Exact Controllability of Structural Acoustic Interactions

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Abstract

In this paper, we work to discern exact controllability properties of two coupled wave equations, one of which holds on the interior of a bounded open domain Ω, and the other on a segment Γ_0 of the boundary $\partial\Omega$. Moreover, the coupling is accomplished through terms on the boundary. Because of the particular physical application involved–the attenuation of acoustic waves within a chamber by means of active controllers on the chamber walls–control is to be implemented on the boundary only. We give here concise results of exact controllability for this system of interactions, with the control functions being applied through $\partial\Omega$. In particular, it is seen that for special geometries, control may be exerted on the boundary segment Γ_0 only. We make use here of microlocal estimates derived for the Neumann-control of wave equations, as well as a special vector field which is now known to exist under certain geometrical situations.

1 Statement of the Problem and Main Results

Throughout, Ω will be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with Lipshitz boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, with each Γ_i nonempty, and $\Gamma_0 \cap \Gamma_1 = \emptyset$. In what follows, we will impose additional assumptions on the Γ_i. On this geometry, we shall consider controllability properties of solutions $[z(t,x), v(t,\tau)]$ to the following PDE model:

$$\begin{cases} z_{tt}(t,x) = \Delta z(t,x) \quad \text{on } (0,T)\times\Omega \\ \begin{cases} \dfrac{\partial}{\partial\nu} z(t,x) = v_t(t,x) \ \text{ on } (0,T)\times\Gamma_0 \\ \dfrac{\partial}{\partial\nu} z(t,x) = u_1 \quad \text{on } (0,T)\times\Gamma_1 \end{cases} \\ [z(0,x),\, z_t(0,x)] = \vec{z}_0 \ \text{ on } \Omega; \end{cases}$$
$$\begin{cases} v_{tt}(t,\tau) = \frac{\partial^2}{\partial\tau^2} v(t,\tau) + u_0(t,\tau) - z_t|_{\Gamma_0} \quad \text{on } (0,T)\times\Gamma_0 \\ \dfrac{\partial v}{\partial n} = 0 \ \text{ on } (0,T)\times\partial\Gamma_0 \\ [v(0,x), v_t(0,x)] = \vec{v}_0 \ \text{ on } \partial\Gamma_0, \end{cases} \tag{1}$$

where $\partial\Gamma_0$ denotes the boundary of the $n-1$ dimensional manifold. Moreover, $\frac{\partial}{\partial\tau}$ and $\frac{\partial}{\partial\nu}$ denote respectively the (unit) tangential and normal derivatives with respect to Γ. $\frac{\partial}{\partial n}$ is here the unit exterior normal derivative with respect to $\partial\Gamma_0$.

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Making the denotations

$$\frac{H^s(\Omega)}{\mathbb{R}} \equiv \left\{f \in H^s(\Omega) : \int_\Omega f d\Omega = 0\right\};$$
$$\frac{H^s(\Gamma_0)}{\mathbb{R}} \equiv \left\{f \in H^s(\Gamma_0) : \int_{\Gamma_0} f d\Gamma_0 = 0\right.$$

we subsequently define the spaces of initial data

$$\begin{aligned} H_1 &\equiv \frac{H^1(\Omega)}{\mathbb{R}} \times \frac{L^2(\Omega)}{\mathbb{R}}; \\ H_0 &\equiv \frac{H^1(\Gamma_0)}{\mathbb{R}} \times \frac{L^2(\Gamma_0)}{\mathbb{R}}; \\ \mathcal{H} &\equiv \mathbf{H}_1 \times \mathbf{H}_0. \end{aligned} \tag{2}$$

In regards to the PDE (1), a straightforward consequence of semigroup theory (see e.g., [1], [24]) provides for continuity of the mapping

$$\{[\vec{z}_0, \vec{v}_0] \in \mathcal{H}, u_1 = 0, u_0 = 0\} \Rightarrow [\vec{z}, \vec{v}] \in C([0,T]; \mathcal{H}).$$

In other words, the problem (1) with $u_i = 0$ is wellposed for initial data in $\mathcal{H}$. Our task here is to ascertain the exact controllability of (1) with boundary controls u_1, u_0 taken in prescribed spaces.

By exact controllability of the PDE (1), we mean the following: we wish to determine if there is a $T^* > 0$ such that for terminal time $T > T^*$, one has the following reachability property: for all initial data $[\vec{z}_0, \vec{v}_0] \in \mathcal{H}$ and preassigned target data $[\vec{z}_T, \vec{v}_T] \in \mathcal{H}$, there exist control functions $[u_1, u_0] \in \mathcal{U}_1 \times \mathcal{U}_0$ (to be specified), such that at terminal time T the corresponding solution $[z, v]$ to (1) satisfies

$$[\vec{z}(T), \vec{v}(T)] = [\vec{z}_T, \vec{v}_T].$$

The PDE system (1) and other coupled PDE models of this type which govern acoustic flow–be they a coupling of *hyperbolic/hyperbolic* vis-à-vis *hyperbolic/parabolic* dynamics–are chiefly characterized as comprising a composite of distinct dynamics, with the coupling being accomplished across boundary interfaces. Examples of these PDE's have long existed in the literature (see e.g., [23], [6],[19]); however, recent innovations in smart material technology, and the potential applications of these innovations in control engineering design, have greatly increased the interest in these structural acoustic models. In particular, much attention has been paid to the PDE's which model the active control of structural acoustic flow within the interior of a chamber Ω (see [1], [2], [4], [7]). In this situation, the boundary segment Γ_1 represents the "hard" walls of the chamber Ω, with Γ_0 being the flexible portion of the chamber wall. The flow within the chamber is assumed to be of acoustic wave type, and hence the presence of the wave equation on Ω, satisfied by z in (1). In this, our initial effort concerning the exact controllability of structural acoustic PDE's, we have exchanged the (damped) Euler beam or Kirchoff plate which would have properly modeled the flexible Γ_0, and replaced it with a wave equation on Γ_0. Although the original structural acoustic model involves a plate equation on Γ_0, many of the mathematical difficulties and challenges associated with the exact controllability problem still prevail with the coupled wave/wave system in (1). However, the more canonical (1) allows the conveyance of main ideas without inundating the reader with a flood of technical details; for this reason then, we will focus our attention on the case where variable v satisfies a wave (rather than a plate) equation. Having considered and set up the solution to the basic problem of controllability for (1), we will subsequently proceed in the future to the fourth order (and thus more problematic) "physical" models which appear in [1] and [7].

In what follows, we shall focus on discerning exact controllability properties in the *finite energy space* $\mathcal{H}$, as defined in (2), while taking into account the particular geometry of the problem. This focus on $\mathcal{H}$ is appropriate, since from the point of view of modeling and applications, it is the only relevant space to be considered. Indeed, $\mathcal{H}$ is exactly the topology where energy conservation occurs for the uncontrolled model (i.e., $u_1 = 0$, $u_0 = 0$ in (1)), and this underlying topology represents the natural energy of the system, as derived from the principle of virtual work. Moreover in the literature, control is implemented on the active wall Γ_0 only, with Γ_1 being inactive. In line with the intended control application, it would then be desirable to correctly formulate, *if possible*, the geometry and control spaces $\mathcal{U}_0$ under which exact controllability can take place. Indeed we say "if possible", for in the structural acoustic control problem stated in [1] and [7], the active Γ_0 comprises one side of the chamber wall only. In consequence, for an *arbitrary* triple $\{\Omega, \Gamma_0, \Gamma_1\}$, one will generally not have exact controllability with control implemented solely on Γ_0, as the necessary geometric conditions will not be satisfied (see [5]).

In this present work then, one of our main results is to find conditions on the geometry and prescribed controls so that with control implemented on Γ_0 *only*, one has exact controllability of (1) for *arbitrary* initial data of finite energy. In contrast with approximate controllability, it is well known that exact controllability of acoustic interactions is a very difficult problem, with most results being of negative type (see [18], [20]). This situation, in fact, should not be too surprising, on account of geometric and topological considerations. (Historically these considerations are brought out in a paper of [29], where it is shown that backwards, well-posed systems *cannot* be exactly controllable under the action of a relatively bounded control operator; see also [25]). Indeed, the hybridization of the two disparate wave components in the acoustic interaction (1) creates a situation where the component $\vec{z}$ on Ω is subject to "smoothing effects" due to the presence of the Neumann boundary data v_t (see e.g. [16], [14]). Therefore, a sole control u_0–that is, $u_1 = 0$ in (1)–acting strictly on the wave component $\vec{v}$, cannot be expected to be strong enough to drive the acoustic variable $\vec{z}$ to an arbitrary state of finite energy. This remark explains the topological difficulty. As for the geometric difficulty, this is due to the propagation of singularities and the conditions of geometric optics needed for exact controllability. For, as we have previously noted, it is now known that in order to control the z-wave equation from the boundary, it is necessary that the support of the control region be sufficiently large (see [5]). Thus precribing control only on Γ_0 may well be insufficient, unless Γ_0 is large relative to Ω. Geometrical configurations, involving rectangular regions with only one side being controlled, are standardly invoked to exemplify the lack of controllability of wave equations, and therefore of structural acoustic interactions (as shown in [21]).

