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THE LIMIT OF THE n-th POWER OF A DENSITY*

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1. Introduction. If $f(\cdot)$ is a bounded density function of an absolutely continuous variate z, then the powers f^2, f^3, ..., can be normalized to define new variates z_2, z_3, ... Typically, z_n will converge in probability to the mode (say m) of $f(\cdot)$, and it is shown below (Corollary 2) that if f is unimodal, $f'(m) = 0$, and $f''(m) \neq 0$, then $y_n = n^{\frac{1}{2}}(z_n - m)$ will tend in distribution to a normal variate with mean equal to zero and variance equal to $-f(m)/f''(m)$. Four examples of this result, relating to gamma, beta, Student's t, and Snedecor's F variates, are given in Section 3. Asymptotic normality is of course well known for these cases.

Our main result, Theorem 1, is more general than Corollary 2 in two respects: (a) The density of z_n is assumed to have the form

$$(1) \qquad c_n \left\{ f(z) \right\}^n k(z)$$

where c_n is a constant and $k(z)$ is bounded. Order statistics have densities of this form, and their asymptotic normality is a consequence (Example 5). (b) The conditions $f'(m) = 0$, $f''(m) \neq 0$, are relaxed to allow more general behavior at the mode. We allow cusps, as exemplified by $f(z) = 1 - |z|$, $(|z| < 1)$, or "flat" maxima for which $f''(m) = 0$. In these cases a limiting density is obtained having the form $c\exp\{-|y|^\gamma\}$ where γ is the order of the first nonvanishing term in the Taylor expansion of $f(z) - f(m)$.

Theorem 2 is a multivariate analog of Theorem 1, applicable for example to the Dirichlet distribution.

Theorem 1 is proved by first expanding f in its Taylor series about the mode, and taking the limit of the n-th power after "standardizing" the variate. The result of this routine calculation is easily anticipated. The difficulty lies in the normalization constants. By truncating the densities and by appealing to the dominated convergence theorem, it is shown without evaluating the normalization constants that these constants converge as desired. Convergence in distribution is then established by Scheffé's theorem.

It may be instructive to cite an example wherein the assumptions are violated in such a way that Scheffé's theorem is inapplicable and the conclusion of Theorem 1 is false. Suppose $f(z)$ has a local maximum at $z = 0$ and an absolute maximum at $z = 1$, with, say, $f(0)=1$, $f''(0)=-1$, $f(1)=2$. If z_n has density proportional to f^n, then certain constant multiples of the densities of $y_n = (2n)^{\frac{1}{2}} z_n$ will approach $\exp\{-y^2/2\}$; but the densities themselves would everywhere approach zero. This is so because the neighborhood of the mode at $z = 1$ accumulates the bulk of the probability as n increases, and this mass of probability tends to infinity on the scale of the variate $(2n)^{\frac{1}{2}} z_n$. Thus it is clear that y_n does not tend to standard normal despite the convergence of the nonnormalized "densities."

2. Main Results.

Lemma 1. If for some $a > 0$, $\delta > 0$, $g(x) = 1 - a^{\gamma}|x|^{\gamma} + o(|x|^{\gamma})$ as $x \to 0$, then $\{g(n^{-1/\gamma} y/a)\}^n \to \exp\{-|y|^{\gamma}\}$ for $-\infty < y < \infty$ as $n \to \infty$.

Proof. Straightforward.

Lemma 2. $\psi(n) = (1 - z^2/n)^n \le e^{-z^2}$ for all $n > z^2 > 0$.

Proof. $\psi'(n) = 0$ is equivalent to $\varphi(x) = 1 - x + x \log x = 0$ where $x = 1 - z^2/n$. But $\varphi(1) = \varphi'(1) = 0$ and $\varphi'(x) > 0$ for $0 < x < 1$. Thus $\psi'(n) \neq 0$ for $n > z^2 > 0$, so that the convergence to the limit

e^{-z^2} is monotone; and it is easily verified that $\psi(n)$ increases with n.

Lemma 3. If $0 \leq g(x) \leq \rho < 1$ for all $x \in S$, and if $\int_S g(x)dx = M < \infty$, then for any $\gamma > 0$, $n^{1/\gamma}\int_S \{g(x)\}^n\, dx \to 0$ as $n \to \infty$.

Proof. For $k = 0,1,\ldots$, define $S_k = \left\{x \mid x \in S \text{ and } 2^{-k-1}\rho < g(x) \leq 2^{-k}\rho\right\}$. Then the Lebesgue measure L of S_k is easly seen to be bounded by $L(S_k) \leq 2^{k+1}\rho^{-1}M$, and

$$\begin{aligned}(1)\qquad \int_S \{g(x)\}^n\, dx &= \sum_k \int_{S_k} \{g(x)\}^n\, dx \leq \sum_k L(S_k)\rho^n 2^{-kn} \\ &\leq 2M\rho^{n-1} \sum_k 2^{k(1-n)} = 2M\rho^{n-1}(1 - 2^{1-n})^{-1},\end{aligned}$$

from which the result follows.

