On the Power Operations of MU.

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Abstract

We study the power opearations of the spectrum MU, the complex cobordism theory.

After reviewing necessary backgrounds, we start by recalling a formula from Justin Noel and Niles Johnson connecting the power operation $P([\mathbb{CP}^n])$ for $[\mathbb{CP}^n] \in MU^*$ with the power operation $P_{\mathbb{CP}}(x)$ for the orientation class $x \in MU^*(\mathbb{CP}^\infty)$.

Next, we find an algorithm calculating the effect of P on a set of polynomial generators $x_i \in MU^*$ from the known formulas coverting $[\mathbb{CP}^n]$'s to x_i's and the fact that under the canonical projection $q : MU^*[\![\alpha]\!]/[p]_F(\alpha) \to MU^*[\![\alpha]\!]/\langle p\rangle_F(\alpha)$, both q_*P and $q_*P_{\mathbb{CP}}$ become ring homomorphisms.

Finally, we display a sample calculation of $P(x_3)$ at $p = 2$ with the help of Maple, and provide an application of our calculation where we put some restrictions on possible E_3 maps from MU to BP(or $BP\langle n\rangle$'s).

Contents

Chapter 1

Introduction

Algebraic topology studies algebraic invariants of topological spaces, such as homology and homotopy groups. These encode a lot of information about spaces and are very powerful tools for understanding them. As an example, homotopy groups completely classify CW-complexes (every manifold is homotopy equivalent to one) up to homotopy equivalence. However, these invariants are not always easy to compute. For instance, the homotopy group of spheres remains largely unknown.

Spectra arise from the study of stable homotopy groups. They also characterize generalized cohomology theories (with mild conditions), such as the classical homology theory, various K-theories and cobordism theories. Spectra are a sequence of spaces in nature, but with a 'smash product' they also bahave like rings in algebra. This allows us to study structured ring spectra (E_∞ spectra), which are commutative up to coherent homotopies.

Power operations are maps (not necessarily ring homomorphisms) $\mathcal{P} : E^*(X) \to E^*(D_j X)$, where $D_j X$ is the extended power construction $E\Sigma_j \ltimes_{\Sigma_j} X^{\wedge j}$. and E is an H_∞ spectra (E_∞ spectra up to homotopy). These are algebraic invariants coming from the coherent homotopies of structured ring spectra. When X is a space, we have a slightly different version $P : E^*(X) \to E^*(BC_p \times X)$ derived from $\mathcal{P}$. The Steenrod operation in mod p cohomology, Dyer-Lashof operation in mod p homology and Adams' $\psi-$operation in complex K-theory essentially can all be realized as power operations defined on them, respectively [BMMS06, Chapter VIII, Section 2-4].

In this thesis, we explicitly compute the power operation P associated to MU^*at $p = 2, 3$ for the first few polynomial generators, where MU is the complex cobordism theory. Detailed results are displayed in section 6.4.

The rest of this thesis is organized as follows:

Chapter 2 will review the basic knowledge of spectra and introduce MU.

Chapter 3 will review relevant notations constructions of operads and E_∞ spectra.

Chapter 4 will review relevant notations of H_∞ spectra and the construction of power operations.

Chapter 5 will review the theory of formal group laws and its relation to complex oriented theories.

Chapter 6 explicitly computes the power opearation of the first few polynomial generators in $\pi_* MU$ for $p = 2, 3$. We also put some restrictions to possible $E_3 - MU$ algebra maps to BP and $BP\langle n\rangle$ as an application.

Chapter 2

Spectra

Spectra are the main objects of study in modern algebraic topology. We briefly review the basic knowledge from [Ada95, Part III].

Definition 2.0.1. A **spectrum** is a sequence of based spaces [1] $\{E_n\}$ together with basepoint preserving maps (called structure maps) $\sigma_n : \Sigma E_n \to E_{n+1}$. Further, if the adjoint of σ_n:

$$E_n \xrightarrow{\tilde{\sigma}_n} \Omega E_{n+1}$$

are weak equivalences for all n's, such a spectrum is called an Ω-spectrum.

Definition 2.0.2. A **function** between spectra E, F is a sequence of maps $f_i : E_i \to F_i$ such that the following diagram strictly commutes:

$$\begin{array}{ccc} \Sigma E_i & \xrightarrow{\Sigma\sigma_i} & E_{i+1} \\ \Big\downarrow{\scriptstyle f_i} & & \Big\downarrow{\scriptstyle f_{i+1}} \\ \Sigma F_i & \xrightarrow{\sigma_i} & F_{i+1} \end{array}$$

A **subspectrum** $E'_n \subset E_n$ is a sequence of subspaces $E'_n \subset E_n$ such that $\sigma_n(\Sigma E'_n) \subset E'_{n+1}$ for each n. It is cofinal if every cell in E_n is eventually mapped into some E'_m after finite suspensions. A **map** between two spectra E, F is a equivalence class of functions $f' : E' \to F$ defined on some cofinal subspectrum $E' \subset E$, with two functions f_1, f_2 defined on E_1, E_2 thought to be equivalent if they coincide on a third cofinal subspectrum.

For a spectrum E we can define its **cylinder** to be $(Cyl\ E)_n = I_+ \wedge E_n$ with structrual maps $1 \wedge \sigma_n$. There are two natural inclusions $i_0, i_1 : E \hookrightarrow Cyl\ E$ corresponding to the two ends of I. Two maps $f_0, f_1 : E \to F$ are **homotopic** if there is a map $F : Cyl\ E \to F$

[1] Throughout this thesis by a topological space we mean a compactly generated weakly Hausdorff space.

such that $F \circ i_0 = f_0, F \circ i_1 = f_1$. We will denote by $[A, B]$ the homotopy class of maps between spectra A and B.

Spectra and maps between them give rise to a category Sp. We list some of its important properties here:

- Sp is additive: there is a unique way to equip $[A,B]$ with an abelian group structure such that composition is bilinear.
- Sp has arbitrary products and coproducts. Finite products coincide with coproducts.
- The operation of forming fibration and cofibration sequences on the space level carry over to spectra. In the homotopy category hSp, fiber sequences coincide with cofiber sequences. This makes hSP a triangulated category where the translation functor is given by suspension.
- Sp is monoidal with respect to the smash product $X \wedge Y$ defined for any two spectra X, Y. It is associative, commutative, and has $\mathbb{S}$ as a two-sided unit, all up to coherent equivalences. Therefore its homotopy category hSp is symmetric monoidal.

The most important reason for studying spectra is Brown's representability theorem:

Theorem 2.0.3. *[Bro62] Any generalized cohomology theory h^* on the homotopy category of based CW-complexes is induced by an Ω-spectrum E with $h^n(X) \cong [X, E_n]$ if it satisfies the following:*

1. *Wedge Axiom: h^* maps coproducts to products: $h^*(\vee_\alpha X_\alpha) \cong \prod_\alpha h^*(X_\alpha)$*
2. *Mayer-Vietoris Axiom: h^* maps pushouts to weak pullbacks: if $u \in h^*(U), v \in h^*(V)$ restrict to the same class in $h^*(U \cap V)$, then there is a class $w \in h^*(U \cup V)$ that restricts to u, v respectively.*

Similar results hold for homology theories if we substitute the wedge axiom with:

$$\varinjlim_\alpha h_*(X_\alpha) \cong h_*(\varinjlim_\alpha X_\alpha),$$

and in that case $h_n(X) = \pi_n(X \wedge E)$.

Therefore the study of various generalized (co)homology theory is tranformed into the study of corresponding spectra. This leads us to define the E**-homology** and E**-cohomology** of a spectrum E to be

$$E^n(X) = [X, \Sigma^n E],$$

$$E_n(X) = [\Sigma^n \mathbb{S}, X \wedge E],$$

respectively. Especilly, the homotopy group of E is defined to be

$$\pi_n E = E_n(S^0) = \varinjlim_k \pi_{n+k} E_k.$$

Just like in the study of spaces, we would like the homology or homotopy groups we are interested in to have more algebraic structure than simply being an abelian group. The first attempt is to ask for a ring structure and this leads to the notion of ring spectra.

Definition 2.0.4. A spectrum E is a **ring spectrum** if it has two maps: a multiplication $\mu : E \wedge E \to E$ and a two-sided unit $\eta : S \to E$. It is commutative if moreover $\mu \simeq \mu \circ c$ where $c : E \wedge E \to E \wedge E$ switches the two factors.

