We conceive of moving from an approximate solution (u_0,λ_0) to an approximate solution (u_1,λ_1), i.e., $F(u_0,\lambda_0) \approx 0$ and $F(u_1,\lambda_1) \approx 0$, with the understanding that u_0 is "easy" to compute. Thus, we answer the question raised at the end of the preceding section by beginning with a calculably small residual, and maintaining it by the predictor-corrector continuation procedure until λ_1 is reached. The approximate Newton corrector iterations are defined as in the previous section (one per λ-mesh point), and the predictor iterations are defined by an approximate Euler method. In this section, we assume the existence of the solution curve.

Let $B = B_r$ denote a closed ball in a Banach space as before: $B := \{v \in X : \|v - v_0\|_X \leqslant r\}$; and let $\lambda_0 < \lambda_1$ be real numbers. We consider a mapping F with (open) domain W containing $B \times [\lambda_0,\lambda_1]$. We assume the following.

(i') F is smooth, i.e., for each $v, w \in B$ and $\lambda, \mu \in [\lambda_0,\lambda_1]$, F has the expansion

$$F(v,\lambda) = F(w,\mu) + F'_w(w,\mu)(v-w) + F'_\mu(w,\mu)(\mu-\lambda) + R(v,w;\lambda,\mu) \quad , \tag{3.1a}$$

where the remainder R satisfies

$$\|R(v,w;\lambda,\mu)\| \leqslant M\left[\|v-w\|_X^2 + (\lambda-\mu)^2\right] \quad , \tag{3.1b}$$

for some positive constant M, not depending on v, w, λ, μ. F, F'_w, and F'_μ are assumed continuous on $B \times [\lambda_0,\lambda_1]$.

(ii') We assume that a partition

$$\lambda_0 < \lambda_{1/N} < \dots < \lambda_{(N-1)/N} < \lambda_1 \ , \qquad N \geqslant 1 \ , \tag{3.2}$$

of $[\lambda_0,\lambda_1]$ is specified. The restrictions $F(\cdot\, , \lambda_{i/N})$, and their derivatives $F'_u(\cdot\, , \lambda_{i/N})$, $i = 0, 1, \dots, N$, are assumed to satisfy (2.1) (compatible with (3.1a) and (2.2).

(iii') For each $(v,\lambda) \in B \times [\lambda_0, \lambda_1]$ there is a linear map $H(v,\lambda) \in L(Z,X)$ such that, for all $z \in Z$,

$$\|[F_v'(v,\lambda)H(v,\lambda) - I]z\|_Z \leqslant M\|z\|_Z \tag{3.3a}$$

and

$$\|H(v,\lambda)z\|_X \leqslant M\|z\|_Z \quad . \tag{3.3b}$$

(iv') There is a solution curve. More precisely, for each $i = 0, 1, \ldots, N$, there is an element $u_i \in B_{r/4}$ satisfying

$$F(u_i, \lambda_{i/N}) = 0 \quad . \tag{3.4}$$

The following result describes the continuation method.

Theorem 3.1. Suppose that hypotheses (i'), (ii'), (iii'), and (iv') are satisfied. Let t^* satisfy

$$Mt^* \leqslant r/4 \ , \qquad t^* \leqslant r \quad . \tag{3.5}$$

We assume, also, that $h \leqslant 1/2$, as defined by (2.16). Finally, suppose the partition (3.2) satisfies the restriction

$$\Delta\lambda = (\lambda_1 - \lambda_0)/N \leqslant c = \left[4M\rho\left(1 + \sup_{v,\lambda}\|F_\lambda'(v,\lambda)\|_Z\right)\right]^{-1} \quad . \tag{3.6}$$

