CONDITIONAL SIGN-SOLVABILITY

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Abstract

Sign-solvable linear systems were introduced by economists in modelling economic systems where only qualitative information is known. We introduce the idea of a conditionally sign-solvable linear system $Ax = b$ by relaxing the definition of sign-solvability so as not to require that each linear system with the same sign pattern as $Ax = b$

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has a solution. Conditional sign-solvability is more widely applicable than sign-solvability. We show that the study of conditional sign-solvability reduces to the study of L-matrices and matrices that we call conditionally S^*-matrices. Some basic properties of conditionally S^*-matrices are obtained.

1 Introduction

The *sign* of a real number a is defined by

$$\text{sign } a = \begin{cases} 0 & \text{if } a = 0 \\ +1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases}$$

The *sign pattern* of an m by n matrix $A = [a_{ij}]$ is the m by n matrix sign $A = [\text{sign } a_{ij}]$ obtained from A by replacing each entry by its sign.

Consider a linear system of equations $Ax = b$ where

$$\text{sign } A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \text{sign } b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1}$$

This linear system may or may not have a solution depending on the magnitudes of the entries of A and b. Suppose $Ax = b$ has a solution $u = (u_1, u_2, u_3)^T$. In order that the first three equations are satisfied, $u_1, u_2,$ and u_3 must have the same sign. In order that the last equation is also satisfied,

$$\text{sign } u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus if $Ax = b$ has a solution, we have determined its sign pattern knowing only the sign patterns of A and b.

The *qualitative class* of a matrix B is the set $\mathcal{Q}(B)$ of all matrices with the same sign pattern as B. We define a linear system of equations $Ax = b$ to be *conditionally sign-solvable* provided

$$\{\tilde{u} : \text{ there exists } \widetilde{A} \in \mathcal{Q}(A) \text{ and } \tilde{b} \in \mathcal{Q}(b) \text{ with } \widetilde{A}\tilde{u} = \tilde{b}\} \tag{2}$$

is a nonempty set which is contained in a single qualitative class $\mathcal{Q}(u)$. If $Ax = b$ is conditionally sign-solvable, then (2) is the *qualitative solution class* of $Ax = b$ and is denoted by $\mathcal{Q}(Ax = b)$. It follows easily that $\mathcal{Q}(Ax = b)$ is the entire qualitative class $\mathcal{Q}(u)$.

The linear system $Ax = b$ is *sign-solvable* provided it is conditionally sign-solvable and $\tilde{A}x = \tilde{b}$ is solvable for each $\tilde{A}$ in $\mathcal{Q}(A)$ and each $\tilde{b}$ in $\mathcal{Q}(b)$. A linear system $Ax = b$ satisfying (1) is conditionally sign-solvable where $\mathcal{Q}(Ax = b) = \mathcal{Q}((1,1,1)^T)$, but it is not sign-solvable.

The study of sign-solvable linear systems was begun by economists [S 47], [L 63], [G 64] and is further studied in [BMQ 68], [KL 81], [KLM 83], and [M 82]. The original motivation was to be able to predict the sign pattern of the solution of a linear system of equations arising from a model of an economic system when only qualitative information is known about its coefficients. Since the linear system arises from a model of a known economic system, it is presumed to have a solution. Hence it seems to us sign-solvability is unnecessarily restrictive, and that a conditional sign-solvable system is a more accurate description of a linear system for which the sign pattern of the solution can be predicted from qualitative information.

Fundamental to the study of sign-solvable linear systems are the classes of L-matrices and of S-matrices. We show that the study of conditionally sign-solvable linear systems reduces to the study of L-matrices and matrices that we call conditionally S^*-matrices. We investigate some of the basic properties of this new class of matrices.

2 Characterization of Conditional Sign-Solvability

Let A be an m by n matrix. Then A is an *L-matrix* provided that each matrix in $\mathcal{Q}(A)$ has linearly independent rows. We say that the matrix A is a *nearly L-matrix* provided that A is not an L-matrix but each of the $m-1$ by n matrices obtained from A by deleting a row is an L-matrix. The matrix A is a *conditionally S^*-matrix*, abbreviated a CS*-matrix provided that

(i) A has no row of all zeros,

(ii) A^T is a nearly L-matrix, and

(iii) $\{\tilde{u} \neq 0 :$ there exists $\tilde{A} \in \mathcal{Q}(A)$ with $\tilde{A}\tilde{u} = 0\} \subseteq \mathcal{Q}(u) \cup \mathcal{Q}(-u)$ for some m by 1 column vector u.