Having understood theses obstacles standing in the way of exact controllability, the idea in this paper is to account for the topologic and geometric problems, so as to construct a scenario for which one can obtain positive results concerning the controllability of (1). In this connection, our first result in Theorem 1 states that *all finite energy states are controlled exactly with controls located on* Γ_0 *alone (with these controls acting only on the* v*-component).* This result does require, however, that the geometry be "appropriate" to the situation; viz., the domain Ω is convex and the "roof" of the acoustic chamber is not too "deep". (See Assumption (A1) and Figure 1). In addition, the control u_0 must be of the appropriate topological strength; i.e., $u_0 \in [H^1(0, T; L^2(\Gamma_0))]'$. So for our first result (Theorem 1(a) below), we assume the following:

Assumption (A1) Assume that Ω is a bounded subset of $\mathbb{R}^n$, with boundary $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, with Γ_0 being flat. Moreover assume the following:

(i) Ω is convex; (ii) there exists a point $x_0 \in \mathbb{R}^2$ such that

$$(x - x_0) \cdot \nu \leq 0 \text{ for all } x \in \Gamma_1.$$

The special vector field which is available, in case that Assumption (A1) holds true–constructed in [17] and denoted below as h in (13)–will be used in the derivation of the observability inequality associated with exact controllability (see (3) below). In particular, this special h appears in the wave multipliers classically used to estimate the energy of the z-wave equation (see e.g., [22],[28],[12],[27],[11]). The behavior of h on the inactive portion of the boundary; i.e., $h \cdot \nu|_{\Gamma_1} = 0$, is a key driver in our first result: With control on Γ_0 only, and under Assumption (A1), the PDE (1) is exactly controllable on $\mathcal{H}$. A discussion concerning the possible configuration of those triples $\{\Omega, \Gamma_0, \Gamma_1\}$ which satisfy Assumption (A1) is given in Appendix C in [17]. A canonical example of such a triple is given in Figure 1.

ftbpFU199.1875pt103.75pt0ptA triple $\{\Omega, \Gamma_0, \Gamma_1\}$ which satisfies Assumption (A1)Figure

On the other hand, if the geometry of the acoustic chamber is unrestricted (see Assumption (A2)), then from our previous discussion, we clearly must have additional control on Γ_1. In this case, the second part of Theorem 1 states that *all finite energy states are controlled exactly with controls* $u_0 \in L^2(0,T;H^{-\frac{1}{4}}(\Gamma_0))$ *(located on* Γ_0*) and* $u_1 \in L^2(0,T;L^2(\Gamma_1))$ *(located on the roof of a chamber of arbitrary geometrical configuration).* So in our second result (Theorem 1(b)below), we make the following assumption:

Assumption (A2) Assume that Ω is either a bounded subset of $\mathbb{R}^n$ with smooth boundary Γ, or else Ω is a parallelepiped. Moreover, assume boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 is flat. No assumptions are made on Γ_1 (See Figure 2).

If Assumption (A2) holds true, then one has exact controllability of (1) for arbitrary initial data of finite energy, with the control region taken to be $\Gamma_0 \cup \Gamma_1$. The point of making the Assumption (A2) is that in this case, one can take a radial vector field h to assist in the multiplier method to be employed to estimate the (acoustic) wave energy. However, since Assumption (A2) is much less restrictive than (A1)–in particular, no impositions are made on the hard walls Γ_1–the corresponding h cannot be expected to help with the high order terms on Γ_1, and hence the need for control on the hard walls. A common feature in both of our main results is the critical use of "sharp" regularity theory which has been developed to handle the tangential derivatives (on the boundary) of solutions to wave equations (see [15] and Lemma 3) below).

ftbpFU355.6875pt103.4375pt0ptTriples $\{\Omega, \Gamma_0, \Gamma_1\}$ which satisfy Assumption (A2)Figure

To the best of our knowledge, Theorem 1(a) and (b) constitute the first exact controllability results for structural acoustic interactions in finite energy spaces, and with general spatial domains Ω. All other results (see [21] and references therein) pertain to controllability on specified subspaces of finite energy–such as those described by the asymptotic behavior of Fourier coefficients–and moreover these results are proved for very special geometries only–a 2D rectangle–with very large classes of controls–$H^{-2}(0,T;L^2(\Gamma_0))$.

From the mathematical point of view, the key ingredients in our proofs are the following:

(i) Sharp trace regularity for the wave equation in the absence of Lopatinski conditions (a distinguishing feature of the Neumann case); see Lemma 4.

(ii) Microlocal analytical estimates which allow the absorption of tangential (wave) traces by time derivatives on the boundary; see Lemma 3.

(iii) A recent result in [17] concerning Carleman's estimates for the wave equation with controlled Neumann part of the boundary. These estimates lead to the aforementioned special vector field h which allows us, in this paper, to handle the uncontrolled portion of the boundary so as to derive the requisite observability estimates.

We now state our main results concisely:

Theorem 1 *(a) Let Assumption (A1) stand, and set*

$$\mathcal{U}_1 = \{0\};$$
$$\mathcal{U}_0 = \left[H^1\left(0,T;L^2(\Gamma_0)\right)\right]'.$$

Then for terminal time T large enough, the problem (1) is exactly controllable on $\mathcal{H}$ within the class of $\mathcal{U}_1 \times \mathcal{U}_0$-controls.

(b) Let Assumption (A2) stand, and set

$$\mathcal{U}_1 = L^2(0,T;L^2(\Gamma_1))$$
$$\mathcal{U}_0 = L^2(0,T;H^{-\frac{1}{4}}(\Gamma_0)).$$

Then likewise for terminal time T large enough, the problem (1) is exactly controllable on $\mathcal{H}$ within the class of $\mathcal{U}_1 \times \mathcal{U}_0$-controls.

Remark 2 *The specification here of natural boundary conditions for the v-component in (1) is not critical in the derivation of our observability results. In fact, one could obtain a similar exact controllability result for (1), with instead $v|_{\Gamma_0} = 0$ on $\partial\Gamma_0$.*

2 The Proof of Exact Controllability

2.1 The Necessary Inequality

With the control spaces $\mathcal{U}_1, \mathcal{U}_0$ as prescribed in Theorem 1(a) or (b), let $\mathcal{L}_T : \mathcal{U}_1 \times \mathcal{U}_0 \supset D(\mathcal{L}_T) \to \mathcal{H}$ denote the *control to terminal state map*

$$\mathcal{L}_T \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} \vec{z}(T) \\ \vec{v}(T) \end{bmatrix}.$$

Then the asserted controllability result Theorem 1(a) (resp. (b)) is equivalent to showing the surjectivity of $\mathcal{L}_T$ as a mapping between the said spaces. In turn, by the classical functional analysis (see e.g. Lemma 3.8.18 and Theorem 6.5.10 of [10]), this ontoness is equivalent to establishing the following inequality for all $\left[\vec{\phi}_0, \vec{\psi}_0\right] \in D(\mathcal{L}_T^*) \subset \mathcal{H}$ (where $\mathcal{L}_T^*$ denotes the Hilbert space adjoint of $\mathcal{L}_T$):

$$\left\| \mathcal{L}_T^* \begin{bmatrix} \vec{\phi}_0 \\ \vec{\psi}_0 \end{bmatrix} \right\|_{\mathcal{U}_1 \times [\mathcal{U}_0]'} \geq C_T \left\| \left[\vec{\phi}_0, \vec{\psi}_0\right] \right\|_{\mathcal{H}}. \tag{3}$$

In PDE terms, this inequality assumes the following form:

Let $\left[\vec{\phi}(t,x), \vec{\psi}(t,\tau)\right]$ denote the solution to the backwards system (*adjoint* with respect to the

PDE (1)):

$$\begin{cases} \phi_{tt}(t,x) = \Delta\phi(t,x) \quad \text{on } (0,T)\times\Omega \\ \begin{cases} \dfrac{\partial}{\partial\nu}\phi(t,x) = \psi_t \text{ on } (0,T)\times\Gamma_0 \\ \dfrac{\partial}{\partial\nu}\phi(t,x) = 0 \text{ on } (0,T)\times\Gamma_1 \end{cases} \\ [\phi(T,x),\ \phi_t(T,x)] = \vec{\phi}_0 \text{ on } \Omega; \end{cases}$$

$$\begin{cases} \psi_{tt}(t,\tau) = \frac{\partial^2}{\partial\tau^2}\psi(t,\tau) - \phi_t|_{\Gamma_0} \quad \text{on } (0,T)\times\Gamma_0 \\ \dfrac{\partial\psi}{\partial n} = 0 \text{ on } (0,T)\times\partial\Gamma_0 \\ [\psi(T,x),\psi_t(T,x)] = \vec{\psi}_0. \end{cases} \tag{4}$$

(By standard semigroup theory, the homogenous system above is wellposed for terminal data $\left[\vec{\phi}_0, \vec{\psi}_0\right] \in \mathcal{H}$.)

To prove then Theorem 1(a), the necessary abstract inequality (3) takes the form

$$\left\| \left[\vec{\phi}_0, \vec{\psi}_0\right] \right\|_{\mathcal{H}}^2 \leq C_T \left\|\psi\right\|_{H^2(0,T;L^2(\Gamma_0))}^2, \tag{5}$$

for all $\left[\vec{\phi}_0, \vec{\psi}_0\right] \in D(\mathcal{L}_T^*)$.

