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On regularity of stationary Stokes and Navier-Stokes equations near boundary

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Abstract. We obtain local estimates of the steady-state Stokes system "without pressure" near boundary. As an application of the local estimates, we prove the partial regularity up to the boundary for the stationary Navier-Stokes equations in a smooth domain in five dimension.

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1. Introduction

The objective of this paper is to study the regularity of stationary Stokes and Navier-Stokes equations near boundary in a domain with smooth boundary. As one of our main results, we obtain local estimates "without pressure" near boundary for the Stokes system. More precisely, let $B_1^+ = \{x \in B_1 | x_n > 0\}$ be the unit half ball in $\mathbb{R}^n$ and let us consider a solution of the Stokes system

$$\left.\begin{array}{c} -\Delta u + \nabla p = f \\ \nabla \cdot u = 0 \end{array}\right\} \text{ in } B_1^+,$$

which vanishes on the flat part of the boundary, i.e. $u(x) = 0$ for $x_n = 0$. Without loss of generality, it is assumed that the mean value of a pressure p is zero in B_1^+. Then for every integer k with $-1 \leq k < \infty$, the following estimate is satisfied;

$$|u|_{W^{k+2,q}(B^+_{\frac{1}{2}})} + |p|_{W^{k+1,q}(B^+_{\frac{1}{2}})} \leq C(|u|_{L^1(B_1^+)} + |f|_{W^{k,q}(B_1^+)}). \tag{1}$$

where $C = C(k, q, n)$ and $1 < q < \infty$. The main point is that the pressure p does not appear in the right-hand side and, in addition, we do not require the control of u everywhere at ∂B_1^+. If we assume $u|_{\partial B_1^+} = 0$, estimates similar to (1) can be found in [13], but the local version described in this paper seems to be new. This result can be extended to general smooth domains.

As an application of (1), we give a proof of the partial regularity for the stationary Navier-Stokes equations up to the boundary in a smooth domain in

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five dimension. The five dimensional steady-state Navier-Stokes equations have been studied in a number of papers (see [9], [10], [11], [12], [28] and [29]). The problem is interesting because dimension five is the smallest dimension where steady-state Navier-Stokes equations is super-critical. Interior partial regularity for *suitable weak solutions* of five dimensional steady-state Navier-Stokes equations was proved in [28]. In [9], it was proved that the boundary value problem of the Navier-Stokes equations

$$\left.\begin{array}{r} -\Delta u + (u\cdot\nabla)u + \nabla p = f \\ \nabla\cdot u = 0 \end{array}\right\} \quad \text{in } \Omega\subset\mathbb{R}^5$$
$$u = 0 \qquad \text{on } \partial\Omega$$

has at least one solution which is regular in the interior provided that f is sufficiently regular. However, the boundary regularity for these solutions remains open. By inspecting the construction in [10] one can see that the constructed solutions are *suitable weak solutions* up to the boundary as defined below (see (28)). Therefore we can conclude that possible singularities of these solutions must lie in a closed set $S\subset\partial\Omega$ of one dimensional Hausdorff measure zero. The partial regularity at the boundary for three dimensional time dependent Navier-Stokes equations was studied in [23]. Very recently, the interior partial regularity results in [3] were extended up to the boundary in [25]. This result improves [23] by showing that, under reasonable assumptions, *suitable weak solutions* are Hölder continuous up to the boundary away from a closed set S with $\mathcal{P}^1(S)=0$ where $\mathcal{P}^1$ is an one dimensional parabolic Hausdorff measure. However, optimal higher regularity of these solutions at $\partial\Omega\times(0,T)\setminus S$, even in spatial variables, is not known.

The plan of this paper is as follows:

In Section 2, notation and definitions are introduced. In addition, we recall some well-known facts needed for our proofs.

In Section 3, we prove the estimate (1) for the Stokes system near boundary.

In Section 4, we briefly review results on the interior partial regularity of *suitable weak solutions* of the Navier-Stokes equations in five dimension and prove that the result of interior case is extended up to the boundary in a smooth domain of five dimensional space.

2. Preliminaries

In this section, we introduce notation and definitions and also recall some well-known results used later. Let us begin with some definitions and notations.

• For $x\in\mathbb{R}^n$ and $r>0$ we denote by $B_{x,r}$ the open ball $\{y\in\mathbb{R}^n : |x-y|<r\}$ and if $x_n=0$, i.e. $x=(x',0)$, then we denote by $B^+_{x,r}$ the half-ball $\{y\in B_{x,r}|y_n>0\}$.

• Let $\Omega \subset \mathbb{R}^n$ be an open set. For $1 \le p \le \infty$, $W^{k,p}(\Omega)$ denote the usual Sobolev space, i.e. $W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), 0 \le |\alpha| \le k\}$. As usual, $W_0^{k,p}(\Omega)$ is defined the completion of $\mathcal{C}_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. We denote by $W^{-1,q}(\Omega)$ the dual space of $W_0^{1,p}(\Omega)$ where $p^{-1} + q^{-1} = 1$. We also denote by $W^{1-\frac{1}{p},p}(\partial\Omega)$ the trace space of $W^{1,p}(\Omega)$.
• For $\alpha \in (0,1)$, we denote by $\mathcal{C}^{k,\alpha}(\bar{\Omega})$ the space of functions whose derivatives up to the k order are Hölder continuous in $\bar{\Omega}$.
• For simplicity, $⨍_\Omega f(x)\,\mathrm{d}x$ indicates the average of a given function $f \in L^1(\Omega)$, namely, $⨍_\Omega f(x)\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega f(x)\,\mathrm{d}x$. If there is no confusion, we use $(f)_\Omega$ instead of $⨍_\Omega f(x)\,\mathrm{d}x$.
• $\Omega' \Subset \Omega$ means the closure of Ω' is compact and contained in Ω.
• The diameter of a measurable set Ω is denoted by $\delta(\Omega)$.
• The capital letter C is used to denote the generic constant, the value of which may change from line to line.

Next we review some well-known facts needed for our purpose. We start with the following Lemma showing a characterization of Hölder continuous functions.

Lemma 1. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$.*

1. *Suppose that $f \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$. Then, for any $x \in \bar{\Omega}$ and for any $\rho > 0$,*
$$⨍_{B_{x,\rho}\cap\Omega} |f - (f)_{x,\rho}|^p \le C\rho^{\alpha p}, \tag{2}$$
where $(f)_{x,\rho} = ⨍_{B_{x,\rho}\cap\Omega} f(y)\,\mathrm{d}y$.
2. *Conversely, assume that $p \ge 1$ and $f \in L^p(\Omega)$. If there exists $C > 0$ such that, for any $x \in \Omega$ and for any $\rho > 0$, (2) holds, then $f \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$.*

Proof. See Theorem 3.1 in [16, page 41]. □

Next, we recall the definition of Hausdorff dimension and elementary Lemma regarding the Hausdorff dimension of a set with some non-Lebesgue points.

Definition 1. *Let $0 < \alpha \le n$ and $\delta > 0$. The Hausdorff measure of a set $E \subset \mathbb{R}^n$ is*
$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0_+} \phi_\delta^\alpha(E)$$
and $\phi_\delta^\alpha(E)$ is defined as follows:
$$\phi_\delta^\alpha(E) = \inf\ \{\ \sum w(\alpha)2^{-\alpha}(\ \mathrm{diam}\ S_j)^\alpha :\ E \subset \cup_j S_j,\ \mathrm{diam}\ S_j < \delta\ \},$$
where $w(\alpha) = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2}+1)}$ denote the volume of unit ball in $\alpha-$dimensional space.

Lemma 2. *Suppose* $0 < \alpha < n$ *and* $f \in L^1(\Omega)$ *where* $\Omega \subset \mathbb{R}^n$. *Then* $\mathcal{H}^\alpha(E) = 0$ *where a set* E *is defined as follows:*

$$E = \{x \in \Omega : \limsup_{r \to 0} \frac{1}{r^\alpha} \int_{B_{x,r}} |f| > 0\}.$$

Proof. See Theorem 3.1 in [6, page 77]. □

We also recall a well-known fact verified easily by iterations.

