# Error Estimates for Finite Difference Solutions of Second-Order Elliptic Equations in Discrete Sobolev Spaces

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## Abstract

We derive a priori $l_p$-estimates for finite difference solutions of second-order elliptic equations with continuous coefficients in the whole space. We shall mainly use the Fefferman-Stein theorem and discrete Sobolev inequalities to establish our purpose. Based on these $l_p$-estimates, we obtain the convergence rate of the approximate solutions and their difference quotients in the sup norm.

# Contents

# Chapter 1

# Introduction

We consider a simple elliptic equation of second order in the whole space

$$Lu(x) - \lambda u(x) = f(x) \quad \text{in} \quad \mathbb{R}^d \tag{1.1}$$

where the operator $L$ is defined by

$$Lu := \sum_{i=1}^{d} a_i(x) D_{x_i x_i} u.$$

We assume $a(x)$ satisfies an ellipticity condition: there is a constant $\kappa > 0$ such that

$$\kappa \le a_i(x) \le \kappa^{-1} \quad \text{for all } x \in \mathbb{R}^d \text{ and } i = 1, ..., d.$$

Also we assume $a(x)$ is uniformly continuous.

$L_p$-solvability to this problem has been intensively studied by numerous authors. For instance, see [1, 4, 15, 16], etc. The $L_p$-theory for elliptic and parabolic equations is still an active research area. Many researchers (see [5, 6, 12, 21]) have tried and are trying to minimize regularity conditions for coefficients, nonhomogeneous data and domains. However, a priori $l_p$-estimates for numerical solutions to elliptic and parabolic difference equations have rarely been touched. It is an important subject because it allows us to measure an error occuring from numerical schemes under mild regularity conditions. Thus, the primary goal of this thesis is to achieve a priori $l_p$-estimates for the second-order elliptic difference equation. Similar methods can be applied to deal with parabolic equations but we omit the details here. The main $l_p$-estimate is stated in Theorem 7.1.1 and the convergence rate of the numerical solutions is stated in Theorem 8.0.3. Before we discuss these results more in depth we introduce some definitions and notations which will be used throughout the thesis.

We first consider a grid over $\mathbb{R}^d$ with a mesh size $h \le 1$:

$$h\mathbb{Z}^d = \{\sum_{i=1}^{d} hn_i e_i : n_i \in \mathbb{Z}\},$$

where $\{e_1, ..., e_d\}$ is a standard basis for $\mathbb{R}^d$. For $\xi \in \mathbb{Z}^d$ and a function $u$ defined in $h\mathbb{Z}^d$ we define a difference quotient of it in the direction of $\xi$:

$$\delta_{h,\xi} u(x) = \frac{u(x + h\xi) - u(x)}{h}.$$

For a multiple vector $\xi = (\xi_1, ..., \xi_n)$ with $\xi_i \in \mathbb{Z}^d$ we define

$$\delta^n_{h,\xi} u(x) = \delta_{h,\xi_1} \cdots \delta_{h,\xi_n} u(x).$$

In the sequel, $\delta^n_h u$ means the set of all $n$th order difference quotients of $u$ where $\xi_i$'s are chosen from $\Lambda = \{\pm e_1, ..., \pm e_d\}$.

Next, we consider the corresponding elliptic difference equation of (1.1):

$$L_h u - \lambda u = f \quad \text{in} \quad h\mathbb{Z}^d, \tag{1.2}$$

where

$$L_h u = -\sum_{i=1}^{d} a_i(x) \delta_{h,e_i} \delta_{h,-e_i} u.$$

We shall look for solutions to (1.1) and (1.2) in Sobolev spaces. Let $k \in \{0, 1, 2, ...\}$ and $p \in [1, \infty]$. Then for an arbitrary domain $D \subset \mathbb{R}^d$, $W^k_p(D)$ stands for the usual Sobolev space [17]. Let us also define discrete Sobolev spaces. Let $p \in [1, \infty)$ and $\Omega$ be a subset of $h\mathbb{Z}^d$. Then by $l_p(\Omega)$ we mean the set of all functions defined in $\Omega$, say $u$, such that

$$\|u\|_{l_p(\Omega)} = (\sum_{\Omega} |u(x)|^p h^d)^{1/p} < \infty.$$

We use $l_\infty(\Omega)$ to represent the set of all bounded functions in $\Omega$. Let us introduce another norm to handle higher order difference quotients of $u$ as follows:

$$[u]_{l_p^1(\Omega)} = \Big( \sum_{x\in\Omega,\xi\in\Lambda,x+h\xi\in\Omega} |\delta_{h,\xi}u(x)|^p h^d\Big)^{1/p}, \tag{1.3}$$

$$\|u\|_{l_p^1(\Omega)} = \|u\|_{l_p(\Omega)} + [u]_{l_p^1(\Omega)}.$$

We emphasize that we exclude some of difference quotients of $u$ if they involve the points outside $\Omega$. Unless otherwise stated (cf. Remarks 6.2.1), we shall use $\|\delta_h u\|_{l_p(\Omega)}$ to denote $[u]_{l_p^1(\Omega)}$ even if $u$ is defined in the whole space $h\mathbb{Z}^d$. In a similar manner, we can define $[\cdot]_{l_p^k(\Omega)}$ and $\|\cdot\|_{l_p^k(\Omega)}$ for $k \in \{2, 3, ...\}$. Also, the subset of functions defined in $\Omega$ which vanish for sufficiently large $x$ is denoted by $l_0(\Omega)$. We often use $\|\cdot\|_{l_p^k}$ in place of $\|\cdot\|_{l_p^k(h\mathbb{Z}^d)}$ if the domain is the whole space and the mesh size $h$ is already specified.

Throughout the thesis, we shall often assume the mesh size is 1. This will simplify most of computations in the sequel. More importantly, many of those estimates are scaling invariant with respect to the mesh size so that we can easily extend them to the other case where it is assumed to be less than one. In the case of $h = 1$, we shall use $\delta_\xi u$ in place of $\delta_{h,\xi}u$.

In order to obtain the $L_p$-solvability of the model equation (1.1), classical theory (see [1, 4, 15, 16]) mainly uses the Calderon-Zygmund decomposition [7] and the Marcinkiewicz interpolation [23]. Given a function on a domain, Calderon and Zygmund [7] introduced a way of splitting this domain into two parts: one where the given function is relatively small and the other which can be partitioned into disjoint cubes where the function is possibly singular but still controllable. Meanwhile, together with this cube decomposition the classical approach essentially uses some properties of the representation formulas for the Laplace or the heat equations to obtain a priori $L_1$ and

$L_2$-estimates. After that, it applies the Marcinkiewicz interpolation theorem to achieve $L_p$-estimates for a general power $p \in (1, 2)$. The case $p \in (2, \infty)$ comes from the duality argument.

Bondesson [3] employed the discrete version of this idea to obtain a priori interior $l_p$-estimates for elliptic and parabolic difference equations. In place of the fundamental solution to the heat equation, he used the fundamental solution to the discrete heat equation of which the useful estimates were developed by Widlund [28]. Shreve [24] independently developed the same interior $l_p$-estimates for elliptic difference equations adapting the methods used in [15, 16] .

However, Krylov [21] recently invented another approach to prove $L_p$-solvability of elliptic and parabolic equations using the Fefferman-Stein and Hardy-Littlewood theorem on sharp and maximal functions. The advantage of Krylov's method is that it uses pointwise estimates of the sharp function of second-order derivatives of a solution and does not use a representation formula of the solution. So it is simple and can be flexibly applied to other situations. Indeed, he has extended the $L_p$ theory to fully nonlinear equations with VMO coefficients by collaboration with Dong and Kim [13, 18] and to fully nonlinear equations under relaxed convexity assumptions [20]. Taking this advantage of flexibility, we pursue to develop discrete $l_p$-estimates using the Fefferman-Stein and Hardy-Littlewood theorem and obtain the rate of convergence of the difference quotients of numerical solutions.

In the literature, Eymard, Gallouet and Herbin [9, 10] developed $l_p$-error estimates for finite volume solutions of convection diffusion equations. Although they successfully estimated the error in the $l_p$ norm, the first-order difference quotients of the error were still estimated in the $l_2$ norm and they did not relax the regularity conditions for nonhomogeneous data.

# Chapter 2

# Review of Classical Approach to $L_p$-Estimates

Before we get into the main subject we shall review the main idea of the traditional approach to obtain the $L_p$-estimates for the model equation (1.1). It will be instructive to compare this to the alternative method utilizing sharp and maximal functions. For those who are familiar with the classical method, which is presented in [7, 11], this chapter may be skipped.

**Theorem 2.0.1.** *Let* $1 < p < \infty$ *and* $f \in L^p(\mathbb{R}^d)$*. Define* $u$ *to be the Newtonian potential of* $f$*:*

$$u(x) = C(d) \int_{\mathbb{R}^d} G(x-y) f(y) dy$$

*where* $G(x) = 1/|x|^{d-2}$ *and* $C(d)$ *is some constant. Then we have* $u \in W_p^2(\mathbb{R}^d)$*,* $\Delta u = f$ *and*

$$\|D^2 u\|_{L_p} \le N \|f\|_{L_p}$$

*where* $N = N(d,p)$*.*

*Proof.* Without loss of generality we may assume $f \in C_0^\infty$, that is, a compactly supported smooth function. We observe

$$\begin{aligned} \|D^2u\|_{L_2}^2 &= \sum_{i,j} \int_{\mathbb{R}^d} |D_{x_i x_j} u|^2 dx \\ &= \sum_{i,j} \int_{\mathbb{R}^d} D_{x_i x_i} u D_{x_j x_j} u dx \quad \text{(integration by parts)} \\ &= \int_{\mathbb{R}^d} |\Delta u|^2 dx = \|f\|_{L_2}^2. \end{aligned} \tag{2.1}$$

Now for each $i$ and $j$, we define $Tf = D_{ij}u$ and try to claim

$$|\{|Tf| > \lambda\}| \le N\lambda^{-1} \int_{\mathbb{R}^d} |f| dx. \tag{2.2}$$

For that purpose take a large cube $C_0$ containing the support of $f$ so that

$$\frac{1}{|C_0|} \int_{C_0} |f| dx \le \lambda.$$

By slicing each edge of $C_0$ in half, we can decompose $C_0$ into $2^d$ disjoint subcubes of the same size. If one of its subcubes, say $C$, satisfies

$$\frac{1}{|C|} \int_C |f| dx > \lambda, \tag{2.3}$$

then we set it aside. Otherwise, we divide $C$ into more refined disjoint subcubes in the same manner. Upon repeating this procedure, we use $\mathcal{C} = \{C_1, C_2, ...\}$ to denote all the subcubes which meet the condition (2.3). Certainly, $C_k$'s ($k \ge 1$) are all disjoint. Let $B = \bigcup_{C_k \in \mathcal{C}} C_k$ and $G = \mathbb{R}^d \backslash B$. Then we have to bear in mind two important properties:

$$|f(x)| \le \lambda, \ \forall x \in G \quad \text{(Lebegue differention theorem)} \tag{2.4}$$

and for each $C_k \in \mathcal{C}$

$$\frac{1}{|C_k|} \int_{C_k} |f| dx \le \frac{|\tilde{C}_k|}{|C_k||\tilde{C}_k|} \int_{\tilde{C}_k} |f| dx \le 2^d \lambda, \tag{2.5}$$

where $\tilde{C}_k$ is the parent of $C_k$, that is, the cube whose subdivision yields $C_k$.

Now we split $f$ into a good function $g$ and a bad function $b$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in G \\ \frac{1}{|C_k|} \int_{C_k} f(x)dx & \text{if } x \in C_k \text{ for some } k \geq 1 \end{cases} \tag{2.6}$$

and

$$b(x) = f(x) - g(x).$$

Then, as for $g$ we have

$$|\{|Tg| > \lambda/2\}| \leq 4\lambda^{-2} \int_{\mathbb{R}^d} |g|^2 dx \leq N\lambda^{-1} \int_{\mathbb{R}^d} |g|dx \quad \text{by (2.4) and (2.5)} \tag{2.7}$$

$$\leq N\lambda^{-1} \int_{\mathbb{R}^d} |f|dx \quad \text{by (2.6).} \tag{2.8}$$

On the other hand, as for $b$ we define $b_k(x) = b(x)I_{C_k}$ for each $k \geq 1$. By the density argument we may assume that $b_k \in C_0^\infty(C_k)$ without losing any properties of the original $b_k$. Then for $x \notin C_k$ we have

$$\begin{aligned} Tb_k(x) &= \int_{C_k} D_{x_i x_j} G(x-y) b(y) dy \\ &= \int_{C_k} (D_{x_i x_j} G(x-y) - D_{x_i x_j} G(x-y_0)) b(y) dy \end{aligned}$$

where $y_0$ is any point in $C_k$. Indeed, this identity holds because $\int_{C_k} b(x)dx = 0$. Then by an easily verified property of Newton potential $G$ we have

$$\begin{aligned} |Tb_k(x)| &\leq Nr \sup_{z \in C_k} |D^3 G(x-z)| \int_{C_k} |b(y)|dy \\ &\leq Nr \sup_{z \in C_k} |x-z|^{-d-1} \int_{C_k} |b(y)|dy, \end{aligned}$$

where $r := d(C_k)$ is the diameter of $C_k$. Let $B_k$ be the ball of radius $r$ which is concentric to $C_k$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B_k} |Tb_k(x)|dx &\le Nr\int_{\mathbb{R}^d\setminus B_k} \mathrm{dist}^{-d-1}(x,C_k)dx\int_{C_k}|b(y)|dy\\ &\le Nr\int_{|x|>r/2}|x|^{-d-1}dx\int_{C_k}|b(y)|dy\\ &\le N\int_{C_k}|b(y)|dy\\ &\le N\int_{C_k}|f(y)|+|g(y)|dy \le N\int_{C_k}|f(y)|dy\end{aligned}$$

by the definition of $g$. Letting $B^* = \bigcup_k B_k$, we have

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B^*}|Tb(x)|dx &\le \sum_k \int_{\mathbb{R}^d\setminus B_k}|Tb_k(x)|dx \le N\sum_k\int_{C_k}|f(y)|dy\\ &\le \int_{\mathbb{R}^d}|f(y)|dy.\end{aligned}$$

This implies

$$|\{|Tb|>\lambda/2\}\cap(B^*)^c| \le 2\lambda^{-1}\int_{(B^*)^c}|Tb(x)|dx \le 2\lambda^{-1}\int_{\mathbb{R}^d}|f(y)|dy.$$

Moreover,

$$\begin{aligned}|B^*| &\le N\sum_k |C_k| \le \lambda^{-1}\sum_k\int_{C_k}|f(y)|dy \quad \text{by (2.3)}\\ &\le N\lambda^{-1}\int_{\mathbb{R}^d}|f(y)|dy.\end{aligned}$$

Consequently,

$$|\{|Tb|>\lambda\}| \le N\lambda^{-1}\int_{\mathbb{R}^d}|f(y)|dy. \tag{2.9}$$

Combining (2.9) and (2.7), we claim that (2.2) is true. Lastly, based on (2.1) and (2.2), Marcinkiewicz interpolation theorem implies

$$\|D^2u\|_{L_p} \le N\|f\|_{L_p}$$

for $p\in(1,2)$. By the duality argument it holds for $p\in(2,\infty)$ as well. The theorem is proved. □

# Chapter 3

# Preliminary Estimates

Through Chapter 3 to Chapter 6 we deal with elliptic equations with constant coefficients. In this case without loss of generality we may further assume $a_i = 1$ for all $i = 1, ..., d$. The starting point towards the $l_p$-estimates is the following $l_2$-estimates.