E_* for a ring spectrum has a desired ring structure induced from μ:

$$E_* \otimes E_* \longrightarrow E_* E \xrightarrow{\mu_*} E_*.$$

The first map here is given by smashing two elements in E_* together.

2.1 Some Examples of Spectra

1. Take $E_n = S^n$ and $\sigma_n : \Sigma S^n \to S^{n+1}$ to be the canonical homeomorphism, we get the **sphere spectrum** $\mathbb{S}$. More generally, we may take any space X and construct its **suspension spectrum** $\Sigma^\infty X$ by $(\Sigma^\infty X)_n = \Sigma^n X$ with identity as structrual maps.

2. For any abelian group G, take $E_n = K(G, n)$ and $\sigma_n : \Sigma K(G, n) \to K(G, n+1)$ to be the adjoint of the homotopy equivalence $K(G, n) \simeq \Omega K(G, n+1)$ we get the **Eilenberg-Maclane spectrum** HG. Up to equivalence it is characterized by having homotopy groups $\pi_0 HG \cong G, \pi_{>0} HG \cong 0$.

3. (See Mil60) MU, the **complex cobordism spectrum**. Let ξ_n be the tautological bundle over $BU(n)$ and $Th(\xi_n)$ be its Thom space. The spectrum MU is defined by $MU_{2n} = Th(\xi_n), MU_{2n+1} = \Sigma MU_{2n}$. For the structrue maps, σ_{2n} is the identity, while $\sigma_{2n+1} : \Sigma^2 Th(\xi_n) \to Th(\xi_{n+1})$ is given by

$$Th(\xi_n) \wedge S^2 \cong Th(\xi_n \oplus \mathbf{1}) \to Th(\xi_{n+1}).$$

Here $\mathbf{1}$ is the trivial complex line bundle over $BU(n)$, and maps between Thom spaces are induced by the pullback square:

$$\begin{array}{ccc} \xi_n \oplus \mathbf{1} & \longrightarrow & \xi_{n+1} \\ \downarrow & & \downarrow \\ BU(n) & \longrightarrow & BU(n+1). \end{array}$$

4. Bousfield (in [Bou79]) defined the notion of localization with respect to any spectrum E. Denote by X_E the E–localization of X. Once we localize MU with respect to $H\mathbb{Z}_{(p)}$, Quillen proved in [Qui07] that there are non-identity idempotent endomorphisms on $MU_{(p)}$, and from that we get a wedge of same (up to suspension) irreducible summands of $MU_{(p)}$. These are called the $p-$**local Brown-Peterson spectrum** BP.[2] We have canonical maps from the splitting endomorphism:

$$BP \xrightarrow{s} MU_{(p)} \xrightarrow{r} BP.$$

For the spectra listed above, $\mathbb{S}$, HR (when R is a ring), MU and BP are all ring spectra. The multiplication on MU comes from maps $Th(\xi_n) \wedge Th(\xi_m) \to Th(\xi_{m+n})$ induced by the pullback square:

$$\begin{array}{ccc} \xi_n \times \xi_m & \longrightarrow & \xi_{n+m} \\ \downarrow & & \downarrow \\ BU(n) \times BU(m) & \longrightarrow & BU(n+m). \end{array}$$

$M\mathbb{Z}_{(p)}$ is also a ring spectrum. In fact, we can take $M\mathbb{Z}_{(p)}$ to be the p–local sphere $\mathbb{S}_{(p)}$ as described in [Sul70, Chapter II], and it inherits a ring structure from $\mathbb{S}$. The two ring spectra above combine to give a ring structure on $MU_{(p)} \cong MU \wedge H\mathbb{Z}_{(p)}$, and the ring structure on BP is induced from $MU_{(p)}$.

[2]We shall omit p hereafter when it is clear from the context.

Chapter 3

Operads and Structured Ring Spectra

The early version of handicrafted smash product as described in [Ada95] (that is also the category Sp we descrbed in chapter I) was found to be symmetric monoidal *only* after passing to homotopy category. In many cases a genuine symmetric monoidal product is preferred since it allows topologists to handle products on a space level. There have been several good categories of spectra equipped with such a smash product. For example, the category of symmetric spectra in [HSS00], orthogonal spectra in [MMSS98] and the S-algebra model in [EKMM]. Once we have a good symmetric monoidal category of spectra, it makes sense to think about commutative ring objects in this category. It turns out that after a zigzag of Quillen equivalences, commutative ring spectra are equivalent to E_∞ spectra introduced below. This extra multiplicative structure brings a lot of convenience to us, and the study of these objects are crucial in homotopy theory. Especially, the power operation we study is based on this.

3.1 Operads

We go through some basic knowledge of operads and define the $E_n(1 \le n \le \infty)$ operads in this section following [May72] and [May77] for later use of defining E_n ring spectra.

Definition 3.1.1. An **operad** $\mathcal{C}$ consists of spaces $\mathcal{C}(j)$, $j \in \mathbb{N}$, with $\mathcal{C}(0)$ a point $*$, together with the following data:

1. Continuous maps $\gamma : \mathcal{C}(k) \times \mathcal{C}(j_1) \ldots \times \mathcal{C}(j_k) \to \mathcal{C}(j), j = \sum_{s=1}^{k} j_s$, such that γ is associative. That is, for any $c \in \mathcal{C}(k), d_s \in \mathcal{C}(j_s), e_t \in \mathcal{C}(i_t)$,

$$\gamma(\gamma(c; d_1, \cdots, d_k); e_1, \cdots, e_j) = \gamma(c; f_1, \cdots, f_k),$$

where $f_s = \gamma(d_s; e_{j_1+\cdots+j_{s-1}+1}, \cdots, e_{j_1+\cdots+j_s})$, and $f_s = *$ if $j_s = 0$.

2. An identity element $1 \in \mathcal{C}(1)$ such that $\gamma(1; d) = d, \gamma(c; 1^k) = c$.

3. A right action of the symmetric group Σ_j on $\mathcal{C}(j)$ such that the following equivariant formulas are satisfied for any $c \in \mathcal{C}(k), d_s \in \mathcal{C}(j_s), \sigma \in \Sigma_k, \tau_s \in \Sigma_{j_s}$:

$$\gamma(c\sigma; d_1, \cdots, d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \cdots, d_{\sigma^{-1}(k)})\sigma(j_1, \cdots j_k)$$

$$\gamma(c; d_1\tau_1, \cdots, d_k\tau_k) = \gamma(c; d_1 \cdots, d_k)(\tau_1 \oplus \ldots \oplus \tau_k).$$

Here $\sigma(j_1, \cdots j_k) \in \Sigma_j$ is the permutation of j letters by permuting k blocks by σ, and $\tau_1 \oplus \ldots \oplus \tau_k$ is the image of $(\tau_1, \cdots, \tau_k)$ under the inclusion $\Sigma_{j_1} \times \ldots \times \Sigma_{j_k} \hookrightarrow \Sigma_j$.

A morphism between operads $\psi : \mathcal{C} \to \mathcal{C}'$ is a sequence of Σ_j–equivariant maps $\psi_j : \mathcal{C}(j) \to \mathcal{C}'(j)$ that preserves 1 and are compatible with γ. It is an equivalence if each ψ_j is a weak equivalence.

Definition 3.1.2. For any based space X its **endomorphism operad** $\mathcal{E}_X$ is the operad with $\mathcal{E}_X(j)$ the based maps $X^j \to X; X^0 = *$. The required data are given by:

1. $\gamma(f; g_1, \cdots, g_k) = f \circ (g_1 \times \ldots \times g_k)$.

2. $1 \in \mathcal{E}_X(1)$ is the identity.

3. $(f\sigma)(y) = f(\sigma y)$, where Σ_j acts on X^j by

$$\sigma(x_1, \cdots, x_j) = (x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(j)}).$$

There is a map $\mathcal{E}_X(j) \times X^j \mapsto X$ that applies a map in $\mathcal{E}_X(j)$ to X^j. An **operation** of an operad $\mathcal{C}$ on a space X is a morphism of operads $\theta : \mathcal{C} \to \mathcal{E}_X$. Such a pair (X, θ) is called a $\mathcal{C}$**−space**. A morphism of $\mathcal{C}$–spaces $f : (X, \theta) \to (X', \theta')$ is a based map $f : X \to X'$ such that $f \circ \theta_j(c) = \theta'_j(c) \circ f^j$ for all j's and $c \in \mathcal{C}(j)$.