On the other hand, the reverse inequality (3) needed to obtain the exact controllability statement Theorem 1(b) takes the explicit form

$$\left\| \left[\vec{\phi}_0, \vec{\psi}_0\right] \right\|_{\mathcal{H}}^2 \leq C_T \left\{ \left\|\phi_t\right\|_{L^2(0,T;L^2(\Gamma_1))}^2 + \left\|\psi_t\right\|_{L^2(0,T;H^{\frac{1}{4}}(\Gamma_0))}^2 \right\}, \tag{6}$$

for all $\left[\vec{\phi}_0, \vec{\psi}_0\right] \in D(\mathcal{L}_T^*)$.

We will use throughout the standard denotations for the "energy" of the system (4):

$$\begin{aligned} \mathcal{E}_\phi(t) &= \int_\Omega \left[\left|\nabla\phi(t)\right|^2 + \left|\phi_t(t)\right|^2 \right] d\Omega; \\ \mathcal{E}_\psi(t) &= \int_{\Gamma_0} \left[\left|\frac{\partial}{\partial\tau}\psi(t)\right|^2 + \left|\psi_t(t)\right|^2 \right] d\Gamma_0; \\ \mathcal{E}(t) &= \mathcal{E}_\phi(t) + \mathcal{E}_\psi(t). \end{aligned} \tag{7}$$

Under the assumptions made above on the geometry Ω, we can assume throughout that there exists a dense set of data corresponding to smooth (enough) solutions to the PDE (4). Indeed, if Ω has smooth boundary (the first part of Assumption (A2)), this assertion follows from classical elliptic and semigroup theory. On the other hand, if Ω is either a parallelepiped (the second part of Assumption (A2)) or else satisfies Assumption (A1) (so that in particular Ω is convex)], then one can appeal to [8]). In this way, one can justify the computations to be done below. Also, we will use the standard denotations

$$\begin{aligned} Q_{\epsilon_0} &= (\epsilon_0, T-\epsilon_0)\times\Omega; \\ \Sigma_0 &= (0,T)\times\Gamma_0; \Sigma_1 = (0,T)\times\Gamma_1. \end{aligned}$$

In addition, with $\vec{\phi} = [\phi, \phi_t]$ and $\vec{\psi} = [\psi, \psi_t]$, we will use below the standard denotation for terms which are "below the level of energy"; namely,

$$\mathsf{l.o.t.}(\vec{\phi}, \vec{\psi}) \equiv C \left\| \left[\vec{\phi}, \vec{\psi}\right] \right\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{1-\epsilon}(\Gamma_0)\times H^{-\epsilon}(\Gamma_0))},$$

for some constant C, where $\epsilon > 0$.

Finally, we will need throughout the cutoff function $\xi(t) \in C_0^\infty(\mathbb{R})$, which is defined by having for arbitrary $\epsilon_0 > 0$,

$$\xi(t) = \begin{cases} 1, \quad \text{for } t \in [\epsilon_0, T-\epsilon_0] \\ \text{a } C^\infty \text{ function with range in } (0,1), \;\; \text{for } t \in (0,\epsilon_0) \cup (T-\epsilon_0, T) \\ 0, \quad \text{for } t \in (-\infty, 0) \cup (T, \infty). \end{cases} \tag{8}$$

2.2 A Preliminary Estimate

Step 1 (The conservation relation). Multiplying the first equation of (4) by ϕ_t, the second by ψ_t, and subsequently integrating in time and space, and integrating by parts we obtain

$$\begin{aligned} \frac{1}{2}\left(\phi_t(\sigma), \phi_t(\sigma)\right)_{L^2(\Omega)}\Big]_{\sigma=s}^{\sigma=t} &= -\frac{1}{2}\left(\nabla\phi(\sigma), \nabla\phi(\sigma)\right)_{L^2(\Omega)}\Big]_{\sigma=s}^{\sigma=t} + \int_s^t \left(\frac{\partial\phi}{\partial\nu}, \phi_t\right)_{L^2(\Gamma_0)} dt; \\ \frac{1}{2}\left(\psi_t(\sigma), \psi_t(\sigma)\right)_{L^2(\Omega)}\Big]_{\sigma=s}^{\sigma=t} &= -\frac{1}{2}\left(\frac{\partial}{\partial\tau}\psi(\sigma), \frac{\partial}{\partial\tau}\psi(\sigma)\right)_{L^2(\Omega)}\Bigg]_{\sigma=s}^{\sigma=t} - \int_s^t \left(\phi_t, \psi_t\right)_{L^2(\Gamma_0)} dt. \end{aligned}$$

Applying the Neumann boundary condition in (4) to the first equation above and the summing the two yields the expected conservation of the system; i.e.,

$$\mathcal{E}(t) = \mathcal{E}(s) \text{ for all } 0 \le s, t \le T.$$

In particular then,

$$\mathcal{E}(s) = \left\| \left[\vec{\phi}_0, \vec{\psi}_0\right] \right\|_{\mathcal{H}}^2 \quad \text{for all } \; 0 \le s \le T. \tag{9}$$

Step 2 (The acoustic wave estimates).

Let h be a $[C^2(\overline{\Omega})]^n$-vector field, which will be eventually specified. With this h, we apply the "classic" wave multipliers (see e.g., [22],[28],[12],[27]). Multiplying the ϕ-wave equation of (4) by $h \cdot \nabla\phi$, integrating in time and space and using the Neumann boundary condition, we have

$$\begin{aligned} \int_{Q_{\epsilon_0}} H\nabla\phi \cdot \nabla\phi dQ_{\epsilon_0} &= \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \psi_t \left(h \cdot \nabla\phi\right) dt d\Gamma_0 + \tfrac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0} \int_\Gamma \phi_t^2 h \cdot \nu dt d\Gamma \\ -\tfrac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0} \int_\Gamma \left|\nabla\phi\right|^2 h \cdot \nu dt d\Gamma &+ \tfrac{1}{2}\int_{Q_{\epsilon_0}} \left(\left|\nabla\phi\right|^2 - \phi_t^2\right) div(h) dQ_{\epsilon_0} - \left[\left(\phi_t, h \cdot \nabla\phi\right)_{L^2(\Omega)}\right]_{\epsilon_0}^{T-\epsilon_0}. \end{aligned} \tag{10}$$

Next, we consider again the first wave equation in (4), this time multiplying by the quantity $\phi div(\tilde{h})$, where $\tilde{h}(x) \in \left[C^2(\overline{\Omega})\right]^n$ is arbitrary. Integrating in time and space, and invoking Green's Theorem and the identity $\nabla\left(\phi div(\tilde{h})\right) = \phi\nabla\left(div(\tilde{h})\right) \cdot \nabla\phi + \left|\nabla\phi\right|^2 div(\tilde{h})$, we obtain

$$\begin{aligned} \int_{Q_{\epsilon_0}} \left(\phi_t^2 - \left|\nabla\phi\right|^2\right) div(\tilde{h}) dQ_{\epsilon_0} &= \left\langle \phi_t, \phi div(\tilde{h}) \right\rangle_{H^{-\epsilon}(\Omega) \times H^{\epsilon}(\Omega)}\Big]_{\epsilon_0}^{T-\epsilon_0} \\ &+ \int_{Q_{\epsilon_0}} \phi\nabla\left(div(\tilde{h})\right) \nabla\phi dQ_{\epsilon_0} - \int_{\epsilon_0}^{T-\epsilon_0} \int_\Gamma \frac{\partial\phi}{\partial\nu} \phi div(\tilde{h}) dt d\Gamma. \end{aligned} \tag{11}$$

Using now the Neumann boundary condition for ϕ in (4), along with Sobolev Trace Theory, we have the following inequality for any C^2-vector field $\tilde{h}$:

$$
\begin{aligned}
&\left|\int_{Q_{\epsilon_0}} \left(\phi_t^2 - |\nabla\phi|^2\right) div(\tilde{h}) dQ_{\epsilon_0}\right| \\
&\leq C \int_{\Sigma_0} \psi_t^2 d\Sigma_0 + \epsilon \int_{Q_{\epsilon_0}} |\nabla\phi|^2 dQ_{\epsilon_0} \\
&\quad + C_{\tilde{h},\epsilon} \left(\int_0^T \|\phi\|^2_{H^{\frac{1}{2}+\epsilon}(\Omega)} dt + \|\phi_t\|^2_{C([0,T]:H^{-\epsilon}(\Omega))} + \|\phi\|^2_{C([0,T]:H^{\epsilon}(\Omega))}\right) \\
&\leq C_\epsilon \int_{\Sigma_0} \psi_t^2 d\Sigma_0 + \epsilon \int_{Q_{\epsilon_0}} |\nabla\phi|^2 dQ_{\epsilon_0} + \text{ l.o.t.} \left(\vec{\phi}, \vec{\psi}\right).
\end{aligned}
\tag{12}
$$

We will now consider the two cases, corresponding to Assumptions (A1) and (A2).