Define sequences $\{v_i\}_1^N$ and $\{w_i\}_0^{N-1}$ as follows. Let w_0 be an element in $B_{r/2}$ with $\|F(w_0, \lambda_0)\|_Z \leqslant (2\rho)^{-1}$. If w_{i-1} has been defined, define the predictor v_i via

$$(v_i - w_{i-1})/\Delta\lambda = -H(w_{i-1}, \lambda_{(i-1)/N})F_\lambda'(w_{i-1}, \lambda_{(i-1)/N}) \tag{3.7}$$

and the corrector, w_i, in analogy with (2.9), via

$$w_i - v_i = -G(v_i, \lambda_{i/N})F(v_i, \lambda_{i/N}) \quad . \tag{3.8}$$

Then, $\{v_i\}_1^N$ and $\{w_i\}_0^{N-1}$ satisfy

(3.9a) $$\|v_i - v_0\| \leqslant 3r/4 \ , \qquad \|w_i - v_0\| \leqslant r/2 \ ,$$

and

(3.9b) $$\|F(v_i, \lambda_{i/N})\|_Z \leqslant \rho^{-1} \ , \qquad \|F(w_i, \lambda_{i/N})\|_Z \leqslant (2\rho)^{-1} \ .$$

Moreover, the element v_N satisfies the hypotheses required of the starting guess in Theorem 2.3, and the corresponding sequence of iterates converges to u_N, where $F(u_N, \lambda_1) = 0$.

Proof: The proof proceeds by induction. Note that w_0 has been selected so that (3.9) holds. Suppose, then, that $\{v_j\}_1^{i-1}$ and $\{w_j\}_0^i$ have been defined for $i \geqslant 0$ and satisfy (3.9). For v_i defined by (3.7),

$$\begin{aligned}
\|v_i - v_0\|_X &\leqslant \|v_i - w_{i-1}\|_X + \|w_{i-1} - v_0\|_X \\
&\leqslant |\Delta\lambda| \|H(w_{i-1}, \lambda_{(i-1)/N})\|_{Z,X} \|F'_\lambda(w_{i-1}, \lambda_{(i-1)/N})\|_Z + r/2 \\
&\leqslant \rho^{-1}/4 + r/2 \qquad (3.10) \\
&\leqslant t^*/4 + r/2 \\
&\leqslant 3r/4 \ ,
\end{aligned}$$

where we have used the induction hypothesis, (3.3b), (3.6), and (3.5). In particular, (3.9a) holds for v_i. In order to verify (3.9b) for v_i, we use (3.1a) to write

(3.11) $$\begin{aligned}
F(v_i, \lambda_{i/N}) = -\Delta\lambda \Big[F'_v(w_{i-1}, \lambda_{(i-1)/N}) H(w_{i-1}, \lambda_{(i-1)/N}) - I\Big] \circ \\
\circ F'_\lambda(w_{i-1}, \lambda_{(i-1)/N}) + R(v_i, w_{i-1}; \lambda_{i/N}, \lambda_{(i-1)/N}) \\
+ F(w_{i-1}, \lambda_{(i-1)/N}) \ ,
\end{aligned}$$

so that, by (3.1b), (3.3), (3.6), (3.7), and the induction hypothesis

(3.12) $$\|F(v_i, \lambda_{i/N})\| \leqslant (4\rho)^{-1} + (4\rho)^{-1} + (2\rho)^{-1} = \rho^{-1} \ .$$

Here we have also used $M\rho^{-1} \leqslant 1$, guaranteed by $h \leqslant 1/2$. Inequality (3.12) establishes (3.9b) for v_i.

The residual condition on the starting guess v_i is satisfied by what has just been proved. Moreover, we have also shown that $v_i \in B_{3r/4}$. It follows that the theory of the previous section applies to the corrector iterate w_i. In particular, we have, by (3.5) and (2.13), with $c = M$,

$$\|w_i - u_i\|_X \leqslant Mt^* \leqslant r/4 \, , \tag{3.13}$$

so that $w_i \in B_{r/2}$ when this inequality is compared to (iv'). Finally, by (2.11a) we have

$$\|F(w_i, \lambda_{i/N})\|_Z \leqslant (2\rho)^{-1} \, , \tag{3.14}$$

since $t_2 - t_1 \leqslant (2\rho)^{-1}$. The induction is completed by (3.13) and (3.14). The final statement is immediate. ∎

4. The Existence of a Solution Curve

In Section 2, Theorem 2.3 expressed approximation results which were sufficiently robust to yield existence of roots. In this section, we shall deduce analogous results relative to the existence of a solution curve depending on the parameter λ. We proceed immediately to the result.