Lemma 3. *Let* $f(t)$ *be a nonnegative bounded function defined in* $[\tau_0, \tau_1]$ *where* $\tau_0 > 0$. *Suppose that for* $\tau_0 \leq t < s \leq \tau_1$ *we have*

$$f(t) \leq [A(s-t)^{-\alpha} + B] + \theta f(s),$$

where A, B, α, *and* θ *are nonnegative constants with* $0 \leq \theta < 1$. *Then for all* $\tau_0 \leq t < s \leq \tau_1$ *we have*

$$f(t) \leq C[A(s-t)^{-\alpha} + B],$$

where C *is a constant depending on* α *and* θ.

Proof. See Lemma 3.1 in [15, page 161]. □

In fact, we will need a slightly more general version of Lemma 3.

Lemma 4. *Let* $f(t)$ *be a nonnegative bounded function defined in* $[\tau_0, \tau_1]$ *where* $\tau_0 \geq 0$ *and* m *be a positive integer. Suppose that for* $\tau_0 \leq t < s \leq \tau_1$, *we have*

$$f(t) \leq [\sum_{i=1}^m A_i(s-t)^{-\alpha_i} + B] + \theta f(s),$$

where A_i, B, α_i *and* θ *are nonnegative constants with* $0 \leq \theta < 1$. *Then for all* $\tau_0 \leq t < s \leq \tau_1$ *we have*

$$f(t) \leq C[\sum_{i=1}^m A_i(s-t)^{-\alpha_i} + B],$$

where C *is a constant depending on* α_i *and* θ.

Proof. This can be proved by modifying the proof of Lemma 3.1 in [15, page 161]. The modification is self-evident, and therefore we omit the details. □

We conclude this section by recalling existence results for divergence equation div $w = f$ in a bounded domain $\Omega \subset \mathbb{R}^n$ (see [2] and [13]).

Lemma 5. *Let $q \in (1,\infty)$. Suppose $\Omega \subset \mathbb{R}^n$ be a domain which contains $B_{x_0,R}$ and is star-shaped with respect to each point of $B_{x_0,R}$. Then for any $f \in L^q(\Omega)$ with $⨍_\Omega f = 0$, there exists a vector field $w \in W_0^{1,q}(\Omega)$ such that*

$$\nabla \cdot w = f \quad \text{in} \quad \Omega \tag{3}$$

and

$$\int_\Omega |\nabla w|^q \le C \int_\Omega |f|^q \tag{4}$$

with $C = C_1(n,q)(1+\frac{\delta(\Omega)}{R})^{n+1}$, where $C_1(n,q)$ depends on n and q, but not Ω.

Proof. See Lemma 3.1 in [13, page 121]. □

3. Estimates for the Stokes system

In this section, we will prove the estimate (1) for the Stokes system mentioned in the introduction. Let x be a boundary point of a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary. For convenience, we denote $\Omega_{x,r} = B_{x,r} \cap \Omega$ with $0 \le r \le r_0$. Here r_0 is the largest positive radius such that $\Omega_{x,r_0} \subset \Omega$ and $\frac{\delta(\Omega_{x,r})}{\delta(B^{x,r})} \le C$ for all $0 < r \le r_0$ where $B^{x,r}$ is the largest ball contained in $\Omega_{x,r}$ and $\delta(A)$ indicates the diameter of A. Since Ω is a Lipschitz domain, there exists a positive $r_0 > 0$ satisfying such properties at a given $x \in \partial\Omega$. If $r_0 = \infty$, then we take $r_0 = 1$ because our concern is local estimate near boundary.

Now we are ready to investigate the local estimate of the Stokes system near the boundary. Suppose $x \in \partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain and as mentioned above, r_0 be the largest positive radius depending on x. We consider the following local problem of the Stokes system:

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega_{x,r_0} \\ \nabla \cdot u = 0 & \text{in } \Omega_{x,r_0} \\ u = 0 & \text{on } B_{x,r_0} \cap \partial\Omega, \end{cases} \tag{5}$$

We emphasize that the homogeneous condition is only assigned on some part of the boundary of Ω_{x,r_0}, not all of $\partial\Omega_{x,r_0}$. Let u be a weak solution solving (5) in a weak sense, in other words, u is in $W^{1,2}(\Omega_{x,r_0})$ and vanishes on $B_{x,r_0} \cap \partial\Omega$ and satisfies

$$\int_{\Omega_{x,r_0}} \nabla u \nabla \xi = \int_{\Omega_{x,r_0}} f\xi \tag{6}$$

$$\text{for all } \xi \in \mathcal{C}_0^\infty(\Omega_{x,r_0}, \mathbb{R}^n) \text{ with } \nabla \cdot \xi = 0,$$

$$\int_{\Omega_{x,r_0}} u \cdot \nabla \zeta = 0 \qquad \text{for all } \zeta \in \mathcal{C}_0^\infty(\Omega_{x,r_0}). \tag{7}$$

Without loss of generality, we may assume $x = 0$ by translation and for simplicity we denote $\Omega_{0,r} = \Omega_r$ for any $0 \le r \le r_0$. Let r, s be any positive numbers with $0 \le r < s \le r_0$ and t is the midpoint of r and s, i.e. $t = \frac{r+s}{2}$. Suppose that η_1 and η_2 be standard cut off functions defined as follows:

$$\eta_1 = \begin{cases} 1 \text{ in } B_r \\ 0 \text{ outside } B_t \end{cases}, \qquad \eta_2 = \begin{cases} 1 \text{ in } B_t \\ 0 \text{ outside } B_s \end{cases}$$

such that η_1 and η_2 are supported in B_t and B_s, respectively and for a fixed constant C they satisfy

$$|\nabla \eta_1| \le \frac{C}{t-r} \le \frac{2C}{s-r}, \quad |\nabla \eta_2| \le \frac{C}{s-t} \le \frac{2C}{s-r}.$$

We first consider the case $f = 0$. We note first that in the weak formulation above, it is easily seen that p is in $L^2(\Omega_{r_0})$ by the variational formulation (see Lemma 1.1 in [13, page 186]). Moreover, we may uniquely take a pressure p satisfying $\int_{\Omega_{r_0}} p = 0$. Multiplying $u\eta_2^2$ to (5) and using the integration by parts, we have

$$\begin{aligned} \int_{\Omega_s} |\nabla u|^2 \eta_2^2 &\le C \int_{\Omega_s} |u|^2 |\nabla \eta_2|^2 + C \int_{\Omega_s} |p\eta_2| \cdot |u \nabla \eta_2| \\ &\le C \int_{\Omega_s} |u|^2 |\nabla \eta_2|^2 + C \int_{\Omega_s \setminus \Omega_t} |p\eta_2|^2 \end{aligned} \tag{8}$$

Using the energy estimate (8), we control the pressure in terms of u.

Lemma 6. *Let u be a weak solution of the Stokes system (5). Then for every r, s with $0 \le r < s \le r_0$, the following estimate holds:*

$$\int_{\Omega_r} |p|^2 \le \frac{C}{(s-r)^2} \int_{\Omega_s} |u|^2. \tag{9}$$

Proof. Let us first consider the divergence problem. According to Lemma 5, there exists $w \in W_0^{1,2}(\Omega_t)$ such that

$$\nabla \cdot w = p\eta_1^2 - (p\eta_1^2)_a \text{ in } \Omega_t, \quad w = 0 \text{ on } \partial \Omega_t, \tag{10}$$

where $(p\eta_1^2)_a = \fint_{\Omega_t} p(y)\eta_1^2(y)\, \mathrm{d}y$ and the following estimate holds

$$\int_{\Omega_t} |\nabla w|^2 \le C \int_{\Omega_t} |p\eta_1^2 - (p\eta_1^2)_a|^2, \tag{11}$$

where C is independent of radius of Ω_t because of Lemma 5 and the choice of r_0. Using the above estimate and $\int_{\Omega_{r_0}} p = 0$, we get

$$\begin{aligned} \int_{\Omega_{r_0}} |p|^2 \eta_1^2 &= \int_{\Omega_{r_0}} p(p\eta_1^2 - (p\eta_1^2)_a) = \int_{\Omega_{r_0}} p \nabla \cdot w \\ &= \int_{\Omega_t} -\Delta u w = \int_{\Omega_t} \nabla u \nabla w \end{aligned}$$

With the aid of the Young's inequality, the right side is estimated as follows:

$$\begin{aligned}\int_{\Omega_t} \nabla u \nabla w &\le (4\epsilon)^{-1}\int_{\Omega_t}|\nabla u|^2 + \epsilon\int_{\Omega_t}|\nabla w|^2\\ &\le (4\epsilon)^{-1}\int_{\Omega_t}|\nabla u|^2 + C\epsilon\int_{\Omega_t}|p\eta_1^2 - (p\eta_1^2)_a|^2\\ &\le (4\epsilon)^{-1}\int_{\Omega_t}|\nabla u|^2 + C\epsilon\int_{\Omega_{r_0}}|p\eta_1|^2.\end{aligned}$$

