A GENERALIZATION OF HELGASON'S SUPPORT THEOREM

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Abstract. We discuss a generalization of Helgason's support theorem for the Radon transform. In this theorem, the assumption of rapid decay of functions is essential. We restrict this rapid decay condition to an open cone and give a generalization. We also mention that our generalization is not possible with no global decay condition, to prove which we construct a counterexample.

Introduction and the main theorem

In this article, we study a generalization of Helgason's support theorem for the Radon transform. For a function f defined on $\mathbb{R}^n$, its Radon transform Rf is defined by

$$Rf(\theta, s) := \int_{\theta^\perp} f(s\theta + y)dy,$$

where $\theta \in S^{n-1}$, $s \in \mathbb{R}$, $\theta^\perp := \{x \in \mathbb{R}^n \ ; \ x \perp \theta\}$, $y \in \theta^\perp$. (θ, s) is identified with the hyperplane

$$\xi = \xi(\theta, s) = \{x \in \mathbb{R}^n \ ; \ x \cdot \theta = s\}.$$

For uniqueness of the exterior problem, the support theorem establish by S. Helgason is well-known.

Theorem 1. (cf. [He]) *Let K be a compact convex set in $\mathbb{R}^n$ and $f \in C(\mathbb{R}^n \setminus K)$. Assume that $Rf(\xi) = 0$ for $\xi \cap K = \emptyset$ and that*

$$|x|^k f(x) \to 0 \text{ as } |x| \to \infty, \quad \text{for } \forall k \in \mathbb{N}. \tag{1}$$

Then $f(x) = 0$ for $x \notin K$.

In this theorem, the condition (1) of rapid decay is essential. Without this condition, it does not hold that $f(x) = 0$ for $x \notin K$. There is a famous counterexample also by S.Helgason [He]. Let $n = 2$ and

$$f(x_1, x_2) := \frac{1}{(x_1 + ix_2)^\alpha},$$

where $\alpha > 1$. Change the values of f in a small compact convex neighborhood K of the origin so that f is smooth in $\mathbb{R}^2$. Consider the integrals along lines which do not intersect K, whose values are zero by Cauchy's integral theorem. By this argument, we conclude that the condition of the type $O(|x|^{-k})$ would not be sufficient for Theorem 1 for any $k \in \mathbb{N}$. Therefore (1) is essential for Theorem 1 to hold. It is interesting to consider the case where the rapid decay condition is restricted to some subset of $\mathbb{R}^n$. In 1993, J.Boman [B2] tried this modification.

Claim. (cf. Cor.4 in [B2]) *Let $f \in C(\mathbb{R}^n \setminus K)$, K be a compact convex set in $\mathbb{R}^n$, Γ be an open convex cone in $\mathbb{R}^n$ and*

$$K_\Gamma := \bigcap_{x \in K} (x + (\Gamma \cup (-\Gamma))).$$

Assume that

$$Rf(\xi) = 0 \quad \text{for } \forall \xi \cap K = \emptyset, \tag{2}$$

$$|x|^k f(x) \to 0 \;\; \text{uniformly as } |x| \to \infty \text{ in } \Gamma, \;\; \text{for } \forall k \in \mathbb{N}, \tag{3}$$

and that

$$f \text{ decays enough at infinity to be integrable on hyperplanes,} \tag{4}$$

for instance, $f(x) = O(|x|^{-n})$ as $|x| \to \infty$. Then $f(x) = 0$ in K_Γ.

Though the condition (4) is assumed in this claim, in Boman's argument [B2], the condition utilized was that $Rf(\xi)$ converges absolutely and is 0 for $\xi \cap K = \emptyset$. The author [T1] proved that this condition is not sufficient for $f(x) = 0$ in K_Γ. In fact, condition (4) itself is not sufficient for $f(x) = 0$ in K_Γ. In the next section, we give a counterexample which shows that (4) is not correct for Boman's claim

The author [T1], [T2] gave an exact decay condition and proved a generalization of Theorem 1.