## 3.1 Some Properties of the simplest equation

**Lemma 3.1.1.** *Let $\lambda \geq 0$ and $k \in \{0, 1, 2, ...\}$. Then for any $u \in l_2(\mathbb{Z}^d)$ it holds that*

$$\|\delta^2 u\|_{l_2^k} + \sqrt{\lambda}\|\delta u\|_{l_2^k} + \lambda\|u\|_{l_2^k} \leq N\|\Delta_1 u - \lambda u\|_{l_2^k},$$

*where $N = N(d, k)$.*

*Proof.* Without loss of generality we may assume that $k = 0$ and $u \in l_0$. For $f = \Delta_1 u - \lambda u$ we observe

$$\begin{aligned}
&\sum_{\mathbb{Z}^d} f(x)^2 = \sum_{\mathbb{Z}^d}(-\sum_{i=1}^{d} \delta_{-e_i}\delta_{e_i}u - \lambda u)^2 \\
=&\sum_{\mathbb{Z}^d}\big(\sum_i(\delta_{-e_i}\delta_{e_i}u)^2 + \sum_{i\neq j}(\delta_{-e_i}\delta_{e_i}u)(\delta_{-e_j}\delta_{e_j}u) + 2\lambda\sum_i u\delta_{-e_i}\delta_{e_i}u + \lambda^2 u^2\big) \\
=&\sum_{\mathbb{Z}^d}\big(\sum_i(\delta_{-e_i}\delta_{e_i}u)^2 + \sum_{i\neq j}|\delta_{e_i}\delta_{e_j}u|^2 + 2\lambda\sum_i|\delta_{e_i}u|^2 + \lambda^2 u^2\big)
\end{aligned}$$

by summation by parts. Therefore, together with identities like

$$\delta_{-e_i}u(x) = -\delta_{e_i}u(x-e_i), \quad \delta_{e_i}\delta_{-e_j}u(x) = -\delta_{e_i}\delta_{e_j}u(x-e_j) \tag{3.1}$$

we conclude

$$\sum_{\mathbb{Z}^d}\left(|\delta^2 u|^2 + \lambda|\delta u|^2 + \lambda^2|u|^2\right) \le N\sum_{\mathbb{Z}^d}|f|^2.$$

The lemma is proved. □

We state $l_2$-solvability of the model equation.

**Lemma 3.1.2.** *For any $\lambda > 0$ and $f \in l_2(\mathbb{Z}^d)$ there exists a unique $u \in l_2(\mathbb{Z}^d)$ such that*

$$\Delta_1 u - \lambda u = f.$$

*Proof.* Uniqueness follows from Lemma 3.1.1. It also implies that the set $\Gamma := (\Delta_1 - \lambda)l_2(\mathbb{Z}^d)$ is a closed linear subset of $l_2(\mathbb{Z}^d)$. Indeed, suppose $\Delta_1 u_n - \lambda u_n = f_n$ converges to $f$ in $l_2$. Then by Lemma 3.1.1 $u_n$ is a Cauchy sequence in $l_2$. Thus, there exists $u \in l_2$ such that $u_n \to u$ in $l_2$. Due to this fact, we certainly have $\Delta_1 u_n - \lambda u_n = f_n$.

Therefore, it remains to show $\Gamma = l_2(\mathbb{Z}^d)$. Assume that it is not the case. Then by the Hahn-Banach theorem there exists $g \in l_2$ with $g \not\equiv 0$ such that

$$\sum_{\mathbb{Z}^d} g(\Delta_1 u - \lambda u) = 0, \quad \forall u \in l_2.$$

In particular,

$$0 = \sum_{\mathbb{Z}^d} g(\Delta_1 g - \lambda g) \le -\sum_{\mathbb{Z}^d}|\delta_{e_i} g|^2 - \lambda\sum_{\mathbb{Z}^d}|g|^2 < 0$$

because $g \not\equiv 0$. This is the desired contradiction and therefore the lemma is proved. □

We observe that the maximum principle also holds for our model equation.

**Lemma 3.1.3.** *Let $\lambda > 0$ and $u \in l_2$. Then we have*

$$\lambda \sup_{\mathbb{Z}^d}|u| \le \sup_{\mathbb{Z}^d}|\Delta_1 u - \lambda u|.$$

*Proof.* It suffices to show that

$$\lambda \sup_{\mathbb{Z}^d} u_+ \leq \sup_{\mathbb{Z}^d}(\lambda u - \Delta_1 u)_+,$$

where $g_+ = \max(g, 0)$. Let $u_+ \not\equiv 0$. Since $u$ decays at infinity, there exists $\bar{x} \in \mathbb{Z}^d$ such that $u(\bar{x}) = \sup_{\mathbb{Z}^d} u > 0$. We observe $\Delta_1 u(\bar{x}) \leq 0$. Therefore,

$$\lambda u(\bar{x}) \leq \lambda u(\bar{x}) - \Delta_1 u(\bar{x}).$$

and the lemma is proved. □

**Theorem 3.1.1.** *Let $\lambda > 0$, $p \in (1, \infty)$. Then the operator $\Delta_1 - \lambda$ is bijective from $l_p$ onto $l_p$. Further, for any $u \in l_p$, we have*

$$\lambda \|u\|_{l_p} \leq N \|\Delta_1 u - \lambda u\|_{l_p}.$$

*Proof.* For any $f \in l_2 \subset l_\infty$, by Lemma 3.1.2 we can find a unique $u \in l_2$ such that

$$\Delta_1 u - \lambda u = f.$$

Define a linear mapping $\Phi : l_2 \to l_2$ by $\Phi(f) = u$. Then we have

$$\lambda \|\Phi f\|_{l_2} \leq N \|f\|_{l_2} \quad \text{(Lemma 3.1.1)}$$

and

$$\lambda \|\Phi f\|_{l_\infty} \leq N \|f\|_{l_\infty} \quad \text{(Lemma 3.1.3)}.$$

The Marcinkiewicz interpolation theorem implies that for any $p \in (2, \infty)$ we can extend $\Phi$ to be defined from $l_p$ to itself and

$$\lambda \|\Phi f\|_{l_p} \leq N \|f\|_{l_p}, \quad \forall f \in l_p. \tag{3.2}$$

Now let $f \in l_p$ and take a sequence $f_n \in l_0$ such that $\|f - f_n\|_{l_p} \to 0$. If we define $u_n = \Phi(f_n) \in l_p \cap l_2$, then by Lemma 3.1.2 we have

$$\Delta_1 u_n - \lambda u_n = f_n.$$

By (3.2) $u_n$ is a Cauchy sequence in $l_p$ and thus there exists a $u \in l_p$ such that

$$\|u - u_n\|_{l_p} \to 0 \quad \text{and thus} \quad \Delta_1 u - \lambda u = f.$$

Further, by (3.2) again, $\|\Phi f - u_n\|_{l_p} = \|\Phi f - \Phi f_n\|_{l_p} \to 0$, which implies $u = \Phi f$. Thus, we finally have

$$\lambda \|u\|_{l_p} \leq N \|\Delta_1 u - \lambda u\|_{l_p}.$$

Therefore, the theorem is proved for $p \in (2, \infty)$ and the duality argument tells us it holds for $p \in (1, 2)$ as well. Hence, the theorem is proved. □

We can also obtain boundary $l_2$-estimates for cubes. For $h \leq 1$, $r \in \{0, h, 2h, ...\}$ and $x \in h\mathbb{Z}^d$ we define:

$$C_{r,h}(x) = \{x + (c_1, ..., c_d) : c_i = 0, \pm h, \pm 2h, ..., \pm r, \ \forall i\},$$

$$\partial C_{r,h}(x) = \{x + (c_1, ..., c_d) \in C_{r,h}(x) : c_i = \pm r \text{ for some } i\},$$

$$\mathring{C}_{r,h}(x) = C_{r,h}(x) \backslash \partial C_{r,h}(x),$$

$$C_{r,h} = C_{r,h}(0), \ \partial C_{r,h} = \partial C_{r,h}(0), \ \mathring{C}_{r,h} = \mathring{C}_{r,h}(0).$$

In case of $h = 1$ we use simpler notations:

$$C_r(x) = C_{r,1}(x), \ \partial C_r(x) = \partial C_{r,1}(x), \ \mathring{C}_r(x) = \mathring{C}_{r,1}(x).$$

**Theorem 3.1.2.** *Let $r > 0$ be an integral multiple of $h$ and suppose*

$$\Delta_h u = f \quad \text{in } \mathring{C}_{r,h} \quad \text{and} \quad u = 0 \quad \text{on } \partial C_{r,h}.$$

*Then we have*

$$\|\delta_h^2 u\|_{l_2(C_{r,h})} \leq N \|f\|_{l_2(\mathring{C}_{r,h})}$$

*where $N = N(d)$.*

**Remark 3.1.1.** *The fact that the constant $N$ is independent of $r$ is crucial.*

*Proof.* Suppose that the theorem is true for the case $h = 1$. Then by observing that

$$\sum_{i=1}^{d} u(x + he_i) - 2u(x) + u(x - he_i) = h^2 \Delta_h u = h^2 f,$$

we conclude

$$\|h^2 \delta_h^2 u\|_{l_2(C_{r,h})} \leq N \|h^2 f\|_{l_2(\mathring{C}_{r,h})}$$

and thus

$$\|\delta_h^2 u\|_{l_2(C_{r,h})} \leq N \|f\|_{l_2(\mathring{C}_{r,h})}.$$

So from now we may assume $h = 1$. We see

$$\begin{aligned} &\sum_{x \in \mathring{C}_r} |f(x)|^2 \qquad (3.3) \\ &= \sum_{x \in \mathring{C}_r} \big( \sum_i -\delta_{-e_i} \delta_{e_i} u(x) \big)^2 \\ &= \sum_{x \in \mathring{C}_r} \Big( \sum_i (-\delta_{-e_i} \delta_{e_i} u(x))^2 + \sum_{i \neq j} (-\delta_{-e_i} \delta_{e_i} u(x))(-\delta_{-e_j} \delta_{e_j} u(x)) \Big). \end{aligned}$$

Here, by summation by parts we have

$$\begin{aligned} &\sum_{i \neq j} \sum_{x \in \mathring{C}_r} (-\delta_{-e_i} \delta_{e_i} u(x))(-\delta_{-e_j} \delta_{e_j} u(x)) \\ =&\sum_{i \neq j} \sum_{\substack{-r \leq x_i < r \\ -r < x_k < r, k \neq i}} \delta_{e_i} u(x) (\delta_{e_i} \delta_{-e_j} \delta_{e_j} u(x)) \end{aligned}$$

because $u = 0$ on $\partial C_r$ and $j \neq i$. Similarly, the last term above is equal to

$$\sum_{i \neq j} \sum_{\substack{-r \leq x_i, x_j < r \\ -r < x_k < r, k \neq i,j}} (\delta_{e_j} \delta_{e_i} u(x))(\delta_{e_i} \delta_{e_j} u(x)).$$

Therefore, together with identities (3.1) we conclude

$$\|\delta^2 u\|_{l_2(C_r)} \leq N \|f\|_{l_2(\mathring{C}_r)}.$$

The theorem is proved. □

**Theorem 3.1.3.** *Let $r > 0$ be an integral multiple of $h$ and*

$$\Delta_h u = f \quad in\ \mathring{C}_{r,h} \quad and \quad u = 0 \quad on\ \partial C_{r,h}.$$

*Then we have*

$$\|u\|_{l_2^1(C_{r,h})} \leq N\|f\|_{l_2(\mathring{C}_{r,h})}$$

*where $N = N(d, r)$.*

*Proof.* We observe

$$-\sum_{\mathring{C}_{r,h}} u(x)\Delta_h u(x) = -\sum_{\mathring{C}_{r,h}} u(x)f(x).$$

Since $u = 0$ in $\partial C_{r,h}$, by summation by parts we have

$$\sum_{i=1}^{d} \sum_{\substack{-r\leq x_i<r \\ -r<x_k<r, k\neq i}} \delta_{h,e_i}u(x)\delta_{h,e_i}u(x) = \sum_{\mathring{C}_{r,h}} u(x)f(x) \tag{3.4}$$

$$\leq \epsilon \sum_{\mathring{C}_{r,h}} |u(x)|^2 + N\epsilon^{-1} \sum_{\mathring{C}_{r,h}} |f(x)|^2.$$

However, we see

$$\sum_{C_{r,h}} |u(x)|^2 = \sum_{x\in C_{r,h}} \big( \sum_{-r\leq y_1\leq x_1} \delta_{h,e_1}u(y_1, x_2, ..., x_d)h\big)^2 \tag{3.5}$$

$$\leq 3r \sum_{x\in C_{r,h}} \sum_{-r\leq y_1\leq x_1} |\delta_{h,e_1}u(y_1, x_2, ..., x_d)|^2 h$$

$$\leq 9r^2 \sum_{x\in C_{r,h}} |\delta_{h,e_1}u(x)|^2.$$

Applying this result to (3.4), we have

$$\|\delta_h u\|_{l_2(C_{r,h})} \leq \epsilon N\|\delta_h u\|_{l_2(C_{r,h})} + N\epsilon^{-1}\|f\|_{l_2(\mathring{C}_{r,h})}.$$

So taking $\epsilon$ sufficiently small, we have

$$\|\delta_h u\|_{l_2(C_{r,h})} \leq N\|f\|_{l_2(\mathring{C}_{r,h})}.$$

Together with (3.5), we conclude

$$\|u\|_{l_2^1(C_{r,h})} \leq N\|f\|_{l_2(\mathring{C}_{r,h})}$$

The theorem is proved. □

# Chapter 4

# Discrete Sobolev Inequalities

Before we investigate interior estimates for the model equation, we need to discuss discrete Sobolev inequalities. For that purpose, it is more appropriate to deal with a general mesh size $h \leq 1$. Suppose $1/h$ is a positive integer. Our underlying space is $h\mathbb{Z}^d$ with a positive measure $\mu_h(\{\cdot\}) = h^d$ at each point in $h\mathbb{Z}^d$.

We first prove the interpolation theorem which will be useful in the sequel.

**Theorem 4.0.4.** *Let $p \in [1, \infty)$. Then for any $u \in l_p(h\mathbb{Z}^d)$ and $\epsilon > 0$ we have*

$$\|\delta_h u\|_{l_p} \leq \epsilon \|\delta_h^2 u\|_{l_p} + N\epsilon^{-1}\|u\|_{l_p}$$

*where $N = N(d,p)$.*

*Proof.* Suppose the assertion is true for $h = 1$. Then it implies that for any $\eta > 0$

$$\|h\delta_h u\|_{l_p} \leq \eta \|h^2 \delta_h^2 u\|_{l_p} + N\eta^{-1}\|u\|_{l_p}.$$

Thus, if we choose $\eta = \epsilon/h$, then it gives the desired result. So from now on we assume $h = 1$. Moreover, if $\epsilon \leq 1$ then trivially we have

$$\|\delta u\|_{l_p} \leq N\epsilon^{-1}\|u\|_{l_p}.$$

So we may further assume that $\epsilon > 1$.