Choosing small ϵ with $C\epsilon < \frac{1}{2}$, we obtain

$$\int_{\Omega_r}|p|^2 \le C\int_{\Omega_t}|\nabla u|^2.$$

According to the energy estimate (8), we have

$$\begin{aligned}\int_{\Omega_r}|p|^2 &\le C\int_{\Omega_s}|u|^2|\nabla\eta_2|^2 + C\int_{\Omega_s\setminus\Omega_t}|p|^2\\ &\le \frac{C}{(s-r)^2}\int_{\Omega_s}|u|^2 + C\int_{\Omega_s\setminus\Omega_t}|p|^2\\ &\le \frac{C}{(s-r)^2}\int_{\Omega_s}|u|^2 + C\int_{\Omega_s\setminus\Omega_r}|p|^2.\end{aligned}$$

Using the hole filling technique, we obtain

$$\int_{\Omega_r}|p|^2 \le \frac{C}{(s-r)^2}\int_{\Omega_s}|u|^2 + \theta\int_{\Omega_s}|p|^2,$$

where $\theta = \frac{C}{C+1}$ which is independent of r, s. Since r, s are arbitrary numbers, due to the Lemma 3, the assertion (9) is completed. □

Since pressure is estimated in terms of velocity, we easily have the following Caccioppoli inequality.

Lemma 7. *Let u be a weak solution of the Stokes system (5). Then for every r, s with $0 \le r < s \le r_0$, the following estimate holds:*

$$\int_{\Omega_r}|\nabla u|^2 \le \frac{C}{(s-r)^2}\int_{\Omega_s}|u|^2.$$

Proof. It is easy consequence of the estimate (8) and (9). Indeed,

$$\begin{aligned}\int_{\Omega_r}|\nabla u|^2 &\le C\int_{\Omega_t}|u|^2|\nabla\eta_1|^2 + C\int_{\Omega_t\setminus\Omega_r}|p\eta_1|^2\\ &\le \frac{C}{(t-r)^2}\int_{\Omega_t}|u|^2 + C\int_{\Omega_t}|p|^2\\ &\le \frac{C}{(t-r)^2}\int_{\Omega_t}|u|^2 + \frac{C}{(s-t)^2}\int_{\Omega_s}|u|^2\\ &\le \frac{C}{(s-r)^2}\int_{\Omega_s}|u|^2.\end{aligned}$$

This completes the proof. □

Remark 1. A standard modification of the above proof gives the following estimate when $f \in W^{-1,2}(\Omega_{r_0})$ and $g \in W^{\frac{1}{2},2}(\Gamma)$ where $\Gamma = \partial\Omega \cap B_{r_0}$;

$$\begin{aligned}||\nabla u||_{L^2(\Omega_r)} + ||p||_{L^2(\Omega_r)} &\le \frac{C}{(s-r)^2}||u||_{L^2(\Omega_s)}\\ &\quad + C(||f||^2_{W^{-1,2}(\Omega_s)} + ||g||^2_{W^{\frac{1}{2},2}(\Gamma)})\end{aligned} \tag{12}$$

for any $0 \le r < s \le r_0$. In what follows we will not use this estimate for $g \ne 0$. The case $f \ne 0$ and $g = 0$ can be proved by an obvious modification of the proof above, and therefore we omit the details. □

So far, we showed that ∇u and pressure p can be controlled by u near the boundary. Now we will show, furthermore, higher derivatives of u and p can be also estimated in terms of u provided that Ω is sufficiently smooth.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a domain of class $\mathcal{C}^{k+2}$ and k be an integer with $-1 \le k < \infty$. Suppose $f \in W^{k,2}(\Omega)$ and u be a weak solution of the Stokes system (5). Then for every r, s with $0 \le r < s \le r_0$, the following local estimate holds:*

$$||u||_{W^{k+2,2}(\Omega_r)} + ||p||_{W^{k+1,2}(\Omega_r)} \le C(||u||_{L^2(\Omega_s)} + ||f||_{W^{k,2}(\Omega_s)}), \tag{13}$$

where $C = C(k, n)$.

Proof. If $k = -1$, the estimate (13) is due to Lemma 6, Lemma 7 and Remark 1. Let $t = \frac{r+s}{2}$. In the case of $k \ge 0$, the following local estimate near boundary was known for a domain of class $\mathcal{C}^{k+2}$ (see Theorem 5.1 [13, page 224-225]). For all $0 \le r < t \le r_0$

$$\begin{aligned}&||u||_{W^{k+2,2}(\Omega_r)} + ||p||_{W^{k+1,2}(\Omega_r)}\\ &\le C(||u||_{W^{1,2}(\Omega_t)} + ||p||_{L^2(\Omega_t)} + ||f||_{W^{k,2}(\Omega_t)}).\end{aligned} \tag{14}$$

Using the estimate (13) when $k = -1$, the right side of (14) is estimated by in terms of u and f, that is,

$$\begin{aligned} &||u||_{W^{1,2}(\Omega_t)} + ||p||_{L^2(\Omega_t)} + ||f||_{W^{k,2}(\Omega_t)} \\ \le\ & C(||u||_{L^2(\Omega_s)} + ||f||_{W^{k,2}(\Omega_s)}). \end{aligned}$$

This completes the proof. □

Remark 2. One can also derive Theorem 1 directly from Lemma 6 and 7 without using results in [13] by the usual technique of taking derivatives of the equations in the tangential directions first and using the equations in the normal direction (This is completely straightforward if Ω is a half-space). □

Remark 3. As is usual in similar situations (see [1], for example), Theorem 1 implies a Liouville-type theorem. Consider the boundary-value problem

$$\begin{cases} -\Delta u + \nabla p = 0 & \text{in } \mathbb{R}^n_+ \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n_+ \\ u = 0 & \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{15}$$

Corollary 1. *Let $u \in W^{1,2}_{loc}(\mathbb{R}^n_+)$ be a weak solution (15). If $u(x) = o(|x|)$ as $|x| \to \infty$, then $u \equiv 0$.*

Proof. This is a standard argument, which we nevertheless sketch for the convenience of the reader. Let $\lambda > 0$ and set $u_\lambda(x) = u(\lambda x)$. By the estimate (13) and imbedding theorem, one has

$$\sup_{B^+_{\frac{1}{2}}} |\nabla u_\lambda| \le C \sup_{B^+_1} |u_\lambda| \tag{16}$$

for each $\lambda > 0$. However, $u(x) = o(|x|)$ as $|x| \to \infty$, then applying (16) as $\lambda \to \infty$, we see that u is constant, and therefore $u \equiv 0$ because $u = 0$ on the $\partial\mathbb{R}^n_+$. This completes the proof. □

Similar theorem was proved in [8] by using the reflection principle for the Stokes system (15). However, it does not seem obvious that the method in [8] could be easily used to obtain local estimates (13). □

In remaining part of this section, using results above, we can verify $W^{k,q}$ estimate (1) for the Stokes system. Let Ω be a smooth domain and $f \in W^{k,q}(\Omega)$ where $1 < q < \infty$ and k is any integer with $-1 \le k < \infty$. For convenience, let us recall the Stokes system (5):

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega_{r_0} \\ \nabla \cdot u = 0 & \text{in } \Omega_{r_0} \\ u = 0 & \text{on } B_{r_0} \cap \partial\Omega. \end{cases}$$

We use the same notations used in (5). We note first that we may assume Ω_{r_0} has a smooth boundary. In fact, it may not be smooth at points on $\partial\Omega\cap B_{r_0}$ (for example, if $\Omega=\mathbb{R}^n_+$, then $\Omega_{r_0}=B^+_{r_0}$), but we can take a smooth domain Ω', instead of Ω_{r_0}, such that $\Omega'\subset\Omega_{r_0}$ and $\Omega_{\frac{r_0}{2}}$ is contained in Ω'. Thus, without loss of generality, Ω_{r_0} is assumed to be smooth. Now we are ready to prove $W^{k,q}$ estimate near boundary for the Stokes system above.

