

COALITIONS, CORE, AND COMPETITION

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We give simplified proofs for a very general form of the basic core result of Vind [12] and Aumann [1]. The method is similar to that of Cornwall [3], [4] but yields a more general result. The framework and the method employed are useful for extensions in several directions.

Aumann showed that, in a particular kind of measure space of traders, the core coincides with the set of competitive allocations. Vind replaced individual traders and their preferences with coalitions (sets) of traders and coalition preferences. Here, as in Cornwall, we remove individual traders further from the scene by taking as coalitions abstract objects whose combinations follow the laws of Boolean algebra. This provides a formal theory of coalitions, rather than one of individuals and coalitions. We show how this results in a more widely applicable theory, which includes the Aumann, Vind, and Cornwall results on identity of core and competitive allocations. Our tools give simpler proofs, and seem promising for extensions in several directions and to coalition research in general.

Coalitions. Intuitively speaking, if we have any two coalitions, it seems reasonable that we should be able to "combine" them to form a new "larger" coalition. And, given any one coalition, it seems natural to consider its "complement" as another coalition.¹ This suggests a Boolean algebra as a natural model for a theory of coalitions. We shall work, then, with a Boolean algebra $\mathfrak{A} = \langle A, \vee, \wedge, -, 0, 1 \rangle$.

(We shall not develop the theory of Boolean algebras, but will make specific references to the standard expositions [7,11,2], whose notation we employ.)

It is sometimes reasonable and usually convenient to assume we can also coalesce any countable set of coalitions a_k into a single coalition $\bigvee a_k$, so we shall at times assume that \mathfrak{A} is a Boolean σ -algebra. (Correspondingly, $\bigwedge a_k$ will denote the infimum of a countable set of coalitions a_k .)

Allocations. Intuitively, we consider an economy with a fixed endowment which can be reallocated among the coalitions. Let $\nu = \langle \nu_1, \dots, \nu_n \rangle$ be a nontrivial countably additive vector measure on \mathfrak{A} , with range included in Ω , the nonnegative orthant of \mathbb{R}^n . We may think of ν as allotting to each coalition $a \in \mathfrak{A}$ a certain endowment $\nu(a)$ of commodities which it may trade. We call any countably additive vector measure α on \mathfrak{A} an allocation if its range is included in Ω ; if also $\alpha(1) = \nu(1)$, then we shall call α feasible. The set of all allocations we denote by M . We shall make frequent reference to the real valued measure $\bar{\nu}$ formed by summing the components ν_i : $\bar{\nu} = \sum_{i=1}^n \nu_i$. When any $\alpha \in M$ has the property that, for all $a \in \mathfrak{A}$, $\bar{\nu}(a) = 0 \Rightarrow \alpha(a) = 0$, then we say that α is $\bar{\nu}$ -continuous. The set of all $\bar{\nu}$ -continuous allocations in M will be called C . Coalitions a with $\bar{\nu}(a) = 0$ will be called null. We assume that $\bar{\nu}(1) > 0$.

To motivate the economic activity of coalitions we assume that each coalition a has a preference relation \succ_a defined on M . Intuitively, $\alpha \succ_a \beta$ has the interpretation that all "members" of coalition a prefer (what they "receive" under) allocation α to allocation β . Under this interpretation, it is natural to postulate that, if $\alpha \succ_a \beta$, then, for any subcoalition b (i.e., $b \leq a$)², we should have $\alpha \succ_b \beta$. Also, for any $a, b \in \mathfrak{A}$, if $\alpha \succ_a \beta$ and $\alpha \succ_b \beta$,

we should have $\alpha \succ_{\mathcal{U}, \nu} \beta$. Formally, then, we postulate that, for any $\alpha, \beta \in M$, the set $P(\alpha, \beta) = \{a \in A: \alpha \succ_a \beta\}$ is an ideal in \mathcal{U} ,³ and say that \succ is an ideal preference.

We also assume that each coalition is concerned only with what each allocation provides it and its subcoalitions. This selfish aspect is expressed by our assumption that, for any $\alpha, \beta, \gamma \in M$ and any $a \in A$, if the restrictions $\alpha|_a$ and $\beta|_a$ of α and β to the subcoalitions algebra $A^a = \{b \in A: b \leq a\}$ are identical, then $\alpha \succ_a \gamma \Leftrightarrow \beta \succ_a \gamma$ and $\gamma \succ_a \alpha \Leftrightarrow \gamma \succ_a \beta$.

Certain allocations are of special economic interest. For any $H \subseteq M$, $\alpha, \beta \in H$ and $a \in A$, we say that α blocks β via a (with respect to $\langle \mathcal{U}, \succ, \nu, H \rangle$) if $\bar{\nu}(a) > 0$ & $\alpha \succ_a \beta$ and $\alpha(a) = \nu(a)$. If α blocks β via some $a \in A$, we say that α blocks β . If there is no $\alpha \in H$ such that α blocks $\beta \in H$, and if β is feasible, then we say that β is in the core (of $\langle \mathcal{U}, \succ, \nu, H \rangle$).