If A is a CS*-matrix, then by (ii) the vector u in (iii) has no zero entries. If each entry of u is positive, then A is a *conditionally S-matrix*, abbreviated

CS-matrix. Thus a matrix A with no zero rows is a CS-matrix if and only if A^T is not an L-matrix and for each $\widetilde{A}$ in $\mathcal{Q}(A)$ the right nullspace of $\widetilde{A}$ is either trivial or is spanned by a vector with only positive entries. If w is a column vector with no zero entries, then the matrix $[w \ -w]$ with two columns is a CS-matrix.

An *S-matrix*, respectively *S^*-matrix*, is a CS-matrix, respectively, CS*-matrix, with the property that every matrix in its qualitative class has a nontrivial right nullspace. Clearly, if A is an m by n CS-matrix, then $n \leq m+1$, and if $n = m+1$, then A is an S-matrix. It is not difficult to show that an S-matrix with m rows has exactly $m+1$ columns [KL 81], [KLM 83]. The matrix

$$\left[\begin{array}{c|c} -1 & \\ -1 & \\ \vdots & I_m \\ -1 & \end{array}\right]$$

is an S-matrix for each positive integer m.

It follows from our introductory example that

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} \tag{3}$$

is a CS-matrix. A matrix obtained from a CS-matrix by repeating rows is a CS-matrix.

A *signing* of order n is a nonzero diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ each of whose diagonal entries d_i equals 0, 1 or -1. A *strict signing* is a signing each of whose diagonal entries equals 1 or -1. Let A be an m by n matrix. If D is a (strict) signing of order m and E is a (strict) signing of order n, then DA is a *(strict) row signing* of A and AE is a *(strict) column signing* of A. A vector u is *unisigned* provided $u \neq 0$ and each of the nonzero entries of u has the same sign. A vector u is *balanced* provided it is not unisigned.

The following lemma is an easy consequence of the definitions.

Lemma 2.1 *Let A be an m by n matrix. Then*

(i) A is an L-matrix if and only if every row signing of A contains a unisigned column [KLM 83]*;*

(ii) A is a nearly L-matrix if and only if there is a signing D such that each column of DA is balanced and every such signing is strict;

(*iii*) *A is a* CS^**-matrix if and only if A has no zero rows and there exists a strict signing D such that AD and* $A(-D)$ *are the only column signings of A each of whose rows is balanced;*

(*iv*) *A is a CS-matrix if and only if A has no zero rows and* $AI_n = A$ *and* $A(-I_n) = -A$ *are the only column signings of A each of whose rows is balanced.*

(*v*) *A is a* CS^**-matrix if and only if there exists a strict signing D such that AD is a CS-matrix.*

By definition the transpose of a CS*-matrix is a nearly L-matrix, but the converse does not hold. For example, consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is easy to verify that the only signings D such that each row of AD is balanced are $\pm\text{diag}(1,-1,1,-1)$ and $\pm\text{diag}(1,-1,-1,1)$. Thus by (ii) of Lemma 2.1 A^T is a nearly L-matrix, and by (iii) A is not a CS*-matrix.

Conditional sign-solvability can also be characterized in terms of signings.

Lemma 2.2 *Let* $Ax = b$ *be a linear system where A is an m by n matrix. Then* $Ax = b$ *is conditionally sign-solvable if and only if there exists a unique signing of the form* $D = diag(d_1, \ldots, d_n, 1)$ *such that each row of* $[A \;\; -b]D$ *is balanced.*

Proof. Let $u = (u_1, u_2, \ldots, u_n)^T$ be a column vector and let D be the signing $\text{diag}(d_1, d_2, \ldots, d_n, 1)$ where sign $u_i = d_i$ for $i = 1, 2, \ldots, n$. Then clearly if there exists $\tilde{A}$ in $\mathcal{Q}(A)$ and $\tilde{b}$ in $\mathcal{Q}(b)$ such that $\tilde{A}u = \tilde{b}$, then each row of $[A \;\; -b]D$ is balanced.