Theorem 2. (cf. [T1],[T2]) *Let* $f \in C(\mathbb{R}^n \setminus K)$*,* K *be a compact convex set in* $\mathbb{R}^n$ *and* Γ *is an open convex cone in* $\mathbb{R}^n$*. Assume* (3),

$$Rf(\xi) = 0 \quad \text{for } \forall \xi (\xi \cap K_\Gamma \neq \emptyset \text{ and } \xi \cap K = \emptyset) \tag{5}$$

$$f(x) = o(|x|^{-n}) \text{ uniformly in } x \text{ as } |x| \to \infty. \tag{6}$$

Then $f(x) = 0$ *in* K_Γ*.*

(6) is an exact global decay condition to replace (4).

A counterexample

In this section, we construct a counterexample to the Claim above. The context of this section is included in [T2]. We construct an entire function $f \not\equiv 0$ on $\mathbb{C}$ satisfying the following conditions. $f(z)$ decays rapidly uniformly outside $\{1/4 < (\mathrm{Re}z)^2 - (\mathrm{Im}z)^2 < 4,\ \mathrm{Re}z < 0,\ \mathrm{Im}z > 0\}$, as $|z| \to \infty$. The Radon transform $Rf(l)$ of f converges absolutely for any line l in $\mathbb{C}$ and $Rf(l) = 0$ for any line l. Let $n = 2$ and we regard $\mathbb{R}^2 \cong \mathbb{C}$. We construct an entire function $g(z)$ defined on $\mathbb{C}$ satisfying the following conditions.

$$\int_l |g'(z)||dz| < \infty, \text{ for } \forall l, \tag{7}$$

$$\begin{aligned} &|z|^k g(z) \to 0 \text{ uniformly in } z \text{ as } |z| \to \infty,\ \forall k > 0, \\ &\text{for } z \in \mathbb{C} \setminus \left\{1/4 < (\mathrm{Re}z)^2 - (\mathrm{Im}z)^2 < 4,\ \mathrm{Re}z < 0,\ \mathrm{Im}z > 0\right\}. \end{aligned} \tag{8}$$

Assume that we obtain an entire function g satisfying (7) and (8). Let $f(z) := g'(z)$. By (7), $\int_l f(z)dz$ converges absolutely for any l. By (8) and Cauchy's integral theorem we obtain

$$\int_l f(z)dz = 0, \text{ for any } l.$$

Let, for example,

$$\Gamma := \left\{\frac{\pi}{3} < \arg z < \frac{2\pi}{3}\right\},$$

then f decreases rapidly in Γ, but f does not vanish in K_Γ. Therefore, what we have to do is the construction of an entire function g satisfying (7) and (8). Let

$$K := \{z \in \mathbb{C}\ ;\ |z| < 5\},$$

$$S := \{1/4 < (\mathrm{Re}z)^2 - (\mathrm{Im}z)^2 < 4,\ \mathrm{Re}z < 0,\ \mathrm{Im}z > 0\},$$

$$M := \mathbb{C} \setminus (K \cup S).$$

Note that M is a closed subset in $\mathbb{C}$ and $\hat{\mathbb{C}} \setminus M$ is connected and arcwise connected at infinity, where $\hat{\mathbb{C}}$ is the one-point compactification of $\mathbb{C}$. We put

$$\varphi(z) := iz^2 - i.$$

Note that we can define $0 < \arg \varphi(z) < 4\pi$ on M and that $\log \varphi(z)$ is defined as a single-valued holomorphic function on M. Consider

$$h(z) := \frac{1}{\varphi(z)^{\log \varphi(z)}} = e^{-(\log \varphi(z))^2} \in C(M) \cap \mathcal{A}(M^{\mathrm{int}}),$$

$$\varepsilon(t) := \frac{1}{(t^2-1)^{\log(t^2-1)}}.$$

Lemma. (cf. [A],[F]) *Let M be such a closed set in $\mathbb{C}$ that $\hat{\mathbb{C}} \setminus M$ is connected and arcwise connected at infinity in $\hat{\mathbb{C}}$. Assume that $\varepsilon(t) \geq 0$ is a decreasing function in t satisfying*

$$\int_1^\infty \frac{\log \varepsilon(t)}{t^{\frac{3}{2}}} dt > -\infty.$$

Then for any $h(z) \in C(M) \cap \mathcal{A}(M^{\mathrm{int}})$ there exists such an entire function $g(z)$ that

$$\Big| h(z) - g(z) \Big| < \varepsilon(|z|), \quad \text{for } \forall z \in M.$$