For any $p \in \mathbb{R}^n$, $a \in A$, $H \subseteq M$, and $\alpha, \beta \in H$, we say that α obstructs β via a (with respect to $\langle p, \mathcal{U}, \succ, \nu, H \rangle$) if $\bar{\nu}(a) > 0$ & $\alpha \succ_a \beta$ and $p \cdot \alpha(a) \leq p \cdot \nu(a)$.⁴ If α obstructs β via some $a \in A$, then we say that α obstructs β . If there is no $\alpha \in H$ which obstructs feasible $\beta \in H$, and if $p \cdot \beta(a) \leq p \cdot \nu(a)$ for all $a \in A$, then we say that $\langle p, \beta \rangle$ is a competitive (trade) equilibrium with respect to $\langle \mathcal{U}, \succ, \nu, H \rangle$. If $\langle p, \beta \rangle$ is a competitive equilibrium for some $p \in \mathbb{R}^n$, we call β a competitive allocation. If in the definition of obstruction we require $p \cdot \alpha(a) < p \cdot \nu(a)$, then we say that α strongly obstructs β , and the resulting equilibrium notion is a quasi-equilibrium.⁵

Our concern here is with the relationship between core allocations and competitive allocations.

Theorem 1. For all $H \subseteq M$, if $\alpha \in M$ and $\langle p, \alpha \rangle$ is a competitive equilibrium with respect to $\langle \mathcal{A}, \succ, v, H \rangle$, then α is in the core of $\langle \mathcal{A}, \succ, v, H \rangle$.

Proof. If $\bar{v}(a) > 0$, $\beta \in H$, and $\beta \succ_a \alpha$, then $p \cdot \beta(a) > p \cdot v(a)$, by the definition of competitive equilibrium. Thus $\beta(a) \neq v(a)$, so β can not block α via a . Since no $\beta \in H$ can block α , α is in the core. Q.E.D.

Remarks. Although our motivation for concepts has occasionally referred to individual traders, or coalition members, we have avoided reference to individuals in all our formal definitions. This is as it should be if it is proper to consider ours a theory of coalitions, and not a theory of individuals and coalitions. And it is as it can be, if we talk not of fields of sets, but of abstract Boolean algebras.⁶

It is reassuring that we can always pass from Boolean algebras to fields of sets and thus obtain individuals. For the Stone Isomorphism Theorem tells us that every Boolean algebra is isomorphic to a field of sets.⁷ Thus it would seem, at first glance, that the algebraic and set theoretic approaches are equivalent, and the choice is one of terminology. In fact, we could then push coalitions further to the background, emphasizing individuals by replacing \bar{v} -continuous allocations⁸ in M by their Radon-Nikodym derivatives defined on the universe of the field of sets. This would make explicit the allocations to individuals, and if we could go from coalition preferences to individual preferences⁹ we would have a more conventional model like Aumann's.

However, not every Boolean σ -algebra is isomorphic to a σ -field of sets, under an isomorphism preserving countable operations. No such isomorphism exists, for example, for the Boolean algebra of Lebesgue measurable subsets of the unit interval $[0,1]$ modulo the ideal of Lebesgue null sets.¹⁰ Indeed, none exists for any nonatomic measure algebra.¹¹ Thus, technically, the Boolean algebraic approach provides greater generality.¹²

The fact that there is no need to mention individuals, and that we can achieve greater generality by not doing so, suggests that the essential concepts and properties of our theory are better exposed by taking Boolean algebras rather than set fields as our starting point. No essential change is required in technique.

One further advantage of Boolean algebras over set algebras is that the former allow us to eliminate null sets. In particular, we are able to concentrate our attention on the quotient algebra \mathfrak{A}/J , where J is the ideal of null coalitions. Intuitively, $\bar{\nu}$ -null coalitions have no endowment, and so can not be expected to play any role in blocking or obstructing any allocation.

Working with Boolean algebras rather than set fields has forced us to express allocations as measures rather than as measurable functions or their integrals (with respect to $\bar{\nu}$). But this again turns out to be an advantage, leading to even greater generality. For we can work with arbitrary measures as allocations, while it is only the $\bar{\nu}$ -continuous ones which correspond to the latter notion of allocation.

We shall say that the preference relation $>$ is monotone (with respect to $\langle \mathfrak{A}, \nu, M \rangle$) if, for all $a \in A$ and all $\alpha, \beta \in M$,

$$[\forall b_b \cong a \ \& \ \bar{\nu}(b) > 0 \ \alpha(b) \cong \beta(b)] \Rightarrow \alpha > \beta.^{13}$$

The following series of propositions shows the critical role played by $\bar{\nu}$ -continuous allocations under our three preference assumptions (the ideal property, selfishness,¹⁴ and monotonicity.