Case I. Assumption (A1) is in force. Given the assumptions on both Γ_0 and Γ_1, it is shown in [17] that there exists a vector field $h(x) = [h_1(x), h_2(x), ..., h_n(x)] \in [C^2(\overline{\Omega})]^n$ such that

$$
\begin{aligned}
&\text{(i) } h \cdot \nu = 0 \text{ on } \Gamma_1; \\
&\text{(ii) The matrix } H(x), \text{ defined by} \\
&\qquad [H(x)]_{ij} = \frac{\partial h_i(x)}{\partial x_j}, \qquad 1 \leq i,j \leq n, \\
&\quad \text{satisfies } H(x) \geq \rho_0 I \text{ on } \Omega, \text{ for some positive constant } \rho_0.
\end{aligned}
\tag{13}
$$

Applying this particular h to (10), we obtain

$$
\begin{aligned}
\rho_0 \int_{Q_{\epsilon_0}} |\nabla\phi|^2 dQ_{\epsilon_0} &\leq \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \psi_t \left(h \cdot \nabla\phi\right) dt d\Gamma_0 + \tfrac{1}{2} \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \phi_t^2 h \cdot \nu dt d\Gamma_0 \\
-\tfrac{1}{2} \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} |\nabla\phi|^2 h \cdot \nu dt d\Gamma_0 &+ \tfrac{1}{2} \int_{Q_{\epsilon_0}} \left(|\nabla\phi|^2 - \phi_t^2\right) div(h) dQ_{\epsilon_0} - \left[(\phi_t, h \cdot \nabla\phi)_{L^2(\Omega)}\right]_{\epsilon_0}^{T-\epsilon_0}.
\end{aligned}
$$

Using the relation $|\nabla\phi|^2 = \left(\psi_t^2 + \frac{\partial\phi}{\partial\tau}^2\right)$ on Γ_0, as well as the conservation relation (9), we obtain from this the majorization

$$
\begin{aligned}
&\rho_0 \int_{Q_{\epsilon_0}} |\nabla\phi|^2 dQ_{\epsilon_0} \\
&\quad \leq C_h \left\{\int_{\Sigma_0} \psi_t^2 d\Sigma_0 + \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \left(\phi_t^2 + \frac{\partial\phi}{\partial\tau}^2\right) dt d\Gamma_0\right\} \\
&\quad +2 \max_{1\leq i\leq n} \|h_i\|_{L^\infty(\Omega)} \mathcal{E}(T) + \tfrac{1}{2} \left|\int_{Q_{\epsilon_0}} \left(\phi_t^2 - |\nabla\phi|^2\right) div(h) dQ_{\epsilon_0}\right|,
\end{aligned}
\tag{14}
$$

where h is the vector field in (13).

Combining (14) and (12) (with $\tilde{h} = h$ therein) yields now

$$
\begin{aligned}
(\rho_0 - \epsilon) \int_{Q_{\epsilon_0}} |\nabla\phi|^2 dQ_{\epsilon_0} \quad &\leq \quad C_{\epsilon,h} \left\{\int_{\Sigma_0} \psi_t^2 d\Sigma_0 + \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \left(\phi_t^2 + \frac{\partial\phi}{\partial\tau}^2\right) dt d\Gamma_0\right\} \\
&\quad +2 \max_{1\leq i\leq n} \|h_i\|_{L^\infty(\Omega)} \mathcal{E}(T) + \text{l.o.t.}(\vec{\phi}, \vec{\psi}).
\end{aligned}
\tag{15}
$$

In turn, the inequalities (12) (taking therein $\tilde{h}$ which satisfies $div(\tilde{h}) = 1$) and (15) (taking therein $\epsilon > 0$ small enough) give the intermediate estimate

$$
\begin{aligned}
\int_{\epsilon_0}^{T-\epsilon_0} \mathcal{E}_\phi(t) dt \quad &\leq \quad C_{\epsilon,h} \left\{\int_{\Sigma_0} \psi_t^2 d\Sigma_0 + \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \left(\phi_t^2 + \frac{\partial\phi}{\partial\tau}^2\right) dt d\Gamma_0\right\} \\
&\quad +C_h \mathcal{E}(T) + \text{ l.o.t.}(\vec{\phi}, \vec{\psi}).
\end{aligned}
\tag{16}
$$

Now in estimating the tangential derivative $\frac{\partial \phi}{\partial \tau}\Big|_{\Gamma_0}$ on the right hand side of (16), there is no appeal to classical Sobolev trace theory. Instead, we recall the following estimate, a product of microlocal machinery.

Lemma 3 *(see [15], Lemma 7.2). Let $\epsilon_0 > 0$ be arbitrarily small. Let w be a solution of the wave equation on $(0,T) \times \Omega$, or more generally, any second-order hyperbolic equation with smooth space dependent coefficients. Then,with $w_c \equiv \xi w$ (with $\xi(t)$ being the cutoff function defined in (8)), we have the estimate*

$$\int_0^T \int_{\Gamma_*} \left(\frac{\partial w_c}{\partial \tau} \right)^2 d\Sigma_0 \leq C_{T,\epsilon_0} \left(\int_0^T \int_{\Gamma_*} \frac{\partial w_c}{\partial t}^2 d\Sigma_0 + \int_0^T \int_{\Gamma} \frac{\partial w}{\partial \nu}^2 d\Sigma \right) + \text{l.o.t.}(w), \tag{17}$$

where Γ_* is a smooth connected segment of boundary Γ.

Invoking then Lemma 3 (with $\Gamma_* = \Gamma_0$ therein), so as to handle the tangential derivative in (16), gives the following: *under Assumption (A1), one has the integral estimate for the energy of the component ϕ of (4),*

$$\begin{aligned} &\int_{\epsilon_0}^{T-\epsilon_0} \mathcal{E}_\phi(t) dt \\ \leq \;& C_{T,\epsilon_0,h} \left\{ \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \phi_t|_{\Gamma_0}^2 \, dt d\Gamma_0 + \int_{\Sigma_0} \xi^2 \; \phi_t|_{\Gamma_0}^2 \, d\Sigma_0 + \int_{\Sigma_0} \psi_t^2 d\Sigma_0 \right\} \\ & + C_h \mathcal{E}(T) + \text{ l.o.t.}(\vec{\phi}, \vec{\psi}), \end{aligned} \tag{18}$$

where the cutoff function ξ is as defined in (8). Here, $C_{T,\epsilon_0,h}$ depends on time T, but C_h does not.

Case II. Assumption (A2) is in force. In this case, since Γ_0 is flat, one can construct a *radial* vector field $h(x) \in [C^2(\overline{\Omega})]^n$ such that

$$h(x) \cdot \nu = 0 \text{ on } \Gamma_0. \tag{19}$$

(Indeed, if $x_0 \in \Gamma_0$, then we can take $h(x) = x - x_0$.) Applying this vector field h to the relation (10), we obtain the inequality

$$\begin{aligned} \int_{Q_{\epsilon_0}} |\nabla \phi|^2 \, dQ_{\epsilon_0} &\leq \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \psi_t (h \cdot \nabla \phi) dt d\Gamma_0 + \frac{1}{2} \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} \phi_t^2 h \cdot \nu dt d\Gamma_1 \\ -\frac{1}{2} \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} |\nabla \phi|^2 \, h \cdot \nu dt d\Gamma_1 &+ \frac{n}{2} \int_{Q_{\epsilon_0}} \left(|\nabla \phi|^2 - \phi_t^2 \right) dQ_{\epsilon_0} - \Big[(\phi_t, h \cdot \nabla \phi)_{L^2(\Omega)} \Big]_{\epsilon_0}^{T-\epsilon_0}. \end{aligned}$$

Using $|\nabla \phi|^2 = \left(\frac{\partial \phi}{\partial \nu}^2 + \frac{\partial \phi}{\partial \tau}^2 \right)$ on Γ, the conservation relation (9), and the fact that radial h is parallel to Γ_0, we obtain

$$\begin{aligned} &\textstyle\int_{Q_{\epsilon_0}} |\nabla \phi|^2 \, dQ_{\epsilon_0} \\ &\quad \leq C_h \left\{ \textstyle\int_{\Sigma_0} \psi_t^2 d\Sigma_0 + \int_{\Sigma_1} \phi_t^2 d\Sigma_1 + \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} \frac{\partial \phi}{\partial \tau}^2 dt d\Gamma_1 \right\} \\ &\quad + C \left| \textstyle\int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \psi_t \frac{\partial \phi}{\partial \tau} dt d\Gamma_0 \right| + 2 \max_{1 \leq i \leq n} \|h_i\|_{L^\infty(\Omega)} \mathcal{E}(T) + \frac{n}{2} \left| \textstyle\int_{Q_{\epsilon_0}} \left(\phi_t^2 - |\nabla \phi|^2 \right) dQ_{\epsilon_0} \right|. \end{aligned} \tag{20}$$

Combining now (20) and (12) (with $\tilde{h} = h$ and $\epsilon = 1$ therein) gives

$$\begin{aligned} \frac{1}{2}\int_{Q_{\epsilon_0}} |\nabla\phi|^2\, dQ_{\epsilon_0} \quad &\leq \quad C_h \left\{ \int_{\Sigma_0} \psi_t^2 d\Sigma_0 + \int_{\Sigma_1} \phi_t^2 d\Sigma_1 + \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} \frac{\partial\phi}{\partial\tau}^2 dt d\Gamma_1 \right\} \\ &\quad + C \left| \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \psi_t \frac{\partial\phi}{\partial\tau} dt d\Gamma_0 \right| + 2 \max_{1\leq i\leq n} \|h_i\|_{L^\infty(\Omega)} \mathcal{E}(T) \\ &\quad + \mathsf{l.o.t.}(\vec{\phi}, \vec{\psi}). \end{aligned} \tag{21}$$