Now let D be a signing of the form $\text{diag}(d_1, d_2, \ldots, d_n, 1)$. If each row of $[A \;\; -b]D$ is balanced, then there exists $\tilde{A}$ in $\mathcal{Q}(A)$ and $\tilde{b}$ in $\mathcal{Q}(b)$ such that $u = (d_1, d_2, \ldots, d_n)^T$ satisfies $\tilde{A}u = \tilde{b}$. The lemma now follows. □

Suppose that $Ax = b$ is conditionally sign-solvable. If $\tilde{A}$ is in $\mathcal{Q}(A)$ and $\tilde{b}$ is in $\mathcal{Q}(b)$, and if $\tilde{A}x = \tilde{b}$ is solvable, then it is easy to see that the columns of $\tilde{A}$ are linearly independent and the solution is unique. Thus if $Ax = b$ is sign-solvable, then A^T is an L-matrix. If $\tilde{A}$ is in $\mathcal{Q}(A)$, then it can happen that there does not exist a $\tilde{b}$ in $\mathcal{Q}(b)$ such that $\tilde{A}x = \tilde{b}$ is solvable. For instance, suppose we take $\tilde{A}$ = sign A in our introductory example (1). Then the leading submatrix of $\tilde{A}$ of order 3 is invertible and hence $\tilde{A}x = \tilde{b}$ does not have a solution for any $\tilde{b}$ in $\mathcal{Q}(b)$. Thus if $Ax = b$ is only conditionally sign-solvable, then we cannot use the above argument to conclude that A^T is an L-matrix. However, the conclusion does hold.

Theorem 2.3 *If $Ax = b$ be a conditionally sign-solvable linear system, then A^T is an L-matrix.*

Proof. Let A be an m by n matrix. Suppose that A^T is not an L-matrix. We prove that $Ax = b$ is not conditionally sign-solvable by showing that there does not exist a unique signing of the form $D = \text{diag}(d_1, \ldots, d_n, 1)$ such that each row of $[A \ -b]D$ is balanced, and then applying Lemma 2.2. We may assume that there exists some diagonal matrix $D = \text{diag}(d_1, \ldots, d_n, 1)$ such that each row of $[A \ -b]D$ is balanced, for otherwise we are done. By (i) of Lemma 2.1 there exists a signing $E = \text{diag}(e_1, \ldots, e_n)$ such that each row of AE is balanced.

After permutations of the columns of A, we may assume that $E = E_1 \oplus O$ where E_1 is a strict signing of order ℓ where $1 \leq \ell \leq n$. After row permutations we may assume that $[A \ -b]$ has the form

$$\left[\begin{array}{c|c|c} A_1 & A_3 & -b^1 \\ \hline O & A_2 & -b^2 \end{array}\right]$$

where A_1 is a k by ℓ matrix with no zero rows. Each row of the matrices

$$[A \ -b]\text{diag}(e_1, \ldots, e_\ell, d_{\ell+1}, \ldots, d_n, 1)$$

and

$$[A \ -b]\text{diag}(-e_1, \ldots, -e_\ell, d_{\ell+1}, \ldots, d_n, 1)$$

is balanced and the theorem follows. □

Corollary 2.4 *Let $Ax = b$ be a linear system where A is a square matrix. Then $Ax = b$ is conditionally sign-solvable if and only if it is sign-solvable.*

Proof. Suppose that $Ax = b$ is conditionally sign-solvable. By Theorem 2.3 A is an L-matrix. Hence each matrix $\tilde{A}$ in $\mathcal{Q}(A)$ is invertible. Thus $\tilde{A}x = \tilde{b}$ is always solvable and $Ax = b$ is sign-solvable. The converse is obvious. □

The main result of this paper is that recognizing the conditional sign-solvability of solvable linear systems is equivalent to recognizing L-matrices and CS*-matrices. First we prove the following lemma.