M, h and ε defined above satisfy the assumption of Lemma, therefore there exists an entire function $g(z)$ satisfying

$$\Big| \frac{1}{\varphi(z)^{\log \varphi(z)}} - g(z) \Big| < \frac{1}{(|z|^2-1)^{\log(|z|^2-1)}} \text{ for } z \in M.$$

Since

$$\left|\frac{1}{\varphi(z)^{\log\varphi(z)}}\right| = \frac{e^{(\arg\varphi(z))^2}}{|\varphi(z)|^{\log|\varphi(z)|}},$$

we have $g(z) \not\equiv 0$. In fact, assume $g \equiv 0$. Taking $z \in \mathbb{R}$ yields contradiction. Also, we have

$$|g(z)| \leq \frac{e^{16\pi^2}+1}{|\varphi(z)|^{\log|\varphi(z)|}} \text{ for } z \in M.$$

Therefore $g(z)$ is rapidly decreasing in M, which implies (8). Let $z \in M^{\text{int}}$ and

$$d = d(z) := \frac{1}{2}\text{dist}(z, \partial M),$$

where ∂M is the boundary of M. Put $L(z) := \max_{|\zeta - z| = d} |g(\zeta)|$ then we have

$$\frac{1}{2}|z| \leq |z| - d \leq |\zeta|, \text{ for } |\zeta - z| = d,$$

since $d(z) \leq \frac{1}{2}|z|$. Hence it holds by that

$$L(z) \leq \max_{|\zeta - z| = d} \frac{e^{16\pi^2}+1}{(|\zeta|^2-1)^{\log(|\zeta|^2-1)}} \leq (e^{16\pi^2}+1)e^{-(\log(\frac{|z|^2}{4}-1))^2}$$

Cauchy's integral formula yields

$$|g'(z)| \leq \frac{L(z)}{d(z)} \leq \frac{e^{16\pi^2}+1}{d(z)}e^{-(\log(\frac{|z|^2}{4}-1)^2}.$$

Since $d(z) = O(1/|z|)$ on the most critical line $\{\text{Im}z = -\text{Re}z\}$ as $-\text{Re}z = \text{Im}z \to \infty$, $|g'(z)|$ is integrable on all lines in $\mathbb{C}$. Thus we have (7). Therefore, $f(z) = g'(z)$ is our counterexample.

Proof of Theorem 2 and remarks

In this section, we give a proof of Theorem 2 and make some remarks. First, note that our counterexample in the previous section is stronger than the one established in [T1]. Our counterexample also suggests that uniqueness of the Radon transform does not hold without any global decay condition even if integrals along any hyperplanes absolutely converge. This fact was first proved by L.Zalcman [Z]. The construction of our counterexample in the previous section a modification of Zalcman's argument in [Z]. For uniqueness to hold, it is sufficient to assume that $f \in L^1$, which was also mentioned in [Z]. It is interesting that similar situation arises when we study uniqueness of the exterior problem. In Theorem 2, we assumed rapid decay of f in Γ which is a localized condition of the sufficient decay condition for Theorem 1, also in Theorem 2, we assumed (6) which is a kind of a localized L^1 condition.

Let us consider the global decay condition (6).

Definition. We imbed $\mathbb{R}^n$ in $S^n/2$ by

$$I : x \to \left(\frac{x}{\sqrt{1+|x|^2}}, \frac{1}{\sqrt{1+|x|^2}} \right),$$

where $x \in \mathbb{R}^n$. By this imbedding, a function $f(x)$ defined on $\mathbb{R}^n$ is regarded as the one $F(s)$ defined on $S^n/2$, that is,

$$F(s) := f(I^{-1}(s)), \quad s = I(x) \in S^n/2.$$

We extend F defined on $S^n/2$ to a function defined on S^n by identifying the antipodal points; i.e., $F(s) = F(-s)$, except on the equator $\{s_{n+1} = 0\}$, where $s = (s_1, \cdots, s_{n+1})$ is the coordinate for S^n. We define $n-1$ form $d\mu(I(\xi), s)$ on $I(\xi)$ by

$$Rf(\xi) = 2 \int_{x \in \xi} F(I(x)) d\mu(I(\xi), I(x)),$$

for f for which $Rf(\xi)$ is well-defined.