Now to handle the term $\int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \psi_t \frac{\partial\phi}{\partial\tau} dt d\Gamma$: again, Sobolev trace theory attaches no meaning to the boundary trace $\left.\frac{\partial\phi}{\partial\tau}\right|_{\Gamma_0}$ for initial data of finite energy. Instead, we deal with this term by the following "sharp" trace regularity result:

Lemma 4 *(See [16], p. 113, Corollary 3.4(b) and Theorem 3.3(a) [with $\alpha = \beta = \frac{3}{4}$ therein]). Let Γ_0 be a flat portion of the boundary, and let w solve the following wave equation*

$$\begin{cases} w_{tt} = \Delta w & \text{on } (0,T)\times\Omega \\ \dfrac{\partial w}{\partial\nu} = g \in L^2(0,T; H^{\frac{1}{4}}(\Gamma_0)) & \text{on } (0,T)\times\Gamma_0 \\ [w(0), w_t(0)] = [w_0, w_1] \in H_1. \end{cases} \tag{22}$$

Then one has the following estimate

$$\left\| \frac{\partial w}{\partial\tau} \right\|_{L^2(0,T;H^{-\frac{1}{4}}(\Gamma_0))} \leq C_T \left(\|g\|_{L^2(0,T;H^{\frac{1}{4}}(\Gamma))} + \|[w_0, w_1]\|_{H_1} \right).$$ [1]

Applying Lemma 4 (taking $g \equiv \psi_t$ therein), along with $ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2$ (taking $\delta \equiv C_T$ of (22)), we have then

$$\begin{aligned} \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \psi_t \frac{\partial\phi}{\partial\tau} dt d\Gamma \quad &= \quad \int_{\epsilon_0}^{T-\epsilon_0} \left\langle \psi_t, \frac{\partial\phi}{\partial\tau} \right\rangle_{H^{\frac{1}{4}}(\Gamma_0)\times H^{-\frac{1}{4}}(\Gamma_0)} dt \\ &\leq \quad \int_{\epsilon_0}^{T-\epsilon_0} \|\psi_t\|_{H^{\frac{1}{4}}(\Gamma_0)} \left\| \frac{\partial\phi}{\partial\tau} \right\|_{H^{-\frac{1}{4}}(\Gamma_0)} dt \\ &\leq \quad \frac{1}{2}\mathcal{E}_\phi(T) + (\frac{C_T}{2} + \frac{1}{2}) \int_{\epsilon_0}^{T-\epsilon_0} \|\psi_t\|^2_{H^{\frac{1}{4}}(\Gamma_0)} dt. \end{aligned} \tag{23}$$

Coupling (23) with (21) yields

$$\begin{aligned} \frac{1}{2}\int_{Q_{\epsilon_0}} |\nabla\phi|^2\, dQ_{\epsilon_0} \quad &\leq \quad C_{\epsilon,h,T} \left\{ \int_0^T \|\psi_t\|^2_{H^{\frac{1}{4}}(\Gamma_0)} dt + \int_{\Sigma_1} \phi_t^2 d\Sigma_1 + \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} \frac{\partial\phi}{\partial\tau}^2 dt d\Gamma_1 \right\} \\ &\quad + C_h \mathcal{E}(T) + \mathsf{l.o.t.}(\vec{\phi}, \vec{\psi}). \end{aligned} \tag{24}$$

In turn, combining this estimate with that in (12) (with again $\tilde{h} = h$), we obtain

$$\begin{aligned} \int_{\epsilon_0}^{T-\epsilon_0} \mathcal{E}_\phi(t) dt \quad &\leq \quad C_{\epsilon,h,T} \left\{ \int_0^T \|\psi_t\|^2_{H^{\frac{1}{4}}(\Gamma_0)} dt + \int_{\Sigma_1} \phi_t^2 d\Sigma_1 + \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} \frac{\partial\phi}{\partial\tau}^2 dt d\Gamma_1 \right\} \\ &\quad C_h \mathcal{E}(T) + \mathsf{l.o.t.}(\vec{\phi}, \vec{\psi}). \end{aligned}$$

A subsequent application of the tangential estimate (17) (with $\Gamma_* = \Gamma_1$) produces the following: *Under Assumption (A2), the ϕ-component of the adjoint system (4) obeys the integral estimate*

$$\int_{\epsilon_0}^{T-\epsilon_0} \mathcal{E}_\phi(t)dt \leq C_{T,\epsilon_0,h} \left\{ \int_0^T \|\psi_t\|^2_{H^{\frac{1}{4}}(\Gamma_0)} dt + \int_{\Sigma_1} \phi_t^2 d\Sigma_1 \right\} + C_h \mathcal{E}(T) + \text{l.o.t.}(\vec{\phi}, \vec{\psi}), \tag{25}$$

where $C_{T,\epsilon_0,h}$ depends on T, but C_h does not.

Step 3 (An estimate for ψ).

Applying the multiplier ψ to the second wave equation in (4), and integrating in time and space yields

$$\begin{aligned} & \left[\int_{\Gamma_0} \psi_t(t)\psi(t) d\Gamma_0 \right]_{t=0}^{t=T} - \int_{\Sigma_0} \psi_t^2 d\Sigma_0 \\ = & -\int_{\Sigma_0} \left(\frac{\partial \psi}{\partial \tau} \right)^2 d\Sigma_0 - \left[\int_{\Gamma_0} \phi(t)|_{\Gamma_0} \psi(t) d\Gamma_0 \right]_{t=0}^{t=T} + \int_{\Sigma_0} \phi|_{\Gamma_0} \psi_t d\Sigma_0. \end{aligned}$$

A rearrangement of terms, combined with a use of Sobolev trace theory yields

$$\begin{aligned} \int_{\Sigma_0} \left(\frac{\partial \psi}{\partial \tau} \right)^2 d\Sigma_0 &= \int_{\Sigma_0} \psi_t^2 d\Sigma_0 + \int_{\Sigma_0} \phi|_{\Gamma_0} \psi_t d\Sigma_0 \\ &\quad - \left[\left\langle \psi_t(t) + \phi(t)|_{\Gamma_0}, \psi(t) \right\rangle_{H^{-\epsilon}(\Gamma_0) \times H^{\epsilon}(\Gamma_0)} \right]_{t=0}^{t=T} \\ &\leq C \left(\int_{\Sigma_0} \psi_t^2 d\Sigma_0 + \int_{\Sigma_0} \phi|^2_{\Gamma_0} d\Sigma_0 \right. \\ &\quad \left. + \left[\|\psi_t(t)\|^2_{H^{-\epsilon}(\Gamma_0)} + \left\| \phi(t)|_{\Gamma_0} \right\|^2_{H^{-\epsilon}(\Gamma_0)} + \|\psi(t)\|^2_{H^{\epsilon}(\Gamma_0)} \right]_{t=0}^{t=T} \right) \\ &\leq C \int_{\Sigma_0} \psi_t^2 d\Sigma_0 + \text{ l.o.t.}(\vec{\phi}, \vec{\psi}). \end{aligned} \tag{26}$$

Combining (18) and (26) (in the case Assumption (A1) holds true), and (25) and (26) (if Assumption (A2) holds true), we have the preliminary estimate for the energy:

Lemma 5 *(a) Under Assumption (A1), the solution to the PDE (4) satisfies the following estimate for all $\epsilon_0 > 0$:*

$$\begin{aligned} & \int_{\epsilon_0}^{T-\epsilon_0} \mathcal{E}(t)dt \\ \leq & C_{T,\epsilon_0} \left\{ \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \phi_t|^2_{\Gamma_0} dt d\Gamma_0 + \int_{\Sigma_0} \xi^2 \phi_t|^2_{\Gamma_0} d\Sigma_0 + \int_{\Sigma_0} \psi_t^2 d\Sigma_0 \right\} \\ & + C\mathcal{E}(T) + \text{ l.o.t.}(\vec{\phi}, \vec{\psi}), \end{aligned} \tag{27}$$

where constant C is independent of time T.

(b) Under Assumption (A2), the solution to the PDE (4) satisfies the following estimate for all $\epsilon_0 > 0$:

$$\int_{\epsilon_0}^{T-\epsilon_0} \mathcal{E}(t)dt$$

$$\leq \quad C_{T,\epsilon_0,h}\left\{\int_0^T \|\psi_t\|^2_{H^{\frac{1}{4}}(\Gamma_0)}\, dt + \int_{\Sigma_1} \phi_t^2 d\Sigma_1\right\} + C\mathcal{E}(T) + \text{ l.o.t.}(\vec{\phi},\vec{\psi}), \tag{28}$$

where constant C is independent of time T.

Recall that our aim is to attain the inequalities (5) (under Assumption (A2)), and (6) (under Assumption (A2)). We can handle the terms $\int_{\epsilon_0}^{T-\epsilon_0} \mathcal{E}(t)dt$ and $\mathcal{E}(T)$ in (27) and (28) by the conservation relation (9), and the lower order terms by a compactness-uniqueness argument. Hence, the "bad terms" appear only in the inequality (27); i.e., $\int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_0} \phi_t|^2_{\Gamma_0} dtd\Gamma_0$ and $\int_{\Sigma_0} \xi^2 \phi_t^2 d\Sigma_0$. Accordingly, we concern ourselves next with $\phi_t|_{\Gamma_0}$.