Lemma 2.5 *Let $Ax = b$ be a linear system of the form*

$$\begin{aligned} A_1x^1 + A_3x^2 &= b^1 \\ A_2x^2 &= 0 \end{aligned} \tag{4}$$

where $[A_1 \ -b^1]$ is a CS-matrix and A_2^T is an L-matrix. Then $Ax = b$ is conditionally sign-solvable.*

Proof. Suppose that A has n columns and A_1 has ℓ columns. Let D be a signing of the form $D = \text{diag}(d_1, \ldots, d_n, 1)$. Since A_2^T is an L-matrix, each row of $[A \ -b]D$ is balanced if and only if $d_{\ell+1} = \cdots = d_n = 0$ and each row of $[A_1 \ -b^1]\text{diag}(d_1, \ldots, d_\ell, 1)$ is balanced. Since $[A_1 \ -b^1]$ is a CS*-matrix, the lemma follows from (iii) of Lemma 2.1 and Lemma 2.2. □

Let B be a p by q matrix. Let α be a subset of $\{1, 2, \ldots, p\}$ and β be a subset of $\{1, 2, \ldots, q\}$. Then the submatrix of B determined by the rows with index in α and the columns with index in β is denoted by $A[\alpha, \beta]$. The submatrix of B determined by the rows with index not in α and columns with index not in β is denoted by $A[\overline{\alpha}, \overline{\beta}]$. If b is a p by 1 column vector, then we shorten $b[\alpha, \{1\}]$ to $b[\alpha]$.

Theorem 2.6 *Let $Ax = b$ be a linear system where $A = [a_{ij}]$ is an m by n matrix and b is an m by 1 column vector. Let $u = (u_1, u_2, \ldots, u_n)^T$ be an n by 1 column vector satisfying $Au = b$, and let*

$$\beta = \{j : u_j \neq 0\} \textit{ and } \alpha = \{i : a_{ij} \neq 0 \textit{ for some } j \in \beta\}.$$

Then $Ax = b$ is conditionally sign-solvable if and only if the matrix

$$\left[\begin{array}{cc} A[\alpha, \beta] & -b[\alpha] \end{array} \right]$$

is a CS-matrix and the matrix*

$$A[\overline{\alpha}, \overline{\beta}]^T$$

is an L-matrix.

Proof. Without loss of generality we may assume that

$$\beta = \{1, 2, \ldots, \ell\} \text{ and } \alpha = \{1, 2, \ldots, k\}$$

for some nonnegative integers k and ℓ. Thus

$$A = \begin{bmatrix} A_1 & A_3 \\ O & A_2 \end{bmatrix}$$

where A_1 is a k by ℓ matrix with no zero rows, and $u_i = 0$ if and only if $\ell + 1 \leq i \leq n$. The linear system $Ax = b$ can be written as

$$\begin{aligned} A_1x^1 + A_3x^2 &= b^1 \\ A_2x^2 &= 0 \end{aligned}$$

where

$$x^1 = x[\{1,2,\ldots,\ell\}] \text{ and } x^2 = x[\{\ell+1,\ldots,n\}],$$

and

$$b^1 = b[\{1,2,\ldots,k\}].$$

It follows from Lemma 2.5 that if $[A_1 \ \ -b^1]$ is a CS*-matrix and A_2^T is an L-matrix, then $Ax = b$ is conditionally sign-solvable.

Now assume that $Ax = b$ is conditionally sign-solvable. By Lemma 2.2 there is a unique signing of the form $D = \text{diag}(d_1,\ldots,d_n,1)$ such that each row of $[A \ \ -b]D$ is balanced. Since $Au = b$ we have $d_i = \text{sign } u_i$ for $i = 1,2,\ldots,n$. If $E = \text{diag}(e_1,\ldots,e_\ell,1)$ is a signing such that each row of $[A_1 \ \ -b^1]E$ is balanced, then each row of

$$[A \ \ -b]\text{diag}(e_1,\ldots,e_\ell,0,\ldots,0,1)$$

is balanced. It now follows from (ii) of Lemma 2.1 that $[A_1 \ \ -b^1]$ is a CS*-matrix. If $F = \text{diag}(f_{\ell+1},\ldots,f_n)$ is a signing such that each row of A_2F is balanced, then each row of

$$[A \ \ -b]\text{diag}(d_1,\ldots,d_\ell,f_{\ell+1},\ldots,f_n,1)$$

is balanced. It follows from (i) of Lemma 2.1 that A_2 is an L-matrix. □

Since a CS*-matrix with the property that every matrix in its qualitative class has a nontrivial right nullspace is an S^*-matrix, Theorem 2.6 immediately implies the following [KLM 83].