3.1.1 E_∞ Operads

Definition 3.1.3. 1. Define $\mathcal{M}(j) = \Sigma_j$ and let e_j be the identity in Σ_j (the space $\mathcal{M}(0)$ has a single element e_0). Define $\gamma(e_k; e_{j_1}, \cdots, e_{j_k}) = e_j, j = \sum j_s$ and extend γ by Σ_k–equivariance, we get an operad $\mathcal{M}$ called the **associative operad**.

2. Define $\mathcal{N}(j) = \{f_j\}$, a single point with trivial Σ_j–action. Define $\gamma(f_k; f_{j_1}, \cdots, f_{j_k}) = f_j, j = \sum j_s$. This way we construct an operad $\mathcal{N}$ called the **commutative operad**.

An operad $\mathcal{C}$ is Σ–**free** if each Σ_j acts freely on $\mathcal{C}(j)$. A morphism $\psi : \mathcal{C} \to \mathcal{C}'$ is a **local equivalence** if each $\psi_j : \mathcal{C}(j) \to \mathcal{C}'(j)$ is a homotopy equivalence. It is a **local** Σ–**equivalence** if each ψ_j is further Σ_j–equivariant. An operad over a discrete operad $\mathcal{D}$ is an operad together with a morphism $\epsilon : \mathcal{C} \to \mathcal{D}$ such that $\pi_o\epsilon : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an isomorphism.

Definition 3.1.4. An A_∞ **operad** is a Σ–free operad over $\mathcal{M}$ such that $\epsilon : \mathcal{C} \to \mathcal{M}$ is a local Σ–equivalence[1]. An E_∞ **operad** is a Σ–free operad over $\mathcal{N}$ such that $\epsilon : \mathcal{C} \to \mathcal{N}$ is a local equivalence. An A_∞ **(resp.** E_∞**) space** is a $\mathcal{C}$–space over any $A_\infty(resp.E_\infty)$ operad $\mathcal{C}$.

Remark 3.1.5. Especially, this means for any E_∞ operad $\mathcal{C}$, $\mathcal{C}(j)$ is a model of $E\Sigma_j$.

3.1.2 E_n Operads

We first define the little cubes operad following [May72, Chapter IV]. E_n operads in general are operads equivalent to this.

Let I^n be the unit n–cube and $(I^n)^\circ$ its interior. A little n–cube is a rectilinear embedding f of $(I^n)^\circ$ into itself, i.e., $f = f_1 \times f_2 \ldots \times f_j$ where all f_i's are linear functions.

Definition 3.1.6. The **little** n**–cubes operad** $\mathcal{C}_n$ is defined as follows:

- $\mathcal{C}_n(j)$ is the j–tuple of pairwise disjoint little n–cubes $(c_1, \ldots c_j)$. $\mathcal{C}_n(0)$ is the unique map from the empty set to $(I^n)^\circ$.
- $\gamma(c; d_1, \cdots d_k) = c \circ (d_1 \sqcup \ldots \sqcup d_k)$ for $c \in \mathcal{C}_n(k), d_i \in \mathcal{C}_n(j_i)$.
- $1 \in \mathcal{C}_n(1)$ is the identity.
- $(c_1, \ldots c_j)\sigma = (c_{\sigma(1)}, \ldots c_{\sigma(j)})$ for $\sigma \in \Sigma_j$.

An E_n **operad** is an operad equivalent to $\mathcal{C}_n$.

The homotopy type of $\mathcal{C}_n(j)$'s has long been known.

Theorem 3.1.7. *[May72, Theorem 4.8]* $\mathcal{C}_n(j)$ *is* Σ_j*–equivariantly homotopy equivalent to* $\mathrm{Conf}(\mathbb{R}^n, j)$*, the configuration space of* j *points in* $\mathbb{R}^n$*.*

Remark 3.1.8. There is a natural map $\mathcal{C}_n \to \mathcal{C}_{n+1}$ that embeds an n–cube into the bottom of an $(n+1)$–cube. The colimit $\mathcal{C}_\infty = \varinjlim_n \mathcal{C}_n$ is an E_∞ operad.

On the space level, E_n operads $(1 \le n \le \infty)$ characterize infinite loop spaces by the following recognition theorem due to May:

Theorem 3.1.9. *(See [May72]) A connected space* X *has the weak homotopy type of an* n*–fold loop space if and only if it is an* E_n *space.*

[1] Actually it is enough to assume ϵ is just a local equivalence, see [May72, Chapter III].

3.2 E_∞ and E_n ring spectra

Roughly speaking, an E_∞ ring spectrum is a spectrum with an E_∞ operad action. Following [May77], we choose the linear isometry operad as our model of E_∞ operad. To introduce E_∞ spectra we need to pass from usual sequencial spectra to coordinate-free spectra, which are indexed by vector spaces instead of integers.

Let $\mathbf{I}$ be the category of finite or countably infinite dimensional real vector spaces and linear isometries[2] between them. Define the **linear isometries operad** $\mathfrak{L}$ by $\mathfrak{L}(j) = \mathbf{I}((\mathbb{R}^\infty)^j, \mathbb{R}^\infty)$ as a suboperad of the endomorphism operad of $\mathbb{R}^\infty$ with base point 0. $\mathfrak{L}$ is an E_∞ operad by [May77, Chapter I, Lemma 1.3]. Denote by $\mathbf{I}_*(\mathbb{R}^\infty)$ the full subcategory of $\mathbf{I}$ whose objects are finite dimensional subspaces of $\mathbb{R}^\infty$.

Let $\mathcal{CG}$ denote the category of compactly generated, non-degenerately based Hausdorff spaces, and $\mathcal{CG}_*$ denote the category of based spaces and based homeomorphisms. For any topological category $\mathfrak{A}$ we will write $h\mathfrak{A}$ for its homotopy category.

Let $t : \mathbf{I}_*(\mathbb{R}^\infty) \to Top$ be the functor that takes a vector space V to the one-point compactification S^V and takes an isometry $f : V \to W$ to the induced map $S^V \to S^W$. Again we take $0 \in V \subseteq tV$ as the base point and note that $tV \wedge tW = t(V \oplus W)$. Define

$$\Sigma^W X = X \wedge tW, \ \Omega^V(X) = F(tV, X).$$

where $F(M, N)$ is the space of based maps $M \to N$.

Definition 3.2.1. A **coordinate-free spectrum** is a functor $T : \mathbf{I}_*(\mathbb{R}^\infty) \to \mathcal{CG}_*$ which induces a functor $T : h\mathbf{I}_*(\mathbb{R}^\infty) \to h\mathcal{CG}$, together with based maps $\sigma : \Sigma^W TV \to T(W + V)$ for $V \perp W$ which satisfy the following:

1. Each adjoint $\tilde{\sigma} : TV \to \Omega^W T(V + W)$ is an inclusion with closed image.

2. The following diagrams commute in $\mathcal{CG}$ where all vector spaces involved are mutually perpendicular to each other:

$$\begin{array}{ccc} TV & \xrightarrow{\cong} & \Sigma^0 TV \\ & \searrow^{\cong} & \downarrow{\scriptstyle \sigma} \\ & & T(V+0) \end{array}$$

$$\begin{array}{ccc} \Sigma^Z \Sigma^W TV & \xrightarrow{\cong} & \Sigma^{W+Z} TV \\ \downarrow{\scriptstyle \Sigma^Z \sigma} & & \downarrow{\scriptstyle \sigma} \\ \Sigma^Z T(V+W) & \to & T(V+W+Z). \end{array}$$

[2] Here an isometry means a distance preserving linear map, it has to be injective but not necessarily surjective.

3. The following diagram commutes in $h\mathcal{CG}$, where $f \in \mathbf{I}_*(V, V')$, $g \in \mathbf{I}_*(W, W')$, $V \perp W$, $V' \perp W'$:

$$\begin{array}{ccc} \Sigma^W TV & \xrightarrow{\sigma} & T(V+W) \\ \downarrow{\scriptstyle Tf\wedge tg} & & \downarrow{\scriptstyle T(f+g)} \\ \Sigma^{W'} TV' & \xrightarrow{\sigma} & T(V'+W'). \end{array}$$

One easily sees $TV \cong TW$ if $dim\ V = dim\ W$. The use of vector space indices is to coherently organize these isomorphisms such that T could be equipped with an action of $\mathfrak{L}$ as decribed below.