Let $\xi_t = \{x_n = t\}$ then

$$Rf(\xi_t) = 2 \int_{I(\xi_t)} F(s) \frac{1}{s_{n+1}^n} ds,$$

where ds is the $n-1$-dimensional surface measure on $\overline{I(\xi_t) \cup (-I(\xi_t))}$. By the definition, the measure $d\mu(I(\xi), s)$ has singularity of the type $1/s_{n+1}^n$ at $s_{n+1} = 0$ for any $I(\xi)$.

Proposition 1. *Assume that $f(x)$ defined on $\mathbb{R}^n$ satisfy* (6) *then*

$$\lim_{s_{n+1} \to 0} \frac{F(s)}{s_{n+1}^n} = 0.$$

The proof is obtained by easy calculation. Take f satisfying (6) and we have $F(s) = F_1(s) s_{n+1}^n$ with $F_1 \in C(S^n/2)$. Since $s_{n+1}^n d\mu(I(\xi), s)$ is an analytic measure in $(I(\xi), s)$ we have

$$\int_{s \in \{s_{n+1}=0\}} F(s) d\mu(\{s_{n+1} = 0\}, s) = 0$$

and

$$Rf(\xi) = \int_{s \in S_\xi} F(s) d\mu(S_\xi, s),$$

where $S_\xi = \overline{I(\xi) \cup (-I(\xi))}$. Note that we identified $d\mu(I(\xi), s) = d\mu(-I(\xi), -s)$. The condition (6) implies that $F(s)$ is not singular on $\{s_{n+1} = 0\}$, which gives a sufficient condition for Theorem 2 to hold.

J.Boman claimed that $f(x) = O(|x|^{-n})$, as $|x| \to \infty$, would be sufficient for Theorem 2 (cf. Claim in Section 1), however, we are not able to judge whether it is true by our argument in this paper.

In Theorem 2, we have assumed that Γ is an open convex cone, however, we can omit this assumption. More precisely, the following theorem holds.

Theorem 3. *Let $f \in C(\mathbb{R}^n \setminus K)$, K be a compact convex set in $\mathbb{R}^n$ and Γ is an open cone in $\mathbb{R}^n$. Assume* (3), (5) *and* (6). *Then $f(x) = 0$ in $K_{\widehat{\Gamma}}$, where $\widehat{\Gamma}$ is the convex hull of Γ.*

By Theorem 2, $f(x) = 0$ in K_Γ. (5) and Holmgren's uniqueness theorem [Hö] yields that $f(x) = 0$ in $K_{\widehat{\Gamma}}$ by the sweeping out method.

Theorem 1 accompanies Theorem 2, moreover, by virtue of Theorem 2, it is sufficient to assume (1) in an open cone whose convex hull is the whole space.

Theorem 4. *Assume that Γ be an open cone whose convex hull $\widehat{\Gamma} = \mathbb{R}^n$. Let K be a compact convex set in $\mathbb{R}^n$, $f \in C(\mathbb{R}^n \setminus K)$. If* (3), (6) *$Rf(\xi) = 0$ for $\xi \cap K = \emptyset$ hold then $f(x) = 0$ for $x \notin K$.*

Theorem 1 is extended for any singular functions [TK], however, we have to assume $f \in C$ for Theorems 2 and 3. If we assume (3) in $\Gamma \cup (-\Gamma)$, not in Γ, then Theorems 2 and 3 also extends for functions with singularities with a little modification (cf Theorem 5 below). We study the relation between the regularity of functions and the sufficient decay condition for uniqueness of the exterior problem. This relation is closely related to uniqueness of functions with microlocal analyticity.