Proposition 1. Let \succ be monotone with respect to (\mathfrak{A}, ν, M) . If α is in the core or is a competitive equilibrium with respect to $(\mathfrak{A}, \succ, \nu, M)$, then α is $\bar{\nu}$ -continuous.

Proof. Suppose α is not $\bar{\nu}$ -continuous; say $\alpha(a) \geq 0$ and $\bar{\nu}(a) = 0$. Then $\alpha(\bar{a}) = \alpha(1) - \alpha(a) = \nu(1) - \alpha(a) \leq \nu(1) - \nu(a) = \nu(\bar{a})$. Thus coalition \bar{a} receives more under its initial endowment than under allocation α . By redistributing this excess uniformly over subcoalitions, we obtain an allocation which, by monotonicity, blocks α . That is, the allocation assigning

$$\alpha(c) + \frac{\bar{\nu}(c)}{\bar{\nu}(\bar{a})} (\nu(\bar{a}) - \alpha(\bar{a}))$$

to each subcoalition $c \subseteq \bar{a}$, and equaling ν elsewhere, blocks α via \bar{a} . So α is not in the core. By Theorem 1, α is not a competitive equilibrium. Q.E.D.

Proposition 2. Suppose \mathfrak{A} is a Boolean σ -algebra, \succ is selfish and ideal, and $\alpha, \beta \in M$, $a \in A$, and $\alpha \succ_a \beta$. Then there exists $b \subseteq a$ with $\nu(b) = \nu(a)$ and there exists $\gamma \in C$ such that $\gamma \succ_b \beta$ and $\gamma(b) = \alpha(b)$.

Proof. Let $\alpha = \alpha_c + \alpha_s$ be the Lebesgue decomposition¹⁵ of α into nonnegative $\bar{\nu}$ -continuous and $\bar{\nu}$ -singular parts α_c and α_s ; thus, for some null e and all $c \in A$, $\alpha_s(c) = \alpha(c \wedge e)$. Let $b = a \wedge \bar{e}$, so $\nu(b) = \nu(a)$. Let $\gamma = \alpha_c$, so γ is $\bar{\nu}$ -continuous and $\gamma(b) = \alpha_c(b) = \alpha(b)$. Since $\alpha \succ_a \beta$ & $b \subseteq a$, $\alpha \succ_b \beta$; but $\alpha_c|_b = \alpha|_b$, so by selfishness $\gamma = \alpha_c \succ_b \beta$. Q.E.D.

Because equality need not hold between $\gamma(b)$ and $\nu(b)$ in the conclusion of Proposition 2, we can not say that γ blocks β . The next proposition will show how β may be blocked in all situations of

interest, but first we give the mathematical Lemma on which it (and a later result) is based.

Lemma. If ν is nonatomic¹⁶ and if α and β are $\bar{\nu}$ -continuous measures on the Boolean σ -algebra \mathfrak{A} into R^m and R^p for some positive integers m and p , and if $a, b \in A$, then, for any $t \in [0, 1]$ there exist $a', b' \in A$ with $a' \wedge b' = 0$, $a' \leq a, b' \leq b$, and $\alpha(a') = t\alpha(a)$ & $\beta(b') = (1 - t)\beta(b)$.

Proof (as in [3, Theorem 1]). Since α and β are $\bar{\nu}$ -continuous and $\bar{\nu}$ is nonatomic, α and β are nonatomic.¹⁷ So also, then, is the vector measure $\gamma = \langle \alpha, \beta \rangle$ on \mathfrak{A} which takes values in the nonnegative orthant of R^{2n} . By Lyapunov's theorem [10,9] we may partition each of $c_1 = a \wedge \bar{b}$, $c_2 = b \wedge \bar{a}$, and $c_3 = a \wedge b$ into two disjoint parts c_{11}, c_{12} and c_{21}, c_{22} and c_{31}, c_{32} with γ -values $\gamma(c_{11}) = t\gamma(c_1)$ and $\gamma(c_{12}) = (1 - t)\gamma(c_1)$. Then, putting the pieces together, we see that $a' = c_{11} \vee c_{31}$ is disjoint from $b' = c_{22} \vee c_{32}$, and $\gamma(a') = t\gamma(a)$ and $\gamma(b') = (1 - t)\gamma(b)$; thus $\alpha(a') = t\alpha(a)$ and $\beta(b') = (1 - t)\beta(b)$. Q.E.D.

Proposition 3. Let \succ be ideal, selfish, and monotone. If \mathfrak{A} is a Boolean σ -algebra, ν is nonatomic, and $\alpha \in M$ blocks feasible $\beta \in C$ for $\langle \mathfrak{A}, \succ, \nu, M \rangle$, then some $\delta \in C$ also blocks β for $\langle \mathfrak{A}, \succ, \nu, M \rangle$.