Theorem 4.1. Suppose that hypotheses (i) and (iii) of Section 3 hold together with the following strengthening of (ii'):

(ii'') The restriction maps $F(\cdot,\lambda)$ and their (partial) derivatives $F'_u(\cdot,\lambda)$ are assumed to satisfy (2.1) and (2.2) for $\lambda_0 \leqslant \lambda \leqslant \lambda_1$.

Suppose also that $B = B_r$ is prescribed, with center $v_0 = u_0$, such that

(4.1a) $$F(\lambda_0, u_0) = 0 \ ,$$

and such that r is sufficiently large relative to M and $\lambda_1 - \lambda_0$:

(4.1b) $$r \geqslant (\lambda_1 - \lambda_0) M \left[1 + \sup_{v,\lambda} \left\| F'_\lambda(v,\lambda) \right\|_Z \right] \ .$$

Then there exists a solution curve $u = u(\lambda)$ such that

(4.2) $$u(\lambda) \in B \ , \quad F\big(u(\lambda), \lambda\big) = 0 \ , \quad \lambda_0 \leqslant \lambda \leqslant \lambda_1 \ .$$

Proof: Let N be the smallest positive integer satisfying

(4.3) $$N \geqslant \max\big(8r(M+M^3), 4, 16M\big) \ .$$

Now let ρ satisfy

(4.4) $$\rho^{-1} = 2r/N \ .$$

The conjunction of (4.4), (2.4), (2.16) and Remark 2.1 show that the choice of N in (4.3) leads to

(4.5) $$h \leqslant 1/2 \ , \quad t^* \leqslant r \ , \quad Mt^* \leqslant r/4 \ .$$

We shall describe a finite inductive process, terminating in at most N steps, in which the solution curve is defined on intervals $[\mu_n, \mu_{n+1}]$ by a predictor/corrector algorithm making reference to $u(\mu_n)$. Suppose, then, that $u(\lambda)$ has been defined for $\lambda \leqslant \mu_n$, with

$$F(\mu_n, u(\mu_n)) = 0 , \tag{4.6}$$

and that

$$\|u_0 - u(\lambda)\|_X \leqslant nr/N , \quad \lambda \leqslant \mu_n . \tag{4.7}$$

Set $\mu_{n+1} = \min(\lambda_1, \mu_n + \Delta\lambda)$, where

$$\Delta\lambda = \left[2M\left(1 + \sup_{v,\lambda} \|F_\lambda'(v,\lambda)\|_Z\right)\right]^{-1} \rho^{-1} \tag{4.8}$$

For each $\mu_n < \lambda \leqslant \mu_{n+1}$, define the predictor v_λ via

$$v_\lambda - u(\mu_n) = -(\lambda - \mu_n) H\big(u(\mu_n), \lambda\big) F_\lambda'\big(u(\mu_n), \mu_n\big) . \tag{4.9}$$

As in (3.10), we conclude that

$$\begin{aligned} \|v_\lambda - u_0\|_X &\leqslant \|v_\lambda - u(\mu_n)\|_X + \|u(\mu_n) - u_0\|_X \\ &\leqslant (2\rho)^{-1} + nr/N \\ &\leqslant (n+1)r/N \end{aligned} \tag{4.10}$$

where we have used (4.4), (4.7), (4.8), and (4.9). Analagously to (3.12), we argue that

$$\|F(v_\lambda, \lambda)\|_Z \leqslant (2\rho)^{-1} + (4\rho)^{-1} , \tag{4.11}$$

where we begin with a relation similar to (3.11), with $F\big(u(\mu_n), \mu_n\big) = 0$, and use this relation in conjunction with (4.8), (4.9), and (3.3). Now we may define $u(\lambda)$, as in Lemma 2.2, as the limit of a sequence of approximate Newton iterates with v_λ as starting guess. This makes use of the strengthened hypothesis (ii'') and the fact that h and t^* satisfy (2.17) with $\alpha = 3/4$ by virtue of (4.5). We shall now show that the sequence terminates in at most N steps. It suffices to show that $N\Delta\lambda \geqslant (\lambda_1 - \lambda_0)$. Using (4.4) as a starting point, we have