Lemma 4. If for some $a > 0$, $\gamma > 0$, $\delta > 0$, $\rho < 1$, $g(x)$ satisfies $\int_{-\infty}^{\infty} g(x)dx < \infty$, and $0 < g(x) < \rho$ for $|x| > \delta$; if $k(x)$ satisfies $0 \leq k(x) \leq K < \infty$ for all x; and if $h_n(\cdot)$ and J_n are defined by

$$(2)\qquad h_n(y) = \{g(n^{-1/\gamma}y/a)\}^n\, k(n^{-1/\gamma}y/a)$$

$$(3)\qquad J_n = \int_{|y| > n^{1/\gamma}a\delta} h_n(y)dy$$

then $J_n \to 0$ as $n \to \infty$.

Proof. Putting $x = n^{1/\gamma}ay$ gives

$$\begin{aligned}(4)\qquad J_n &= n^{1/\gamma}a\int_{|x| > \delta} \{g(x)\}^n k(x)dx \\ &\leq n^{1/\gamma}aK\int_{|x| > \delta} \{g(x)\}^n dx\end{aligned}$$

which tends to zero by Lemma 3.

Theorem 1. Let $f(\cdot)$ and $k(\cdot)$ be nonnegative functions satisfying $f(0) > 0$, $k(0) > 0$, $k(x)$ continuous at $x = 0$, $k(x) \leq K < \infty$ and $\int_{-\infty}^{\infty} f(x)dx < \infty$. Define $g(x) = f(x)/f(0)$, and assume that for some

$\gamma > 0, \quad a > b > 0, \quad \delta > 0, \quad 0 < \rho < 1,$

$$(5) \qquad g(x) = 1 - a^\gamma |x|^\gamma + o(|x|^\gamma) \qquad \text{as} \quad x \to 0$$

$$(6) \qquad g(x) \leqq 1 - b^\gamma |x|^\gamma \qquad \text{for} \quad |x| < \delta$$

$$(7) \qquad g(x) \leqq \rho \qquad \text{for} \quad |x| > \delta .$$

Let x_n denote a random variable whose density is proportional to $\{f(\cdot)\}^n k(\cdot)$. Then the distribution of $y_n = n^{1/\gamma} a x_n$ converges to the distribution whose density is proportional to $\exp\{-|y|^\gamma\}$.

<u>Proof</u>. The density of y_n is proportional to (2). As $n \to \infty$, the second factor approaches $k(0)$ for all y, by the continuity of $k(x)$ at $x = 0$, and by (5) and Lemma 1,

$$(8) \qquad h_n(y) \to k(0) \exp\{-|y|^\gamma\} \qquad \text{as} \quad n \to \infty.$$

Let us define

$$g^{(0)}(x) = \begin{cases} g(x) & |x| < \delta \\ 0 & |x| \geqq \delta \end{cases}$$

$$h_n^{(0)}(y) = \{g^{(0)}(n^{-1/\gamma} y/a\}^n k(n^{-1/\gamma} y/a).$$

Then $\lim h_n^{(0)}(y) = \lim h_n(y) = k(0)\exp\{-|y|^\gamma\}$, and by (6), $k(x) \leqq K$, and Lemma 2, the functions $h_n^{(0)}(y)$ are uniformly dominated by an integrable function, namely,

$$h_n^{(0)}(y) \leqq K \exp\{-|by/a|^\gamma\}, \quad n = 1,2,\ldots, \quad -\infty < y < \infty.$$

We now consider the normalization constants,

$$I_n = \int_{-\infty}^{\infty} h_n(y)dy = J_n + K_n$$

where J_n and K_n are integrals over $|y| > n^{1/\beta} a\delta$ and $|y| < n^{1/\beta} a\delta$, respectively. Appealing to Lemma 4 and the dominated convergence theorem, we have

$$\lim I_n = \lim K_n = \lim \int_{-\infty}^{\infty} h_n^{(0)}(y)dy = \int_{-\infty}^{\infty} \lim h_n^{(0)}(y)dy = k(0)\int_{-\infty}^{\infty} \exp\{-|y|^{\gamma}\}dy,$$

showing that the densities converge to a density, so that convergence in distribution follows from Scheffé's (1947) theorem.

Corollary 1 (unimodal case). The theorem remains true if (6) and (7) are replaced by the assumption that $f(x)$ is unimodal, that is, nondecreasing for a negative x and nonincreasing for positive x.

Proof. Choose $b = a/2$. The monotony assumptions and (5) imply that $\delta > 0$ and $\rho < 1$ can be found such that (6) and (7) are satisfied.

Corollary 2 (unimodal normal case). If assumptions (5), (6) and (7) are replaced by

$f(x)$ nondecreasing (or nonincreasing) for $x < 0$ (or $x > 0$)

$f'(0) = 0, \quad f''(0) < 0,$

then the random variable $n^{1/2}x_n$ will tend in distribution to a normal variate with mean equal to zero and variance equal to $-f(0)/f''(0)$.

We now state a multivariate analog which can be proved by the same methods.