Proof. Suppose $\alpha \succ_a \beta$ & $\alpha(a) = \nu(a)$ & $\bar{\nu}(a) > 0$. Then by Proposition 2 there exists $\gamma \in C$ and $b \leq a$ with $\gamma \succ_b \beta$ & $\gamma(b) = \alpha(b) \leq \alpha(a) = \nu(a) = \nu(b)$. If $\gamma(b) = \nu(b)$ we are done, so suppose $\gamma(b) < \nu(b)$. Let ζ be the measure $\langle \beta - \nu, \bar{\nu} \rangle$ on \mathfrak{A} with values in R^{n+1} , and let η be the vector measure with values in R^n defined by $\eta = \gamma - \nu$. Since $\beta, \gamma \in C$, ζ and η are $\bar{\nu}$ -continuous, so by the Lemma there exist nonnull $a' \leq 1$ and $b' \leq b$ with

$a' \wedge b' = 0$ and $\zeta(a') = \frac{1}{2}\zeta(1)$ & $\eta(b') = \frac{1}{2}\eta(b)$. Thus $\beta(a') - v(a') = 0$ & $\bar{v}(a') = \frac{1}{2}\bar{v}(a) > 0$; and, since $v(b) \geq \gamma(b)$, we have $v(b') - \gamma(b') \geq 0$. Because $b' \leq b$, $\gamma > \beta$. Now on a' , let δ distribute the "excess," $v(b') - \gamma(b') \geq 0$, as follows

$$\forall_{c \leq a'} \delta(c) = \beta(c) + \frac{\bar{v}(c)}{v(a')} (v(b') - \gamma(b')),$$

and let $\delta|_{b'} = \gamma|_{b'}$ and $\delta = v$ elsewhere. Then monotonicity implies that $\delta \succ_a \beta$, and selfishness implies that $\delta \succ_b \beta$. Thus $\delta \succ_{a \vee b} \beta$ and $\delta(a' \vee b') = v(a' \vee b') \geq 0$, so \bar{v} -continuous δ blocks β . Q.E.D.

Proposition 4. Let \succ be ideal, selfish, and monotone. If \mathfrak{A} is a Boolean σ -algebra and v is nonatomic, then for any α and β , α is in the core of $\langle \mathfrak{A}, \succ, v, M \rangle$ if and only if α is in the core of $\langle \mathfrak{A}, \succ, v, C \rangle$; $\langle p, \beta \rangle$ is a competitive equilibrium for $\langle \mathfrak{A}, \succ, v, M \rangle$ if and only if $\langle p, \beta \rangle$ is a competitive equilibrium for $\langle \mathfrak{A}, \succ, v, C \rangle$; and a quasi-equilibrium for $\langle \mathfrak{A}, \succ, v, C \rangle$ is a quasi-equilibrium for $\langle \mathfrak{A}, \succ, v, M \rangle$.

Proof. If α and β are core and competitive equilibrium allocations for $\langle \mathfrak{A}, \succ, v, M \rangle$, then by Proposition 1, $\alpha, \beta \in C$. If no allocation in M blocks α , then a fortiori no allocation in $C \subseteq M$ blocks α ; and any allocation in C which might obstruct β with respect to $\langle \mathfrak{A}, \succ, v, C \rangle$ would also obstruct with respect to $\langle \mathfrak{A}, \succ, v, M \rangle$.

Conversely, suppose α is in the core for $\langle \mathfrak{A}, \succ, v, C \rangle$. Then the same is true with respect to $\langle \mathfrak{A}, \succ, v, M \rangle$, since any blocking by elements of M could also be effected by elements C , according to Proposition 3. And if $\langle p, \beta \rangle$ is a competitive or quasi-equilibrium for $\langle \mathfrak{A}, \succ, v, C \rangle$, then note that $p \geq 0$. For if, say $p_1 < 0$, then $\gamma(b) = \beta(b) + \langle \bar{v}(b), 0, \dots, 0 \rangle$ defines a $\gamma \in C$ with $\gamma \succ_1 \beta$ (by monotonicity). But

$p \cdot \gamma(1) < p \cdot \beta(1) = p \cdot \bar{v}(1)$, obstructing β ; so $p \geq 0$. Then, it follows from Proposition 2 that any obstructing or (strong obstructing) by elements of M could also be effected by elements of C . Q.E.D.

Proposition 4 allows us to focus our attention on just the \bar{v} -continuous allocations.¹⁸ The subset of M consisting of these allocations we have denoted by C . We now can see that allocations on \mathfrak{A} in C induce allocations in the quotient algebra \mathfrak{A}/J .