Theorem 2. *Let $\Omega\subset\mathbb{R}^n$ be a domain of class $\mathcal{C}^{k+2}$ and k be an integer with $-1\le k<\infty$ and $1<q<\infty$. Suppose $f\in W^{k,q}(\Omega_{r_0})$ and $u\in W^{1,q}(\Omega_{r_0})$ solve the Stokes system (5) in a weak sense. Then for any $0\le r<s<r_0$ the following local estimate holds:*

$$||u||_{W^{k+2,q}(\Omega_r)}+||p||_{W^{k+1,q}(\Omega_r)}\le C(||f||_{W^{k,q}(\Omega_{r_0})}+||u||_{L^1(\Omega_s)}),\quad (17)$$

where $C=C(k,n,q)$.

Proof. We first investigate a priori estimate for smooth solutions. The idea is to split u as sum of v and w, which solves the Stokes system with nonzero external force and zero boundary, and with zero external force, respectively. First we consider the following Stokes system:

$$\begin{cases}-\Delta v+\nabla p_1=f & \text{in } \Omega_{r_0}\\ \nabla\cdot v=0 & \text{in } \Omega_{r_0}\\ v=0 & \text{on } \partial\Omega_{r_0}.\end{cases}$$

It is well known that the following estimate holds (see Theorem 6.1 in [13, page 231–232]).

$$||v||_{W^{k+2,q}(\Omega_{r_0})}+||p_1||_{W^{k+1,q}(\Omega_{r_0})}\le C||f||_{W^{k,q}(\Omega_{r_0})},\quad (18)$$

where p_1 is the normalized pressure, i.e. $\int_{\Omega_{r_0}}p_1=0$. Here we set $w=u-v$ and $p_2=p-p_1$. Then w,p_2 solve

$$\begin{cases}-\Delta w+\nabla p_2=0 & \text{in } \Omega_{r_0}\\ \nabla\cdot w=0 & \text{in } \Omega_{r_0}\\ w=0 & \text{on } B_{r_0}\cap\partial\Omega.\end{cases}$$

Using L^2 estimate (13) of the Theorem 1, for any $-1\le m<\infty$ we have

$$||w||_{W^{m+2,2}(\Omega_r)}+||p_2||_{W^{m+1,2}(\Omega_r)}\le C||w||_{L^2(\Omega_t)},\quad (19)$$

where t is the midpoint of r and s, i.e. $t=\frac{r+s}{2}$. Using the estimate (19) and iteration method, for any ρ_1,ρ_2 with $0\le\rho_1<\rho_2\le r_0$, we can easily prove

$$\int_{\Omega_{\rho_1}}|w|^2\le\frac{C}{(\rho_2-\rho_1)^n}(\int_{\Omega_{\rho_2}}|w|)^2.\quad (20)$$

Combining (19) and (20), we have

$$||w||_{W^{k+2,q}(\Omega_r)} + ||p_2||_{W^{k+1,q}(\Omega_r)} \le C||w||_{L^1(\Omega_s)}. \tag{21}$$

To sum up estimates above, we have the following a priori estimate

$$||u||_{W^{k+2,q}(\Omega_r)} + ||p||_{W^{k+1,q}(\Omega_r)} \le C(||f||_{W^{k,q}(\Omega_{r_0})} + ||u||_{L^1(\Omega_s)}). \tag{22}$$

Indeed, using estimates (18) and (21), we have

$$\begin{aligned}
&||u||_{W^{k+2,q}(\Omega_r)} + ||p||_{W^{k+1,q}(\Omega_r)} \le C(||f||_{W^{k,q}(\Omega_{r_0})} + ||w||_{L^1(\Omega_s)}) \\
&\le C(||f||_{W^{k,q}(\Omega_{r_0})} + ||u||_{L^1(\Omega_s)} + ||v||_{L^1(\Omega_s)}) \\
&\le C(||f||_{W^{k,q}(\Omega_{r_0})} + ||u||_{L^1(\Omega_s)} + ||v||_{W^{k+2,q}(\Omega_s)}) \\
&\le C(||f||_{W^{k,q}(\Omega_{r_0})} + ||u||_{L^1(\Omega_s)}).
\end{aligned}$$

Next we approximate solution u by smooth solutions. For convenience, we denote $\Gamma \equiv \partial\Omega_{r_0}$. We note first if $f \in W^{k,q}(\Omega_{r_0})$, then there exists a sequence $f_n \in \mathcal{C}^\infty(\Omega_{r_0})$ such that f_n converges to f in $W^{k,q}(\Omega_{r_0})$. On the other hand, since $u \in W^{1,q}(\Omega_{r_0})$, we have $u|_\Gamma \in W^{1-\frac{1}{q},q}(\Gamma)$. Thus, there exists a sequence $\phi_n \in \mathcal{C}^\infty(\Gamma)$ such that they approximate $u|_\Gamma$ in $W^{1-\frac{1}{q},q}(\Gamma)$. In particular, we can choose ϕ_n such that $\phi_n = 0$ on $B_{r_0} \cap \partial\Omega$. Now for given smooth data f_n and ϕ_n, there exists smooth solutions u_k and normalized pressures p_k such that they solve the following Stokes system:

$$\begin{cases} -\Delta u_k + \nabla p_k = f_k & \text{in } \Omega_{r_0} \\ \nabla \cdot u_k = 0 & \text{in } \Omega_{r_0} \\ u_k = \phi_k & \text{on } \Gamma \end{cases} \tag{23}$$

Since they are smooth, u_n and p_n satisfy the estimate (22), that is

$$||u_n||_{W^{k+2,q}(\Omega_r)} + ||p||_{W^{k+1,q}(\Omega_r)} \le C(||f_n||_{W^{k,q}(\Omega_{r_0})} + ||u_n||_{L^1(\Omega_{r_0})}). \tag{24}$$

On the other hand, using the linearity of the system above, we also have the following estimate

$$\begin{aligned}
&||u_l - u_m||_{W^{1,q}(\Omega_{r_0})} + ||p_l - p_m||_{L^q(\Omega_{r_0})} \\
&\le C(||f_l - f_m||_{W^{-1,q}(\Omega_{r_0})} + ||\phi_l - \phi_m||_{W^{1-\frac{1}{q},q}(\Gamma)}).
\end{aligned} \tag{25}$$

Observing $||f_l - f_m||_{W^{-1,q}(\Omega_{r_0})} \le C||f_l - f_m||_{W^{k,q}(\Omega_{r_0})}$ when $k \ge -1$, we obtain that u_k and p_k are Cauchy sequences in $W^{1,q}(\Omega_{r_0})$ and $L^q(\Omega_{r_0})$. Let us write $\tilde{u}$ and $\tilde{p}$ as limits of u_k and p_k, respectively. It is easily seen that $\tilde{u}$ and $\tilde{p}$ solves the following stokes system in a weak sense:

$$\begin{cases} -\Delta \tilde{u} + \nabla \tilde{p} = f & \text{in } \Omega_{r_0} \\ \nabla \cdot \tilde{u} = 0 & \text{in } \Omega_{r_0} \\ \tilde{u} = u & \text{on } \Gamma = \partial\Omega_{r_0} \end{cases} \tag{26}$$

We note that by approximation argument, $\tilde{u}$ and $\tilde{p}$ satisfy (22) due to (24) and (25). On the other hand, it is obvious that $\tilde{u} = u$ and $\tilde{p} = p$ in Ω_{r_0} because of the uniqueness of boundary value problem of the Stokes system. This completes the proof. □

Remark 4. Estimate (17) will play an important role in the proof of the partial regularity result up to the boundary for the Navier-Stokes equations at the boundary in next section. □

4. Partial regularity for the Navier-Stokes equations

In this section, we first review the partial regularity of weak solutions of the Navier-Stokes equations away from the boundary in five dimension (see [28] and [30]). Next we prove that the result of interior partial regularity is extended up to the boundary in a smooth domain of 5-dimensional space.