Theorem 2. Let $f(\cdot)$ and $k(\cdot)$ be nonnegative functions of the vector $\underset{\sim}{x} = (x_1,\ldots,x_p)'$ satisfying $f(\underset{\sim}{0}) > 0$, $k(\underset{\sim}{0}) > 0$, $k(\underset{\sim}{x})$ continuous at $\underset{\sim}{x} = \underset{\sim}{0}$, $k(\underset{\sim}{x}) \leq K < \infty$, $\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\underset{\sim}{x})d\underset{\sim}{x} < \infty$. Define $g(\underset{\sim}{x}) = f(\underset{\sim}{x})/f(\underset{\sim}{0})$, $r^2 = \underset{\sim}{x}'\underset{\sim}{x}$, and assume that for some positive definite matrices A, B and for some $\gamma > 0$, $\delta > 0$, $0 < \rho < 1$,

$$g(\underset{\sim}{x}) = 1 - \underset{\sim}{x}'A\underset{\sim}{x} + o(r^2) \qquad \text{as } r \to 0$$

$$g(\underset{\sim}{x}) \leq 1 - \underset{\sim}{x}'B\underset{\sim}{x} \qquad \text{for } r < \delta$$

$$g(\underset{\sim}{x}) \leq \rho \qquad \text{for } r > \delta$$

Let x_n denote a random vector whose density of proportional to $\{f(\cdot)\}^n k(\cdot)$. Then the distribution of $\underset{\sim}{y}_n = (2n)^{1/2} \underset{\sim}{x}_n$ converges to the multivariate normal distribution with density proportional to $\exp\{-\frac{1}{2}\, \underset{\sim}{y}' A \underset{\sim}{y}\}$.

3. Examples.

Example 1 (gamma density). Let $x = z-1$, $f(x) = ze^{-z}$, $(z > 0)$, and let z_n be a variate with density proportional to $z^n e^{-nz}$, $(z > 0)$. According to Corollary 2, $n^{1/2}(z_n - 1)$ tends in distribution to standard normal. Of course z_n is a gamma variate, and the more familiar argument appealing to the central limit theorem gives the equivalent result that $n(n+1)^{-1/2}(z_n - 1 - n^{-1})$ tends to standard normal.

Example 2 (beta density). For $\alpha > 0$, $\beta > 0$, $m = \alpha/(\alpha+\beta)$, $x = z-m$, let $f(x) = z^{\alpha}(1-z)^{\beta}$, $0 < z < 1$, and let z_n be a beta variate whose density is proportional to $z^{n\alpha}(1-z)^{n\beta}$. Corollary 2 implies that $\sigma^{-1} n^{\frac{1}{2}}(z_n - m)$ tends to standard normal, where $\sigma^2 = -f(0)/f''(0) = \alpha\beta/(\alpha+\beta)^3$.

Example 3 (Student's t). Let $f(x) = (1 + x^2)^{-\frac{1}{2}}$, and let x_n have density proportional to $(1 + x^2)^{-n/2}$. Then $t = n^{\frac{1}{2}} x_n$ has Student's distribution with n degrees of freedom, and is asymptotically standard normal by Corollary 2.

Example 4 (Snedecor's F). For $\beta > \alpha > 0$, $m = \alpha/(\beta - \alpha)$, $x = z - m$, let $f(x) = z^{\alpha}(1+z)^{\beta}$ for $z > 0$, and let z_n be a variate whose density is proportional to $z^{n\alpha}(1+z)^{-n\beta}$ for $z > 0$. Corollary 2 implies that $\sigma^{-1} n^{\frac{1}{2}}(z_n - m)$ tends to standard normal, where $\sigma^2 = \alpha\beta(\beta-\alpha)^5$. If $k_1 = 2n\alpha + 2$, $k_2 = 2n(\beta - \alpha) - 2$, then $F = k_2 z_n / k_1$ has Snedecor's distribution with k_1 and k_2 degrees of freedom, showing that the F distribution with large degrees of freedom is approximately normal.

<u>Example 5</u> (order statistics). Let α and β be nonnegative integers, and let $h(\cdot)$ and $H(\cdot)$ be a bounded density and the corresponding cumulative distribution function. Assume that there is a unique m such that $h(m) > 0$, $h'(m)$ exists, and $H(m) = \alpha/(\alpha+\beta)$. Let $x = z - m$, and define $k(x) = h(z)$ and $f(x) = \{H(z)\}^{\alpha}\{1 - H(z)\}^{\beta}$. Then $f(x)$ is nondecreasing (or nonincreasing) for $x < 0$ (or $x > 0$), and $f'(0) = 0$. The function $\{f(x)\}^{n}k(x)$ is proportional to the density of $x_n = z_n - m$ where z_n is the $(\alpha n+1)$-th order statistic in a sample of size $(\alpha+\beta)n+1$ from the density $h(\cdot)$. By Corollary 2, $\sigma^{-1}n^{\frac{1}{2}}(z_n - m)$ tends in distribution to standard normal as n tends to infinity where $\sigma^2 = -f(0)/f''(0) = (\alpha+\beta)/\alpha\beta\{h(m)\}^2$.

REFERENCE

Scheffé, H. (1947). A useful convergence theorem for probability distributions. Annals of Mathematical Statistics. 18, 434-438.