For every $\alpha \in C$, we define a measure $\underline{\alpha}$ (also written as α/J) on \mathfrak{A}/J with values in Ω as follows: for any $a/J \in \mathfrak{A}/J$, $\underline{\alpha}(a/J) = \alpha(a)$. Since all elements a' of the equivalence class a/J differ from a only on null sets, and since α is \bar{v} -continuous, it is apparent that $\underline{\alpha}$ is well-defined. Conversely, every allocation on \mathfrak{A}/J arises thus from some (unique) allocation in C . We denote, then, by C/J the set of all allocations on \mathfrak{A}/J .

We next define the preference $>/J$ induced on \mathfrak{A}/J by $>$ on \mathfrak{A} . For any $\underline{\alpha}, \underline{\beta} \in C/J$ and any $\underline{a} = a/J$, we say that $\underline{\alpha} ((>/J)_{\underline{a}}) \underline{\beta}$ if there exists $a' \in A$ such that $\underline{\alpha} = \alpha/J$, $\underline{\beta} = \beta/J$, $\underline{a} = a'/J$, and $\alpha >_{a'} \beta$. It is easily verified that any ideal, monotonicity, or selfishness properties of $>$ are preserved by $>/J$.

Proposition 4 permitted us to pass from $\langle \mathfrak{A}, >, v, M \rangle$ to $\langle \mathfrak{A}, >, v, C \rangle$ and back. We now show that we can further pass between $\langle \mathfrak{A}, >, v, C \rangle$ and $\langle \mathfrak{A}/J, >/J, v/J, C/J \rangle$.

Proposition 5. For any $\alpha, \beta \in C$, α is in the core and $\langle p, \beta \rangle$ is a competitive equilibrium for $\langle \mathfrak{A}, >, v, C \rangle$ if and only if $\underline{\alpha}$ is in the core and $\langle p, \underline{\beta} \rangle$ is a competitive equilibrium for $\langle \mathfrak{A}/J, >/J, v/J, C/J \rangle$.

Proof. The "if" assertions are trivial. The "only if" may be proved as follows. Let α and β be core and competitive allocations, respectively. If γ blocks α via a/J , then, for some b with $\bar{v}(b) = \bar{v}(a)$ we have $\gamma >_b \alpha$ and $\gamma(b) = v(b)$, contradicting that α is in the core. A similar proof by contradiction shows that $\langle p, \underline{\beta} \rangle$ is a competitive equilibrium. Q.E.D.

Proposition 4 will simplify the proof of our next theorem. This theorem shows, in a sense, that when individual traders are small relative to the market, the core allocations are the quasi-competitive ones. Of course, we cannot talk of individual traders in our Boolean algebraic model, for we only have coalitions. However, the economic size of an individual trader is intuitively represented by his initial endowment, and we can express through this means that traders are small. We assume, in particular, that ν is a nonatomic vector measure; thus, for any coalition $a \in A$ with $\nu_i(a) > 0$ for some component ν_i of ν , there exists a subcoalition $b \subseteq a$ with $\nu_i(a) > \nu_i(b) > 0$. This prevents any individual from being a (atomic) coalition by himself, and provides a weak expression of the negligibility of individuals.

Remark. If \mathfrak{A} is a Boolean σ -algebra and we choose units of measurement so that $\bar{\nu}(1) = 1$, then $\langle \mathfrak{A}/J, \bar{\nu}/J \rangle$ has become a probability algebra.¹⁹ If in addition $\bar{\nu}$ is nonatomic, and if we assume \mathfrak{A} is "minimal" in the sense that it is σ -generated²⁰ by a countable set of coalitions, then $\langle \mathfrak{A}/J, \bar{\nu}/J \rangle$ has a simple representation. It is isomorphic to the probability algebras $\langle \mathfrak{B}/N, \mu/N \rangle$ and $\langle \mathfrak{C}/N', \mu'/N' \rangle$, where \mathfrak{B} and \mathfrak{C} are the σ -fields of Borel and Lebesgue measurable subsets, respectively, of the unit interval $[0,1]$, and where N and N' are the families of null sets in \mathfrak{B} and \mathfrak{C} respectively, with respect to Lebesgue measure μ on \mathfrak{B} and μ' on \mathfrak{C} .²¹ Our model is determined then up to null sets, which are economically irrelevant under the above definitions of core and competitive equilibrium. Thus, the particular probability space used by Aumann, far from requiring justification, is essentially forced upon us by nonatomicity and minimality. If we drop the minimality condition, we still have, up to null sets, some nonatomic probability space, which Aumann observed²² would do as well.

We shall say that \succ is lower semicontinuous for $\langle \mathfrak{A}, M \rangle$ if for all nonnull $a \in A$ and all $\alpha \in M$, $\{\beta(a) : \beta \in M \text{ \& } \beta \succ_a \alpha\}$ is open relative to Ω .

Theorem 2. Let \mathfrak{A} be a Boolean σ -algebra, and let ν be nonatomic with $\nu(1) > 0$. Let \succ be ideal, selfish, monotone, and lower semicontinuous for $\langle \mathfrak{A}, \nu, M \rangle$. Let $\alpha \in M$. If α is in the core for $\langle \mathfrak{A}, \succ, \nu, M \rangle$, then for some $p \in \Omega$, $\langle p, \alpha \rangle$ is a competitive equilibrium for $\langle \mathfrak{A}, \succ, \nu, M \rangle$.