Corollary 2.7 *Let $Ax = b$ be a linear system where $A = [a_{ij}]$ is an m by n matrix and b is an m by 1 column vector. Let $u = (u_1,u_2,\ldots,u_n)^T$ be an n by 1 column vector satisfying $Au = b$, and let*

$$\beta = \{j : u_j \neq 0\} \text{ and } \alpha = \{i : a_{ij} \neq 0 \text{ for some } j \in \beta\}.$$

Then $Ax = b$ is sign-solvable if and only if the matrix

$$\left[\begin{array}{cc} A[\alpha,\beta] & -b[\alpha] \end{array} \right]$$

is an S^*-matrix and the matrix

$$A[\overline{\alpha},\overline{\beta}]^T$$

is an L-matrix.

3 BCS *-matrices

Let M be an m by n CS-matrix. If u is a 1 by n nonzero row vector, then clearly

$$\left[\frac{M}{u} \right]$$

is a CS-matrix if and only if u is balanced. Thus it is natural to consider CS-matrices, respectively CS*-matrices, for which each matrix obtained by deleting a row is not a CS-matrix, respectively not a CS*-matrix. We call such matrices *barely* CS- and *barely* CS**-matrices*, abbreviated BCS- and BCS*-matrices. Thus M is a BCS*-matrix if and only if M is a CS*-matrix and for each $i = 1, 2, \ldots, m$ there exists a signing D_i such that row i is the only row of MD_i which is not balanced. In particular, if M is a BCS*-matrix, then no two rows of sign M are equal or opposites. Any S^*-matrix, respectively S-matrix, is an example of a BCS*-matrix, respectively BCS-matrix.

An L-matrix A is a *barely L-matrix* [BCS 92] provided each matrix obtained from A by deleting a column is not an L-matrix. An L-matrix A is *L-decomposable* [BCS 92] provided there exist permutation matrices P and Q such that

$$A = P \left[\begin{array}{cc} A_1 & O \\ A_3 & A_2 \end{array} \right] Q,$$

where A_1 and A_2 are (non-vacuous) L-matrices. If A is not L-decomposable, then A is *L-indecomposable.* The L-indecomposable, barely L-matrices are fundamental in the study of L-matrices. The following result is contained in [BCS 92].

Lemma 3.1 *Let A be an m by n L-matrix. Then A is an L-indecomposable, barely L-matrix if and only if for each integer j with $1 \leq j \leq n$ there exists a strict row signing of A whose only unisigned column is column j.*

A BCS*-matrix with n columns can have exponentially many rows. For example, let B be a 2^{n-1} by n matrix whose rows are all the 1 by n row vectors of 1's and -1's with first entry equal to 1 and whose first row is the all 1's vector. It is easy to check that B^T is an L-indecomposable, barely L-matrix. Let A be the $2^{n-1}-1$ by n matrix obtained from B by deleting its first row. Then it is also easy to check that A is a CS-matrix. If $v = (v_1, v_2, \ldots, v_n)$ is a row of A, then $D = \text{diag}(v_1, v_2, \ldots, v_{n-1}, v_n)$ is a signing such that vD is the only row of AD which is not balanced. Therefore A is a BCS-matrix. In general, any matrix obtained from B by deleting a row of B is a BCS*-matrix. This observation suggests that the problem of recognizing matrices which are not CS-matrices is an NP-complete problem.[1] In contrast, there is a polynomial algorithm to recognize whether or not a matrix is an S-matrix [KLM 83], [M 82].

We now generalize the above construction.