2.3 Analysis of $\phi_t|_{\Gamma_0}$

Our main estimate here is the following:

Lemma 6 *Let $\epsilon > 0$ be arbitrary. Then the ψ-component of the system (4) satisfies the inequality*

$$\int_{\Sigma_0} \xi^2 \psi_{\tau\tau}^2 d\Sigma_0 dt \leq C\,\|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} + \text{ l.o.t.}(\vec{\phi},\vec{\psi}),$$

where ξ is the cutoff function defined in (8).

Since

$$\phi_t|_{\Gamma_0} = \psi_{\tau\tau} - \psi_{tt} \quad \text{on } (0,T)\times\Gamma_0,$$

Lemma 6 gives immediately the following necessary estimate:

Corollary 7 *Let $\epsilon > 0$ be arbitrary. Then the ϕ-component of the system (4) satisfies the inequality*

$$\int_{\Sigma_0} \xi^2 \ \phi_t|^2_{\Sigma_0} d\Sigma_0 \leq C\,\|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} + \text{ l.o.t.}(\vec{\phi},\vec{\psi}).$$

2.3.1 Proof of Lemma 6

The proof of Lemma 6 follows from sequence of propositions. In proving these propositions, we will have need of the cutoff function $\xi(t)$ defined above in (8). With $\xi(t)$, we set $[\phi_c(t),\psi_c(t)] \equiv [\xi(t)\phi(t),\xi(t)\psi(t)]$, so that

$$[\phi_c(t),\psi_c(t)] = \begin{cases} [\phi(t),\psi(t)] & \text{for } \epsilon_0 \leq t \leq T-\epsilon_0 \\ 0 & \text{outside } (0,T). \end{cases} \tag{29}$$

Proposition 8 *The ψ-component of the PDE (4) satisfies the following estimate for arbitary $\epsilon_1,\epsilon_2 > 0$:*

$$\int_{\Sigma_0}\left(\xi\frac{\partial\psi_t}{\partial\tau}\right)^2 d\Sigma_0 \leq C\,\|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} + \epsilon_1\int_{\Sigma_0}\xi^2\phi_t^2 d\Sigma_0 + \epsilon_2\int_{\Sigma_0}\xi^2\left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0. \tag{30}$$

Proof of Proposition 8: We note from (4) that $[\phi_c, \psi_c]$ satisfies

$$\begin{cases} \frac{\partial^2}{\partial t^2}\phi_c = \Delta\phi_c + \xi''\phi + 2\xi'\phi_t \quad \text{on } (0,T)\times\Omega \\ \begin{cases} \dfrac{\partial}{\partial\nu}\phi_c(t,x) = \xi\psi_t \quad \text{on } (0,T)\times\Gamma_0 \\ \dfrac{\partial}{\partial\nu}\phi_c(t,x) = 0 \quad \text{on } (0,T)\times\Gamma_1 \end{cases} \end{cases}$$

$$\begin{cases} \frac{\partial^2}{\partial t^2}\psi_c = \frac{\partial^2}{\partial\tau^2}\psi_c - \xi\phi_t|_{\Gamma_0} + 2\xi'\psi_t + \xi''\psi \quad \text{on } (0,T)\times\Gamma_0 \\ \dfrac{\partial\psi_c}{\partial n} = 0 \quad \text{on } (0,T)\times\partial\Gamma_0 \end{cases} \tag{31}$$

Time-differentiating the ψ_c-wave equation above, we have

$$\begin{array}{l} \frac{\partial^3}{\partial t^3}\psi_c = \\ \quad \xi\frac{\partial^2}{\partial\tau^2}\psi_t - \xi\ \phi_{tt}|_{\Gamma_0} + 2\xi'\psi_{tt} + \xi'\frac{\partial^2}{\partial\tau^2}\psi - \xi'\phi_t|_{\Gamma_0} + 3\xi''\psi_t\ + \xi'''\psi \ \text{ in } (0,T)\times\Gamma_0. \end{array} \tag{32}$$

Using the multiplier $(\xi\psi_t)$, and integrating in time and space, we obtain

$$\begin{array}{l} -\int_{\Sigma_0}\xi^2\dfrac{\partial^2\psi_t}{\partial\tau^2}\psi_t d\Sigma_0 = \\ \quad \int_{\Sigma_0}\left(-\frac{\partial^3}{\partial t^3}\psi_c - \xi\ \phi_{tt}|_{\Gamma_0} + 2\xi'\psi_{tt} + \xi'\frac{\partial^2}{\partial\tau^2}\psi - \xi'\phi_t|_{\Gamma_0} + 3\xi''\psi_t\ + \xi'''\psi\right)\xi\psi_t d\Sigma_0 \\ \le \left|\int_{\Sigma_0}\left(-\xi\ \phi_{tt}|_{\Gamma_0} + \xi'\frac{\partial^2}{\partial\tau^2}\psi - \xi'\phi_t|_{\Gamma_0}\right)\xi\psi_t d\Sigma_0\right| + C\left\|\psi\right\|^2_{H^2(0,T;L^2(\Gamma_0))}. \end{array} \tag{33}$$

To estimate the first term on the right hand side above, we note that an integration by parts and the fact that $\xi(0) = \xi(T) = 0$ gives

$$\begin{aligned} & \int_{\Sigma_0}\left(\xi'\phi_t|_{\Gamma_0} + \xi\ \phi_{tt}|_{\Gamma_0}\right)\xi\psi_t d\Sigma_0 \\ = & \ -\int_{\Sigma_0}\left(\xi'\psi_t + \xi\psi_{tt}\right)\xi\phi_t d\Sigma_0 \\ \le & \ \epsilon_1\int_{\Sigma_0}\xi^2\phi_t^2 d\Sigma_0 + C_{\epsilon_1}\left\|\psi\right\|^2_{H^2(0,T;L^2(\Gamma_0))}. \end{aligned} \tag{34}$$

In addition,

$$\int_{\Sigma_0}\xi'\frac{\partial^2\psi}{\partial\tau^2}\xi\psi_t d\Sigma_0 \le \epsilon_2\int_{\Sigma_0}\xi^2\left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0 + C_{\epsilon_2}\left\|\psi\right\|^2_{H^2(0,T;L^2(\Gamma_0))}. \tag{35}$$

Combining (33)-(35), we obtain

$$-\int_{\Sigma_0}\xi^2\frac{\partial^2\psi_t}{\partial\tau^2}\psi_t d\Sigma_0 \le C\left\|\psi\right\|^2_{H^2(0,T;L^2(\Gamma_0))} + \epsilon_1\int_{\Sigma_0}\xi^2\phi_t^2 d\Sigma_0 + \epsilon_2\int_{\Sigma_0}\xi^2\left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0. \tag{36}$$

Applying now Green's Theorem on the left hand side of (36) (using implicitly the fact that $\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Gamma_0} = 0$) gives now the asserted result. ∎

Proposition 9 *The ψ-component of the PDE system (4) obeys the estimate*

$$\begin{aligned}
&\int_{\Sigma_0} \xi^2 \left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0 \\
\leq\ & \{\epsilon_1 C_{\epsilon_5} + \epsilon_5 C_T\} \int_{\Sigma_0} \xi^2 \left.\phi_t\right|^2_{\Gamma_0} d\Sigma_0 \\
&+ (\epsilon_2 C_{\epsilon_5} + \epsilon_3 + \epsilon_4) \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0 \\
&+ C\,\|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} + \mathsf{l.o.t.}(\vec{\phi},\vec{\psi}),
\end{aligned} \tag{37}$$

where the ϵ_i, $i = 1,..5$, are arbitrarily small.

Proof: Multiplying the second wave equation in (31) by $\xi\psi_{\tau\tau}$, and integrating in time and space yields

$$\begin{aligned}
&\int_{\Sigma_0} \xi^2 \left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0 \\
=\ & \int_{\Sigma_0} \xi^2 \left.\phi_t\right|_{\Gamma_0} \frac{\partial^2\psi}{\partial\tau^2} d\Sigma_0 + \int_{\Sigma_0} \xi \frac{\partial^2}{\partial t^2}(\xi\psi) \frac{\partial^2\psi}{\partial\tau^2} d\Sigma_0 - \int_{\Sigma_0} (2\xi'\psi_t + \xi''\psi)\,\xi \frac{\partial^2\psi}{\partial\tau^2} d\Sigma_0 \\
\leq\ & \int_{\Sigma_0} \xi^2 \left.\phi_t\right|_{\Gamma_0} \frac{\partial^2\psi}{\partial\tau^2} d\Sigma_0 + C_{\epsilon_3}\,\|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} + \epsilon_3 \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0.
\end{aligned} \tag{38}$$

For the first term on the right hand side of this inequality, we can integrate by parts, first in time, and then in space (using implicitly $\xi(0) = \xi(T) = 0$), so as to have:

$$\begin{aligned}
\int_{\Sigma_0} \xi^2 \left.\phi_t\right|_{\Gamma_0} \frac{\partial^2\psi}{\partial\tau^2} d\Sigma_0 \ =\ & -2\int_{\Sigma_0} \left.\phi\right|_{\Gamma_0} \frac{\partial^2\psi}{\partial\tau^2}\xi\xi' d\Sigma_0 - \int_{\Sigma_0} \xi^2 \left.\phi\right|_{\Gamma_0} \frac{\partial^2\psi_t}{\partial\tau^2} d\Sigma_0 \\
=\ & \int_{\Sigma_0} \left.\frac{\partial\phi}{\partial\tau}\right|_{\Gamma_0} \frac{\partial\psi_t}{\partial\tau}\xi^2 d\Sigma_0 - 2\int_{\Sigma_0} \left.\phi\right|_{\Gamma_0} \frac{\partial^2\psi}{\partial\tau^2}\xi\xi' d\Sigma_0 \\
\leq\ & \int_{\Sigma_0} \left.\frac{\partial\phi}{\partial\tau}\right|_{\Gamma_0} \frac{\partial\psi_t}{\partial\tau}\xi^2 d\Sigma_0 \\
& +\epsilon_4 \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0 + \mathsf{l.o.t.}(\vec{\phi},\vec{\psi}).
\end{aligned} \tag{39}$$

(In obtaining (39), we are also using implicitly Sobolev trace theory and the inequality $ab \leq \epsilon a^2 + C_\epsilon b^2$.)