We take a cut-off function $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta \geq 0$, $\zeta(s) = 1$ for all $s \in (-2, 2)$ and $\int_{\mathbb{R}} \zeta(t)dt = 1$. Now for each $t \in \mathbb{Z}$ we define

$$\xi(t) = \int_t^{t+1} \zeta(s/\epsilon)ds. \tag{4.1}$$

We observe that for $t \in \{1, 2, 3, ...\}$

$$\begin{aligned} &\delta_{e_1} u(x + te_1)\xi(t) - \delta_{e_1} u(x + (t-1)e_1)\xi(t-1) \\ &= -\delta_{-e_1}\delta_{e_1} u(x + te_1)\xi(t) + \delta_{e_1} u(x + (t-1)e_1)(\xi(t) - \xi(t-1)) \\ &= -\delta_{-e_1}\delta_{e_1} u(x + te_1)\xi(t) + A + B \end{aligned} \tag{4.2}$$

where

$$A = u(x + te_1)(\xi(t+1) - \xi(t)) - u(x + (t-1)e_1)(\xi(t) - \xi(t-1))$$

and

$$B = -u(x + te_1)(\xi(t+1) - 2\xi(t) + \xi(t-1)).$$

So if we sum (4.2) over $t \in \{1, 2, 3, ...\}$, then we have

$$\delta_{e_1} u(x)\xi(0) = \sum_{t=1}^{\infty} \big[ -\delta_{e_1}\delta_{e_1} u(x + te_1)\xi(t) \big] - u(x)(\xi(1) - \xi(0)) - \sum_{t=1}^{\infty} u(x + te_1)(\xi(t+1) - 2\xi(t) + \xi(t-1)).$$

Here, we observe

$$\xi(0) = \int_0^1 \zeta(s/\epsilon)ds = 1 \quad \text{and} \quad \xi(1) = \int_1^2 \zeta(s/\epsilon)ds = 1$$

because $\epsilon > 1$ and $\zeta = 1$ for $t \in (-2, 2)$. Then by Minkowski's inequality

$$\|\delta_{e_1} u\|_{l_p} \leq \|\delta_{-e_1}\delta_{e_1} u\|_{l_p} \sum_{t=1}^{\infty} |\xi(t)| + \|u\|_{l_p} \sum_{t=1}^{\infty} |\xi(t+1) - 2\xi(t) + \xi(t-1)|. \tag{4.3}$$

But we observe that (4.1) implies

$$\sum_{t=1}^{\infty} |\xi(t)| \leq \int_1^{\infty} \zeta(s/\epsilon)ds \leq \epsilon$$

and

$$\begin{aligned}
&\sum_{t=1}^{\infty} |\xi(t+1) - 2\xi(t) + \xi(t-1)| \\
&= \frac{1}{2}\sum_{t=1}^{\infty} \Big| \int_t^{t+1} \xi''(s)(t+1-s)ds + \int_{t-1}^{t} \xi''(s)(s-t+1)ds \Big| \\
&\leq N \int_0^{\infty} |\xi''(s)|ds \\
&= N \int_0^{\infty} \epsilon^{-1} \big|\zeta'\big((s+1)/\epsilon\big) - \zeta'(s/\epsilon)\big|ds \quad \text{by (4.1)} \\
&\leq N\epsilon^{-1} \int_0^{\infty} \int_{s/\epsilon}^{(s+1)/\epsilon} |\zeta''(r)|drds \\
&= N\epsilon^{-1} \int_0^{\infty} \int_{r\epsilon-1}^{r\epsilon} |\zeta''(r)|dsdr \leq N\epsilon^{-1}.
\end{aligned}$$

Applying these results to (4.3) we obtain

$$\|\delta_{e_1} u\|_{l_p} \leq \epsilon \|\delta_{-e_1}\delta_{e_1} u\|_{l_p} + N\epsilon^{-1}\|u\|_{l_p}.$$

The theorem is proved. □

## 4.1 Gagliardo-Nirenberg inequality

Now let us prove the discrete analogue to the Gagliardo-Nirenberg inequality.

**Lemma 4.1.1.** *Let $q = \frac{d}{d-1}$. Here, $q$=$\infty$ if $d$=1. Then for any $u \in l_1(h\mathbb{Z}^d)$ we have*

$$\|u\|_{l_q} \leq \prod_{i=1}^{d} \|\delta_{e_i,h} u\|_{l_1}^{1/d} \leq \|\delta_h u\|_{l_1}. \tag{4.4}$$

*Proof.* Without loss of generality we may assume $u \in l_0$. Also, by a scaling argument we can assume $h = 1$. We shall use an induction argument for $d$. If $d = 1$, then by using the fact $u$ is compactly supported we have: for any $x \in \mathbb{Z}$

$$u(x) = \sum_{y<x, y\in\mathbb{Z}} \delta u(y),$$

$$|u(x)| \leq \sum_{y\in\mathbb{Z}} |\delta u(y)|.$$

Now suppose $d > 1$. Then by the above result, for any $x_1$ and $y = (x_2, ..., x_d)$ we have

$$|u(x_1, y)| \leq \sum_{x_1 \in \mathbb{Z}} |\delta_{e_1} u(x_1, y)| =: \phi(y). \tag{4.5}$$

We assume that the lemma is true for the smaller dimension $d - 1$. Then we have

$$\begin{aligned} \sum_{\mathbb{Z}^d} |u(x)|^{d/(d-1)} &= \sum_{x_1 \in \mathbb{Z}} \sum_{y \in \mathbb{Z}^{d-1}} |u(x_1, y)|^{1/(d-1)} |u(x_1, y)| \\ &\leq \sum_{x_1 \in \mathbb{Z}} \sum_{y \in \mathbb{Z}^{d-1}} |\phi(y)|^{1/(d-1)} |u(x_1, y)| \quad \text{by (4.5)} \\ &\leq \sum_{x_1 \in \mathbb{Z}} \big( ( \sum_{y \in \mathbb{Z}^{d-1}} |\phi(y)|)^{1/(d-1)} \|u(x_1, \cdot)\|_{l_{(d-1)/(d-2)}} \big) \end{aligned}$$

by Hölder's inequality. Then by the induction hypothesis we further have that the last term above is less than

$$\begin{aligned} &\|\phi\|_{l_1(\mathbb{Z}^{d-1})}^{1/(d-1)} \sum_{x_1 \in \mathbb{Z}} \prod_{i=2}^{d} \|\delta_{e_i} u(x_1, \cdot)\|_{l_1(\mathbb{Z}^{d-1})}^{\frac{1}{d-1}} \\ &= ( \sum_{x \in \mathbb{Z}^d} |\delta_{e_1} u(x)|)^{1/(d-1)} \sum_{x_1 \in \mathbb{Z}} \prod_{i=2}^{d} \|\delta_{e_i} u(x_1, \cdot)\|_{l_1(\mathbb{Z}^{d-1})}^{\frac{1}{d-1}} \quad \text{by (4.5)} \\ &\leq \|\delta_{e_1} u\|_{l_1}^{1/(d-1)} \prod_{i=2}^{d} \big( \sum_{x_1 \in \mathbb{Z}} \|\delta_{e_i} u(x_1, \cdot)\|_{l_1(\mathbb{Z}^{d-1})} \big)^{\frac{1}{d-1}} \end{aligned}$$

by the generalized Hölder's inequality. Since the last term is equal to

$$\prod_{i=1}^{d} \|\delta_{e_i} u\|_{l_1}^{1/(d-1)},$$

we conclude that inequality (4.4) is true. The theorem is proved. □

We can extend the above lemma as follows.

**Lemma 4.1.2.** *Let $1 \leq p < d$ and $q = \frac{dp}{d-p}$. Then for any $u \in l_p$ we have*

$$\|u\|_{l_q} \leq N \|\delta_h u\|_{l_p} \tag{4.6}$$

*where $N = N(d, p)$.*

*Proof.* We may assume $u \in l_0$ and $h = 1$. We only need to prove the lemma for the case $1 < p < d$. We apply Lemma 4.1.1 to $v = |u|^\gamma$, where $\gamma = p(d-1)/(d-p) > 1$. Then we observe

$$\begin{aligned}
\big(\sum_{\mathbb{Z}^d} |u|^q\big)^{\frac{d-1}{d}} = \big(\sum_{\mathbb{Z}^d} |u|^{\gamma d/(d-1)}\big)^{\frac{d-1}{d}} \le \sum_{\mathbb{Z}^d} \big|\delta |u|^\gamma\big| \qquad (4.7)\\
\le \sum_{\mathbb{Z}^d} \gamma \big| |u(x+e_i)| - |u(x)| \big| \big( |u(x)|^{\gamma-1} + |u(x+e_i)|^{\gamma-1} \big)
\end{aligned}$$

by the mean value theorem. Further, the last term above is less than

$$\begin{aligned}
&\sum_{\mathbb{Z}^d} \gamma |\delta u(x)| \big( |u(x)|^{\gamma-1} + |u(x+e_i)|^{\gamma-1} \big) \qquad (4.8)\\
&\le 2\gamma \big( \sum_{\mathbb{Z}^d} |\delta u|^p \big)^{\frac{1}{p}} \big( \sum_{\mathbb{Z}^d} |u|^{\frac{(\gamma-1)p}{p-1}} \big)^{\frac{p-1}{p}} \quad \text{(Hölder's inequality)}\\
&= 2\gamma \big( \sum_{\mathbb{Z}^d} |\delta u|^p \big)^{\frac{1}{p}} \big( \sum_{\mathbb{Z}^d} |u|^q \big)^{\frac{p-1}{p}}.
\end{aligned}$$

Therefore, by (4.7) and (4.8) we have

$$\big(\sum_{\mathbb{Z}^d} |u|^q\big)^{\frac{1}{q}} \le 2\gamma \big(\sum_{\mathbb{Z}^d} |\delta u|^p\big)^{\frac{1}{p}}.$$

The theorem is proved. □

We can achieve similar inequalities for bounded domains using the so-called extension theorem. Here, we only prove the theorem for the case where the domains are assumed to be cubes. We introduce a new notation: for $t \in h\mathbb{Z}$ and $x = (x_1, ..., x_d) \in h\mathbb{Z}^d$

$$|t|_h = \max(|t|, h),$$

$$|x|_h = \max(|x_1|_h, ..., |x_d|_h).$$

This modification is needed to deal with a cube of radius 0, that is, a single point cube.

**Lemma 4.1.3.** *Let* $1 \le p < d$, $q = \frac{dp}{d-p}$ *and* $r \in \{0, h, 2h, ...\}$. *Then for any* $u$ *defined in* $C_{r,h}$ *we have*

$$\|u\|_{l_q(C_{r,h})} \le N|r|_h^{-1} \|u\|_{l_p(C_{r,h})} + N\|\delta_h u\|_{l_p(C_{r,h})} \qquad (4.9)$$

*where* $N = N(d,p)$.

*Proof.* For $r = 0$, we have

$$\|u\|_{l_q(C_{r,h})} = |u(0)|h^{d/q}$$

and

$$|r|_h^{-1}\|u\|_{l_p(C_{r,h})} = h^{-1}|u(0)|h^{d/p} = |u(0)|h^{d/q}.$$

Thus (4.9) holds for $r = 0$. We now assume $r \neq 0$. We shall use the so-called extension theorem. For that purpose, we extend $u$ to be defined in $h\mathbb{Z}^d$ as follows. First, define a periodic function $f$ with period $4r$:

$$f(t) = \begin{cases} -t - 2r, & \text{if } -2r \le t \le -r \\ t, & \text{if } -r \le t \le r \\ -t + 2r, & \text{if } r \le t \le 2r. \end{cases}$$

Second, define

$$\hat{u}(x_1, ..., x_d) = u(f(x_1), ..., f(x_d)).$$

Last, take a cut-off function $\zeta(x)$, which is 1 in $C_{r,h}$ and zero outside $C_{2r,h}$, such that

$$|\delta_h \zeta| \le 1/r.$$

Then for $\bar{u} := \zeta\hat{u} \in l_0$ we have

$$\begin{aligned} \|\delta_{h,e_i}\bar{u}\|_{l_p(h\mathbb{Z}^d)} &\le \|(\delta_{h,e_i}\zeta)\hat{u}\|_{l_p(h\mathbb{Z}^d)} + \|\zeta(\cdot + he_i)\delta_{h,e_i}\hat{u}\|_{l_p(h\mathbb{Z}^d)} \\ &\le Nr^{-1}\|u\|_{l_p(C_{r,h})} + N\|\delta_h u\|_{l_p(C_{r,h})}. \end{aligned} \tag{4.10}$$

Hence, we have

$$\begin{aligned} \|u\|_{l_q(C_{r,h})} &\le \|\bar{u}\|_{l_q(h\mathbb{Z}^d)} \\ &\le N\|\delta_h\bar{u}\|_{l_p(h\mathbb{Z}^d)} \quad \text{(Lemma 4.1.2)} \\ &\le Nr^{-1}\|u\|_{l_p(C_{r,h})} + N\|\delta_h u\|_{l_p(C_{r,h})} \quad \text{by (4.10)}. \end{aligned}$$

The lemma is proved. □

In conclusion, we have the following Sobolev inequality. The discrete Sobolev inequality of this type for more general domains was originally obtained by Sobolev himself [25].

**Theorem 4.1.1.** *Let* $r \in \{0, h, 2h, ...\}$, $k \in \{1, 2, ...\}$, $m \in \{0, 1, ..., k\}$ *and* $k - m < d/p$. *Set* $q$ *such that*

$$k - \frac{d}{p} = m - \frac{d}{q}.$$

*Then for any* $u$ *defined in* $C_{r,h}$ *we have*

$$\|u\|_{l_q^m(C_{r,h})} \leq N\|u\|_{l_p^k(C_{r,h})}$$

*where* $N = N(k, d, p, |r|_h)$. *More precisely,*

$$\|u\|_{l_q^m(C_{r,h})} \leq N_0\big(|r|_h^{m-k}\|u\|_{l_p(C_{r,h})} + \sum_{n=1}^{k-1}(1 + |r|_h^{-1})|r|_h^{n-k+1}\|\delta_h^n u\|_{l_p(C_{r,h})} + \|\delta_h^k u\|_{l_p(C_{r,h})}\big)$$

*where* $N_0 = N_0(k, d, p)$.

*Proof.* If $m = k - 1$, then by Lemma 4.1.3 we have

$$\|u\|_{l_q^m(C_{r,h})} \leq N\|u\|_{l_p^{m+1}(C_{r,h})} = N\|u\|_{l_p^k(C_{r,h})}.$$

Now we prove the theorem for other cases $m \leq k - 2$. For each $i \in \{m, m + 1, ..., k\}$ we define $q_i$ as follows:

$$k - \frac{d}{p} = m - \frac{d}{q} = i - \frac{d}{q_i}.$$

Then we have

$$\|u\|_{l_q^m(C_{r,h})} \leq N\|u\|_{l_{q_{m+1}}^{m+1}(C_{r,h})} \leq ... \leq N\|u\|_{l_{q_i}^i(C_{r,h})} \leq ... \leq N\|u\|_{l_p^k(C_{r,h})}.$$

The second assertion is easy to check by tracing the constant $N$ above more carefully. The proof is complete. □

## 4.2 Embedding $l_p^1 \subset l_\infty$

Now we shall estimate the sup norm of $u$. The main theorem is stated in Theorem 4.3.1. Although it is briefly mentioned in [3, 24, 27], we could not find a detailed proof of it in the literature. Thus, we provide the detailed proof here.