Theorem 3.2 *Let $B = [b_{ij}]$ be an m by n matrix such that B^T is an L-indecomposable, barely L-matrix. Let i be an integer such that each entry in row i of B is different from zero. Then the matrix A obtained from B by deleting row i is a BCS*-matrix, and $D = diag(b_{i1}, b_{i2}, \ldots, b_{in})$ is a strict signing such that each row of AD is balanced.*

Proof. Since B^T is an L-indecomposable, barely L-matrix, it follows from Lemma 3.1 that for each row i' of A there exists a strict column signing of A whose only unisigned row is row i'. Thus it suffices to show that A is a CS*-matrix. Let E be a signing such that each row of AE is balanced. Suppose that E is not a strict signing. We may assume that $E = E_1 \oplus O$ where E_1 is a strict signing of order $\ell < n$. We may further assume that $i = 1$ and

$$B = \begin{bmatrix} B_1 & B_3 \\ O & B_2 \end{bmatrix}$$

where B_1 has ℓ columns and no zero rows. Since B^T is L-indecomposable, B_2^T is not an L-matrix. Hence there exists a signing F of order $n - \ell$ such that each row of B_2F is balanced. Since row 1 of B has no zero entries, one of the column signings $B(E_1 \oplus F)$ and $B(E_1 \oplus -F)$ of B has all of its rows balanced. This contradicts the assumption that B^T is an L-matrix. Hence E is a strict signing. Since B^T is an L-matrix and each entry of the first row

[1] It is easy to see that this problem is in the class NP: certify that a matrix A is not a conditionally S-matrix by giving either an unbalanced row of A or a signing $D \neq \pm I_n$ such that all rows of AD are balanced.

of B is different from zero, $E = \pm D$. Since B^T is a barely L-matrix, each row of AD is balanced and hence A is a CS*-matrix. □

We now show that every BCS*-matrix can be obtained by the above construction.

Theorem 3.3 *Let A be an m by n BCS*-matrix and let $D = diag(d_1, d_2, \ldots, d_n)$ be a signing such that each row of AD is balanced. Let*

$$B = \left[\begin{array}{cccc} & A & & \\ \hline d_1 & d_2 & \cdots & d_n \end{array}\right].$$

Then each entry of the last row of B is different from zero, and B^T is an L-indecomposable, barely L-matrix.

Proof. It follows from (i) and (iii) of Lemma 2.1 that B^T is an L-matrix and each entry of the last row of B is different from zero. By Lemma 3.1 it suffices to show that for each integer i with $1 \leq i \leq m+1$ that there is a strict column signing of B whose only unisigned row is row i. If $i = m+1$, BD is such a strict column signing of B. Now assume that $1 \leq i \leq m$. Since A is a BCS*-matrix, there exists a signing E such that row i is the only unisigned row of AE. First suppose that E is not a strict signing. Without loss of generality we may assume that $E = E_1 \oplus O$ where E_1 is a strict signing of order $\ell < n$. Further by permuting rows of A we may assume that $i = 1$ and that A has the form

$$\left[\begin{array}{cc} A_1 & A_3 \\ O & A_2 \end{array}\right],$$

where A_1 is a k by ℓ matrix with no zero rows. By (iii) of Lemma 2.1 there exists a strict signing D such that each row of AD is balanced. Let $D = D_1 \oplus D_2$ where D_1 and D_2 are strict signings of orders ℓ and $n - \ell$, respectively. Both $A(E_1 \oplus D_2)$ and $A(E_1 \oplus (-D_2))$ are strict column signings of A all of whose rows, except possibly the first row, are balanced. Since A is a CS*-matrix, the first row of one of these strict column signings of A must be unisigned. Hence we may assume that E is a strict signing. Since E is a strict signing, $E \neq \pm D$ and hence row i is the only unisigned row of BE. □.

It follows from Theorems 3.2 and 3.3 that the study of BCS*-matrices is essentially equivalent to the study of L-indecomposable, barely L-matrices having a column with no zero entries.

4 Concluding Remarks

Let A be an m by n matrix whose columns are the vectors $v_1, v_2, \ldots, v_n$. Then $v_1, v_2, \ldots, v_n$ are the vertices of an $(n-1)$-simplex whose relative interior contains the origin if and only if the right nullspace of A is spanned by a vector u each of whose entries is positive. It follows that A is a CS-matrix if and only if there is a matrix in $\mathcal{Q}(A)$ with linearly dependent columns, and for each matrix $\tilde{A}$ in $\mathcal{Q}(A)$ with linearly dependent column vectors, its column vectors are the vertices of a $(n-1)$-simplex whose relative interior contains the origin. This generalizes the observation in [KL 81] that an m by $m+1$ matrix is an S-matrix if and only if for each $\tilde{A}$ in $\mathcal{Q}(A)$, the columns of $\tilde{A}$ are the vertices of a simplex whose interior contains the origin.