Given an E_∞ operad $\mathcal{G}$ and a map $\xi : \mathcal{G} \to \mathfrak{L}$. We will abuse our notation and use the same letter for an element in $\mathcal{G}$ and its image in $\mathfrak{L}$ under this map (which is a linear isometry).

Definition 3.2.2. A $\mathcal{G}-$**spectrum** is a spectrum E with maps

$$\xi_j(g) : EV_1 \wedge \cdots \wedge EV_j \to Eg(V_1 \oplus \cdots \oplus V_j)$$

for $j \geq 0, g \in \mathcal{G}(j)$, where $\xi_0(*)$ is the unit map $e : S^0 \to E_0$, and that the following conditions hold:

1. For $g \in \mathcal{G}(k), h_r \in \mathcal{G}(j_r)$ $(1 \leq r \leq k), j = j_1 + \cdots + j_r$, the following diagram commutes:

$$\begin{array}{ccc} EV_1 \wedge \cdots \wedge EV_j & \xrightarrow{\xi_j(\gamma(g,h_1,\cdots,h_k))} & E\gamma(g,h_1 \cdots h_k)(V_1 \oplus \cdots \oplus V_j) \\ \downarrow{\scriptstyle \xi_{j_1}(h_1)\wedge\cdots\wedge\xi_{j_k}(h_k)} & & \downarrow{\scriptstyle id} \\ EW_1 \wedge \cdots \wedge EW_k & \xrightarrow{\xi_k(g)} & Eg(W_1 \oplus \cdots \oplus W_k) \end{array}$$

where $W_r = h_r(V_{j_1+\cdots+j_{r-1}+1} \oplus \cdots \oplus V_{j_1+\cdots j_r})$.

2. $\xi_1(1) : EV \to EV$ is the identity.

3. For $g \in \mathcal{G}(j), \tau \in \Sigma_j$ the following diagram commutes:

$$\begin{array}{ccc} EV_1 \wedge \cdots \wedge EV_j & \xrightarrow{\xi_j(g\tau)} & Eg\tau(V_1 \oplus \cdots \oplus V_j) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle id} \\ EV_{\tau^{-1}(1)} \wedge \cdots \wedge EV_{\tau^{-1}(j)} & \xrightarrow{\xi_j(g)} & Eg(V_{\tau^{-1}(1)} \oplus \cdots \oplus V_{\tau^{-1}(j)}) \end{array}$$

4. For fixed V_i's and W, ξ_j is continuous in g as g ranges over the subspace of $\mathcal{G}(j)$ such that $g(V_1 \oplus \cdots \oplus V_j) = W$.

5. For $g \in \mathcal{G}(j), V_i \perp W_i$, the following diagram commutes:

$$\begin{array}{ccc}
EV_1 \wedge tW_1 \wedge \cdots \wedge EV_j \wedge tW_j & \xrightarrow{\sigma\wedge\cdots\wedge\sigma} & E(V_1+W_1)\wedge\cdots\wedge E(V_j+W_j) \\
\downarrow \cong & & \downarrow \xi_j(g) \\
EV_1 \wedge \cdots \wedge EV_j \wedge t(W_1 \oplus \cdots \oplus W_j) & & Eg((V_1+W_1)\oplus\cdots\oplus(V_j+W_j)) \\
\downarrow \xi_j(g)\wedge tg & & \| \\
Eg(V_1\oplus\cdots\oplus V_j)\wedge tg(W_1\oplus\cdots\oplus W_j) & \xrightarrow{\sigma} & E(g(V_1\oplus\cdots\oplus V_j)+g(W_1\oplus\cdots\oplus W_j)).
\end{array}$$

6. If $g \in \mathcal{G}(1)$, then $\xi_1(g) : EV \to EgV$ is a homeomorphism in the homotopy class $E(g|_V)$, and every morphism in $\mathbf{I}_*(\mathbb{R}^\infty)$ is obtained from some $g \in \mathcal{G}(1)$.

An E_∞ **(ring) spectrum** is a $\mathcal{G}$–spectrum over any E_∞ operad $\mathcal{G}$ with a given morphism of operads $\mathcal{G} \to \mathfrak{L}$.

Therefore an E_∞ spectrum E is one with coherently defined products $\mathcal{G}(j) \ltimes_{\Sigma_j} (E^j) \to E$ for some E_∞ operad $\mathcal{G}$ and each $j \geq 1$. The contractibility of $\mathcal{G}(j)$ ensures that we have a coherent choice of homotopy H between $\mu\circ(\mu\wedge 1) \simeq \mu\circ(1\wedge\mu)$. Moreover, different chioces of H are themselves homotopic, and similarly for all higher homotopies. The Σ_j equivariance further ensures that these products are not only associative, but also commutative up to coherent choices of homotopies.

If we replace $\mathcal{G}$ with an E_n operad instead, we have the notion of E_n **(ring) spectrum**. These are intermediate stages between general ring spectra and E_∞ ones, for which we may to choose coherent homotopies up to the $(n-2)$th stage ([May72, Corollary 4.5]), but not necessarily any higher.

Remark 3.2.3. MU has a canonical E_∞ structure. For any complex vector space V with $dim_{\mathbb{C}} V = n$, write $U(V)$ for the corresponding unitary automorphism on V, $BU(V)$ for the classifying space for V–bundles, and $Th(V)$ for the Thom space. $\forall g \in \mathcal{G}(j)$ viewed as a linear isometry, let $\xi_j(g) = Th(g)$ be the induced isomorphism $Th(V_1)\wedge\cdots\wedge Th(V_j) \cong Th(g(V_1 \oplus \cdots \oplus V_j))$. We also have a canonical map $\sigma : S^V \to Th(V)$ by applying the Thom construction to the $V-$ bundle map $* \to BU(V)$. By [May77, Chapter IV, Lemma 2.2], these data are enough to define an E_∞ structure on MU. This argument works for many usual cobordism theories, for example MO, MSp and $MSpin$.

From now on, whenever we mention MU as an E_∞ ring spectra, it is equipped with this canonical E_∞ structure.

Chapter 4

H_∞ Structure and Power Operations

For a space X, The extended power construction D_jX is defined to be $E\Sigma_j \ltimes_{\Sigma_j} X^{\wedge j}$ where Σ_j acts on $E^{\wedge j}$ by permutation. We can also construct D_jE from a ring spectrum E with similar geometric properties. In this chapter we construct D_jE and define the associated power operation following [BMMS06].

4.1 Construction of D_jE

We briefly review the construction of D_jE for a spectrum E following [BMMS06, Chapter VII].

Recall from section 3.2 the linear isometry operad $\mathfrak{L}$. Each $\mathfrak{L}(j)$ is a model of $E\Sigma_j$. Fix a j and choose an increasing and exhaustive filtration $\{W_i\}$ of $\mathfrak{L}(j)$ by finite Σ_j–CW complexes. The compactness of W_i guarantees that for a finite dimensional subspace $V \subset (\mathbb{R}^\infty)^j$, $\bigcup\limits_{w\in W_i} w(V)$ is contained in a finite dimensional subspace of $\mathbb{R}^\infty$. Especially if we take $V = (\mathbb{R}^i)^j$, there is a finite dimensional subspace A_i such that $\forall w \in W_i, w((\mathbb{R}^i)^j) \subset A_i$. We may assume $\{A_i\}$ form an increasing sequence, and write B_i and B_i' for the orthogonal complement of $\mathbb{R}^i$ in $\mathbb{R}^{i+1}$ and A_i in A_{i+1}, respectively.

Consider the bundle map over W_i:

$$W_i \times (\mathbb{R}^i)^j \longrightarrow W_i \times A_i$$

$$(w, x_1 \cdots x_j) \longrightarrow (w, w(x_1 \oplus \cdots x_j))$$

It is an embedding of bundles. Let η_i be the bundle of the orthogonal complement and $T\eta_i$ be its Thom space. It becomes a Σ_j–map if we let Σ_j act on $(\mathbb{R}^i)^j$ by permutation and on A_i trivially, and let Σ_j act diagonally on $W_i \times (\mathbb{R}^i)^j$ and $W_i \times A_i$.