**Lemma 4.2.1.** *Let $p > d$ and $r \in \{0, h, 2h, ...\}$. Then for any $u \in l_p(h\mathbb{Z}^d)$ we have*

$$\|u\|_{l_\infty(h\mathbb{Z}^d)} \le N\|u\|_{l_p^1(h\mathbb{Z}^d)} \tag{4.11}$$

*and*

$$\|u\|_{l_\infty(C_{r,h})} \le N(|r|_h^{-1} + 1)\|u\|_{l_p(C_{r,h})} + N\|\delta_h u\|_{l_p(C_{r,h})}$$

*where $N = N(p, d)$.*

*Proof.* For $\rho \in \mathbb{R}$ we use $Q_\rho$ to denote a continuous cube in $\mathbb{R}^d$:

$$Q_\rho = \{(c_1, ..., c_d) : -\rho \le c_i \le \rho,\ c_i \in \mathbb{R},\ \forall i\}.$$

And $\partial Q_\rho$ means its continuous boundary in $\mathbb{R}^d$. We extend $u(x)$ to be defined in $\mathbb{R}^d$ as follows: for $x \in \mathbb{R}^d$

$$u(x) := u(\lfloor x \rfloor_h) = u(\lfloor x_1 \rfloor_h, ..., \lfloor x_d \rfloor_h),$$

where $\lfloor \cdot \rfloor_h = \lfloor \cdot/h \rfloor h$ and $\lfloor \cdot \rfloor$ is the integer part. Take a cut-off function $\zeta(t)$, $t \in \mathbb{R}$, such that its support is included in $(-1, 1)$ and $\zeta(0) = 1$. Let $\mathbb{Z}_0 = \{0, 1, 2, ...\}$. Then for any $y \in \partial Q_1$ we have

$$\begin{aligned} u(0) &= -\sum_{t \in h\mathbb{Z}_0} \big(u((t+h)y)\zeta(t+h) - u(ty)\zeta(t)\big) \\ &= -\sum_{t \in h\mathbb{Z}_0} \Big(u(ty)\delta_h\zeta(t) + \frac{1}{h}\big(u(ty + hy) - u(ty)\big)\zeta(t+h)\Big)h. \end{aligned}$$

We integrate both sides over the surface $y \in \partial Q_1$ and divide the resulting terms by $|\partial Q_1|$:

$$u(0) = \frac{-1}{|\partial Q_1|} \sum_{t \in h\mathbb{Z}_0} \int_{\partial Q_1} \Big(u(ty)\delta_h\zeta(t) + \frac{1}{h}\big(u(ty + hy) - u(ty)\big)\zeta(t+h)\Big)h\,dS(y) \tag{4.12}$$

where $dS(y)$ is the surface measure.

For $t \neq 0$, by the change of variables $x = ty$, we have

$$dS(y) = \frac{1}{t^{d-1}} dS(x) = \frac{1}{|x|_h^{d-1}} dS(x)$$

and

$$\big|u(ty + hy) - u(ty)\big|/h = \big|u(x + hy) - u(x)\big|/h \qquad (4.13)$$
$$= \big|u(\lfloor x + hy \rfloor_h) - u(\lfloor x \rfloor_h)\big|/h.$$

Since $|x + hy - x|_h = h$, we have

$$\big|u(\lfloor x + hy \rfloor_h) - u(\lfloor x \rfloor_h)\big|/h \leq |\delta_h u(\lfloor x \rfloor_h)| + \sum_{\xi \in \Lambda} |\delta_h u(\lfloor x \rfloor_h + h\xi)|. \qquad (4.14)$$

Now let $\mathbb{Z}_1 = \{1, 2, 3, ...\}$. Then, (4.12) can be estimated as follows:

$$\begin{aligned}
|u(0)| \leq\ & N\big(|u(0)\delta_h\zeta(0)| + \big(|\delta_h u(0)| + \sum_{\xi\in\Lambda} |\delta_h u(h\xi)|\big)|\zeta(h)|\big)h + \\
& N \sum_{t \in h\mathbb{Z}_1} \int_{\partial Q_t} \frac{1}{|x|_h^{d-1}} \Big(|u(\lfloor x \rfloor)\delta_h\zeta(t)| + \big(|\delta_h u(\lfloor x \rfloor_h)| + \\
& \sum_{\xi\in\Lambda} |\delta_h u(\lfloor x \rfloor_h + h\xi)|\big)|\zeta(t+h)|\Big) h dS(x) \quad \text{by (4.13) and (4.14)} \\
\leq\ & N \sum_{t \in h\mathbb{Z}_0} \sum_{x \in \partial C_{t,h}} \frac{1}{|x|_h^{d-1}} \big(|u(x)\delta_h\zeta(t)| + \big(|\delta_h u(x)| + \\
& \sum_{\xi\in\Lambda} |\delta_h u(x + h\xi)|\big)|\zeta(t+h)|\big) h^d \\
\leq\ & N \sum_{x \in C_{2,h}} \frac{1}{|x|_h^{d-1}} \big(|u(x)| + |\delta_h u(x)|\big) h^d
\end{aligned}$$

because $\zeta$ is supported in $(-1, 1)$. Using Hölder's inequality with $q = \frac{p}{p-1}$, we know that the last term above is less than

$$\begin{aligned}
& N\Big(\sum_{x \in C_{2,h}} \big(|u(x)|^p + |\delta_h u(x)|^p\big)h^d\Big)^{1/p} \Big(\sum_{x \in C_{2,h}} \frac{1}{|x|_h^{(d-1)q}} h^d\Big)^{1/q} \\
& \leq N\|u\|_{l_p^1} \Big(\int_{Q_2} \frac{1}{|x|^{(d-1)q}} dx\Big)^{1/q} \\
& \leq N\|u\|_{l_p^1}
\end{aligned}$$

because $(d-1)q < d$. Thus we have proved

$$|u(0)| \le N\|u\|_{l_p^1} \quad \text{and thus} \quad \|u\|_{l_\infty} \le N\|u\|_{l_p^1}. \tag{4.15}$$

The second assertion is an easy consequence of the first one. Indeed, in case $r = 0$ we have

$$|r|_h^{-1}\|u\|_{l_p(C_{r,h})} = h^{-1}|u(0)|h^{d/p} \ge |u(0)| = \|u\|_{l_\infty(C_{r,h})}$$

because $p > d$ and $h \le 1$. For $r \ne 0$, we recall the definition of $\bar{u}$ in the proof of Lemma 4.1.3. Then we have

$$\begin{aligned} \|u\|_{l_\infty(C_{r,h})} &\le \|\bar{u}\|_{l_\infty} \le N\|\bar{u}\|_{l_p^1} \quad \text{by (4.11)} \\ &\le N(|r|_h^{-1}+1)\|u\|_{l_p(C_{r,h})} + N\|\delta_h u\|_{l_p(C_{r,h})} \quad \text{by (4.10).} \end{aligned}$$

The lemma is proved. □

## 4.3 General Discrete Sobolev Inequalities

Consequently, we obtain the following Sobolev inequality.

**Theorem 4.3.1.** *Let $r \in \{0, h, 2h, ...\}$, $k \in \{1, 2, ...\}$, $p > 1$ and $k - \frac{d}{p} > 0$. Then for any $u$ defined in $C_{r,h}$ we have*

$$\|u\|_{l_\infty^{k-\lfloor d/p \rfloor -1}(C_{r,h})} \le N\|u\|_{l_p^k(C_{r,h})}$$

*where $N = N(k, p, d, |r|_h)$. Moreover, for any $u \in l_p(h\mathbb{Z}^d)$ it also holds that*

$$\|u\|_{l_\infty^{k-\lfloor d/p \rfloor -1}(h\mathbb{Z}^d)} \le N\|u\|_{l_p^k(h\mathbb{Z}^d)}$$

*where $N = N(k, p, d)$.*

*Proof.* First, suppose $d/p$ is not an integer. Then there exists an integer $m$ such that

$$m < k - \frac{d}{p} < m + 1.$$

We define $q$ as

$$k-\frac{d}{p}=m+1-\frac{d}{q}$$

so that $d<q<\infty$. Then we have

$$\begin{aligned}\|u\|_{l_\infty^{k-\lfloor d/p\rfloor-1}(C_{r,h})}=\|u\|_{l_\infty^m(C_{r,h})}&\le N\|u\|_{l_q^{m+1}(C_{r,h})}\quad\text{(Theorem 4.2.1)}\\ &\le N\|u\|_{l_p^k(C_{r,h})}\quad\text{(Theorem 4.1.1).}\end{aligned}$$

For the case where $\frac{d}{p}$ is an integer, choose $s<p$ so that

$$k-\frac{d}{p}-\frac{1}{2}=k-\frac{d}{s}.$$

Then we have

$$\|u\|_{l_\infty^{k-\lfloor d/p\rfloor-1}(C_{r,h})}=\|u\|_{l_\infty^{k-\lfloor d/s\rfloor-1}(C_{r,h})}\le N\|u\|_{l_s^k(C_{r,h})}\le N\|u\|_{l_p^k(C_{r,h})}$$

because $C_{r,h}$ is a bounded domain.

This consequence immediately implies the second assertion. The theorem is proved.

□

# Chapter 5

# Tools from Harmonic Analysis

The $l_p$-estimates of the present thesis is based on the Hardy-Littlewood theorem and the Fefferman-Stein theorem. To start with, let us define maximal and sharp functions on $\mathbb{Z}^d$. Let $\mathbb{C}$ be the collection of all cubes $C_r(x)$ over $\mathbb{Z}^d$, where $x \in \mathbb{Z}^d$ and $r \in \{0, 1, 2, ...\}$. From now on, for the sake of notational convenience we shall often use an integral sign with a counting measure $\mu$ in $\mathbb{Z}^d$ in place of a summation sign. The maximal and sharp functions are defined by

$$\mathbb{M}g(x) := \sup_{C\in\mathbb{C}, x\in C} \fint_C |g(y)| d\mu(y) := \sup_{C\in\mathbb{C}, x\in C} \frac{\sum_C |g(y)|}{\sum_C 1},$$

$$g^\sharp(x) = \sup_{C\in\mathbb{C}, x\in C} \fint_C |g(y) - g_C| d\mu(y)$$

where

$$g_C = \fint_C g(y) d\mu(y).$$

## 5.1 Hardy-Littlewood Theorem

Under this setting to prove the Hardy-Littlewood theorem, we need to see whether Vitali's covering argument (for instance, see [26]) works for discrete cubes.

**Theorem 5.1.1.** *If $p \in (1,\infty)$, $f \in l_p$, then*

$$\|\mathbb{M}f\|_{l_p} \leq N\|f\|_{l_p}, \tag{5.1}$$

*where $N = N(d,p)$.*

*Proof.* We assume $f \in l_1 \cap l_p$. For each $\lambda > 0$ we shall claim that

$$|\{x : \mathbb{M}f(x) > \lambda\}| \leq N\lambda^{-1} \sum_{x \in \{\mathbb{M}f > \lambda\}} |f|(x) \tag{5.2}$$

where $N = N(d)$. Choose a finite subset $K = \{x_1, ..., x_n\}$ of $\{\mathbb{M}f > \lambda\}$. For each $x_i \in K$ there exists a cube $C_i \ni x$ such that

$$⨍_{C_i} |f| d\mu(y) > \lambda. \tag{5.3}$$

Also, by this property for any $y \in C_i$ we have

$$\mathbb{M}f(y) > \lambda$$

and this implies $C_i \subset \{\mathbb{M}f > \lambda\}$.

To apply Vitali's covering procedure, we introduce a new notation: for any $C = C_r(x) \in \mathbb{C}$ we denote $\tilde{C} = C_{3r}(x)$. Now from the above $C_i$'s we choose the largest one and denote it by $Q_1$. Next, from the remaining cubes we choose the largest one which does not intersect $Q_1$. Denote it by $Q_2$. We continue this procedure: From the family $\{C_1, ..., C_n\} \backslash \{Q_1, ..., Q_k\}$ choose the largest one which does not have an intersection with $\{Q_1, ..., Q_k\}$ until such a choice is no more available. Then according to this algorithm we may easily see that

$$K \subset \bigcup_{i=1}^{n} C_i \subset \bigcup_{i=1}^{k} \tilde{Q}_i.$$

Therefore we have

$$\begin{aligned} |K| &\leq \sum_{i=1}^{k} |\tilde{Q}_i| \leq N \sum_{i=1}^{k} |Q_i| \leq N\lambda^{-1} \sum_{i=1}^{k} \sum_{x \in Q_i} |f|(x) \quad \text{by (5.3)} \\ &\leq N\lambda^{-1} \sum_{x \in \{\mathbb{M}f > \lambda\}} |f|(x) \end{aligned}$$

because $Q_i$'s are disjoint and included in $\{\mathbb{M}f > \lambda\}$ as mentioned above. Last, since $|\{\mathbb{M}f > \lambda\}|$ is the supremum of such $|K|$, our claim (5.2) is proved.

There is a well-known argument to prove (5.1) based on inequality (5.2). We only describe the main idea here without concerning issues like $\mathbb{M}f = \infty$. A rigorous argument can be found, for instance, in [17].

$$\begin{aligned}
\sum_{\mathbb{Z}^d} \mathbb{M}^p f &= \int_0^\infty |\{\mathbb{M}^p f > \lambda\}| d\lambda = \int_0^\infty |\{\mathbb{M}f > \xi\}| p\xi^{p-1} d\xi \\
&\leq N \int_0^\infty p\xi^{p-2} \sum_{\{\mathbb{M}f > \xi\}} |f|(x) d\xi \\
&= N \sum_{x \in \mathbb{Z}^d} |f|(x) \int_0^{\mathbb{M}f(x)} p\xi^{p-2} d\xi \quad \text{(Fubini's theorem)} \\
&= Np/(p-1) \sum_{x \in \mathbb{Z}^d} |f|(x) \mathbb{M}^{p-1} f(x) \\
&\leq N \|f\|_{l_p} \big(\sum_{\mathbb{Z}^d} \mathbb{M}^p f\big)^{(p-1)/p} \quad \text{(Hölder's inequality).}
\end{aligned}$$

Hence (5.1) follows. By the density argument we can drop the condition $f \in l_1$ and the theorem is proved. □

## 5.2 Fefferman-Stein Theorem

On the other hand, the Fefferman-Stein theorem is proved in [17] or [21] for a large class of measure spaces. Our discrete space also belongs to such a class. However, since its probabilistic proof is instructive, we provide a detailed proof here.

First, we define a filtration $\{\mathcal{F}_n\}$, where $n \in \{0, -1, -2, ...\}$, consisting of dyadic cubes in $\mathbb{Z}^d$ as follows. For any $(a_1, ..., a_d) \in \mathbb{Z}^d$ we define

$$Q_n(a_1, ..., a_d) = [a_1 2^{-n}, (a_1+1)2^{-n}) \times \cdots \times [a_d 2^{-n}, (a_d+1)2^{-n}) \cap \mathbb{Z}^d.$$

Let $\mathbb{Q}_n$ be the collection of those $Q_n(a_1, ..., a_d)$. Then $\mathcal{F}_n$ is defined by

$$\mathcal{F}_n = \sigma(\mathbb{Q}_k,\ k \leq n).$$

For a function $f$ on $\mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ we introduce

$$f_{|n}(x) = \fint_{Q_{n,x}} f(y)d\mu(y)$$

where $Q_{n,x}$ is *the* cube $Q_n(a_1, ..., a_d) \in \mathbb{Q}_n$ containing $x$. In particular, we define $f_{|\infty}(x) = f(x)$.