Remark. The literature offers us two routes to follow in proving Theorem 2. Vind's method employs slightly stronger assumptions, and requires development of properties of set-measures. On the other hand, it is suitable for our present framework in which preferences are defined for coalitions, rather than individuals.

A second approach would involve using Debreu's results [5] to pass from coalition preferences to individual preferences, and from measure-allocations to their Radon-Nikodym derivatives, and then using Aumann's method of proof. This would require much stronger assumptions on preference.²³

We follow, therefore, a third, hybrid route, very similar to Cornwall's [3]. We show how Aumann's method works in our Vind-like framework.

Proof of Theorem 2. By Proposition 4 we may assume that α belongs to the core of $\langle \mathfrak{A}, \succ, \nu, C \rangle$ and it suffices to show that α is a competitive equilibrium with respect to $\langle \mathfrak{A}, \succ, \nu, C \rangle$. For each nonnull $a \in A$, let $G(a) = \{\beta(a) - \nu(a) : \beta \in C \text{ \& } \beta \succ_a \alpha\}$. Let $G = \bigcup \{G(a) : a \in A \text{ \& } \bar{\nu}(a) > 0\}$.

Clearly, $0 \notin G$: otherwise α would be blocked via some $a \in A$ with respect to $\langle \mathfrak{A}, \succ, \nu, C \rangle$.

We show that G is convex. For suppose $x_1 = \beta_1(a_1) - \nu(a_1) \in G(a_1)$ & $x_2 = \beta_2(a_2) - \nu(a_2) \in G(a_2)$ and $x = tx_1 + (1-t)x_2$ for $0 < t < 1$.

Let ζ and η be the \bar{v} -continuous measures $\beta_1 - v$ and $\beta_2 - v$ on \mathcal{A} into \mathbb{R}^n . Then by the Lemma there exist $a_1', a_2' \in A$ such that $a_1' \preceq a_1$ & $a_2' \preceq a_2$ & $a_1' \wedge a_2' = 0$ & $\zeta(a_1') = t\zeta(a_1)$ & $\eta(a_2') = (1-t)\eta(a_2)$. Thus $\beta_1(a_1') - v(a_1') = tx_1$ and $\beta_2(a_2') - v(a_2') = (1-t)x_2$ and, since $a_1' \preceq a_1$, $\beta_1 \succ_{a_1} \alpha$. Then letting $a = a_1' \vee a_2'$, and defining $\beta = \beta_1$ on A^* and $\beta = v$ elsewhere, we have $\beta \in C$ and $\beta \succ_a \alpha$ and $\beta(a) - v(a) = tx_1 + (1-t)x_2$. Thus $tx_1 + (1-t)x_2 \in G(a)$, so $tx_1 + (1-t)x_2 \in G$.

Since 0 is not in convex G , there is a hyperplane with normal $p \neq 0$ weakly separating 0 from G : $p \cdot x \geq 0$ for all $x \in G$. We note that, for any $a \in A$ and $\beta \in C$, $\bar{v}(a) > 0$ & $\beta \succ_a \alpha$ implies $\beta(a) - v(a) \in G(a)$, so $p \cdot \beta(a) \geq p \cdot v(a)$. Finally, we see that $p \cdot \alpha(a) \leq p \cdot v(a)$ for all $a \in A$ as follows (cf. [1, p. 46]). By monotonicity of preferences, for each $a \in A$ there are $\beta \in C$ with $\beta(a)$ arbitrarily closed to $\alpha(a)$ and preferred to $\alpha(a)$, so $p \cdot \alpha(a) \geq p \cdot v(a)$. Thus, if $p \cdot \alpha(a) > p \cdot v(a)$ for some nonnull $a \in A$, then $p \cdot \alpha(1) > p \cdot v(1)$, contradicting the feasibility of the core allocation α . This proves that $\langle p, \alpha \rangle$ is a quasi-equilibrium for $\langle \mathcal{A}, \succ, v, C \rangle$.

Note that $p \geq 0$. For if, say, $p_1 < 0$, then $\gamma(b) = \alpha(b) + \langle \bar{v}(b), 0, \dots, 0 \rangle$ defines a $\gamma \in C$ with $\gamma \succ_1 \alpha$ (by monotonicity) and $p \cdot \gamma(1) < p \cdot v(1)$, contradicting the preceding paragraph.

Note also that, according to Proposition 4, $\langle p, \alpha \rangle$ is a quasi-equilibrium for $\langle \mathcal{A}, \succ, v, M \rangle$ as well as for $\langle \mathcal{A}, \succ, v, C \rangle$.