There is a commutative diagram of bundle embeddings:

$$\begin{array}{ccc} W_i \times (\mathbb{R}^i)^j & \longrightarrow & W_{i+1} \times (\mathbb{R}^i)^j \\ \downarrow & & \downarrow \\ W_i \times A_i & & W_{i+1} \times (\mathbb{R}^{i+1})^j \\ \downarrow & & \downarrow \\ W_i \times A_{i+1} & \longrightarrow & W_{i+1} \times A_{i+1} \end{array} .$$

Taking the orthogonal complement we have a bundle map $\eta_i \oplus B_i' \to \eta_{i+1} \oplus (B_i)^j$ over $W_i \hookrightarrow W_{i+1}$. The induced map of Thom spaces $T\eta_i \wedge S^{B_i'} \to T\eta_{i+1} \wedge (S^{B_i})^j$ is a Σ_j–map.

For a spectrum E, define a prespectrum $\tilde{D}$ by the following: let $a_i = dim A_i$ and $\tilde{D}_{a_i}$ be the space $T\eta_i \wedge_{\Sigma_j} (E_i)^j$ and the structural map σ_i be the composition:

$$\begin{array}{ccccc} \Sigma^{a_{i+1}-a_i} T\eta_i \wedge_{\Sigma_j} (E_i)^j & \xrightarrow{\cong} & (T\eta_i \wedge S^{B_i'}) \wedge_{\Sigma_j} (E_i)^j & \longrightarrow & (T\eta_{i+1} \wedge (S^{B_i})^j) \wedge_{\Sigma_j} (E_i)^j \\ \big\downarrow \sigma_i & & & & \big\downarrow \cong \\ T\eta_{i+1} \wedge_{\Sigma_j} (E_{i+1})^j & \longleftarrow & T\eta_{i+1} \wedge_{\Sigma_j} (\Sigma E_i)^j & \xrightarrow{\cong} & T\eta_{i+1} \wedge_{\Sigma_j} (S^{B_i} \wedge E_i)^j . \end{array}$$

Filling the kth space $\tilde{D}_k$ for $a_i < k < a_{i+1}$ with $\Sigma^{k-a_i} T\eta_i \wedge_{\Sigma_j} (E_i)^j$ and the structral maps with identities, we get a prespectrum $\tilde{D}$. The associated spectrum is defined to be $D_j E$.

4.2 Power Operations

Definition 4.2.1. A ring spectrum E is an H_∞ **ring spectrum** if there are maps $\xi_j : D_j E \to E$ for all integers j, where $\xi_1 = id$ and the following diagrams commute up to homotopy:

$$\begin{array}{ccccc} D_j E \wedge D_k E & & \xrightarrow{\alpha_{j,k}} & & D_{j+k} E \\ \big\downarrow \xi_j \wedge \xi_k & & & & \big\downarrow \xi_{j+k} \\ E \wedge E & \xrightarrow{\iota_2} & D_2 E & \xrightarrow{\xi_2} & E \end{array}$$

$$\begin{array}{ccc} D_j D_k E & \xrightarrow{\beta_{j,k}} & D_{jk} E \\ \big\downarrow D_j \xi_k & & \big\downarrow \xi_{jk} \\ D_j E & \xrightarrow{\xi_j} & E, \end{array}$$

where $\alpha_{j,k}$ is the natural map

$$(E\Sigma_j \times_{\Sigma_j} E^{\wedge j}) \wedge (E\Sigma_k \times_{\Sigma_k} E^{\wedge k}) \cong (E\Sigma_j \times E\Sigma_k) \times_{\Sigma_j \times \Sigma_k} (E^{\wedge j} \wedge E^{\wedge k}) \to E\Sigma_{j+k} \times_{\Sigma_{j+k}} E^{\wedge (j+k)},$$

ι_2 is the natural map

$$E \wedge E \longrightarrow E\Sigma_2 \times_{\Sigma_2} E^{\wedge 2}$$

that maps into the second factor, and $\beta_{j,k}$ is the natural map

$$E\Sigma_j \times_{\Sigma_j} (E\Sigma_k \times_{\Sigma_k} E^{\wedge k})^{\wedge j} \cong (E\Sigma_j \times (E\Sigma_k)^j) \times_{\Sigma_j \int \Sigma_k} (E^{\wedge k})^{\wedge j} \to E\Sigma_{jk} \times_{\Sigma_{jk}} E^{\wedge jk}.$$

Here $\Sigma_j \int \Sigma_k$ is the wreath product, namely $\Sigma_j \times (\Sigma_k)^j$ with product

$$(\sigma, \mu_1, \cdots, \mu_j)(\tau, \nu_1, \cdots, \nu_j) = (\sigma\tau, \mu_{\tau(1)}\nu_1, \cdots, \mu_{\tau(j)}\nu_j).$$

This structure gives rise to power operations $\mathcal{P}_j : E^0X \to E^0D_jX$ as follows:

$$(\; X \xrightarrow{f} E \;) \mapsto (\; D_jX \xrightarrow{D_jf} D_jE \xrightarrow{\xi_j} E \;).$$

However, this only defines a power operation on E^0X. A moment's reflection about Steenrod operations would remind us power operations in nonzero degrees are more useful, and therefore we introduce the following:

Definition 4.2.2. An H^d_∞ **ring spectrum** is a ring spectrum E together with maps $\xi_{j,i} : D_j\Sigma^{di}E \to \Sigma^{dij}E$ for all integers i, j such that $\xi_{1,i} = id$ and the following diagrams commute:

$$\begin{array}{ccc} D_j\Sigma^{di}E \wedge D_k\Sigma^{di}E & \xrightarrow{\alpha_{j,k}} & D_{j+k}\Sigma^{di}E \\ \downarrow{\scriptstyle \xi_{j,i}\wedge\xi_{k,i}} & & \downarrow{\scriptstyle \xi_{j+k,i}} \\ \Sigma^{dij}E \wedge \Sigma^{dik}E & \xrightarrow{\phi} & \Sigma^{di(j+k)}E \end{array}$$

$$\begin{array}{ccc} D_jD_k\Sigma^{di}E & \xrightarrow{\beta_{j,k}} & D_{jk}\Sigma^{di}E \\ \downarrow{\scriptstyle D_j\xi_{k,i}} & & \downarrow{\scriptstyle \xi_{jk,i}} \\ D_j\Sigma^{dki}E & \xrightarrow{\xi_{j,ki}} & \Sigma^{djki}E \end{array}$$

$$\begin{array}{ccccc} D_j(\Sigma^{dh}E \wedge \Sigma^{di}E) & & \xrightarrow{\delta_j} & & D_j\Sigma^{dh}E \wedge D_j\Sigma^{di}E \\ \downarrow{\scriptstyle D_j\phi} & & & & \downarrow{\scriptstyle \xi_{j,h}\wedge\xi_{j,i}} \\ D_j\Sigma^{d(h+i)}E & \xrightarrow{\xi_{j,h+i}} & \Sigma^{dj(h+i)}E & \xleftarrow{\phi} & \Sigma^{djh}E \wedge \Sigma^{dji}E \end{array}$$

Here δ_j is the natural map

$$D_j(X \wedge Y) = E\Sigma_j \times_{\Sigma_j} (X \wedge Y)^{\wedge j} \to (E\Sigma_j \times_{\Sigma_j} X^{\wedge j}) \wedge (E\Sigma_j \times_{\Sigma_j} Y^{\wedge j}) = D_jX \wedge D_jY.$$

Taking $i = 0$ we see every H^d_∞ spectrum is H_∞. But just as expected, this time we have operations in nonzero degrees $\mathcal{P}_{i,j} : E^{di}(X) \to E^{dij}(D_jX)$, defined similarly to the H_∞ case.

H_∞ (resp.H^d_∞) maps between H_∞ (resp.H^d_∞) spectra are those respecting the ξ operations.

Remark 4.2.3. E_∞ spectra always have an induced H_∞ structure. In fact, for any E_∞ operad $\mathcal{G}, \mathcal{G}(j)$ is a model of $E\Sigma_j$. The $\xi_j(g)$ used in the definition of E_∞ operad gives the evaluation map $\xi_j : \mathcal{G}(j) \times_{\Sigma_j} E^{\wedge j} \simeq E\Sigma_j \times_{\Sigma_j} E^{\wedge j} \to E$. In particular, MU is an H_∞ spectrum.