Also a function $\tau : \mathbb{Z}^d \to \{0, -1, -2, ...\} \cup \{\infty\}$ is called a stopping time relative to $\{\mathcal{F}_n\}$ if it satisfies: for each $n \in \{0, -1, -2, ...\}$ we have

$$\{\tau \leq n\} \in \mathcal{F}_n.$$

In fact, since our filtration $\{\mathcal{F}_n\}$ is generated by a certain type of *partitions*, it is easy to see that the above definition is equivalent to:

$$\{\tau = n\} = \bigcup_k Q_k, \quad \text{for some } \{Q_k\} \subset \mathbb{Q}_n. \tag{5.4}$$

Now we define the maximal and sharp function of $f$ relative to $\{\mathcal{F}_n\}$ as follows:

$$\mathcal{M}f(x) = \sup_{n \leq 0} |f|_{|n}(x)$$

and

$$f^{\#}(x) = \sup_{n \leq 0} \fint_{Q_{n,x}} |f(y) - f_{|n}(y)| d\mu(y).$$

Last, we take a constant $N_0 = N_0(d)$ such that

$$|Q'| \leq N_0 |Q|$$

for any $Q' \in \mathbb{Q}_{n-1}$ and $Q \in \mathbb{Q}_n$.

**Lemma 5.2.1.** *For any $f \in l_1(\mathbb{Z}^d)$ and a stopping time $\tau$ we have*

$$\sum_{\mathbb{Z}^d} f_{|\tau}(x) = \sum_{\mathbb{Z}^d} f(x).$$

*Proof.* Since $f_{|\infty}(x) = f(x)$, we only need to prove

$$\sum_{\mathbb{Z}^d} f_{|\tau}(x) I_{\tau<\infty} = \sum_{\mathbb{Z}^d} f(x) I_{\tau<\infty}.$$

For that purpose, we observe

$$\begin{aligned}\sum_{\mathbb{Z}^d} f_{|\tau}(x) I_{\tau<\infty} = \sum_{n\le 0} \sum_{\{x:\tau(x)=n\}} f_{|n}(x) &= \sum_{n\le 0}\sum_{k}\sum_{x\in Q_k} f_{|n}(x) \quad \text{by (5.4)}\\ &= \sum_{n\le 0}\sum_{k}\sum_{x\in Q_k} f(x)\end{aligned}$$

because for each $x \in Q_k$ we have $Q_{n,x} = Q_k$. And we proceed further:

$$\sum_{n\le 0}\sum_{k}\sum_{x\in Q_k} f(x) = \sum_{n\le 0} \sum_{\{x:\tau(x)=n\}} f(x) = \sum_{\mathbb{Z}^d} f(x) I_{\tau<\infty}.$$

The lemma is proved. □

**Lemma 5.2.2.** *Let $\lambda > 0$ and $\alpha = \frac{1}{2N_0}$. Then for any $f \in l_1(\mathbb{Z}^d)$ we have*

$$|\{x : |f|(x) > \lambda\}| \le 4\lambda^{-1} \sum_{\{\mathcal{M}f>\alpha\lambda\}} f^{\#}(x).$$

*Proof.* We observe

$$\begin{aligned}|f|^{\#}(x) &= \sup_{n\le 0} \fint_{Q_{n,x}} \big||f|(y) - |f|_{|n}(y)\big| d\mu(y)\\ &\le \sup_{n\le 0} \fint_{Q_{n,x}} \fint_{Q_{n,x}} \big||f|(y) - |f|(z)\big| d\mu(z) d\mu(y)\\ &\le \sup_{n\le 0} \fint_{Q_{n,x}} \fint_{Q_{n,x}} \big|f(y) - f(z)\big| d\mu(z) d\mu(y) \le 2f^{\#}(x).\end{aligned}$$

So we may assume $f \ge 0$ and aim to prove

$$|\{x : f(x) > \lambda\}| \le 2\lambda^{-1} \sum_{\{\mathcal{M}f>\alpha\lambda\}} f^{\#}(x).$$

Define $\tau(x) = \inf\{n \le 0 : f_{|n}(x) > \alpha\lambda\}$, where $\inf(\emptyset) = \infty$. We suppose $f(x) > \lambda$. Then since

$$f_{|0}(x) = f(x) > \lambda > \alpha\lambda,$$

we have $\tau(x) < \infty$. Let $\tau(x) = n$. Then

$$\alpha\lambda \geq \frac{1}{|Q_{n-1,x}|} \sum_{Q_{n-1,x}} f(y) \geq \frac{|Q_{n,x}|}{|Q_{n-1,x}|} \fint_{Q_{n,x}} f(y) d\mu(x)$$
$$\geq N_0^{-1} \fint_{Q_{n,x}} f(y) d\mu(x)$$

and thus

$$f_{|\tau}(x) = f_{|n}(x) \leq \lambda/2.$$

Therefore, we have

$$\begin{aligned}
|\{x : f(x) > \lambda\}| &= |\{x : f(x) > \lambda,\ f_{|\tau}(x) \leq \lambda/2,\ \tau < \infty\}| \\
&\leq |\{x : |f(x) - f_{|\tau}(x)| \geq \lambda/2,\ \tau < \infty\}| \\
&\leq 2\lambda^{-1} \sum_{\{\tau<\infty\}} |f(x) - f_{|\tau}(x)| \\
&= 2\lambda^{-1} \sum_{\{\tau<\infty\}} |f - f_{|\tau}|_{|\tau}(x) \quad \text{(Lemma 5.2.1)} \\
&\leq 2\lambda^{-1} \sum_{\{\tau<\infty\}} f^{\#}(x) = 2\lambda^{-1} \sum_{\{\mathcal{M}f>\alpha\lambda\}} f^{\#}(x).
\end{aligned}$$

The lemma is proved. □

**Theorem 5.2.1.** *If $p \in (1, \infty)$, $f \in l_p$, then*

$$\|f\|_{l_p} \leq N \|f^{\sharp}\|_{l_p},$$

*where $N = N(d, p)$.*

*Proof.* First we assume $f \in l_1 \cap l_p$. Then we have

$$\begin{aligned}
\sum_{\mathbb{Z}^d} |f|^p &\le \int_0^\infty |\{|f|^p > \xi\}| d\xi = \int_0^\infty |\{|f| > \lambda\}| p\lambda^{p-1} d\lambda \\
&\le N \int_0^\infty \sum_{\{\mathcal{M}f > \alpha\lambda\}} f^{\#}(x) \lambda^{p-2} d\lambda \quad \text{(Lemma 5.2.2)} \\
&= N \sum_{\mathbb{Z}^d} f^{\#}(x) \int_0^{\mathcal{M}f(x)/\alpha} \lambda^{p-2} d\lambda \\
&= N \sum_{\mathbb{Z}^d} f^{\#}(x) \mathcal{M}^{p-1} f(x) \le N \|f^{\#}\|_{l_p} \|\mathcal{M}f\|_{l_p}^{p-1} \quad \text{(Hölder)} \\
&\le N \|f^{\sharp}\|_{l_p} \|\mathbb{M}f\|_{l_p}^{p-1} \quad \text{(obvious)} \\
&\le N \|f^{\sharp}\|_{l_p} \|f\|_{l_p}^{p-1} \quad \text{(Theorem 5.1.1).}
\end{aligned}$$

Therefore, we conclude

$$\|f\|_{l_p} \le N \|f^{\sharp}\|_{l_p}.$$

By the density argument we can drop the assumption $f \in l_1$. The theorem is proved. □

# Chapter 6

# Equations with Constant Coefficients

## 6.1 Interior Estimates

Now we investigate interior estimates. We turn to the assumption that the mesh size $h = 1$ whenever it simplifies relevant computations.

**Lemma 6.1.1.** *Let $\lambda \geq 0$ and $0 \leq r < R$ be integers. Then for any $u$ defined in $C_R$ we have*

$$\lambda \|u\|_{l_2(C_r)} + \|\delta^2 u\|_{l_2(C_r)} \leq N\big(\|f\|_{l_2(C_R)} + (R-r)^{-2}\|u\|_{l_2(C_R)}\big), \tag{6.1}$$

*where $f = \Delta_1 u - \lambda u$ and $N = N(d)$.*

*Proof.* For $m = 0, 1, ..., M$, where $M$ is to be specified later, let $s_0 = r$ and

$$s_m = r + (R-r)\sum_{i=1}^{m} 2^{-i}.$$

We choose $M$ such that

$$s_{M-1} < R - 1 \leq s_M. \tag{6.2}$$

Define $r_m = \lfloor s_m \rfloor$, $r_{M+1} = R$ and $C(m) = C_{r_m}$. We take cut-off functions $\zeta_m$ defined in $\mathbb{Z}^d$ such that

$$\zeta_m = 1 \quad \text{in} \quad C(m), \quad \zeta_m = 0 \quad \text{in} \quad \mathbb{Z}^d \backslash \mathring{C}(m+1), \tag{6.3}$$

$$|\delta \zeta_m| \le N 2^m (R-r)^{-1}, \quad |\delta^2 \zeta_m| \le N 2^{2m} (R-r)^{-2}.$$

In particular, we take $\zeta_{M+1} \equiv 1$ in $C_R$ and $\zeta_{M+1} \equiv 0$ outside $C_R$.

Notice that $\zeta_m u$ satisfies

$$\Delta_1(\zeta_m u) - \lambda \zeta_m u = \zeta_m f + (\Delta_1 \zeta_m) u(x - e_i) - \delta_{e_i} \zeta_m \delta_{-e_i} u(x + e_i) -$$
$$\delta_{-e_i} \zeta_m (x + e_i) \delta_{e_i} u(x - e_i).$$

Here, by the property (6.3) we observe that

$$\Delta_1 \zeta_m(x) = 0 \quad \text{in } \mathbb{Z}^d \backslash C(m+1)$$

and

$$\delta_{e_i} \zeta_m(x) = 0, \quad \delta_{-e_i} \zeta_m(x + e_i) = 0 \quad \text{if } x_i \ge r_{m+1}.$$

Similarly, according to a slightly different expression like

$$\Delta_1(\zeta_m u) - \lambda \zeta_m u = \zeta_m f + (\Delta_1 \zeta_m) u(x + e_i) - \delta_{-e_i} \zeta_m \delta_{e_i} u(x - e_i) -$$
$$\delta_{e_i} \zeta_m (x - e_i) \delta_{-e_i} u(x + e_i),$$

we have

$$\delta_{-e_i} \zeta_m(x) = 0, \quad \delta_{e_i} \zeta_m(x - e_i) = 0 \quad \text{if } x_i \le -r_{m+1}.$$

So for each $m$ we have

$$\begin{aligned} P_m &:= \lambda \|(\zeta_m u)\|_{l_2} + \|\delta^2(\zeta_m u)\|_{l_2} \\ &\le N \|\Delta_1(\zeta_m u) - \lambda \zeta_m u\|_{l_2} \quad \text{(Theorem 3.1.1).} \\ &\le N\big(\|f\|_{l_2(C_R)} + 2^{2m}(R-r)^{-2}\|u\|_{l_2(C_R)} + 2^m (R-r)^{-1} \|\delta u\|_{l_2(C_{m+1})}\big) \end{aligned} \tag{6.4}$$

by property (6.3). To estimate $\|\delta u\|_{l_2(C_{m+1})}$ we shall use the interpolation inequality as follows: assuming $u$ is defined to be 0 outside $C_R$, we have for any $\epsilon > 0$

$$\|\delta u\|_{l_2(C_{m+1})} \le \|\delta(\zeta_{m+1}u)\|_{l_2(\mathbb{Z}^d)}$$
$$\le \epsilon 2^{-m}(R-r)\|\delta^2(\zeta_{m+1}u)\|_{l_2} + N\epsilon^{-1}2^m(R-r)^{-1}\|(\zeta_{m+1}u)\|_{l_2}.$$

So by putting $Q_m = \|\delta^2(\zeta_m u)\|_{l_2}$, we can further estimate (6.4) as

$$P_m \le N\big(\|f\|_{l_2(C_R)} + \epsilon^{-1}2^{2m}(R-r)^{-2}\|u\|_{l_2(C_R)} + \epsilon Q_{m+1}\big).$$

After that we have

$$P_0 + \sum_{m=1}^{M} \epsilon^m Q_m \le \sum_{m=0}^{M} \epsilon^m P_m$$
$$\le N \sum_{m=0}^{M} \epsilon^m \big(\|f\|_{l_2(C_R)} + \epsilon^{-1}2^{2m}(R-r)^{-2}\|u\|_{l_2(C_R)} + \epsilon Q_{m+1}\big).$$

Hence, for sufficiently small $\epsilon$, say 1/8, we have

$$\lambda\|u\|_{l_2(C_r)} + \|\delta^2 u\|_{l_2(C_r)} \le P_0$$
$$\le N\big(\|f\|_{l_2(C_R)} + (R-r)^{-2}\|u\|_{l_2(C_R)} + 8^{-M-1}Q_{M+1}\big).$$

But, since the mesh size is assumed to be 1, we have

$$Q_{M+1} = \|\delta^2(\zeta_{M+1}u)\|_{l_2} \le N\|u\|_{l_2(C_R)}.$$

Further, by condition (6.2) we have

$$r + (R-r)\sum_{i=1}^{M} 2^{-1} \ge R - 1,$$
$$2^{-M} \le (R-r)^{-1} \quad \text{and thus} \quad 8^{-M-1} \le (R-r)^{-2}.$$

Consequently, we achieve

$$\lambda\|u\|_{l_2(C_r)} + \|\delta^2 u\|_{l_2(C_r)} \le N\big(\|f\|_{l_2(C_R)} + (R-r)^{-2}\|u\|_{l_2(C_R)}\big).$$

The lemma is proved. □

By a simple induction argument we have the following result as well.

**Lemma 6.1.2.** *Let $\lambda \geq 0$, $k \in \{0, 1, 2, ...\}$ and $0 \leq r < R$ be integers. Then for any $u$ defined in $C_R$ we have*

$$\lambda \|\delta^k u\|_{l_2(C_r)} + \|\delta^{k+2} u\|_{l_2(C_r)} \leq N\big(\sum_{n=0}^{k} (R-r)^{-k+n} \|\delta^n f\|_{l_2(C_R)} + (R-r)^{-k-2} \|u\|_{l_2(C_R)} + (R-r)^{-k-1} \|\delta u\|_{l_2(C_R)}\big) \tag{6.5}$$

*where $f = \Delta_1 u - \lambda u$ and $N = N(d, k)$.*

*Proof.* Without loss of generality, we may assume $R - r > k$. Indeed, let us suppose $R - r \leq k$. Then

$$\begin{aligned} \lambda \|\delta^k u\|_{l_2(C_r)} + \|\delta^{k+2} u\|_{l_2(C_r)} &\leq \|\lambda \delta^k u - \Delta_1 \delta^k u\|_{l_2(C_r)} + \|\delta^{k+2} u\|_{l_2(C_r)} \\ &\leq \|\delta^k f\|_{l_2(C_r)} + N\|u\|_{l_2(C_r)} \\ &\leq \|\delta^k f\|_{l_2(C_r)} + N(\frac{k}{R-r})^{k+2} \|u\|_{l_2(C_r)}. \end{aligned}$$

This is the desired result. Therefore, we now assume $R - r > k$.