To estimate the first term on the right hand side of (39), we use in sequence the inequality $ab \leq \epsilon a^2 + C_\epsilon b^2$, the estimate (30) (for $\frac{\partial\psi_t}{\partial\tau}$) and the Lemma 3 (for $\left.\frac{\partial\phi}{\partial\tau}\right|_{\Gamma_0}$):

$$\begin{aligned}
\int_{\Sigma_0} \left.\frac{\partial\phi}{\partial\tau}\right|_{\Gamma_0} \frac{\partial\psi_t}{\partial\tau}\xi^2 d\Sigma_0 \ \leq\ & \epsilon_5 \int_{\Sigma_0} \left.\frac{\partial\phi_c}{\partial\tau}\right|^2_{\Gamma_0} d\Sigma_0 + C_{\epsilon_5} \int_{\Sigma_0} \xi^2 \frac{\partial\psi_t}{\partial\tau}^2 d\Sigma_0 \\
\leq\ & \epsilon_1 C_{\epsilon_5} \int_{\Sigma_0} \xi^2 \left.\phi_t\right|^2_{\Gamma_0} d\Sigma_0 + \epsilon_2 C_{\epsilon_5} \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0 \\
& +\epsilon_5 C_T \int_{\Sigma_0} \xi^2 \left.\phi_t\right|^2_{\Gamma_0} d\Sigma_0 + C\,\|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} \\
& + \mathsf{l.o.t.}(\vec{\phi},\vec{\psi}).
\end{aligned} \tag{40}$$

Combining now the estimates (39) and (40), we obtain

$$\begin{aligned}
&\int_{\Sigma_0} \xi^2 \left.\phi_t\right|_{\Gamma_0} \frac{\partial^2\psi}{\partial\tau^2} d\Sigma_0 \\
\leq\ & \{\epsilon_1 C_{\epsilon_5} + \epsilon_5 C_T\} \int_{\Sigma_0} \xi^2 \left.\phi_t\right|^2_{\Gamma_0} d\Sigma_0 \\
&+ (\epsilon_2 C_{\epsilon_5} + \epsilon_4) \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0 \\
&+ C \|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} + \mathsf{l.o.t.}(\vec{\phi},\vec{\psi}). \qquad (41)
\end{aligned}$$

Applying (41) to the right hand side of (38) concludes the derivation of the estimate (37). □

Proposition 10 *The ϕ-component of the solution to the system (4) obeys the following estimate:*

$$\begin{aligned}
\int_{\Sigma_0} \xi^2 \left.\phi_t\right|^2_{\Gamma_0} d\Sigma_0 \quad \leq \quad & (\epsilon_2 C + \epsilon_3 + \epsilon_4) \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0 \\
& + C \|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} + \mathsf{l.o.t.}(\vec{\phi},\vec{\psi}), \qquad (42)
\end{aligned}$$

where the ϵ_i, $i = 2,3,4$ are arbitrarily small.

Proof of Proposition 10: We have upon squaring both sides of the ψ_c-wave equation in (31), and subsequently integrating in time and space,

$$\begin{aligned}
\int_{\Sigma_0} \xi^2 \left.\phi_t\right|^2_{\Gamma_0} d\Sigma_0 \quad = \quad & \int_{\Sigma_0} \left(\frac{\partial^2\psi_c}{\partial\tau^2} + 2\xi'\psi_t + \xi''\psi - \frac{\partial^2\psi_c}{\partial t^2}\right)^2 d\Sigma_0 \\
\leq \quad & 2\int_{\Sigma_0} \left(\frac{\partial^2\psi_c}{\partial\tau^2}\right)^2 d\Sigma_0 + C \|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} \\
\leq \quad & \{\epsilon_1 C_{\epsilon_5} + \epsilon_5 C_T\} \int_{\Sigma_0} \xi^2 \left.\phi_t\right|^2_{\Gamma_0} d\Sigma_0 \\
& + (\epsilon_2 C_{\epsilon_5} + 2\epsilon_3 + 2\epsilon_4) \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0 \\
& + C \|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} + \mathsf{l.o.t.}(\vec{\phi},\vec{\psi}), \qquad (43)
\end{aligned}$$

where in obtaining the last inequality, we have invoked Proposition 9. Taking now ϵ_5 small enough so that, say, $1 - \epsilon_5 C_T > \frac{3}{4}$, we have

$$\begin{aligned}
&\frac{3}{4}\int_{\Sigma_0} \xi^2 \left.\phi_t\right|^2_{\Gamma_0} d\Sigma_0 \\
\leq\ & \epsilon_1 C_{\epsilon_5} \int_{\Sigma_0} \xi^2 \left.\phi_t\right|^2_{\Gamma_0} d\Sigma_0 \\
&+ (\epsilon_2 C_{\epsilon_5} + 2\epsilon_3 + 2\epsilon_4) \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2\psi}{\partial\tau^2}\right)^2 d\Sigma_0 \\
&+ C \|\psi\|^2_{H^2(0,T;L^2(\Gamma_0))} + \mathsf{l.o.t.}(\vec{\phi},\vec{\psi}).
\end{aligned}$$

Subsequently taking ϵ_1 small enough so that $\frac{3}{4} - \epsilon_1 C_{\epsilon_5} > \frac{1}{2}$ gives now

$$\begin{aligned} & \frac{1}{2} \int_{\Sigma_0} \xi^2 \left. \phi_t \right|_{\Gamma_0}^2 d\Sigma_0 \\ \leq \; & (\epsilon_2 C_{\epsilon_5} + 2\epsilon_3 + 2\epsilon_4) \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2 \psi}{\partial \tau^2} \right)^2 d\Sigma_0 \\ & + C \left\| \psi \right\|^2_{H^2(0,T;L^2(\Gamma_0))} + \; \mathsf{l.o.t.}(\vec{\phi}, \vec{\psi}), \end{aligned}$$

which is the asserted estimate (42), upon a rescaling of ϵ_3 and ϵ_4. $\square$

Proof proper of Lemma 6: Squaring both sides of the ψ_c-wave equation in (31), integrating in time and space, and applying Cauchy-Schwartz, we obtain the inequality

$$\int_{\Sigma_0} \xi^2 \left(\frac{\partial^2 \psi}{\partial \tau^2} \right)^2 d\Sigma_0 \leq 2 \int_{\Sigma_0} \xi^2 \left(\left. \phi_t \right|_{\Gamma_0} \right)^2 d\Sigma_0 + C \left\| \psi \right\|^2_{H^2(0,T;L^2(\Gamma_0))} . \tag{44}$$

Applying now the trace estimate (42) gives

$$\begin{aligned} & \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2 \psi}{\partial \tau^2} \right)^2 d\Sigma_0 \\ \leq \; & 2 \left(\epsilon_2 C + \epsilon_3 + \epsilon_4 \right) \int_{\Sigma_0} \xi^2 \left(\frac{\partial^2 \psi}{\partial \tau^2} \right)^2 d\Sigma_0 \\ & + C \left\| \psi \right\|^2_{H^2(0,T;L^2(\Gamma_0))} + \; \mathsf{l.o.t.}(\vec{\phi}, \vec{\psi}). \end{aligned}$$

Taking now the ϵ_i small enough yields the estimate

$$\int_{\Sigma_0} \xi^2 \left(\frac{\partial^2 \psi}{\partial \tau^2} \right)^2 d\Sigma_0 \leq C \left\| \psi \right\|^2_{H^2(0,T;L^2(\Gamma_0))} + \; \mathsf{l.o.t.}(\vec{\phi}, \vec{\psi}).$$

This completes the proof of Lemma 6.

2.4 Proof Proper of Theorem 1

2.4.1 Completion of the Proof of Theorem 1(a)

The Tainted Observability Inequality Recall that Theorem 1(a), which provides for exact controllability of the PDE system (1) under Assumption (A1), is equivalent to obtaining the inequality (5) for the adjoint system (4).