Remark 4.2.4. MU admits an H_∞^2 structure. To see this, let V_k be the permutation representation of Σ_k on $\mathbb{C}^k$, then $E\Sigma_k \times_{\Sigma_k} (V_k \otimes \mathbb{C}^i) \to E\Sigma_k \times_{\Sigma_k} * = B\Sigma_k$ is a vector bundle over $B\Sigma_k$. Note that $D_k S^{2i} \cong Th(V_k \otimes \mathbb{C}^i)$.

For any complex oriented (see definition 5.1.1 below) cohomology theory E, we have a Thom isomorphism [May77, Chapter III]: $E^*(Th(V_k \otimes \mathbb{C}^i)) \cong E^*(\Sigma^{2ki} B\Sigma_k)$. Let $\mu_{i,k}$: $Th(V_k \otimes \mathbb{C}^i) \cong D_k S^{2i} \to \Sigma^{2ki} E$ be the map representing the Thom class. McClure shows in [BMMS06, Chapter III] that $\mu_{i,k}$ combined with the H_∞ map $\mu_k : D_k MU \to MU$ gives an H_∞^2 structure on MU. The structural map is given by:

$$D_k(\Sigma^{2i} MU) \longrightarrow D_k S^{2i} \wedge D_k MU \xrightarrow{\mu_{i,k} \wedge \mu_k} \Sigma^{2ki} MU \wedge MU \xrightarrow{\phi} \Sigma^{2ki} MU.$$

There is a parallel theory of the extended power $D_\pi E$ for any subgroup $\pi < \Sigma_j$ and power operations associated to that. Namely, we can define P_π as:

$$(\ X \xrightarrow{f} E\ ) \mapsto (\ D_\pi X \xrightarrow{D_\pi f} D_\pi E \longrightarrow D_j E \xrightarrow{\xi_j} E\ ).$$

Here the map in middle is induced from the inclusion $\pi \hookrightarrow \Sigma_j$.

For a space X, we have a diagonal map $B\pi \times X \xrightarrow{1 \times \Delta} E\pi \times X^j = D_\pi X$.

Upon applying $\Sigma_+^\infty D_\pi X \cong D_\pi(\Sigma_+^\infty X)$ and taking $\pi = C_p < \Sigma_p, E = MU, X = *$, these together define power operations $P : MU^* \to MU^*(BC_p)$. We will study this operation in chapter 6.

Chapter 5

Formal Group Laws

The theory of formal group laws plays an important role in our calculation. We review the necessary background here from [Rav03, Appendix A2][Ada95].

Definition 5.0.1. A (one-dimensional commutative) **formal group law** on a commutative unital ring R is a power series $F(x, y) \in R[\![x, y]\!]$ such that

1. $F(0, x) = F(x, 0) = x$,
2. $F(x, y) = F(y, x)$,
3. $F(x, F(y, z)) = F(F(x, y), z)$.

The reason for this terminology and the source of properties come from this: suppose we have a one-dimensional Lie group and $g : U \to \mathbb{R}$ is a coordinate chart on a neighborhood of the identity e such that $g(e) = 0$, then the multiplication on the Lie group can be expressed as a germ of real analytic function near 0 satisfying the properties above by the axioms of an (abelian) group.

With the properties in hand it is easy to show that $F(x, y) = x + y + O(2)$ and that there is a power series $i(x)$ such that $F(x, i(x)) = x$. We call $i(x)$ the **formal inverse** of F.

Definition 5.0.2. A **morphism** between two formal group laws F, G is a power series $f(x) \in R[\![x]\!]$ such that $f(0) = 0$, $f(F(x, y)) = G(f(x), f(y))$. It is an **isomorphism** if it is invertible under composition, and a **strict isomorphism** if further $f'(0) = 1$. A strict isomorphism from F to the additive formal group law $G(x, y) = x + y$ is called a **logarithm** of F.

Not all formal group laws admit a logarithm (i.e. are strictly isomorphic to the additive one). However the next theorem shows it is always the case when we are working over $\mathbb{Q}$–algebras.

Theorem 5.0.3 (See [Rav03] A2.1.6)**.** *Suppose F is a formal group law defined over a ring R. Then F admits a logarithm when condidered as a formal group law over $R \otimes \mathbb{Q}$.*

Proof. Consider

$$f(x) = \int_0^x \frac{dt}{F_y(t,0)}$$

where $F_y = \frac{\partial F}{\partial y}$. We show $f(x)$ is a logarithm of F by checking $w = f(F(x,y)) - f(x) - f(y) \equiv 0$. Differentiating $F(x, F(y,z)) = F(F(x,y), z)$ with respect to z and setting $z = 0$ we have

$$F_y(F(x,y),0) = F_y(x,y)F(y,0).$$

On the other hand, $w_y = \frac{\partial w}{\partial y} = f'(F(x,y))F_y(y,0) - f'(y)$, and $f'(y) = \frac{1}{F_y(y,0)}$ by its definition. Therefore

$$w_y = \frac{F_y(x,y)}{F_y(F(x,y),0)} - \frac{1}{F_y(y,0)} = 0.$$

Similarly $\frac{\partial w}{\partial x} = 0$ and therefore w is a constant. Taking $x = y = 0$ we know $w \equiv 0$. □

For any ring map $f : R \to S$ with a formal group law $F(x,y) = \Sigma a_{ij}x^i y^j$ defined over R, there is an induced formal group law f^*F defined on S as $f^*F(x,y) = \Sigma f(a_{ij})x^i y^j$.

Theorem 5.0.4 (Lazard[Laz55])**.** *There is a ring L (called the **Lazard ring**) and a formal group law F (called the **universal formal group law**) defined over it with the universal property that for any formal group law G on some ring R there is a ring homomorphism $\theta : L \to R$ such that $\theta^*F = G$.*

Actually we can take $L = \mathbb{Z}[a_{ij}]/I$ where I is the ideal determined purely by the defining properties in definition 5.0.1 (for example, (2) tells us $a_{ij} = a_{ji}$).

Definition 5.0.5. A formal group law over a torsion-free $\mathbb{Z}_{(p)}$–algebra is p–**typical** if its logarithm has the form $\sum_{i\geq 0} l_i x^{p^i}$ with $l_0 = 1$.

Theorem 5.0.6. *There is also a universal p-typical formal group law (universal among p-typical formal group laws) defined over $\mathbb{Z}_{(p)}[v_1, v_2, \cdots]$.*

5.1 Complex oriented theory

Now we connect formal group laws to topology and demonstrate that it naturally shows up in cohomology theories with an complex orientation.

Definition 5.1.1. For a commutative ring spectrum E, a **complex orientation** of E is an element $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ such that $i^*x = 1 \in \tilde{E}^2(\mathbb{C}P^1) \cong \tilde{E}^2(S^2) = \pi_0 E$. Here $i : \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$ is the natural inclusion. Such a spectrum E is said to be **complex oriented**.

Theorem 5.1.2. *If E is complex oriented, $E^*(\mathbb{C}P^\infty) \cong E^*[\![x]\!]$, $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[\![x_1, x_2]\!]$.*

Proof. We have a commutative diagram of Atiyah-Hirzebruch Spectral Sequences calculating $E^*(\mathbb{C}P^n)$ and $E^*(\mathbb{C}P^1)$:

$$\begin{array}{ccc} H^s(\mathbb{C}P^n, E^t) & \Longrightarrow & E^{s+t}(\mathbb{C}P^n) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ H^s(\mathbb{C}P^1, E^t) & \Longrightarrow & E^{s+t}(\mathbb{C}P^1). \end{array}$$

The orientation class $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ lifts to a class $\bar{x} \in H^s(\mathbb{C}P^\infty, E^t)$. We will abuse our notations to call its image in the $\mathbb{C}P^n$ spectral sequence x and $\bar{x}$ as well. The second spectral sequence is concentrated in columns $s = 0, 2$ and collapses, so all differentials vanish on the $s = 0$ column (i.e. commutes with E^t coefficients). Being the unit in $\pi_0 E$, $i^*\bar{x}$ must have bidegree (2,0) and can be taken as the generator of $\tilde{H}^2(\mathbb{C}P^1)$. Therefore $\bar{x}$ has to be the generator of $H^2(\mathbb{C}P^n, E^*)$ and a permanent cycle in the first spectral sequence. Extending this using the multiplicative structure, we know all differentials vanish in the first spectral sequence. The extension problem is solved by observing that each column in $H^s(\mathbb{C}P^n, E^*)$ is a free E^*–module, so there cannot be non-trivial extensions when restricted to $H^s(\mathbb{C}P^n, E^*)$ for each n and hence $E^*(\mathbb{C}P^n) \cong E^*[x]/(x^n)$. Finally, since $E^*(\mathbb{C}P^{n+1})$ maps surjectively onto $E^*(\mathbb{C}P^n)$ for each n, the sequence $\{E^*(\mathbb{C}P^n)\}$ satisfies Mittag-Leffler condition to give:

$$E^*(\mathbb{C}P^\infty) = E^*(\varprojlim_n \mathbb{C}P^n) \cong \varinjlim_n E^*(\mathbb{C}P^n) \cong \varinjlim_n E^*[x]/(x^n) = E^*[\![x]\!].$$

The proof for $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is similar. □

Let $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \mapsto \mathbb{C}P^\infty$ be the classifying map of the tensor product of two tautological line bundles $\xi_1 \otimes \xi_2$, then $m^*x = \mu(x_1, x_2) = \sum_{i,j} a_{ij} x_1^i x_2^j$ for some $a_{ij} \in \pi_* E$. It follows from properties of tensor product of bundles that μ is a formal group law on $\pi_* E$.