We shall use an induction argument. The case $k = 0$ is justified by Lemma 6.1.1. Suppose the lemma is true for a general $k$ and now we prove the statement for $k + 1$. For each $i = 1, ..., d$ define $v = \delta_{e_i} u$ and observe

$$\Delta_1 v - \lambda v = \delta_{e_i} f.$$

Then by the induction hypothesis for $R_1 = r + \lfloor (R - r)k/(k + 1) \rfloor$ we have

$$\lambda \|\delta^k v\|_{l_2(C_r)} + \|\delta^{k+2} v\|_{l_2(C_r)} \leq N\big(\sum_{n=0}^{k} (R_1 - r)^{-k+n} \|\delta^{n+1} f\|_{l_2(C_{R_1})} + (R_1 - r)^{-k-2} \|v\|_{l_2(C_{R_1})} + (R_1 - r)^{-k-1} \|\delta v\|_{l_2(C_{R_1})}\big). \tag{6.6}$$

Also, by Lemma 6.1.1

$$\|\delta v\|_{l_2(C_{R_1})} \leq \|\delta^2 u\|_{l_2(C_{R_1})} \leq \|f\|_{l_2(C_R)} + (R - R_1)^{-2} \|u\|_{l_2(C_R)}.$$

Applying this to (6.6) and recalling $v = \delta_{e_i} u$, we have

$$\lambda \|\delta^{k+1} u\|_{l_2(C_r)} + \|\delta^{k+3} u\|_{l_2(C_r)} \leq N\big(\sum_{n=0}^{k+1} (R-r)^{-k-1+n} \|\delta^n f\|_{l_2(C_R)} + (R-r)^{-k-3} \|u\|_{l_2(C_R)} + (R-r)^{-k-2} \|\delta u\|_{l_2(C_R)}\big). \quad (6.7)$$

The lemma is proved. ∎

Since (6.5) is scaling invariant and $N$ does not depend on $\lambda$, we obtain the same estimate for the case of a general mesh size $h$.

**Lemma 6.1.3.** *Let $\lambda \geq 0$, $k \in \{0, 1, 2, ...\}$ and $0 \leq r < R$ be integral multiples of $h$. Then for any $u$ defined in $C_{R,h}$ we have*

$$\lambda \|\delta_h^k u\|_{l_2(C_{r,h})} + \|\delta_h^{k+2} u\|_{l_2(C_{r,h})} \leq N\big(\sum_{n=0}^{k} (R-r)^{-k+n} \|\delta_h^n f\|_{l_2(C_{R,h})} + (R-r)^{-k-2} \|u\|_{l_2(C_{R,h})} + (R-r)^{-k-1} \|\delta_h u\|_{l_2(C_{R,h})}\big),$$

*where $f = \Delta_h u - \lambda u$ and $N = N(d, k)$.*

*Proof.* We observe

$$h^2 \Delta_h u - h^2 \lambda u = h^2 f(x), \quad \forall x \in h\mathbb{Z}^d.$$

Then Lemma 6.1.2 implies

$$h^2 \lambda \|h^k \delta_h^k u\|_{l_2(C_{r,h})} + \|h^{k+2} \delta_h^{k+2} u\|_{l_2(C_{r,h})} \leq N\big(\sum_{n=0}^{k} (\frac{R-r}{h})^{-k+n} \|h^{n+2} \delta_h^n f\|_{l_2(C_{R,h})} + (\frac{R-r}{h})^{-k-2} \|u\|_{l_2(C_{R,h})} + (\frac{R-r}{h})^{-k-1} \|h \delta_h u\|_{l_2(C_{R,h})}\big)$$

and thus

$$\lambda \|\delta_h^k u\|_{l_2(C_{r,h})} + \|\delta_h^{k+2} u\|_{l_2(C_{r,h})} \leq N\big(\sum_{n=0}^{k} (R-r)^{-k+n} \|\delta_h^n f\|_{l_2(C_{R,h})} + (R-r)^{-k-2} \|u\|_{l_2(C_{R,h})} + (R-r)^{-k-1} \|\delta_h u\|_{l_2(C_{R,h})}\big),$$

The lemma is proved. ∎

Based on the previous lemma and the discrete Sobolev inequalities we can estimate difference quotients of $u$ in the sup norm.

**Theorem 6.1.1.** *Let* $\lambda \geq 0$, $k$, $m \in \{0, 1, 2, ...\}$, $2(k-m) > d$ *and* $0 \leq r < R$ *be integral multiples of* $h$. *Then for any* $u$ *defined in* $C_{R,h}$ *we have*

$$\|\delta_h^2 u\|_{l_\infty^m(C_{r,h})} \leq N\big(\|f\|_{l_2^{k+1}(C_{R,h})} + \|u\|_{l_2(C_{R,h})} + \|\delta_h u\|_{l_2(C_{R,h})}\big), \tag{6.8}$$

*where* $f = \Delta_h u - \lambda u$ *and* $N = N(k, d, |r|_h, R)$.

*Proof.* We have

$$\|\delta_h^2 u\|_{l_\infty^m(C_{r,h})} \leq N\|\delta_h^2 u\|_{l_2^k(C_{r,h})} \quad \text{(Theorem 4.3.1)}$$
$$\leq N\big(\|f\|_{l_2^k(C_{R,h})} + \|u\|_{l_2(C_{R,h})} + \|\delta_h u\|_{l_2(C_{R,h})}\big) \quad \text{(Theorem 6.1.3)}.$$

The proof is complete. □

## 6.2 Estimates of Integral Oscillations

We aim to estimate integral oscillations of a function in terms of difference quotients of it. For that purpose, the following discrete Poincaré inequality plays a crucial role.

**Theorem 6.2.1.** *Given positive integers* $r_i$ *for* $1 \leq i \leq d$, *let* $\Omega = \{(c_1, ..., c_d) : 0 \leq c_i \leq r_i, c_i \in \mathbb{Z}\}$ *and* $p \in [1, \infty)$. *Then for any* $u$ *defined on* $\Omega$ *we have*

$$\sum_{x\in\Omega}\sum_{y\in\Omega} |u(x) - u(y)|^p \leq N d^p(\Omega)|\Omega|\|\delta u\|_{l_p(\Omega)}^p, \tag{6.9}$$

*where* $d(\Omega) = \max_i\{r_i\}$ *and* $N = N(d)$.

*Proof.* Define a continuous cube $Q$ as

$$Q = \{(c_1, ..., c_d) : 0 \leq c_i \leq r_i,\ c_i \in \mathbb{R}\}.$$

Extend $u$ to be defined in $Q$ as follows: for $x \in Q$ we define

$$u(x) = u(\lfloor x_1 \rfloor, ..., \lfloor x_d \rfloor).$$

Given any $x, y \in \Omega$, by a telescopic sum we have

$$|u(y) - u(x)| \leq \sum_{t\in[0,1]} \big(|u((1-t^+)x + t^+y) - u((1-t)x + ty)| + \\ |u((1-t)x + ty) - u((1-t^-)x + t^-y)|\big) \quad (6.10)$$

where we mean

$$u((1-t^+)x + t^+y) := \lim_{s\downarrow t} u((1-s)x + sy) \quad \text{and}$$

$$u((1-t^-)x + t^-y) := \lim_{s\uparrow t} u((1-s)x + sy).$$

Moreover, we see that the number of nonzero sums in (6.10) is less than

$$\sum_{i=1}^{d} |y_i - x_i| \leq Nd(\Omega).$$

So by Hölder's inequality we have

$$|u(y) - u(x)|^p \leq Nd^{p-1}(\Omega) \sum_{t\in[0,1]} \big|u((1-t^+)x + t^+y) - u((1-t)x + ty)\big|^p + \\ \big|u((1-t^-)x + t^-y) - u((1-t)x + ty)\big|^p.$$

Now let

$$\hat{\Omega} = \{(c_1, ..., c_d) : 0 \leq c_i < r_i, c_i \in \mathbb{Z}\}.$$

Then we see

$$\sum_{x\in\hat{\Omega}} \sum_{y\in\hat{\Omega}} |u(x) - u(y)|^p \leq \int_Q \int_Q |u(x) - u(y)|^p dydx \leq \quad (6.11)$$

$$Nd^{p-1}(\Omega) \sum_{t\in[0,1]} \int_Q \int_Q \big|u((1-t^\pm)x + t^\pm y) - u((1-t)x + ty)\big|^p dydx$$

$$\leq 2Nd^{p-1}(\Omega) \sum_{t\in[1/2,1]} \int_Q \int_Q \big|u((1-t^\pm)x + t^\pm y) - u((1-t)x + ty)\big|^p dydx.$$

Under the change of variables $z = (1-t)x + ty$, we have

$$z \in Q \quad \text{(by convexity of } Q\text{)}, \quad dy = t^{-d}dz \leq 2^d dz$$

and

$$\big|u((1-t^{\pm})x+t^{\pm}y)-u((1-t)x+ty)\big| \le |\delta u(z)| + \sum_{y\in\Lambda}|\delta u(z+y)|.$$

Of course, in the right hand side above some difference quotients of $u$ which involve the points outside $\Omega$ are excluded. Then again since the number of nonzero sums in $\sum_{t\in[1/2,1]}$ is less than $Nd(\Omega)$, we have

$$\sum_{x\in\hat\Omega}\sum_{y\in\hat\Omega}|u(x)-u(y)|^p \le Nd^p(\Omega)\int_Q\int_Q\big(|\delta u(z)|^p+\sum_{\xi\in\Lambda}|\delta u(z+\xi)|^p\big)dzdx$$

$$\le Nd^p(\Omega)|\Omega|\|\delta u\|^p_{l_p(\Omega)}.$$

By modifying $\hat\Omega$ appropriately, we can easily obtain

$$\sum_{x\in\hat\Omega}\sum_{y\in\hat\Omega}|u(x)-u(y)|^p \le Nd^p(\Omega)|\Omega|\|\delta u\|^p_{l_p(\Omega)}.$$

The theorem is proved. □

We are ready to discuss the integral oscillation of a function.

**Lemma 6.2.1.** *Let $p\in[1,\infty)$ and $r\ge 0$ be an integer. Then for any $u$ defined in $C_r$ we have*

$$\int_{C_r}|u(x)-u_{C_r}|^p d\mu \le Nr^p\int_{C_r}|\delta u|^p d\mu,$$

*where $N=N(p,d)$.*

*Proof.* Choose a cut-off function $\zeta\ge 0$ such that (i)$\zeta=0$ outside $C_{\lfloor r/2\rfloor}$, (ii)$\int_{C_r}\zeta(x)=1$ and (iii)$|\zeta(x)|\le\frac{N}{r^d}$, $|\delta\zeta(x)|\le\frac{N}{r^{d+1}}$. Let

$$\bar u=\int_{C_r}\zeta(y)u(y)d\mu(y).$$

Then by Holder's inequality we have

$$\int_{C_r}|u(x)-\bar u|^p d\mu=\int_{C_r}\big|\int_{C_r}(u(x)-u(y))\zeta(y)d\mu(y)\big|^p d\mu(x)$$

$$\le N\int_{C_r}\Big(\int_{C_r}|u(x)-u(y)|^p d\mu(y)\big(\int_{C_r}|\zeta(y)|^q d\mu(y)\big)^{p/q}\Big)d\mu(x),$$

where $q = p/(p-1)$,

$$\leq Nr^{(-dq+d)p/q} \int_{C_r} \int_{C_r} |u(x) - u(y)|^p d\mu d\mu$$
$$\leq Nr^p \int_{C_r} |\delta u(x)|^p d\mu \quad \text{(Theorem 6.2.1)}.$$

Therefore, we have

$$\int_{C_r} |u(x) - u_{C_r}|^p d\mu \leq N \int_{C_r} |u(x) - \bar{u}|^p d\mu \leq Nr^p \int_{C_r} |\delta u|^p d\mu.$$

The proof is complete. □

**Lemma 6.2.2.** *Let $p \in [1, \infty)$ and $r$ be an integral multiple of $h$. Then for any $u$ defined in $C_{r,h}$ we have*

$$\int_{C_{r,h}} |\delta_i u(x) - (\delta_i u)_{C_{r,h}}|^p d\mu_h \leq Nr^p \int_{C_{r,h}} |\delta^2 u|^p d\mu_h \tag{6.12}$$

*and*

$$\int_{C_{r,h}} |u(x) - u_{C_{r,h}} - \sum_{i=1}^{d} x_i (\delta_i u)_{C_{r,h}}|^p d\mu_h \leq Nr^{2p} \int_{C_{r,h}} |\delta^2 u|^p d\mu_h \tag{6.13}$$

*where $\delta_i u(x)$ is interpreted as*

$$\delta_i u(x) = \begin{cases} -\delta_{h,-e_i} u(x) & \textit{if} \quad x_i = r \\ \delta_{h,e_i} u(x) & \textit{otherwise} \end{cases} \tag{6.14}$$

*and $N = N(p, d)$.*

*Proof.* Since both (6.12) and (6.13) are scaling invariant, we may assume $h = 1$. Denote $C = C_r$. The first assertion (6.12) is an immediate consequence of Lemma 6.2.1. To prove (6.13), define $v(x) = u(x) - u_C - \sum_i x_i (\delta_i u)_C$. Observe that

$$v_C = 0, \quad \delta_i v = \delta_i u - (\delta_j u)_C, \quad \delta^2 v = \delta^2 u.$$

Thus

$$v - v_C = u(t, x) - u_C - \sum_i x_i (\delta_i u)_C.$$

So by Lemma 6.2.1 we obtain

$$\int_C |u(t,x) - u_C - \sum_i x_i(\delta_i u)_C|^p d\mu = \int_C |v - v_C|^p d\mu$$
$$\le Nr^p \int_C |\delta v|^p d\mu = Nr^p \int_C |\delta u - (\delta u)_C|^p d\mu$$
$$\le Nr^{2p} \int_C |\delta^2 u|^p d\mu \quad \text{by (6.12).}$$

The lemma is proved. □

Using these results, we can estimate the sup norm of higher order difference quotients of $u$ in terms of the $l_2$ norm of $\delta^2 u$.