(4.12) $$\rho = c_1 N , \qquad c_1 = 1/(2r) .$$

By (4.8) and (4.12),

(4.13a) $$N\Delta\lambda = c_2 N\rho^{-1} = c_2 N(c_1 N)^{-1} = c_2 c_1^{-1}$$

where

(4.13b) $$c_2 = \left[2M\left(1 + \sup_{v,\lambda} \|F'_\lambda(v,\lambda)\|_Z\right)\right]^{-1} .$$

It is elementary to see that the inequality

(4.14) $$c_2 c_1^{-1} \geqslant (\lambda_1 - \lambda_0)$$

is equivalent to (4.1b). In particular, $N\Delta\lambda \geqslant (\lambda_1 - \lambda_0)$. ∎

5. Compact Perturbations of the Identity

In this section we consider mappings Φ, of Z into Z, which are of the form

(5.1) $$\Phi(z) = z + K(z)$$

where K is a compact, smooth mapping of a region of Z into Z. The precise hypotheses are described below. This situation is motivated by mappings F of the form

(5.2) $$F(v) = Lv + \psi(v) ,$$

where L is an isomorphism of X onto Z, ψ is a smooth map, acting boundedly, from X into Z, and the injection map of X into Z is compact. In this case, we may identify Φ with $L^{-1}F$. In the abstract, at least, are included certain nonlinear systems of elliptic partial differential equations.

We shall assume that there is a compact subset $\mathcal{V}$ of Z such that

(i) K is Lipschitz continuously differentiable on a ball of radius r containing $\mathcal{V}$;

(ii) $K'(v)$ is compact for each $v \in \mathcal{V}$.

Then (cf. Anselone [1, p. 112])

(5.3) $$\{K'(v) : v \in \mathcal{V}\} \text{ is collectively compact.}$$

However, (i), (ii) and (iv) to follow are precisely the hypotheses needed to assert that (cf. the argument of Taylor [21, pp. 280-1]) there exists a uniformly bounded right inverse $\Gamma(v)$ for $I+K'(v)$ from its closed range, i.e.,

(5.4) $$[I + K'(v)]\,\Gamma(v) = I , \quad \|\Gamma(v)\| \leq M , \quad v \in \mathcal{V} .$$

In the application to Φ, the set $\mathcal{V}$ would be selected to be the image of a bounded subset of Z under a fixed compact linear mapping Λ. $\mathcal{V}$ should contain all the iterates of the Newton Sequence.

Thus, we assume the following.

(iii) There is a neighborhood $\mathcal{N}$ of 0 such that, if $v \in \mathcal{V}$, and $\Phi(v) \in \mathcal{N}$, then $v - \Gamma(v)\Phi(v) \in \mathcal{V}$.

(iv) $I+K'(v)$ is surjective for $v \in \mathcal{V}$.

It is clear that hypotheses (i), (ii), (iii) and (iv) ensure that the Newton iteration sequence, based on exact inverses, may be defined as in Section 1. We could easily substitute finite rank approximations for $K'(v)$ in (5.4), thereby obtaining approximate right inverses. In the case that $K'(v)$ is symmetric, we can give explicit definitions in terms of orthogonal expansions for the exact and approximate right inverses, and for $\mathcal{V}$.

Finally, note that continuation may proceed, based upon the simple homotopy

$$\Phi(v,\lambda) = v + \lambda K(v) , \qquad 0 \leqslant \lambda \leqslant 1 . \tag{5.5}$$

In this case, $\Phi(0,0) = 0$. A slight strengthening of (iii) is required for the homotopy, also known in the numerical analysis literature as damped Newton iteration (cf. [2]). We shall elaborate on this summary section in a sequel to this paper. We shall also show how (iv) can be omitted. This is the interesting case, of course.

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