4.1. Interior partial regularity

Let Ω be a bounded or unbounded domain with smooth boundary in $\mathbb{R}^5$. We consider the stationary Navier-Stokes equations:

$$\begin{cases} -\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega \\ \nabla\cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{27}$$

where u and p are unknown velocity field and pressure and f is a prescribed external force. As usual, a weak solution $u : \Omega\to\mathbb{R}^5$ means that it is in the class $W_0^{1,2}(\Omega,\mathbb{R}^5)$ and satisfies the following:

$$\int_\Omega \nabla u\nabla\xi + (u\cdot\nabla)u\xi = \int_\Omega f\xi$$
$$\text{for all } \xi\in\mathcal{C}_0^\infty(\Omega,\mathbb{R}^5) \text{ with } \nabla\cdot\xi = 0 \text{ and}$$
$$\int_\Omega u\cdot\nabla\zeta = 0 \text{ for all } \zeta\in\mathcal{C}_0^\infty(\Omega,\mathbb{R}).$$

Zero boundary condition should be understood in sense of traces. In addition, u is assumed to satisfy the local energy estimate up to the boundary:

$$\begin{aligned}\int_\Omega |\nabla u|^2\phi^2\,\mathrm{d}y \le C\{&\int_\Omega |u|^2|\nabla\phi|^2\,\mathrm{d}y \\ &+\int_\Omega |u|(\frac{|u|^2}{2}+|p|)|\nabla\phi^2|\,\mathrm{d}y + \int_\Omega |f\cdot u|\phi^2\,\mathrm{d}y\},\end{aligned} \tag{28}$$

where $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^5)$. We say a weak solution u satisfying the local energy estimate (28) a *suitable weak solution.* The main step in the partial regularity of the interior case is to prove $\psi(x,r) \le Cr^{3\alpha}$ for every $B_{x,r} \Subset \Omega$ where

$$\psi(x,r) = \fint_{B_{x,r}} |u - (u)_{B_{x,r}}|^3 \, \mathrm{d}x.$$

Next Lemma gives a sufficient condition for the decay of ψ.

Lemma 8. *For any $M > 0$, there exist $C(M)$ and $\epsilon_1(M)$ such that for any $\tau \in (0, \frac{1}{2})$, if $\psi(x,r) \le \frac{\epsilon_1}{r^3}$ and $|(u)_{B_{x,r}}| \le \frac{M}{r}$, then $\psi(x,\tau r) \le C\tau^3\psi(x,r)$.*

The sketch of proof. The proof can be done by a standard "blow-up" method, which is similar to Lin's argument in [20] (see also [18] and [28]). In [18] and [20], the argument is used for the time dependent 3-dimensional Navier-Stokes equations. The 5-dimensional steady-state Navier-Stokes equations is in fact simpler, because the pressure is much easier to handle. Although this argument is standard, for clarity, we will sketch the proof briefly. Suppose the assertion above is not true. Then there exists sequence of weak solutions u_k, x_k, r_k and $\epsilon_k \searrow 0$ such that $\psi_{u_k}(x_k, r_k) = \frac{\epsilon_k^3}{r_k^3}$ and $|(u_k)_{x_k,r_k}| \le \frac{M}{r_k}$, but $\psi_{u_k}(x_k, \tau r_k) > C\tau^3\psi_{u_k}(x_k, r_k)$. Let $v_k(y) \equiv r_k u_k(x_k + r_k y)$. Then we get $\fint_{B_1} |v_k - (v_k)_{B_1}|^3 = \epsilon_k^3$ and $|(v_k)_{B_1}| \le M$. Let us define w_k by $w_k \equiv \epsilon_k^{-1}(v_k - (v_k)_{B_1})$. For simplicity, we denote $\mathbf{a}_k = (v_k)_{B_1}$. Then it is easily checked that $\int_{B_1} w_k = 0$ and $\int_{B_1} |w_k|^3 = 1$. In addition, w_k solves the following system in a weak sense:

$$\left.\begin{array}{c} -\Delta w_k + \mathbf{a}_k \nabla w_k + \epsilon_k w_k \nabla w_k + \nabla q_k = 0 \\ \nabla \cdot w_k = 0 \end{array}\right\} \text{ in } B_1. \tag{29}$$

By the assumption, we know that

$$\fint_{B_\tau} |w_k - (w_k)_{B_\tau}|^3 > C\tau^3. \tag{30}$$

Here we claim that $\int_{B_{1-\delta}} |\nabla w_k|^2$ is uniformly bounded. Indeed, note first that $L^{\frac{3}{2}}$ norm of pressure q_k can be estimated by L^3 norm of w_k. More precisely, $q_k = \tilde{q_k} + h_k$ where $\tilde{q_k} = \epsilon_k \Delta^{-1} \partial x_j \partial x_i (w_k^i w_k^j)$ and h_k is a harmonic function, which is also controllable in terms of w_k. To sum up, we have $||q_k||_{L^{\frac{3}{2}}(B_{1-\delta})} \le C||w||^2_{L^3(B_1)}$ for a sufficiently small $\delta > 0$. Once pressure is controlled in terms of u, then we control L^2 norm of ∇w_k in terms of L^3 norm of w_k by using the energy inequality. Since $|\mathbf{a}_k| \le M$ for all k and $W^{1,2}$ norm of w_k is uniformly bounded, a pair of subsequences of $\mathbf{a}_k$ and w_k, denoted by $\mathbf{a}_k$ and w_k again, converges point-wisely to $\mathbf{a}$ and converges weakly in $W^{1,2}_{loc}(B_1)$ to w, which solves the following system in a weak sense in B_1:

$$-\Delta w + (\mathbf{a} \cdot \nabla) w + \nabla q = 0, \quad \nabla \cdot w = 0. \tag{31}$$

We note also strong convergence of w_k in $L^3(B_{\frac{1}{2}})$ by the imbedding theorem. Passing (30) to the limit, strong convergence implies $⨍_{B_\tau} |w-(w)_{B_\tau}|^3 \geq C\tau^3$. On the other hand, w is smooth in $B_{\frac{1}{2}}$. Thus there exists an absolute constant $\tilde{C}$ such that $⨍_{B_\tau} |w-(w)_{B_\tau}|^3 \leq \tilde{C}\tau^3$, which is contrary to $⨍_{B_\tau} |w-(w)_{B_\tau}|^3 \geq C\tau^3$. To be precise, at the beginning we can choose $C > 2\tilde{C}$, which leads to a contradiction. □

By the appropriate choice of parameters, we get the Hölder continuity. Indeed, let us set $C\tau^3 = \tau^{3\alpha}$ where $0 < \alpha < 1$. Iterating the above process, we get $\psi(x, \tau^k r) \leq (\tau^k)^{3\alpha}\psi(x,r)$. This shows that $\psi(x,\rho) \leq C\rho^{3\alpha}$, which implies $u \in \mathcal{C}^{0,\alpha}$. More precisely, we can consider $\psi(x,r)$ as a function of x for fixed r. Then since it is continuous with respect to x, if $x_0 \in \Omega$ such that it satisfies the assumptions

$$\psi(x_0, r) < \frac{\epsilon}{r^3} \text{ and } |(u)_{B_{x_0,r}}| < \frac{M}{r}, \tag{32}$$

then there exists an open set O containing x_0 such that for all $x \in O$ the assumptions (32) holds. According to Lemma 8, it is Hölder continuous in a small neighborhood $O' \Subset O$, i.e. $u \in \mathcal{C}^{0,\alpha}(O')$ in a neighborhood of x_0. Furthermore, u is regular because Hölder continuity implies smoothness by using the standard argument for the elliptic system. To sum up, the sufficient condition is found to guarantee that u is smooth.

Theorem 3. *For any $M > 0$, there exists $\epsilon_1(M)$ such that if $\psi(x,r) \leq \frac{\epsilon_1}{r^3}$ and $|(u)_{B_{x,r}}| \leq \frac{M}{r}$, then u is smooth in a neighborhood of x.*

We say that a point $x \in \Omega$ is an *interior regular point* if (32) is satisfied for some $r > 0$. To be precise, it is defined as follows.

Definition 2. *$x \in \Omega$ is called an* interior regular point *if*

$$\liminf_{r\to 0} \frac{1}{r^2}\int_{B_{x,r}} |u(y)-(u)_{B_{x,r}}|^3 \,\mathrm{d}y = 0,$$
$$\liminf_{r\to 0} \frac{1}{r^4}\left|\int_{B_{x,r}} u(y)\,\mathrm{d}y\right| < \infty.$$

Next Lemma enable us to investigate the size of interior regular points.

Lemma 9. *Suppose that for a given $x \in \Omega$*

$$\limsup_{r\to 0} \frac{1}{r}\int_{B_{x,r}} |\nabla u|^2 = 0.$$

Then x is an interior regular point.