There is a significant computational difference between the relationship of S^*-matrices to S-matrices and that of CS*-matrices to CS-matrices. If A is an m by $m+1$ S^*-matrix, then to find a strict signing D such that AD is an S-matrix we need only find a nonzero vector $u = (u_1, u_2, \ldots, u_{m+1})^T$ such that $Au = 0$ and let

$$D = \text{diag}(\text{sign } u_1, \text{sign } u_2, \ldots, \text{sign } u_{m+1}).$$

Now let A be an m by n CS*-matrix. There may not exist an n by 1 nonzero vector u such that $Au = 0$. Finding a strict signing D such that AD is a CS-matrix is equivalent to first finding a matrix $\tilde{A}$ in $\mathcal{Q}(A)$ with rank less than n and then finding a nonzero solution of $\tilde{A}x = 0$. Finding an $\tilde{A}$ in $\mathcal{Q}(A)$ with rank less than n seems to be a difficult problem in general. This is suggested by the next theorem and the observation that if a matrix $\tilde{A}$ in $\mathcal{Q}(A)$ has rank n, then there is an open subset X of $\mathcal{Q}(A)$ containing $\tilde{A}$ such that all matrices in X have rank n.

Theorem 4.1 *Let A be an m by n CS*-matrix which is not an S^*-matrix. Then the set of all matrices $\tilde{A}$ in $\mathcal{Q}(A)$ such that the rank of $\tilde{A}$ equals n is a dense subset of $\mathcal{Q}(A)$.*

Proof. Suppose that there exists a matrix $\tilde{A}$ and an open subset X of $\mathcal{Q}(A)$ containing $\tilde{A}$ such that X contains only matrices of rank $n-1$. Let u be an n by 1 nonzero column vector with $\tilde{A}u = 0$. Since A is a CS*-matrix, u has no zero entries. Without loss of generality

$$\tilde{A} = \begin{bmatrix} Y \\ Z \end{bmatrix}$$

where Y is an $n-1$ by n matrix of rank $n-1$. For any matrix $\widetilde{Z}$ in $\mathcal{Q}(Z)$ sufficiently close to Z, the matrix

$$\begin{bmatrix} Y \\ \widetilde{Z} \end{bmatrix}$$

has rank $n-1$, and hence $\widetilde{Z}u = 0$. Since u has no zero entries, we conclude that Z has no nonzero entries. Since a CS*-matrix has no zero rows, it follows that Z is vacuous. Hence A is an S^*-matrix contradicting the hypothesis of the theorem. □

If A is an m by n nearly L-matrix and Y is a m by p matrix, then

$$\begin{bmatrix} A & Y \end{bmatrix}$$

is a nearly L-matrix if and only if it is not an L-matrix. Thus it is also natural to consider nearly L-matrices for which each matrix obtained by deleting a column is not a nearly L-matrix. Let A be such a nearly L-matrix. While the left nullspace of a matrix $\widetilde{A}$ in $\mathcal{Q}(A)$ is trivial or contains nonzero vectors of only one sign pattern or its negative, the number of different sign patterns as $\widetilde{A}$ varies over the qualitative class of A can be exponentially large. For example, let A_k be the matrix of order $2k$ defined by

$$\begin{bmatrix} G & O & \cdots & O & F \\ F & G & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & G & O \\ O & O & \cdots & F & G \end{bmatrix}$$

where

$$G = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \text{ and } F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is easy to verify that the only signings D for which each column of DA_k is balanced are those of the form $D = \text{diag}(c_1, c_1, c_2, c_2, \ldots, c_k, c_k)$ where each c_i equals ± 1. Thus A_k is a nearly L-matrix and there are 2^k different sign patterns in the left nullspaces of matrices in $\mathcal{Q}(A_k)$. For each column of A_k there is a nonstrict signing which balances each of the other columns of A_k. For instance, if $E = \text{diag}(-1, 0, 1, \ldots, 1)$, then only column 1 of EA_k is not balanced, while if $E' = \text{diag}(0, 0, 1, \ldots, 1)$ then only column 2 of $E'A_k$ is not balanced. Hence each matrix obtained from A_k by deleting a column is not a nearly L-matrix.

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