**Lemma 6.2.3.** *Suppose that $u$ and $v$ are defined in $C_r$ and satisfy*

$$\Delta_1 v = 0 \quad \text{in } \mathring{C}_r \quad \text{and} \quad v = u \quad \text{on } \partial C_r.$$

*Then we have*

$$\|\delta^{m+2} v\|^2_{l_\infty(C_{\lfloor r/4 \rfloor})} \le N r^{-2m-d} \int_{C_r} |\delta^2 u|^2 d\mu$$

*where $N = N(d,m)$.*

*Proof.* Let $h = 1/r$ and define $\tilde{v}(x) = v(x/h)$ and $\tilde{u}(x) = u(x/h)$ for each $x \in h\mathbb{Z}^d$. Take a cut-off function $\zeta$ which is equal to 0 in $C_{\lfloor 3r/4 \rfloor h,h}$ but 1 outside $C_{\lfloor 3r/4 \rfloor h,h}$. Then $\tilde{w} := \tilde{v} - \zeta\tilde{u}$ satisfies

$$\begin{cases} \Delta_h \tilde{w} &= -\Delta_h(\zeta\tilde{u}) =: f \quad \text{in } \mathring{C}_{1,h} \\ \tilde{w} &= 0 \quad \text{on } \partial C_{1,h}. \end{cases} \tag{6.15}$$

Further, we have

$$\Delta_h \tilde{w} = 0 \quad \text{in } C_{\lfloor r/2 \rfloor h,h}.$$

Thus,

$$\|\delta_h^{m+2}\tilde{v}\|_{l_\infty(C_{\lfloor r/4 \rfloor h,h})} = \|\delta_h^{m+2}\tilde{w}\|_{l_\infty(C_{\lfloor r/4 \rfloor h,h})}$$
$$\le N\big(\|\tilde{w}\|_{l_2(C_{\lfloor r/2 \rfloor h,h})} + \|\delta_h \tilde{w}\|_{l_2(C_{\lfloor r/2 \rfloor h,h})}\big) \quad \text{(Theorem 6.1.1)}$$
$$\le N\|f\|_{l_2(\mathring{C}_{\lfloor r/2 \rfloor h,h,h})} \quad \text{(Theorem 3.1.3)}$$
$$\le N\|\tilde{u}\|_{l_2^2(\mathring{C}_{1,h})}.$$

Now we define

$$g = \tilde{v} - \tilde{u}_{\mathring{C}_{1,h}} - \sum_i x_i(\delta_{h,e_i}\tilde{u})_{\mathring{C}_{1,h}}.$$

Then the fact that $\delta_h^2 g = \delta_h^2 \tilde{v}$ and $g = \tilde{u} - \tilde{u}_{\mathring{C}_{1,h}} - \sum_i x_i(\delta_i\tilde{u})_{\mathring{C}_{1,h}}$ on $\partial C_{1,h}$ implies

$$\begin{aligned}
\|\delta_h^{m+2}\tilde{v}\|_{l_\infty(C_{\lfloor r/4\rfloor h,h})} &= \|\delta_h^{m+2} g\|_{l_\infty(C_{\lfloor r/4\rfloor h,h})} \\
&\le N\|\tilde{u} - \tilde{u}_{\mathring{C}_{1,h}} - \sum x_i(\delta_{h,e_i}\tilde{u})_{\mathring{C}_{1,h}}\|_{l_2^2(\mathring{C}_{1,h})} \\
&\le N\big(\|\tilde{u} - \tilde{u}_{\mathring{C}_{1,h}} - \sum x_i(\delta_{h,e_i}\tilde{u})_{\mathring{C}_{1,h}}\|_{l_2(\mathring{C}_{1,h})} + \\
&\qquad \|\delta\tilde{u} - (\delta\tilde{u})_{\mathring{C}_{1,h}}\|_{l_2(\mathring{C}_{1,h})} + \|\delta^2\tilde{u}\|_{l_2(C_{1,h})}\big) \\
&\le N\|\delta_h^2\tilde{u}\|_{l_2(C_{1,h})} \quad \text{(Lemma 6.2.2).}
\end{aligned}$$

So far $N$ depends only on $d$ and $m$ because the underlying domain is a unit cube. However, according to the dilation $h = 1/r$ we obtain

$$\begin{aligned}
h^{-2m-4}\|\delta^{m+2}v\|^2_{l_\infty(C_{\lfloor r/4\rfloor})} &= \|\delta_h^{m+2}\tilde{v}\|^2_{l_\infty(C_{\lfloor r/4\rfloor h,h})} \\
&\le N\int_{C_{1,h}} |\delta_h^2\tilde{u}|^2 d\mu_h = Nh^d\int_{C_r} |\delta^2 u|^2 h^{-4} d\mu.
\end{aligned}$$

Therefore,

$$\|\delta^{m+2}v\|^2_{l_\infty(C_{\lfloor r/4\rfloor})} \le Nh^{2m+d}\int_{C_r} |\delta^2 u|^2 d\mu = Nr^{-2m-d}\int_{C_r} |\delta^2 u|^2 d\mu.$$

The lemma is proved. □

**Theorem 6.2.2.** *Let $\nu \ge 8$. Suppose $u$ and $v$, which are defined in $h\mathbb{Z}^d$, satisfy*

$$\begin{cases} \Delta_1 v &= 0 \quad \text{in } \mathring{C}_{\nu r} \\ \quad v &= u \quad \text{on } \partial C_{\nu r}. \end{cases}$$

*Then we have*

$$\fint_{C_r} |\delta^2 v(x) - (\delta^2 v)_{C_r}|^2 d\mu \le N\nu^{-2}\fint_{C_{\nu r}} |\delta^2 u|^2 d\mu, \tag{6.16}$$

*where $N = N(d)$.*

**Remark 6.2.1.** *Here, we do not exclude some of difference quotients of $v$ and $u$ which involve the points outside $C_r$ and $C_{\nu r}$, respectively. In fact, this adjustment is needed to apply the Fefferman-Stein and Hardy-Littlewood theorem in the proof of Theorem 6.3.1.*

*Proof.* We see

$$\begin{aligned}
⨍_{C_r} |\delta^2 v(x) - (\delta^2 v)_{C_r}|^2 d\mu &\le r^2 \|\delta^3 v\|^2_{l_\infty(C_{r+1})} \\
&\le r^2 \|\delta^3 v\|^2_{l_\infty(C_{\lfloor \nu r/4 \rfloor})} \quad \text{(because } \nu \ge 8) \\
&\le N r^2 (\nu r)^{-2-d} \int_{C_{\nu r}} |\delta^2 u|^2 d\mu \quad \text{(Lemma 6.2.3)} \\
&\le N \nu^{-2} ⨍_{C_{\nu r}} |\delta^2 u|^2 d\mu.
\end{aligned}$$

The proof is complete. □

**Theorem 6.2.3.** *Let $\nu \ge 8$. Then for any $u$ defined in $\mathbb{Z}^d$ we have*

$$⨍_{C_r} |\delta^2 u(x) - (\delta^2 u)_{C_r}|^2 d\mu \le N \nu^d ⨍_{C_{\nu r}} |f|^2 d\mu + N \nu^{-2} ⨍_{C_{\nu r}} |\delta^2 u|^2 d\mu,$$

*where $f := \Delta_1 u$ and $N = N(d)$.*

*Proof.* It is a classical fact that there exists a unique solution $v$ to

$$\begin{cases} \Delta_1 v &= 0 \quad \text{in } \mathring{C}_{\nu r} \\ \quad v &= u \quad \text{on } \partial C_{\nu r}. \end{cases}$$

Then Theorem 6.2.2 implies

$$⨍_{C_r} |\delta^2 v - (\delta^2 v)_{C_r}|^2 d\mu \le N \nu^{-2} ⨍_{C_{\nu r}} |\delta^2 u|^2 d\mu. \tag{6.17}$$

On the other hand, $w := u - v$ satisfies

$$\begin{cases} \Delta_1 w &= f \quad \text{in } \mathring{C}_{\nu r} \\ \quad v &= 0 \quad \text{on } \partial C_{\nu r}. \end{cases}$$

So by Theorem 3.1.2

$$\int_{C_r} |\delta^2 w|^2 d\mu \le N \int_{C_{\nu r}} |f|^2 d\mu$$

and thus

$$\fint_{C_r} |\delta^2 w|^2 d\mu \le N\nu^d \fint_{C_{\nu r}} |f|^2 d\mu. \tag{6.18}$$

Therefore, we have

$$\begin{aligned}
&\fint_{C_r} |\delta^2 u(x) - (\delta^2 u)_{C_r}|^2 d\mu \\
&\le 2 \fint_{C_r} |\delta^2 w(x) - (\delta^2 w)_{C_r}|^2 d\mu + 2 \fint_{C_r} |\delta^2 v(x) - (\delta^2 v)_{C_r}|^2 d\mu \\
&\le 2 \fint_{C_r} |\delta^2 w|^2 d\mu + N\nu^{-2} \fint_{C_{\nu r}} |\delta^2 u|^2 d\mu \quad \text{by (6.17)} \\
&\le N\nu^d \fint_{C_{\nu r}} |f|^2 d\mu + N\nu^{-2} \fint_{C_{\nu r}} |\delta^2 u|^2 d\mu \quad \text{by (6.18).}
\end{aligned}$$

The proof is complete. □

## 6.3 $l_p$-Estimates

Now we are ready to state a crucial a priori $l_p$-estimate for elliptic equations.

**Theorem 6.3.1.** *Let $p \in (1, \infty)$, $\lambda \ge 0$. Then for any $u \in l_p$ we have*

$$\|\delta^2 u\|_{l_p} + \sqrt{\lambda}\|\delta u\|_{l_p} + \lambda\|u\|_{l_p} \le N\|\Delta_1 u - \lambda u\|_{l_p} \tag{6.19}$$

*where $N = N(d, p)$.*

*Proof.* Since we have already obtained the desired result for $p = 2$, we shall only deal with the case $p > 2$. The case $p < 2$ follows from the duality argument. For any $\nu \ge 8$, $x \in \mathbb{Z}^d$ and $r \in \{0, 1, 2, ...\}$, let $f = \Delta_1 u$, $C = C_r(x)$ and $C' = C_{\nu r}(x)$. Then Theorem 6.2.3 implies

$$\begin{aligned}
\Big( \fint_C |\delta^2 u(x) - (\delta^2 u)_C| d\mu\Big)^2 &\le \fint_C |\delta^2 u(x) - (\delta^2 u)_C|^2 d\mu \\
&\le N\nu^d \fint_{C'} |f|^2 d\mu + N\nu^{-2} \fint_{C'} |\delta^2 u|^2 d\mu.
\end{aligned}$$

Recalling the definition of sharp and maximal functions, we have

$$(\delta^2 u)^\sharp(x) \le N\nu^{d/2}\mathbb{M}^{1/2}|f|^2(x) + N\nu^{-1}\mathbb{M}^{1/2}|\delta^2 u|^2(x). \tag{6.20}$$

By the Hardy-Littlewood theorem with $p/2 > 1$, it holds that

$$\|\mathbb{M}^{1/2}|f|^2\|_{l_p} = \|\mathbb{M}|f|^2\|_{l_{p/2}}^{1/2} \le N\|f\|_{l_p},$$
$$\|\mathbb{M}^{1/2}|\delta^2 u|^2\|_{l_p} \le N\|\delta^2 u\|_{l_p}.$$

Using this result and applying the Fefferman-Stein theorem to (6.20), we have

$$\|\delta^2 u\|_{l_p} \le N\|(\delta^2 u)^\sharp\|_{l_p} \le N\nu^{d/2}\|f\|_{l_p} + N\nu^{-1}\|\delta^2 u\|_{l_p}.$$

Thus, if we take $\nu$ sufficiently large, we have

$$\|\delta^2 u\|_{l_p} \le N\|f\|_{l_p}.$$

Consequently,

$$\begin{aligned}\|\delta^2 u\|_{l_p} + \lambda\|u\|_{l_p} &\le N(\|\Delta_1 u\|_{l_p} + \lambda\|u\|_{l_p})\\ &\le N(\|\Delta_1 u - \lambda u\|_{l_p} + \lambda\|u\|_{l_p})\\ &\le N\|\Delta_1 u - \lambda u\|_{l_p} \quad \text{(Theorem 3.1.1)}.\end{aligned}$$

Last, $\sqrt{\lambda}\|\delta u\|_{l_p} \le \|\Delta_1 u - \lambda u\|_{l_p}$ comes from the interpolation inequality. The theorem is proved. □

Since (6.19) is scaling invariant and $N$ is independent of $\lambda$, we can obtain the following result easily.

**Theorem 6.3.2.** *Let $\lambda > 0$ and $m \in \{0, 1, 2, ...\}$. Then for any $f \in l_p(h\mathbb{Z}^d)$ there exists a unique solution $u \in l_p(h\mathbb{Z}^d)$ to*

$$\Delta_h u - \lambda u = f \quad \text{in } h\mathbb{Z}^d. \tag{6.21}$$

*Moreover, we have*

$$\|\delta_h^2 u\|_{l_p^m} + \sqrt{\lambda}\|\delta_h u\|_{l_p^m} + \lambda\|u\|_{l_p^m} \le N\|f\|_{l_p^m}$$

*where $N = N(d, m, p)$.*

*Proof.* Solvability comes from Thereom 3.1.1. The second assertion comes from Theorem 6.3.1 by the scaling argument. The theorem is proved. □

# Chapter 7

# Equations with Continuous Coefficients

From now on we consider an elliptic difference equation with variable coefficients:

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d \tag{7.1}$$

where $Lu(x) = \sum_{i=1}^{d} a_i(x) D_{x_i x_i} u(x)$. The operator $L$ satisfies the ellipticity condition: there is a constant $\kappa > 0$ such that

$$\kappa \le a_i(x) \le \kappa^{-1} \quad \text{for all } x \in \mathbb{R}^d \text{ and } i = 1, ..., d.$$

Further, we assume that $a_i(x)$ is uniformly continuous: there is a function $\omega(\epsilon)$, called modulus of continuity, such that $\omega(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$ and

$$|a_i(x) - a_i(y)| \le \omega(|x-y|).$$

We define an elliptic difference equation corresponding to (7.1) as

$$L_h u - \lambda u = f \quad \text{in } h\mathbb{Z}^d \tag{7.2}$$

where $L_h u(x) = -\sum_{i=1}^{d} a_i(x) \delta_{h,e_i} \delta_{h,-e_i} u(x)$.

## 7.1 $l_p$-Estimates

To obtain $l_p$ estimates of type (6.19) for the difference equation (7.2) with variable coefficients, we first consider the case where $u$ is supported in a sufficiently small cube.

**Lemma 7.1.1.** *We can find a constant $\epsilon < 1$, depending only on $d, \kappa, \omega$, such that for any $\lambda \geq 0$, $h < \epsilon/4$ and $u \in l_p(h\mathbb{Z}^d)$ supported in $C_{\lfloor \epsilon/2h \rfloor h,h}$ we have*

$$\|\delta_h^2 u\|_{l_p(h\mathbb{Z}^d)} + \sqrt{\lambda}\|\delta_h u\|_{l_p(h\mathbb{Z}^d)} + \lambda\|u\|_{l_p(h\mathbb{Z}^d)} \leq N\|L_h u - \lambda u\|_{l_p(h\mathbb{Z}^d)} \tag{7.3}$$

*where $N = N(d, \kappa, p)$.*

*Proof.* Let us freeze the leading coefficients as follows

$$L_{h,0}u(x) = -\sum_{i=1}^{d} a_i(0)\delta_{h,e_i}\delta_{h,-e_i}u(x).$$

Then by Theorem 6.3.2 we have

$$\|\delta_h^2 u\|_{l_p(h\mathbb{Z}^d)} + \sqrt{\lambda}\|\delta_h u\|_{l_p(h\mathbb{Z}^d)} + \lambda\|u\|_{l_p(h\mathbb{Z}^d)} \leq N\|L_{h,0} u - \lambda u\|_{l_p(h\mathbb{Z}^d)}.$$

Since $u$ is supported in $C_{\lfloor \epsilon/2h \rfloor h,h}$, we have

$$\|L_{h,0}u - L_h u\|_{l_p(h\mathbb{Z}^d)} \leq \|L_{h,0}u - L_h u\|_{l_p(C_{\lfloor \epsilon/h \rfloor h,h})} \leq \omega(2\sqrt{d}\epsilon)\|\delta_h^2 u\|_{l_p(h\mathbb{Z}^d)}.$$

If we choose $\epsilon$ so small that $\omega(2\sqrt{d}\epsilon) \leq 1/2$, then we have the desired result (7.3). The lemma is proved. $\square$

**Theorem 7.1.1.** *Take a mesh size $h$ from Lemma 7.1.1. Let $m \in \{0, 1, 2, ...\}$. Then there exists a constant $\lambda_0 \geq 1$, depending only on $d, \kappa, \omega$, such that for any $\lambda \geq \lambda_0$ and $u \in l_p(h\mathbb{Z}^d)$ we have*

$$\|\delta_h^2 u\|_{l_p^m(h\mathbb{Z}^d)} + \sqrt{\lambda}\|\delta_h u\|_{l_p^m(h\mathbb{Z}^d)} + \lambda\|u\|_{l_p^m(h\mathbb{Z}^d)} \leq N\|L_h u - \lambda u\|_{l_p^m(h\mathbb{Z}^d)}$$