Proof. This can be proved by a variant of Lin's argument in [20] (see Theorem 3.3). See also Theorem 2.2 in [18] and Theorem 1.2 in [28]. $\square$

We note first that $\mathcal{H}^1(S) = 0$ where $S = \{x : \lim_{r\to 0} \sup \frac{1}{r} \int_{B_{x,r}} |\nabla u|^2 > 0\}$ by Lemma 2. Combining Theorem 3 and Lemma 9, it is easily checked that u is smooth in a neighborhood of every regular point. To sum up all results, we conclude this section by stating the well known interior partial regularity result.

Theorem 4. *Let u be a* suitable weak solution *of the Navier-Stokes equations (27). Then there exists closed set $S \subset \Omega$ with $\mathcal{H}^1(S) = 0$ such that u is smooth in $\Omega \setminus S$.*

4.2. Boundary partial regularity

The obstacle in showing partial regularity up to the boundary lies in difficulty with controlling the pressure associated with u at the boundary. In the interior case, it can be easily controlled in terms of u because the pressure can be decomposed as the sum of controllable term and a harmonic function. However, such approach seems not to be applicable near the boundary because of difficulty in estimating the harmonic function up to the boundary. However, with the aid of the Stokes estimate near boundary we established in section 3, we can extend the partial regularity result up to the boundary.

Let u be a *suitable weak solution* of Navier-Stokes equations (27). Here the domain Ω we consider has a smooth boundary and, for simplicity, we assume $f = 0$. To characterize the behavior of u near the boundary, let us introduce $\phi(x, r)$, for every boundary point $x \in \partial\Omega$ and every $r > 0$, defined as follows:

$$\phi(x, r) \equiv \frac{1}{r^5} \int_{\Omega_{x,r}} |u(y)|^3 \, \mathrm{d}y,$$

where $\Omega_{x,r} = B_{x,r} \cap \Omega$. In a similar manner as the interior case, next Lemma establishes the decay of averaged L^3 norm of u.

Lemma 10. *Let u be a* suitable weak solution *of Navier-Stokes equations (27). Then there exists an absolute constant C such that the following statement holds. Let x be a boundary point of Ω. For a given $\epsilon > 0$, there exists $r_0 = r_0(\epsilon)$ such that if $r < r_0$ and $\phi(x, r) < \frac{\epsilon}{r^3}$, then*

$$\phi(x, \tau r) \leq C\tau^3 \phi(x, r) \quad \textit{for any} \quad \tau \in (0, \frac{1}{2}).$$

Proof. Suppose not. Then for fixed $\tau \in (0, \frac{1}{2})$, there are sequences of u_k, x_k, r_k and ϵ_k with $r_k \searrow 0$ and $\epsilon_k \searrow 0$ such that $\phi_{u_k}(x_k, r_k) = \frac{\epsilon_k^3}{r_k^3}$, but $\phi_{u_k}(x_k, \tau r_k) \geq C\tau^3 \phi_{u_k}(x_k, r_k)$ for some C, which will be specified later. Let us denote by p_k

the pressure associated with u_k. Without loss of generality we may assume that $(p_k)_{\Omega_{x_k,r_k}} = 0$. Let us scale u_k and p_k by changing variables in the following manner,

$$w_k(y) = \epsilon_k^{-1} r_k u_k(x_k + r_k y), \quad q_k(y) = \epsilon_k^{-1} r_k^2 p_k(x_k + r_k y)$$

and we denote the transformed domain by Ω_1^k and Ω_τ^k corresponding to Ω_{x_k,r_k} and $\Omega_{x_k,\tau r_k}$, respectively. Then simple computation shows

$$\phi_{w_k}(0,1) = \int_{\Omega_1^k} |w_k|^3 = 1 \tag{33}$$

$$\phi_{w_k}(0,\tau) = \frac{1}{\tau^5} \int_{\Omega_\tau^k} |w_k|^3 \geq C\tau^3 \tag{34}$$

and it solves, in a weak sense,

$$\begin{cases} -\Delta w_k + \nabla q_k = -\epsilon_k (w_k \cdot \nabla) w_k & \text{in } \Omega_1^k \subset B_1 \\ \nabla \cdot w_k = 0 & \text{in } \Omega_1^k \subset B_1 \\ w_k = 0 & \text{on } B_1 \cap \partial\Omega_1^k \end{cases}$$

We note first that Ω_1^k is more and more close to a half ball B_1^+ as k increases because Ω is smooth and $r_k \searrow 0$. Secondly, we observe that $(w_k \cdot \nabla) w_k \in W^{-1,\frac{3}{2}}(\Omega_1^k)$. Thus, using that L^3 norm of w_k is uniformly bounded due to (33), for any sufficiently small δ with $0 < \delta < 1$, by L^p estimate (17) of Theorem 2 it follows that q_k is uniformly bounded in the norm of $L^{\frac{3}{2}}$ in a smaller region $\Omega_{1-\frac{\delta}{4}}^k$. Moreover, L^2 norm of ∇w_k is also uniformly bounded in $\Omega_{1-\frac{\delta}{2}}^k$ by the local energy estimate (28). Now we extend w_k into a unit ball B_1 by assigning 0 in $B_1 \setminus \Omega_1^k$ and we denote such extension of w_k by $\tilde{w}_k$. Then $\tilde{w}_k \in W^{1,2}(B_1)$ and L^2 norm of $\nabla \tilde{w}_k$ is also uniformly bounded in $B_{1-\frac{\delta}{2}}$. Hence using the compactness argument, we have the strong convergence of subsequence $\tilde{w}_{k_j}$ of $\tilde{w}_k$ in $L^3(B_{1-\delta})$. In the same manner as above, q_{k_j} is extended to B_1 and thus we also have a weak convergence for $L^{\frac{3}{2}}$ norm of $\tilde{q}_{k_j}$, which is the extension of q_{k_j} in $B_{1-\delta}$. To sum up the argument above, first take the sequence w_{k_j}, denoted again by w_k without any confusion, instead of w_k at the beginning and consider its extension $\tilde{w}_k$ and a pressure $\tilde{q}_k$ to a unit ball. Using the Stokes estimate and standard compactness argument, it follows that there are $w \in W^{1,2}(B_{1-\delta})$ and $q \in L^{\frac{3}{2}}(B_{1-\delta})$ such that followings hold:

$$\tilde{w}_k \rightharpoonup w \text{ in } W^{1,2}(B_{1-\delta}) \quad \text{weakly}$$
$$\tilde{w}_k \to w \text{ in } L^3(B_{1-\delta}) \quad \text{strongly}$$
$$\tilde{q}_k \rightharpoonup q \text{ in } L^{\frac{3}{2}}(B_{1-\delta}) \quad \text{weakly.}$$

In addition, w and q solve the following Stokes system in a half ball in a distribution sense

$$\begin{cases} -\Delta w + \nabla q = 0 & \text{in } B^+_{1-\delta} \\ \nabla \cdot w = 0 & \text{in } B^+_{1-\delta} \\ w = 0 & \text{on } B_{1-\delta} \cap \{x_5 = 0\}. \end{cases}$$

because, as mentioned earlier, Ω^k_1 and $\partial\Omega^k_1 \cap B_1$ become B^+_1 and $B_1 \cap \{x_5 = 0\}$, respectively as $k \to \infty$. Thus, it is easily checked that w and q are smooth in $B^+_{\frac{1}{2}}$ (see Theorem 2). Since w_k converges to w strongly in $L^3(B^+_{\frac{1}{2}})$, by passing (34) to the limit we get

$$\phi_w(0,\tau) = \frac{1}{\tau^5}\int_{B^+_\tau} |w(y)|^3 \, \mathrm{d}y \geq C\tau^3.$$

However, there is a different constant $\tilde{C}$ independent of the choice of sequences w_k and q_k such that the following inequality holds

$$\frac{1}{\tau^5}\int_{B^+_\tau} |w|^3 \leq \tilde{C}\tau^3, \tag{35}$$

which leads to a contradiction because we can choose the constant C bigger than $2\tilde{C}$ at the beginning. Hence it suffices to prove (35). Indeed, let x_p be the projection point of x onto boundary, $B_{\frac{1}{2}} \cap \{x_5 = 0\}$. After simple computations, we obtain

$$\begin{aligned} \frac{1}{\tau^5}\int_{B^+_\tau} |w(x)|^3 \, \mathrm{d}x &= \frac{1}{\tau^5}\int_{B^+_\tau} |w(x) - w(x_p)|^3 \, \mathrm{d}x \text{ because } w(x_p) = 0 \\ &\leq \frac{1}{\tau^5}\int_{B^+_\tau} \Big| \int_0^1 \frac{d}{dt} w(tx + (1-t)x_p) \, \mathrm{d}t \Big|^3 \, \mathrm{d}x \\ &\leq \frac{1}{\tau^5}\int_{B^+_\tau} \Big| \int_0^1 |\nabla w(tx + (1-t)x_p)| \cdot |x - x_p| \, \mathrm{d}t \Big|^3 \, \mathrm{d}x \\ &\leq \frac{1}{\tau^5}\int_{B^+_\tau} (\sup_{B^+_{\frac{1}{2}}} |\nabla w| \tau)^3 \, \mathrm{d}x \leq (\sup_{B^+_{\frac{1}{2}}} |\nabla w|)^3 \tau^3 \end{aligned}$$

In fact, $\tilde{C} = (\sup_{B^+_{\frac{1}{2}}} |\nabla w|)^3$, which is an absolute constant. This completes the proof. □

Definition 3. *$x \in \partial\Omega$ is called a* boundary regular point *provided that* $\liminf_{r\to 0} r^3 \phi(x,r) = 0$, *i.e.*

$$\liminf_{r\to 0} \frac{1}{r^2}\int_{\Omega_{x,r}} |u(y)|^3 \, \mathrm{d}y = 0.$$

To show the Hölder continuity up to the boundary, let us start some observations concerning a boundary regular point.