Combining the estimates from Lemma 5 and Corollary 7, along with the definition of the cutoff function ξ in (8), we obtain

$$\begin{aligned} & \int_{\epsilon_0}^{T-\epsilon_0} \mathcal{E}(t) dt \\ \leq \; & C_{T,\epsilon_0} \left\| \psi \right\|^2_{H^2(0,T;L^2(\Gamma_0))} + C \mathcal{E}(T) + \mathsf{l.o.t.}(\vec{\phi}, \vec{\psi}), \end{aligned} \tag{45}$$

where again C_{T,ϵ_0} depends on time T, but C does not. Using the conservation relation (9), we have

$$\begin{aligned} & (T - 2\epsilon_0) \, \mathcal{E}(T) \\ \leq \; & C_T \left\| \psi \right\|^2_{H^2(0,T;L^2(\Gamma_0))} + C \mathcal{E}(T) + \mathsf{l.o.t.}(\vec{\phi}, \vec{\psi}). \end{aligned}$$

From this inequality, we have then

Lemma 11 *Under Assumption (A1), the solution of (4) obeys the following estimate for terminal time T large enough:*

$$\left\| \left[\vec{\phi}_0, \vec{\psi}_0\right] \right\|_{\mathcal{H}}^2 \leq C \left\|\psi\right\|^2_{H^2(0,T;L^2(\Gamma_0))} + \text{l.o.t.}(\vec{\phi}, \vec{\psi}). \tag{46}$$

Removal of Lower Order Terms The proof of Theorem 1(a) will be completed upon the derivation of the following estimate which can derived by a compactness/uniqueness, akin to that employed in [15] and [2]. In the statement of the result, we recall the following implication from Holmgren's Uniqueness Theorem:

$$\begin{cases} p_{tt} = \Delta p \ \text{ in } (0,T) \times \Omega \\ \frac{\partial p}{\partial \nu} = 0 \quad \text{on } (0,T) \times \Gamma \\ p|_{\Gamma_0} = 0 \quad \text{on } (0,T) \times \Gamma_0 \\ \text{If } T \text{ is sufficiently large, then necessarily } p = 0. \end{cases} \tag{47}$$

(see [9], Theorem 5.33, p. 129).

Lemma 12 *Let T be large enough so that the uniqueness property in (47) and the inequality (46) hold true. Then there exists a constant $C_T > 0$ such that the solution of (4) satisfies the following inequality:*

$$\left\| \left[\vec{\phi}, \vec{\psi}\right] \right\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{1-\epsilon}(\Gamma_0)\times H^{-\epsilon}(\Gamma_0))} \leq C_T \left\|\psi\right\|^2_{H^2(0,T;L^2(\Gamma_0))} \tag{48}$$

Proof: This runs by a classical compactness/uniqueness argument. Suppose the lemma is false. Then there exists a sequence of initial data $\left\{\left[\vec{\phi}_0^{(n)}, \vec{\psi}_0^{(n)}\right]\right\} \subset D(\mathcal{L}_T^*)$, and a corresponding solution sequence $\left\{\left[\vec{\phi}^{(n)}, \vec{\psi}^{(n)}\right]\right\}$ of the PDE (4) which satisfies

$$\begin{aligned} \left\|\vec{\phi}^{(n)}, \vec{\psi}^{(n)}\right\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{1-\epsilon}(\Gamma_0)\times H^{-\epsilon}(\Gamma_0))} &= 1 \ \forall n, \\ \left\|\psi^{(n)}\right\|^2_{H^2(0,T;L^2(\Gamma_0))} &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{49}$$

Consequently, the existence of the inequality (46) for T large enough, and the convergence in (49), imply that the sequence $\left\{\left\|\left[\vec{\phi}_0^{(n)}, \vec{\psi}_0^{(n)}\right]\right\|_{\mathcal{H}}\right\}_{n=1}^{\infty}$ is bounded uniformly in n, whence there exists a subsequence, still denoted by $\left\{\left[\vec{\phi}_0^{(n)}, \vec{\psi}_0^{(n)}\right]\right\}_{n=1}^{\infty}$, and $\left[\vec{\phi}_0^*, \vec{\psi}_0^*\right] \in \mathcal{H}$ such that

$$\left[\vec{\phi}_0^{(n)}, \vec{\psi}_0^{(n)}\right] \to \left[\vec{\phi}_0^*, \vec{\psi}_0^*\right] \quad \text{in } \mathcal{H} \text{ weakly.} \tag{50}$$

If we denote $\left[\vec{\phi}^*, \vec{\psi}^*\right]$ as the solution pair of (4), corresponding to terminal data $\left[\vec{\phi}_0^*, \vec{\psi}_0^*\right]$, then *a fortiori*

$$\left\{\left[\vec{\phi}^{(n)}, \vec{\psi}^{(n)}\right]\right\} \to \left[\vec{\phi}^*, \vec{\psi}^*\right] \ \text{ in } L^\infty(0,T;\mathcal{H}) \text{ weak star.} \tag{51}$$

Moreover, reading off the ϕ-wave equation in (4), we can use elliptic theory, the weak convergence of $\left\{\phi^{(n)}\right\}$, and the convergence of the sequence in (49), so as to deduce the estimate

$$\left\|\phi_{tt}^{(n)}\right\|_{C([0,T];[H^1(\Omega)]')} \leq C, \ \text{ uniformly in } n. \tag{52}$$

From (49), we have in addition

$$\left\|\psi_{tt}^{(n)}\right\|_{L^2(0,T;L^2(\Gamma_0))} \le C, \text{ uniformly in } n. \tag{53}$$

From (51)–(53), we conclude then, from a classic compactness result of J. Simon in [26], that

$$\left[\vec{\phi}^{(n)}, \vec{\psi}^{(n)}\right] \to \left[\vec{\phi}^{*}, \vec{\psi}^{*}\right] \quad \text{in } C([0,T]; H^{1-\epsilon}(\Omega) \times H^{-\epsilon}(\Omega) \times H^{1-\epsilon}(\Gamma_0) \times H^{-\epsilon}(\Gamma_0)) \text{ strongly.}$$

In consequence, we have from (49) the equality

$$\left\|\vec{\phi}^{*}, \vec{\psi}^{*}\right\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{1-\epsilon}(\Gamma_0)\times H^{-\epsilon}(\Gamma_0))} = 1. \tag{54}$$

Furthermore, as we noted earlier in equating (3) and (5), we have the representation

$$\mathcal{L}_T^{*}\begin{bmatrix} \vec{\phi}_0^{(n)} \\ \vec{\psi}_0^{(n)} \end{bmatrix} = \psi_t^{(n)} \text{ for all } n. \tag{55}$$

Given the convergence in (49), we conclude that the weak limit $\left[\vec{\phi}^{*}, \vec{\psi}^{*}\right]$ is in $D(\mathcal{L}_T^{*})$, and

$$\mathcal{L}_T^{*}\begin{bmatrix} \vec{\phi}_0^{*} \\ \vec{\psi}_0^{*} \end{bmatrix} = 0. \tag{56}$$

We will use (54) and (56) now to obtain a contradiction. From (56), we conclude that

$$\psi_t^{*} = 0. \tag{57}$$

Hence ψ^* is constant on Σ_0, and since either *a priori* either $\psi^*|_{\partial\Gamma_0} = 0$ or $\psi^* \in \frac{H^1(\Gamma_0)}{\mathbb{R}}$, we deduce that

$$\psi^{*} = 0. \tag{58}$$

From the ψ-wave equation in (4), this implies that $\phi_t^*|_{\Gamma_0} = 0$. Making the change of variable $p = \phi_t^*$, then from (4) and (57), we see that p satisfies the wave equation in (47). Since T is sufficiently large to secure the Holmgren uniqueness property, we have necessarily that

$$\phi_t^{*} = 0. \tag{59}$$

So ϕ^* is a constant on Q, and moreover since $\phi^* \in \frac{H^1(\Omega)}{\mathbb{R}}$, we deduce that

$$\phi^{*} = 0. \tag{60}$$

The relations (57), (58), (59) and (60) contradict the equality (54). $\square$

For T sufficiently large, the proof of Theorem 1 is now completed by combining the estimates in Lemmas 11 and 12.

2.4.2 Completion of the Proof of Theorem 1(b)

Applying the conservation relation (9) to the inequality (28), valid under Assumption (A2), we have

$$\begin{aligned} & (T-2\epsilon_0)\mathcal{E}(T) \\ \leq \quad & C_{T,\epsilon_0,h}\left\{\int_0^T \|\psi_t\|^2_{H^{\frac{1}{4}}(\Gamma_0)}\,dt + \int_{\Sigma_1}\phi_t^2 d\Sigma_1\right\} + C\mathcal{E}(T); \end{aligned} \tag{61}$$

whence the inequality

$$\mathcal{E}(T) \leq C_{T,\epsilon_0,h}\left\{\int_0^T \|\psi_t\|^2_{H^{\frac{1}{4}}(\Gamma_0)}\,dt + \int_{\Sigma_1}\phi_t^2 d\Sigma_1\right\} + \text{ l.o.t.}(\vec{\phi},\vec{\psi}). \tag{62}$$

By an argument virtually identical to that used to prove Lemma 12, we can eliminate the lower order terms in (62) to thereby obtain the requisite reverse inequality (6). This completes the proof of Theorem 1(b).

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