First, let us review the proof of Theorem 2 (equivalently, the proofs of Theorems 3 and 4) shortly. Assume (3). By (5) and (6), $F(s)$ and its derivatives of all orders along normal directions to $\{s_{n+1} = 0\}$ tend to 0 as $s_{n+1} \to +0,\ s \in I(\Gamma)$. $\int_{s \in S_\xi} F(s) d\mu(S_\xi, s) = 0$ for any $\xi \cap K = \emptyset$ yields that $WF_A(F) \cap N^*(S_\xi) = \emptyset$ for any $\xi \cap K = \emptyset$ (cf. [B2]). Then we apply a local vanishing theorem for distributions (cf. [B1] and [TT]).

Proposition 2. *Let $S \subset \mathbb{R}^n$ be a real analytic submanifold. Assume f be a distribution satisfying*

$$N^*(S) \cap WF_A(f) = \phi,$$

where $WF_A(f)$ is the analytic wave front set of f and $N^*(S)$ is the conormal bundle of S. Assume that the restrictions to S of f and all its derivatives along the conormal direction to S vanish, that is,

$$\partial_\xi^\alpha f|_S = 0 \quad \text{for all } \alpha,$$

where $(x, \xi) \in N^*(S)$. Then $f = 0$ in some neighborhood of S.

Remark that when f is continuous, it is sufficient for this proposition to assume that the boundary values from one side to S of f and all its derivatives vanish. Therefore, F vanishes in a neighborhood of $\{s_{n+1} = 0\} \cap \overline{(I(\Gamma) \cup (-I(\Gamma)))}$. Holmgren's uniqueness theorem (cf. [Hö]) gives the theorems.

Note that for hyperfunctions, any decay condition would not imply that they are regular at infinity. Hence we had to assume

$$\int_{s_{n+1}=0} F(s) \frac{1}{s_{n+1}^n} ds = 0 \tag{9}$$

for hyperfunctions.

Theorem 5. *Assume the same assumptions on Γ and K as Theorem 4. Let f be a Fourier hyperfunction with defining function of the order $o(|z|^{-n})$. Suppose that as a Fourier hyperfunction on $\Gamma \cup (-\Gamma)$, f decays exponentially. If $Rf(\xi) = 0$ for $\xi \cap K = \emptyset$ and (9) hold then* $\mathrm{supp} f$ *is contained in the complement of $K_{\widehat{\Gamma}}$.*

For the decay condition of hyperfunctions, confer [K2] and [TK]. We introduce the idea of the proof of Theorem 5. $Rf(\xi) = 0$ for $\xi \cap K = \emptyset$ and (9) imply that F is micro analytic at $\{s_{n+1} = 0\}$ in its conormal directions. Exponential decay of f yields that $P(D')F|_{s_{n+1}=0} = 0$, where $P(D')$ is any differential operator along conormal direction to $\{s_{n+1} = 0\}$ with constant coefficients, the symbol of P is an infra-exponential entire function. Hence uniqueness of hyperfunction with analytic parameter [K1] and Kashiwara-Kawai's theorem (Theorem 4.4.1 in [K2]) prove the theorem. Note that we have to assume exponential decay on $\Gamma \cup (-\Gamma)$, which yields the restriction to $\{s_{n+1} = 0\} \cap \overline{(I(\Gamma) \cup (-I(\Gamma)))}$ of the derivatives with any local operator vanish. This condition is very important for hyperfunctions.

Remark. i) Since Proposition 2 holds for non-quasi-analytic ultradistributions, Theorem 3 is extended for non-quasi-analytic ultradistributions with the assumptions (3) in $\Gamma \cup (-\Gamma)$, (5), (6) and (9).

ii) Proposition 2 does not hold for quasi-analytic ultradistributions [B3] (then for hyperfunctions). For hyperfunctions it is sufficient to assume that f decays exponentially in $\Gamma \cup (-\Gamma)$ because of the uniqueness of hyperfunctions with analytic parameters established by A.Kaneko [K1]. This condition is replaced by the one expressed in the terms of associated functions when we study quasi-analytic ultradistributions. It is interesting to study whether the decay condition in $\Gamma \cup (-\Gamma)$ in Theorem 5 is weakened, which is left open.

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