*where $N = N(d, \kappa, p, m)$.*

*Proof.* Without loss of generality we may assume $m = 0$. And take $\epsilon$ from Lemma 7.1.1. For the sake of notational convenience we may assume that $\epsilon$ is an integral multiple of $h$. We choose a cut-off function $\zeta$ such that it is supported in $C_{\epsilon/2,h}$ and

$$\sum_{x \in h\mathbb{Z}^d} |\zeta(x)|^p h^d = 1.$$

We observe that

$$|\delta_{h,e_i}\delta_{h,e_j} u(x)|^p = \sum_{y \in h\mathbb{Z}^d} |\delta_{h,e_i}\delta_{h,e_j} u(x)|^p |\zeta(x-y)|^p h^d \tag{7.4}$$

and by the product rule (with respect to $x$)

$$\begin{aligned}
&|\delta_{h,e_i}\delta_{h,e_j} u(x)|^p |\zeta(x-y)|^p \\
&= \big|\delta_{h,e_i}\delta_{h,e_j}(u(x)\zeta(x-y)) - \delta_{h,e_i} u(x+he_j)\delta_{h,e_j}\zeta(x-y) - \\
&\quad \delta_{h,e_j} u(x+he_i)\delta_{h,e_i}\zeta(x-y) - u(x+he_i+he_j)\delta_{h,e_i}\delta_{h,e_j}\zeta(x-y)\big|^p \\
&\le N|\delta_{h,e_i}\delta_{h,e_j}(u(x)\zeta(x-y))|^p + N(|\delta_{h,e_i} u(x+he_i)|^p + |\delta_{h,e_j} u(x+he_j)|^p + \\
&\quad |u(x+he_i+he_j)|^p)(|\delta_h\zeta(x-y)|^p + |\delta_h^2\zeta(x-y)|^p).
\end{aligned}$$

We sum both sides of (7.4) over $x \in h\mathbb{Z}^d$ and change the order of summations to have

$$\|\delta_h^2 u\|_{l_p}^p \le N \sum_{y \in h\mathbb{Z}^d} \|\delta_h^2(u(\cdot)\zeta(\cdot - y))\|_{l_p}^p h^d + N\|\delta_h u\|_{l_p}^p + N\|u\|_{l_p}^p. \tag{7.5}$$

Since $u(\cdot)\zeta(\cdot - y)$ is supported in $C_{\epsilon/2,h}(y)$, Lemma 7.1.1 implies

$$\|\delta_h^2(u(\cdot)\zeta(\cdot - y))\|_{l_p}^p \le N\|(L_h - \lambda)(u(\cdot)\zeta(\cdot - y))\|_{l_p}^p$$

for each $\lambda \ge 0$. Then again by applying the product rule in

$$(L_h - \lambda)(u(\cdot)\zeta(\cdot - y))$$

we have

$$\begin{aligned}
\sum_{y \in h\mathbb{Z}^d} \|\delta_h^2(u(\cdot)\zeta(\cdot - y))\|_{l_p}^p h^d &\le N\|L_h u - \lambda u\|_{l_p}^p \sum_{y \in h\mathbb{Z}^d} |\zeta(y)|^p h^d + \\
&\quad N(\|u\|_{l_p}^p + \|\delta_h u\|_{l_p}^p) \sum_{h\mathbb{Z}^d} (|\delta_h\zeta|^p + |\delta_h^2\zeta|^p) h^d \\
&\le N(\|L_h u - \lambda u\|_{l_p}^p + \|u\|_{l_p}^p + \|\delta_h u\|_{l_p}^p).
\end{aligned}$$

Thus, by (7.5)

$$\begin{aligned}\|\delta_h^2 u\|_{l_p}^p &\le N(\|L_h u - \lambda u\|_{l_p}^p + \|u\|_{l_p}^p + \|\delta_h u\|_{l_p}^p) \\ &\le N\|L_h u - \lambda u\|_{l_p}^p + N\|u\|_{l_p}^p + \frac{1}{4}\|\delta_h^2 u\|_{l_p}^p \quad \text{(Theorem 4.0.4).}\end{aligned} \tag{7.6}$$

Likewisely,

$$\lambda^p |u(x)|^p = \lambda^p \sum_{y \in h\mathbb{Z}^d} |u(x)|^p |\zeta(x-y)|^p h^d$$

and thus

$$\begin{aligned}\lambda^p \|u\|_{l_p}^p \le \sum_{y \in h\mathbb{Z}^d} \lambda^p \|u(\cdot)\zeta(\cdot - y)\|_{l_p}^p h^d &\le N \sum_{y \in h\mathbb{Z}^d} \|(L_h - \lambda)(u(\cdot)\zeta(\cdot - y))\|_{l_p}^p h^d \\ &\le N\|L_h u - \lambda u\|_{l_p}^p + N\|u\|_{l_p}^p + \frac{1}{8}\|\delta_h^2 u\|_{l_p}^p.\end{aligned}$$

By choosing $\lambda_0$ so large that $\lambda_0^p > 2N$, we obtain

$$\lambda \|u\|_{l_p}^p \le N\|L_h u - \lambda u\|_{l_p}^p + \frac{1}{4}\|\delta_h^2 u\|_{l_p}^p.$$

for each $\lambda \ge \lambda_0$. Together with (7.6),

$$\|\delta_h^2 u\|_{l_p}^p + \lambda \|u\|_{l_p}^p \le N\|L_h u - \lambda u\|_{l_p}^p.$$

The theorem is proved. □

## 7.2 Method of Continuity

Now we prove the $l_p$-solvability of the elliptic difference equation using the so-called method of continuity.

**Theorem 7.2.1.** *Take a mesh size $h$ and $\lambda_0$ from Theorem 7.1.1. Then for any $\lambda \ge \lambda_0$ and $f \in l_p(h\mathbb{Z}^d)$ there exists a unique solution $u \in l_p(h\mathbb{Z}^d)$ to (7.2).*

*Proof.* For each $t \in [0, 1]$ we define

$$L_t = t(\Delta_h - \lambda) + (1-t)(L_h - \lambda).$$

And suppose that for some $t_0$ there exists a unique solution $u \in l_p$ to

$$L_{t_0} u = f \quad \text{in } h\mathbb{Z}^d.$$

For this $t_0$ define a linear mapping $\mathcal{T}_{t_0} : l_p \to l_p$ by $\mathcal{T}_{t_0}(f) = u$. We shall claim that there is a small constant $\epsilon > 0$ such that for each $t \in (t_0 - \epsilon, t_0 + \epsilon) \cap [0, 1]$

$$L_t u = f$$

also has a unique solution $u$ in $l_p$.

To establish this assertion we observe that $L_t u = f$ is equivalent to

$$L_{t_0} u = (L_{t_0} - L_t)u + f.$$

Thus, we need to prove that a mapping $\mathcal{F}_t : l_p \to l_p$, defined by $\mathcal{F}_t(u) = \mathcal{T}_{t_0}((L_{t_0} - L_t)u + f)$, has a fixed point. We see that the leading coefficients of

$$L_{t_0} = t_0 \Delta_h + (1 - t_0) L_h - \lambda$$

satisfy the ellipticity and uniform continuity conditions. Here, the ellipticity constant and the modulus of continuity do not depend on $t_0$. Therefore, Theorem 7.1.1 implies

$$\|\mathcal{T}_{t_0}(f)\|_{l_p} \leq N \|f\|_{l_p}.$$

It leads us to

$$\begin{aligned}
\|\mathcal{F}_t(u) - \mathcal{F}_t(v)\|_{l_p} &= \|\mathcal{T}_{t_0}((L_{t_0} - L_t)(u - v))\|_{l_p} \leq N \|(L_{t_0} - L_t)(u - v))\|_{l_p} \\
&\leq N |t_0 - t| \|(\Delta_h - L_h)(u - v)\|_{l_p} \\
&\leq N |t_0 - t| h^{-2} \|(u - v)\|_{l_p}.
\end{aligned}$$

So if we choose $\epsilon$ sufficiently small, then $\mathcal{F}_t$ becomes a contraction mapping. Thus, it has a fixed point.

To complete the proof, we recall that Theorem 3.1.1 allows us to begin with $t_0 = 1$. Then by applying the above argument finitely many times, we reach the conclusion that

$$Lu - \lambda u = L_0 u = f$$

has a unique solution in $l_p$. Hence, the theorem is proved. □

# Chapter 8

# $l_p$-Error Estimates

We aim to estimate the error between the original solution and the numerical solution to the model equation (7.1) in the $l_p$ norm. $L_p$-solvability for this equation is a classical result. See [11, 17].

**Theorem 8.0.2.** *Let $m \in \{0, 1, 2, ...\}$. There exists $\lambda_0 = \lambda_0(d, \kappa, \omega) \geq 1$ such that for any $\lambda \geq \lambda_0$ and $f \in W_p^m(\mathbb{R}^d)$ there is a unique solution $u \in W_p^{m+2}(\mathbb{R}^d)$ to equation (7.1). Moreover, we have*

$$\|D^2 u\|_{W_p^m} + \sqrt{\lambda}\|Du\|_{W_p^m} + \lambda\|u\|_{W_p^m} \leq N\|f\|_{W_p^m}$$

*where $N = N(d, \kappa, m, p)$.*

Its discrete counterpart has been discussed in Theorem 7.1.1 and Theorem 7.2.1. The lemma below implies that if $m > d/p$, then the given function $f \in W_p^m(\mathbb{R}^d)$ is continuous and belongs to $l_p(h\mathbb{Z}^d)$ as well.

**Lemma 8.0.1.** *Let $m > d/p$ and $h \leq 1$. Then for any $f \in W_p^m$ we have*

$$\|f\|_{l_p(h\mathbb{Z}^d)} \leq N\|f\|_{W_p^m(\mathbb{R}^d)}, \tag{8.1}$$

*where $N = N(d, m, p)$.*

*Proof.* By the standard Sobolev inequality [8] we know $f$ is a continuous function (or at least it has a continuous version). For any $x \in h\mathbb{Z}^d$ we have

$$\begin{aligned} |f(x)|^p &\le \sup_{z \in B_1(0)} |f(x+hz)|^p \\ &\le N \sum_{|\alpha| \le m} \int_{B_1(0)} |D_z^\alpha f(x+hz)|^p dz \quad \text{(Sobolev inequality)}, \end{aligned}$$

where $B_r(y)$ stands for the open ball in $\mathbb{R}^d$ of radius $r$ centered at $y$,

$$\begin{aligned} &\le N \sum_{|\alpha| \le m} h^{p|\alpha|} \int_{B_1(0)} |(D^\alpha f)(x+hz)|^p dz \\ &\le N \sum_{|\alpha| \le m} h^{p|\alpha|-d} \int_{B_h(x)} |D^\alpha f(y)|^p dy \\ &\le N \sum_{|\alpha| \le m} h^{-d} \int_{B_h(x)} |D^\alpha f(y)|^p dy \quad \text{(because } h \le 1\text{)}. \end{aligned}$$

Thus,

$$\sum_{x \in h\mathbb{Z}^d} |f(x)|^p h^d \le N \sum_{|\alpha| \le m} \sum_{x \in h\mathbb{Z}^d} \int_{B_h(x)} |D^\alpha f(y)|^p dy \le N \|f\|_{W_p^m}^p.$$

The lemma is proved. □

As another auxiliary fact, we state the following lemma.

**Lemma 8.0.2.** *Let $m \ge 0$ be an integer and $\psi \in W_p^{m+4}(\mathbb{R}^d)$. Then*

$$\|L_h\psi - L\psi\|_{W_p^m} \le Nh^2 \|D^4\psi\|_{W_p^m}$$

*where $N = N(d, \kappa)$.*

*Proof.* By Taylor's formula

$$L_h\psi(x) - L\psi(x) \le \frac{1}{6h^2} \sum_{i=1}^{d} \int_{-1}^{1} |a_i(x) D_{x_i}^4 \psi(x + hse_i)| h^4 ds.$$

Thus,

$$\|L_h\psi - L\psi\|_{W_p^m} \le Nh^2 \|D^4\psi\|_{W_p^m}.$$

The theorem is proved. □

Finally, we are able to state the main $l_p$-error estimate.

**Theorem 8.0.3.** *Let* $m, n \in \{0, 1, 2, ...\}$, $p \in (1, \infty)$, $m > n + d/p$. *Take* $\lambda_0$ *and* $h$ *from Theorem 7.1.1. For given* $\lambda \geq \lambda_0$ *and* $f \in W_p^{m+2}$, *let* $u \in W_p^{m+4}$ *and* $u_h \in l_p(h\mathbb{Z}^d)$ *be unique solutions to (7.7.1) and (7.7.2), respectively. Define*

$$r(x) = u(x) - u_h(x) \quad \text{for } x \in h\mathbb{Z}^d.$$

*Then we have*

$$\|\delta_h^2 r\|_{l_p^n(h\mathbb{Z}^d)} + \sqrt{\lambda}\|\delta_h r\|_{l_p^n(h\mathbb{Z}^d)} + \lambda\|r\|_{l_p^n(h\mathbb{Z}^d)} \leq h^2 N\|f\|_{W_p^{m+2}(\mathbb{R}^d)} \tag{8.2}$$

*where* $N = N(d, \kappa, m, p)$. *Moreover, if* $n > d/p$, *then we have*

$$\|\delta_h^2 r\|_{l_\infty} + \sqrt{\lambda}\|\delta_h r\|_{l_\infty} + \lambda\|r\|_{l_\infty} \leq h^2 N\|f\|_{W_p^{m+2}}$$

*where* $N = N(d, \kappa, m, p)$.

*Proof.* For the first assertion, we observe that the error term $r$ satisfies

$$\begin{aligned} L_h r - \lambda r &= L_h u - \lambda u - L_h u_h + \lambda u_h + Lu - Lu \\ &= L_h u - Lu \quad \text{in } h\mathbb{Z}^d. \end{aligned}$$

Lemma 8.0.1 implies that $u \in W_p^{m+4} \subset l_p$ and thus $r = u - u_h \in l_p$. So by Theorem 7.1.1 we have

$$\begin{aligned} \|\delta_h^2 r\|_{l_p^n} + \sqrt{\lambda}\|\delta_h r\|_{l_p^n} + \lambda\|r\|_{l_p^n} &\leq N\|L_h u - Lu\|_{l_p^n} \\ &\leq N\|L_h u - Lu\|_{W_p^m} \quad \text{(Lemma 8.0.1)} \\ &\leq h^2 N\|D^4 u\|_{W_p^m} \ \text{(Lemma 8.0.2)} \\ &\leq h^2 N\|f\|_{W_p^{m+2}} \quad \text{(Theorem 8.0.2)}. \end{aligned}$$

The second assertion comes from Theorem 4.3.1 and (8.2). The theorem is proved. □

# Chapter 9

# Further Discussion

In the above work, we assumed that the off-diagonal entries of the operator $L(x)$ are all zeros. To cover more general operators of the form:

$$L(x)u(x) = \sum_{i,j=1}^{d} a_{ij}(x) D_{x_i x_j} u(x),$$

we need to use the following fact: there exist $\{l_k : l_k \in \mathbb{R}^d\}_{k=1}^m$ and $\{\lambda_k(x) : \lambda_k(x) \in \mathbb{R}\}_{k=1}^m$ such that

$$a(x) = \sum_{k=1}^{m} \lambda_k(x) l_k l_k^*$$

and

$$\sum_{i,j=1}^{d} a_{ij}(x) D_{x_i x_j} u(x) = \sum_{k=1}^{m} \lambda_k(x) D_{l_k l_k} u(x).$$

Then, there is a chance that the methods used in the previous chapters would work as well. One of the challenges arising from this approach is that the number of vectors $\{\lambda_k\}$, $m$, is usually strictly larger than $d$ and thus we may not be able to obtain the main results in a straightforward manner. Thus, I am planning to investigate this idea in more detail. Furthermore, it would also be meaningful to study $l_p$-error estimates for elliptic and parabolic equations under relaxed regularity conditions.

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