We have a canonical orientation on MU, namely $x^{MU} \in MU^2(\mathbb{C}P^\infty) = [\mathbb{C}P^\infty, Th(\xi_1)]$ given by the quotient map $\mathbb{C}P^\infty = BU(1) \simeq D(\xi_1) \to D(\xi_1)/S(\xi_1) = Th(\xi_1)$, which is actually a homotopy equivalence since the sphere bundle $S(\xi_1)$ is contractible. This determines a formal group law on $\pi_* MU$. Milnor and Novikov computed $\pi_* MU$ and Quillen discovered the miraculous relation between the complex cobordism and formal group laws:

Theorem 5.1.3 (Milnor[Mil60], Novikov[Nov67]). *$\pi_* MU \cong \mathbb{Z}[x_1, x_2 \cdots]$ where deg $x_i = 2i$.*

Theorem 5.1.4 (Quillen[Qui07]). *The classifying map of the formal group law on $\pi_* MU$ is an isomorphism $L \cong \pi_* MU$.*

There is a similar story for p–typical formal group laws and BP.

Theorem 5.1.5 (See [Haz78]). *Let $V = \mathbb{Z}_{(p)}[v_1, v_2 \cdots]$. There is a universal p–typical formal group law defined on V, in the sense that if we denote the $p-$typical formal group* *law on V by F, then for any $p-$typical formal group law G defined over a commutative $\mathbb{Z}_{(p)}-$algebra R, there is a unique ring homomorphism $\theta : V \to R$ such that $G = \theta^* F$.*

Theorem 5.1.6 (Brown&Peterson[BP65], Quillen[Qui71b])**.** *There is a canonical isomorphism $\pi_* BP \cong \mathbb{Z}_{(p)}[v_1, v_2 \cdots]$ where deg $v_i = 2(p^i - 1)$. Further, the canonical splitting $MU_{(p)} \to BP$ carries the p–localized universal formal group law to the universal p–typical formal group law.*

The universal p–typical formal group law does not admit a logarithm itself, but theorem 5.0.3 tells us it does admit one rationally. Different choices of v_i lead to different formulas calculating the coefficients l_i's in the p–typical logarithm. The one we use below is the **Hazewinkel generator** from [Haz78] where l_i's are determined by the formula

$$pl_n = \Sigma_{0 \le i < n} l_i v_{n-i}^{p^i}.$$

Chapter 6

Computing Power Operations on MU

Fix a prime number p. Recall at the end of chapter 4 we defined power operation $P : MU^* \to MU^*(BC_p)$. We will compute this operation explicitly in this chapter.

6.0.1 Notations

Throughout this chapter we will write $\mathrm{coeff}(f(x), x^n)$ for the coefficient of x^n in a power series $f(x)$. We will denote by $x +_F y$ the power series $F(x, y)$ defined by a formal group law F (to be made clear in the context). We will write $[p]_F(\alpha)$ for the p-fold sum $\alpha +_F \alpha ... +_F \alpha = F(\alpha, F(\alpha, ...F(\alpha, \alpha)))$ using the formal group law F, and $\langle p\rangle_F(\alpha)$ for $\frac{[p]_F(\alpha)}{\alpha}$.

6.1 Computations of $P([\mathbb{C}P^n])$'s

$MU^*(BC_p)$, the target of our power operation P, is known to be isomorphic to $MU^*[\![\alpha]\!]/[p]_F(\alpha)$ by the work of Landweber [Lan70], and has a canonical projection $q : MU^*[\![\alpha]\!]/[p]_F(\alpha) \to MU^*[\![\alpha]\!]/\langle p\rangle_F(\alpha)$.

The E_∞ structure on MU induces one on $F(\mathbb{C}P^\infty, MU)$ and hence there is a power operation (take $X = F(\mathbb{C}P^\infty, MU)$) $P_{\mathbb{C}P} : MU^*(\mathbb{C}P^\infty) \to MU^*(BC_p \times \mathbb{C}P^\infty)$. $MU^*(\mathbb{C}P^\infty) \cong MU^*[\![x]\!]$ for a tautological class x by theorem 5.1.2. Moreover, $MU^*(BC_p \times \mathbb{C}P^\infty) \cong MU^*[\![x, \alpha]\!]/[p]_F(\alpha)$ by [Lan76, 3.1]. The effect of $P_{\mathbb{C}P}$ on x was calculated by Quillen:

Theorem 6.1.1. *[Qui71a, 3.15]*

$$P_{\mathbb{C}P}(x) = \prod_{i=0}^{p-1} (x +_F [i]_F(\alpha)).$$

We can write $P_{\mathbb{C}P}(x) = x \sum_{i\geq 0} a_i x^i$ for some $a_i \in MU^*[\![\alpha]\!]/[p]_F(\alpha)$.

Recall that geometrically, MU_* is isomorphic to the complex cobordism ring via the Pontrjagin-Thom isomorphism, which classifies manifolds with stably complex tangent bundles.

Definition 6.1.2. Define $c_n = [\mathbb{C}P^n] \in MU_{2n}$, the canonically defined elements representing the bordism class of $\mathbb{C}P^n$ with its tangent bundle.

Based on the work of Quillen [Qui71a] and Novikov (see [Ada95, Theorem I.8.1]), Johnson and Noel calculated the operations q_*P in terms of c_n's.

Theorem 6.1.3. *[JN10, Theorem 5.19]*
$q_*\chi^{2n}P(c_n) = \chi^{2n+1}\sum_{k=0}^{n} c_{n-k}\cdot \mathrm{coeff}((\sum_{i\geq 0} a_i z^i)^{-(n+1)}, z^k)$,
where $\chi = \prod_{i=1}^{p-1}[i]_F(\alpha) \in MU^*[\![\alpha]\!]/[p]_F(\alpha)$.

6.2 Polynomial Generators of MU^*

The c_n's are shown to be rational generators of MU_* using the method of [Tho54], but not integral ones. Hazewinkel found a way to produce integral generators from those.

Theorem 6.2.1. *[Haz78, 34.4.3] Let* $l_i = \frac{c_i}{i+1}$ *and* p *be a prime, We can find a set of polynomial generators* $x_i \in MU_*$ *from the formula*

$$v(n)l_{n-1} = x_{n-1} + \sum_{d|n, d\neq 1,n} \frac{\mu(n,d)v(n)}{v(d)} l_{\frac{n}{d}-1} x_{d-1}^{\frac{n}{d}}. \tag{6.2.1}$$

Here

$$v(n) = \begin{cases} p & \text{if} \quad n = p^i, i \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

$$\mu(n,d) = \prod_{p|n} c(p,d),$$

where $c(p,d) = 1$ *if* $v(d) = 1$ *or* p*;* $c(p,d) \equiv 1 \pmod p$*,* $c(p,d) \equiv 0 \pmod{p'}$ *if* $v(d) = p' \neq p$*. These generators map to Hazewinkel generators of* BP_* *under the canonical map which sends* x_{p^i-1} *to* v_i *and* x_j*'s to 0 if* $j \neq p^i - 1$ *for any* i*.*

We list the first few x_i's obtained from the formula above as a corollary (for later use we express c_i's in terms of x_i's):

Corollary 6.2.2.

$$c_1 = x_1$$
$$c_2 = x_2$$
$$c_3 = 2x_3 + x_1^3$$
$$c_4 = x_4$$
$$c_5 = 6x_5 + 3x_2^2x_1 + 4x_2x_1^3$$
$$c_6 = x_6$$
$$c_7 = 4x_7 + 2x_3^2x_1 + 2x_3x_1^4 + x_1^7$$
$$\ldots$$

Both P and $P_{\mathbb{C}P}$ are multiplicative by [BMMS06, Chapter VIII, Proposition 1.4 (iii)] but neither is additive. However, by [JN10, Proposition 5.14] both q_*P and $q_*P_{\mathbb{C}P}$ are ring homomorphisms. This, combined with theorem 6.1.3, allows us to compute the efffect of power operation on x_i's.