Lemma 11. *Let x be a boundary regular point. For given $\epsilon > 0$, there exist $\delta = \delta(x,\epsilon)$ and $r = r(x,\epsilon)$ such that for any $\tau \in (0, \frac{1}{2})$*

$$\phi(z,\tau r) \le C\tau^3\phi(z,r) < C\tau^3\frac{\epsilon}{r^3} \quad \textit{for all} \quad z \in B_{x,\delta}\cap\partial\Omega$$

where C is the absolute constant in the Lemma 10.

Proof. Since x is a boundary regular point, for given $\epsilon > 0$, there is a positive radius $r > 0$ such that $r^3\phi(x,r) < \frac{\epsilon}{2}$. Without loss of generality, we may take $r < r_0$ where r_0 is the positive constant in Lemma 10. Since $r^3\phi(x,r)$ is absolutely continuous with respect to x, there exists a neighborhood $B_{x,\delta}\cap\partial\Omega$ of x such that $r^3\phi(z,r) < \epsilon$ for all $z \in B_{x,\delta}\cap\partial\Omega$. By the Lemma 10, we have the decay of averaged L^3 norm of u for all $z \in B_{x,\delta}\cap\partial\Omega$. This completes the proof. □

In a similar way as the interior case, let us introduce

$$\psi(x,r) \equiv ⨍_{\Omega_{x,r}} |u-(u)_{\Omega_{x,r}}|^3 \text{ for any } x\in\bar{\Omega},$$

where $(u)_{\Omega_{x,r}} = \frac{1}{|\Omega_{x,r}|}\int_{\Omega_{x,r}} u(y)\,\mathrm{d}y$. Let us first observe the decay property of ψ near a boundary regular point.

Corollary 2. *Let $x \in \partial\Omega$ be a boundary regular point. For given $\epsilon > 0$, there exist $\delta = \delta(x,\epsilon)$ and $r = r(x,\epsilon)$ such that for any $\tau \in (0,\frac{1}{2})$*

$$\psi(z,\tau r) \le C\tau^3\phi(z,r) < C\tau^3\frac{\epsilon}{r^3} \quad \textit{for all} \quad z \in B_{x,\delta}\cap\partial\Omega$$

where C is the absolute constant.

Proof. According to Lemma 11, for given $\epsilon > 0$ there exists $\delta = \delta(x,\epsilon)$ and $r = r(x,\epsilon)$ such that $\phi(z,\tau r) \le C\tau^3\phi(z,r) < C\tau^3\frac{\epsilon}{r^3}$ for all $z \in B_{x,\delta}\cap\partial\Omega$. With the aid of the result, we have

$$\begin{aligned}\psi(z,\tau r) &= ⨍_{\Omega_{z,\tau r}} |u(y)-(u)_{\Omega_{z,\tau r}}|^3\,\mathrm{d}y \\ &\le C(⨍_{\Omega_{z,\tau r}} |u(y)|^3\,\mathrm{d}y) \le C\phi(z,\tau r) \\ &\le C\tau^3\phi(z,r) < C\tau^3\frac{\epsilon}{r^3}.\end{aligned}$$

This completes the proof. □

Up to now, we showed the decay of $\psi(z,\tau r)$ at every boundary point in a neighborhood of a boundary regular point x. The next goal is to show the same estimate at every interior point in a neighborhood of the boundary regular point x. For convenience, we denote by z' the nearest boundary point from z, in other words $|z-z'| = \mathrm{dist}(z,\partial\Omega)$. For convenience, we call z' the projection point of $z\in\Omega$ onto boundary $\partial\Omega$. Instead of showing $\psi(z,\tau r)\le C\tau^3\phi(z',r)$, we shall prove $\psi(z,\frac{1}{2^j}r)\le C(\frac{1}{2^j})^3\phi(z',r)$ for all positive integer j at each interior point z where C is an absolute constant. Next Lemma shows this assertion.

Lemma 12. *Let $x\in\partial\Omega$ be a boundary regular point. Then for given $\epsilon>0$, there exists $\delta=\delta(x,\epsilon)$ and $r=r(x,\epsilon)$ such that,*

$$\psi(z,\tau r)\le C\tau^3\phi(z',r) < \frac{C\tau^3\epsilon}{r^3} \textit{ for all } z\in\Omega_{x,\delta} \tag{36}$$

for any $\tau\in(0,\frac{1}{2})$ where C and z' are an absolute constant and the projection point of z onto $\partial\Omega$, respectively.

Proof. In this proof, without loss of generality, ϵ is assumed to be less than 1. In fact, the smallness of ϵ will be specified later. According to the Lemma 11, there is a neighborhood $\Omega_{x,\delta}$ of x for some $\delta=\delta(x,\epsilon)>0$ such that for some $r=r(x,\epsilon)$, the following inequality holds

$$\phi(z,\tau r)\le C\tau^3\phi(z,r) < C\tau^3\frac{\epsilon}{r^3} \text{ for all } z\in B_{x,\delta}\cap\partial\Omega. \tag{37}$$

Without loss of generality, we may take $\delta<\frac{r}{2}$. Instead of (36), we shall show $\psi(z,\frac{1}{2^j}r)\le C(\frac{1}{2^j})^3\phi(z',r)$ for all positive integer j and at every interior point $z\in\Omega_{x,\delta}$. Then the last inequality of (36) follows from (37). Let z be a point in $\Omega_{x,\delta}$ and $k=k(z)$ be the smallest positive integer such that $B_{z,\frac{r}{2^k}}\cap\partial\Omega=\emptyset$ but $B_{z,\frac{r}{2^{k-1}}}\cap\partial\Omega\neq\emptyset$. Let us first prove that

$$\psi(z,\frac{r}{2^j})\le C(\frac{1}{2^j})^3\phi(z',r) \text{ for all } j=1,2,...,k. \tag{38}$$

Indeed, note first that $z\in B_{z',\frac{r}{2^{j-1}}}$ for all $1\le j\le k$. Thus, using the estimate (37) and $\Omega_{z,\frac{r}{2^j}}\subset\Omega_{z',\frac{r}{2^{j-2}}}$ for $2\le j\le k$ (note that $\Omega_{z,\frac{r}{2}}\subset\Omega_{z',r}$ for $j=1$), we have

$$\begin{aligned}
\psi(z,\frac{r}{2^j}) &= \fint_{\Omega_{z,\frac{r}{2^j}}} |u-(u)_{\Omega_{z,\frac{r}{2^j}}}|^3\,\mathrm{d}y \le C\fint_{\Omega_{z,\frac{r}{2^j}}}|u(y)|^3\,\mathrm{d}y \\
&\le C\fint_{\Omega_{z',\frac{r}{2^{j-2}}}}|u(y)|^3\,\mathrm{d}y = C\phi(z',\frac{r}{2^{j-2}}) \\
&\le C(\frac{1}{2^{j-2}})^3\phi(z',r)\le C(\frac{1}{2^j})^3\phi(z',r)
\end{aligned}$$