As usual, the passage from quasi-equilibrium to competitive equilibrium will use lower semicontinuity. First we show $p > 0$. We already know that $p \geq 0$; so if, say, $p_1 = 0$, then suppose $p_2 > 0$. We saw above that $\gamma \succ_1 \alpha$, so $\beta(1) = \gamma(1) - \langle 0, \varepsilon, 0, \dots, 0 \rangle$ for some real $\varepsilon > 0$ and $\beta \in M$ with $\beta \succ_1 \alpha$, by lower semicontinuity and the fact that $\gamma(1) \geq \alpha(1) = v(1) > 0$. By the preceding paragraph, $p \cdot v(1) \leq p \cdot \beta(1) = p \cdot \alpha(1) - p_2 \varepsilon = p \cdot v(1) - p_2 \varepsilon < p \cdot v(1)$, a contradiction; so $p > 0$. To show $\delta \succ_a \alpha$ & $\bar{v}(a) > 0 \Rightarrow p \cdot \delta(a) > p \cdot v(a)$, note that if $p \cdot \delta(a) = p \cdot v(a)$, then by lower semicontinuity and the positivity of p , $p \cdot \eta(a) < p \cdot v(a)$ for some $\eta \in M$ with $\eta \succ_a \alpha$. This contradicts the preceding paragraph.

Q.E.D.

Boolean Rings. It may happen that we do not wish to consider the complement of every coalition as a legitimate coalition. If we still can consider, for any two coalitions a complement of each with respect to the other, then it is natural to formally consider a relatively complemented distributive lattice with 0 (but not necessarily with a unit: a Boolean ring, or generalized Boolean algebra).²⁴

Assuming still that the set of coalitions contains a supremum for every countable set of coalitions, we have a Boolean σ -ring. As we no longer require $1 \in A$, we must change the definition of feasibility. First we note that, for any allocation α , $z_i = \sup\{\alpha_i(a) : a \in A\}$ is finite for each component α_i (for the sup can be approached from below by a countable sequence $\alpha_i(a_k) \uparrow z_i$, and the sup of the a_k 's belongs to A : $\bigvee a_k \in A$). Thus any allocation α assumes in A its maximum $z(\alpha) \in \Omega$. We say that an allocation α is feasible if $z(\alpha) = z(v)$. With this generalization of our earlier notion of feasibility, we can define the core and competitive notions as before. We assume that $z(v) > 0$.

We shall briefly indicate why our propositions and theorems still hold true for this extension from Boolean algebras to Boolean rings.

Theorem 1 clearly requires no change in its proof.

In the proof of Proposition 1, let b_1 be an element of A on which v assumes its maximum, let b_2 be an element of A on which α assumes its maximum, and let $b = b_1 \vee b_2 \vee a$. Then replacing 1 by b , and \bar{a} by $b - c$ (the complement of c relative to b), the proof goes through as before.

Proposition 2 requires no change in its proof for the case of σ -rings.

The Lemma also requires no change in its proof for the case of σ -rings. This is because Lyapunov's theorem is true for the case of σ -rings as well as for Boolean algebras. To see this, we note that we may pass from the σ -ring \mathcal{U} to the measure ring \mathcal{U}/Δ , where Δ is the ideal of \bar{v} -null sets; but this measure ring is also a Boolean

algebra (since all sets on which $\bar{\nu}$ attains its maximum differ only by null sets), so Lyapunov's theorem applies, and the result may be transferred back to the original σ -ring \mathfrak{A} .

Propositions 3, 4, and 5 require no change in their proofs.

To prove Theorem 2 for \mathfrak{A} a σ -ring, we note that, by Proposition 5 we may pass to the measure ring \mathfrak{A}/J , which (as in the above explanation concerning the Lemma) is also a Boolean σ -algebra (in fact, a complete Boolean algebra) -- and then the proof proceeds as earlier.

FOOTNOTES

¹This assumption is later relaxed to relative complementation.

² $b \leq a$ is the usual Boolean notation meaning $b = b \wedge a$. There should be no confusion with the relation \leq on R^A .

³That is:

$$\begin{aligned} a, b \in P(\alpha, \beta) &\Rightarrow a \vee b \in P(\alpha, \beta), \text{ and} \\ a \in P(\alpha, \beta) \ \& \ b \in A &\Rightarrow a \wedge b \in P(\alpha, \beta). \end{aligned}$$

Cf. [12, p. 169; 4, p. 358]. In [12], $P(\alpha, \beta)$ is a principal ideal (that is, it is generated by a single element of A), though that is stronger than necessary.

⁴We sometimes use "for" in place of "with respect to." When it is clear from the context what set $H \subseteq M$ is intended, we dispense with the qualification "with respect to $\langle p, \mathcal{U}, \succ, \nu, H \rangle$."

⁵Note that, if $\langle p, \alpha \rangle$ is a quasi-equilibrium, then the cost of obtaining \succ -superior allocations is at least as great as the cost of α , for any allocation, even though it may not maximize the \succ -satisfaction obtainable with the cost of α .