where C is an absolute constant. Next we consider the other case $j > k$. Our goal is to show the same estimate as the previous case, that is to say, $\psi(z, \frac{1}{2^j}r) \leq C(\frac{1}{2^j})^3\phi(z', r)$ for all $j > k$. The main tool is to use the Lemma 8 in the case $j > k$ because $B_{z,\frac{1}{2^j}}$ is away from the boundary. We note first that $B_{z,\frac{1}{2^k}}$ is strictly contained in Ω because k was chosen to satisfy that property. Let us first consider $\psi(z, \frac{1}{2^j}r)$ in case of $j = k$. From the estimates of (38), we already have known that

$$\psi(z, \frac{r}{2^k}) \leq C(\frac{1}{2^k})^3\phi(z', r), \tag{39}$$

where C is an absolute constant. Next we will show that $(u)_{\Omega_{(z,\frac{r}{2^k})}} \leq \frac{C}{(\frac{r}{2^k})}$. Indeed, since z', the projection point of z onto the boundary, is a regular point on the boundary, using the estimate (37), we have

$$\begin{aligned}(u)_{\Omega_{(z,\frac{r}{2^k})}} &\leq C(\fint_{\Omega_{z',\frac{r}{2^{k-2}}}} |u(y)|^3\, \mathrm{d}y)^{\frac{1}{3}} \leq C(\phi(z', \frac{r}{2^{k-2}}))^{\frac{1}{3}} \\ &\leq C(\frac{1}{2^k})(\phi(z', r))^{\frac{1}{3}} \leq C(\frac{1}{2^k})\frac{\epsilon^{\frac{1}{3}}}{r} \\ &\leq \frac{C}{2^k r} \leq \frac{C}{(\frac{r}{2^k})},\end{aligned} \tag{40}$$

where C is an absolute constant and we used $\epsilon < 1$. Recalling Lemma 8, let us choose $M = 2C$ where C is the absolute constant in (40). According to Lemma 8, for given $M = 2C$, there exists $\epsilon = \epsilon(M)$ and $C = C(M)$ such that if $\psi(z, r) < \frac{\epsilon}{r^3}$ and $(u)_{z,B_r} < \frac{M}{r}$, then we have $\psi(z, \tau r) \leq C\tau^3\psi(z, r)$, At the beginning, ϵ can be given to satisfy $2C\epsilon < \epsilon(M)$ where C is a absolute constant in (39). To apply Lemma 8, it is enough to check if the assumptions $\psi(z, r) < \frac{\epsilon(M)}{r^3}$ and $(u)_{z,B_r} < \frac{M}{r}$ are satisfied. The second one is due to (40) because

$$(u)_{B_{(z,\frac{r}{2^k})}} \leq \frac{C}{(\frac{r}{2^k})} < \frac{M}{(\frac{r}{2^k})}. \tag{41}$$

On the other hand, from (39), we have

$$\psi(z, \frac{r}{2^k}) \leq C(\frac{1}{2^k})^3\phi(z', r) < C(\frac{1}{2^k})^3\frac{\epsilon}{r^3} < C\frac{\epsilon}{(\frac{r}{2^k})^3} < \frac{\epsilon(M)}{(\frac{r}{2^k})^3}. \tag{42}$$

The estimates of (41) and (42) enable us to use the result of the Lemma 8. Using the interior decay property, we obtain for all $j > k$,

$$\psi(z, \frac{r}{2^j}) \leq C(\frac{1}{2^j})^3\phi(z', r).$$

Indeed, with the aid of Lemma 8 and (39), for every j with $j > k$, we have

$$\begin{aligned}\psi(z, \frac{r}{2^j}) &= \psi(z, \frac{1}{2^{j-k}}\frac{r}{2^k}) \le C(M)(\frac{1}{2^{j-k}})^3\psi(z, \frac{r}{2^k}) \\ &\le C(M)(\frac{1}{2^{j-k}})^3 C(\frac{1}{2^k})^3\phi(z', r) \\ &\le C(\frac{1}{2^j})^3\phi(z', r), \end{aligned} \tag{43}$$

where C is also absolute constant. Taking a maximum absolute constant C in (38) and (43), we conclude $\psi(z, \frac{r}{2^j}) \le C(\frac{1}{2^j})^3\phi(z', r)$ for every positive integer j. This completes the proof. ◻

Lemma 12 and Corollary 2 implies that u is regular near the boundary.

Corollary 3. *If $x \in \partial\Omega$ be a boundary regular point, then u is Hölder continuous in a neighborhood of x. Therefore it is smooth in the neighborhood of x.*

Proof. According to Corollary 2 and Lemma 12, for a sufficiently small ϵ, there exists $\delta = \delta(x.\epsilon)$ and $r = r(x, \epsilon)$ such that for every $\tau \in (0, \frac{1}{2})$

$$\psi(z, \tau r) \le C\tau^3\frac{\epsilon}{r^3} \quad \text{for all} \quad z \in B_{x,\delta} \cap \bar{\Omega}$$

where C is an absolute constant. Let α be a positive number with $\alpha \in (0, 1)$. Since ϵ and r are fixed constant, there exists $\tau_0 = \tau_0(\epsilon, r, \alpha)$ such that $\tau^3\frac{\epsilon}{r^{3+3\alpha}} \le (\tau)^{3\alpha}$ for all $0 < \tau \le \min\{\frac{1}{2}, \tau_0\}$. Hence $\psi(z, \tau r) \le C(\tau r)^{3\alpha}$ for all $z \in B_{x,\delta} \cap \bar{\Omega}$, which implies the Hölder continuity in $B_{x,\frac{\delta}{2}} \cap \bar{\Omega}$ by the second assertion of Lemma 1. The full regularity follows from the standard argument of an elliptic system. This completes the proof. ◻

Now we investigate the size of a singular set.

Lemma 13. *Suppose $x \in \partial\Omega$ be a boundary point satisfying*

$$\lim_{r\to 0}\sup\frac{1}{r}\int_{\Omega_{x,r}}|\nabla u|^2 = 0. \tag{44}$$

Then x is a boundary regular point.

Proof. Let x be a boundary point satisfying (44). To prove the above statement, we should show that there exists $r > 0$ such that $\phi(x, r) < \frac{\epsilon}{r^3}$ for any given $\epsilon > 0$. For convenience, let us recall $\phi(x, r)$ defined by

$$\phi(x, r) = \frac{1}{r^5}\int_{\Omega_{x,r}}|u(y)|^3\,\mathrm{d}y.$$

We shall show that each term $r^3\phi(x,r)$ converges to 0 as $r \to 0$. Indeed, using the imbedding theorem and the assumption (44), we have

$$\begin{aligned} r^3\phi(x,r) &= \frac{1}{r^2}\int_{\Omega_{x,r}} |u(y)|^3\,\mathrm{d}y \le \frac{C}{r^2}\{\int_{\Omega_{x,r}} |u(y)|^{\frac{10}{3}}\,\mathrm{d}x\}^{\frac{9}{10}}|\Omega_{x,r}|^{\frac{1}{10}} \\ &\le \frac{C}{r^2}\{\int_{\Omega_{x,r}} |u(y)|^{\frac{10}{3}}\,\mathrm{d}x\}^{\frac{9}{10}} r^{\frac{1}{2}} \le \frac{C}{r^{\frac{3}{2}}}(\int_{\Omega_{x,r}} |\nabla u|^2\,\mathrm{d}x)^{\frac{3}{2}} \\ &\le C(\frac{1}{r}\int_{\Omega_{x,r}} |\nabla u|^2\,\mathrm{d}x)^{\frac{3}{2}} \to 0 \text{ as } r \to 0. \end{aligned} \tag{45}$$

Hence $\phi(x,r)$ is less than $\frac{\epsilon}{r^3}$ for a given ϵ by taking a sufficiently small r provided that (44) is satisfied. Hence x is a boundary regular point. This completes the proof. □

Summing up the results above, we obtain the main theorem.

Theorem 5. *Let u be a* suitable weak solution *of the Navier-Stokes equations (27). Then u is smooth up to the boundary in Ω except for a possible singular closed set S with $\mathcal{H}^1(S) = 0$.*

Proof. According to the Lemma 13, every boundary point is a regular point if it holds (44). Thus, $S = \{x \in \partial\Omega : \limsup_{r\to 0} \frac{1}{r}\int_{\Omega_{x,r}} |\nabla u|^2 > 0\}$ is the possible singular set at boundary. It is easily verified that S is closed in $\partial\Omega$ and $\mathcal{H}^1(S) = 0$. Indeed, every boundary regular point has an open neighborhood where all point are regular, too. The last assertion is due to Lemma 2. This completes the proof. □

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