⁶The abstract model will be weakened later from a Boolean algebra to a Boolean ring. We shall show, however, that this generalization can be reduced to the Boolean algebra framework.

⁷See [11, p. 24, Theorem 8.2; 7, p. 78, Theorem 6; 2, p. 227, Theorem 11].

⁸In Propositions 2, 3, and 4, we shall see that they are the only ones of interest.

⁹Debreu gives some sufficient conditions for this in [5].

¹⁰For the definition of a quotient \mathfrak{B}/J of a Boolean algebra \mathfrak{B} modulo an ideal J , see [11, p. 29, and 7, p. 53]. Basically, elements are identified which differ only on elements of J .

¹¹[7, pp. 100-101; p. 67, Lemma 3]. A measure algebra is a Boolean σ -algebra together with a real-valued countably additive measure which is positive everywhere but at 0. See [7, p. 67] and [11, p. 201].

¹²While Vind's results [12] do not encompass abstract Boolean algebras, Cornwall's [4, Theorem 2] (which is proved using Theorem 1 of [3]), on the other hand, does not encompass σ -fields of sets (applying, essentially, only to measure algebras). We might observe that every Boolean σ -algebra is "almost" isomorphic to a σ -field of sets. By the Loomis-Sikorski Theorem [11, p. 117, Theorem 29.1], every Boolean σ -algebra is isomorphic to a σ -field \mathfrak{F} of sets modulo a σ -ideal of \mathfrak{F} . In those cases for which the σ -ideal can be taken as the family of $\bar{\nu}$ -null sets, and where (as in Aumann [1]) modifications on null sets are irrelevant, such an isomorphism is satisfactory. Propositions 3 and 4 will give general conditions under which this is the case.

¹³For any $x, y \in \Omega$, $x \geq y$ means $x \geq y$ & $x \neq y$. Note that our monotonicity assumption, though analogous to Aumann's (after correcting the misprint in his hypothesis 2.2 [1, p. 43], is slightly weaker than Vind's hypothesis (II) [12, p. 170]. It improves the similarity of the two models.

¹⁴Cf. page 3.

¹⁵Cf. [6, p. 132, Theorem 14], where it is proved for real-valued measures on σ -fields of sets. This extends easily to vector-valued measures. Essentially the same methods prove the result for Boolean σ -algebras.

¹⁶An atom of a measure α is an element $a \in A$ with $\alpha(a) \neq 0$ and such that, for every $b \leq a$, $\alpha(b) = \alpha(a)$ or $\alpha(b) = 0$. Nonatomic measures are those with no atoms.

¹⁷If, for example, α had an atom a , then $\bar{\alpha}(a) > 0$ & $\bar{\nu}(a) > 0$ with $\bar{\alpha}(b) = \bar{\alpha}(a)$ or $\bar{\alpha}(b) = 0$ for all $b \leq a$ (where $\bar{\alpha}$ is the sum $\sum \alpha_i$ of the components α_i of α). By Lyapunov's theorem, we can partition a as $a = a_1 \vee a_2$ with $\bar{\nu}(a_1) = \frac{1}{2}\bar{\nu}(a)$ & $\bar{\alpha}(a_1) = \bar{\alpha}(a)$. Repeating this, we obtain $\bar{\alpha}(a_k) = \bar{\alpha}(a)$ & $\bar{\nu}(a_k) \downarrow 0$, a contradiction of $\bar{\nu}$ -continuity, since $\bar{\alpha}(\bigwedge a_k) = \bar{\alpha}(a)$ & $\bar{\nu}(\bigwedge a_k) = 0$.

¹⁸Proposition 4 also motivates the attention given in [5] to $\bar{\nu}$ -continuous allocations.

¹⁹A probability algebra is a measure algebra with measure of 1 equal to 1.

²⁰A discussion of σ -generators is in [11, §23].

²¹Since \mathfrak{U}/J is also countably σ -generated, it follows from [2, p. 261, Lemma 3 and Theorem 8] that the (normalized) measure algebra of $\langle \mathfrak{U}, \bar{\nu} \rangle$ is isomorphic to $\langle \mathfrak{B}/N, \mu/N \rangle$ and $\langle \mathfrak{C}/N', \mu/N' \rangle$. See also [8, p. 73, Theorem C].

²²[1, p. 44].

²³We would need "upper semicontinuity" and, while Debreu's transitivity could be weakened to something only slightly stronger than Vind's assumption (II)[12, p. 170], it is not clear that it can be entirely eliminated. (Although Aumann assumes upper semicontinuity in his assumption (2.3)[1, p. 243], it is not used in his proof.) These stronger hypotheses would be more appropriate for a proof of existence of equilibrium than for the present proof.

²⁴[2, p. 48; 8, p. 19, p. 165].

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