Remark 6.2.3. In fact when $p = 2, P(x+y) = P(x)+P(y)+xy\langle 2\rangle_F(\alpha)$. One can see this from the fact that q_*P is a ring homomorphism and $P(x) \equiv x^2 \bmod \alpha$ by [BMMS06, Chapter VIII, Proposition 1.4 (i)]. We can deduce additive formulas for odd p's similarly using $P(x) \equiv x^p \bmod \alpha$ instead.

6.3 A Sample Calculation

We give a sample calculation of $P(x_3)$ (assuming the knowledge of $P(x_1)$, which is a lot easier) modulo α^9 for $p = 2$ here to illustrate the procedure since this is the first case where $c_n \neq x_n$. The machine computations are done by Maple.

The logarithm $l(x)$ of the universal formal group law on MU_* is well-known by the work of Mischenko:

Theorem 6.3.1. *[Nov67, Appendix 1] On MU_*, the universal formal group law has logarithm*

$$l(x) = \sum_{n\geq 1} \frac{c_{n-1}}{n} x^n.$$

We will denote the exponential (i.e. the inverse power series of $l(x)$) by $e(x)$, so by definition $e(l(x)) = 1$. Modulo α^9 we have:

$$
\begin{aligned}
e(\alpha) = {} & \alpha - \frac{c_1\alpha^2}{2} + \left(-\frac{c_2}{3} + \frac{{c_1}^2}{2}\right)\alpha^3 + \left(-\frac{c_3}{4} + \frac{5\,c_1c_2}{6} - \frac{5\,{c_1}^3}{8}\right)\alpha^4 \\
& + \left(-\frac{c_4}{5} + \frac{3\,c_1c_3}{4} - \frac{7\,{c_1}^2c_2}{4} + \frac{{c_2}^2}{3} + \frac{7\,{c_1}^4}{8}\right)\alpha^5 \\
& + \left(-\frac{c_5}{6} + \frac{7\,c_1c_4}{10} - \frac{7\,{c_1}^2c_3}{4} - \frac{14\,c_1{c_2}^2}{9} + \frac{7\,c_2c_3}{12} + \frac{7\,c_2{c_1}^3}{2} - \frac{21\,{c_1}^5}{16}\right)\alpha^6 \\
& + \left(-\frac{9\,{c_1}^2c_4}{5} + 5\,{c_1}^2{c_2}^2 + \frac{15\,{c_1}^3c_3}{4} - \frac{55\,{c_1}^4c_2}{8} + \frac{2\,c_1c_5}{3} + \frac{8\,c_2c_4}{15} - 3\,c_1c_2c_3 - \frac{c_6}{7}\right. \\
& \qquad \left. - \frac{4\,{c_2}^3}{9} + \frac{33\,{c_1}^6}{16} + \frac{{c_3}^2}{4}\right)\alpha^7 \\
& + \left(-\frac{15\,{c_1}^2c_5}{8} - \frac{45\,c_1{c_3}^2}{32} - \frac{5\,{c_2}^2c_3}{4} + \frac{55\,c_1{c_2}^3}{18} + \frac{33\,{c_1}^3c_4}{8} - \frac{495\,{c_1}^4c_3}{64} - \frac{55\,{c_1}^3{c_2}^2}{4}\right. \\
& \left. + \frac{429\,{c_1}^5c_2}{32} + \frac{9\,c_1c_6}{14} + \frac{c_2c_5}{2} + \frac{9\,c_3c_4}{20} - 3\,c_1c_2c_4 + \frac{165\,c_2{c_1}^2c_3}{16} - \frac{c_7}{8} - \frac{429\,{c_1}^7}{128}\right)\alpha^8.
\end{aligned}
$$

$$
\begin{aligned}
\langle 2\rangle_F(\alpha) & = e(2l(x)) \\
& = 2 - c_1\alpha + \left(2\,{c_1}^2 - 2\,c_2\right)\alpha^2 + \left(8\,c_1c_2 - \frac{9\,{c_1}^3}{2} - \frac{7\,c_3}{2}\right)\alpha^3 \\
& \quad + \left(11\,{c_1}^4 - 28\,{c_1}^2c_2 + 15\,c_1c_3 + 8\,{c_2}^2 - 6\,c_4\right)\alpha^4 \\
& \quad + \left(-\frac{111\,{c_1}^2c_3}{2} - 58\,c_1{c_2}^2 + \frac{283\,c_2{c_1}^3}{3} + 28\,c_1c_4 + 30\,c_2c_3 - \frac{31\,c_5}{3} - \frac{57\,{c_1}^5}{2}\right)\alpha^5 \\
& \quad + \left(-110\,{c_1}^2c_4 + 302\,{c_1}^2{c_2}^2 + 195\,{c_1}^3c_3 - \frac{941\,{c_1}^4c_2}{3} + \frac{158\,c_1c_5}{3} + 56\,c_2c_4\right. \\
& \qquad \left. - 229\,c_1c_2c_3 - 18\,c_6 - 40\,{c_2}^3 + 77\,{c_1}^6 + 28\,{c_3}^2\right)\alpha^6 \\
& \quad + \left(-\frac{659\,{c_1}^2c_5}{3} - \frac{1793\,c_1{c_3}^2}{8} - 236\,{c_2}^2c_3 + 428\,c_1{c_2}^3 + 404\,{c_1}^3c_4 - \frac{1339\,{c_1}^4c_3}{2}\right. \\
& \qquad - \frac{4114\,{c_1}^3{c_2}^2}{3} + \frac{6229\,{c_1}^5c_2}{6} + 100\,c_1c_6 + \frac{316\,c_2c_5}{3} + 104\,c_3c_4 - 452\,c_1c_2c_4 \\
& \qquad \left. + \frac{2479\,c_2{c_1}^2c_3}{2} - \frac{127\,c_7}{4} - \frac{1717\,{c_1}^7}{8}\right)\alpha^7 \\
& \quad + \left(\frac{383\,c_1c_7}{2} + 200\,c_2c_6 + \frac{584\,c_3c_5}{3} - \frac{2698\,c_1c_2c_5}{3} - 877\,c_1c_3c_4 + 2550\,c_2{c_1}^2c_4\right. \\
& \qquad + 2618\,c_1{c_2}^2c_3 - \frac{17399\,{c_1}^3c_2c_3}{3} - 442\,{c_1}^2c_6 - 464\,{c_2}^2c_4 - \frac{921\,c_2{c_3}^2}{2} \\
& \qquad + \frac{2530\,{c_1}^3c_5}{3} + \frac{5031\,{c_1}^2{c_3}^2}{4} - 1435\,{c_1}^4c_4 - 2980\,{c_1}^2{c_2}^3 + \frac{17350\,{c_1}^4{c_2}^2}{3} \\
& \qquad \left. + 2272\,{c_1}^5c_3 - \frac{20591\,{c_1}^6c_2}{6} - \frac{170\,c_8}{3} + \frac{680\,{c_2}^4}{3} + \frac{2451\,{c_1}^8}{4} + 96\,{c_4}^2\right)\alpha^8.
\end{aligned}
$$

When $p = 2$, $P_{\mathbb{C}P}(x) = x(x +_F \alpha)$, and it is easy to see $\chi = a_0 = \alpha$. By theorem 6.1.3,

$$
\begin{aligned}
a_0^6 P(c_3) & = a_0^7 \sum_{k=0}^{3} c_{3-k} \cdot \mathrm{coeff}((\sum_{i\geq 0} a_i z^i)^{-4}, z^k) \\
& = -4\,c_1{a_0}^2a_2 + 10\,c_1a_0{a_1}^2 - 4\,c_2{a_0}^2a_1 + c_3{a_0}^3 - 4\,{a_0}^2a_3 + 20\,a_0a_1a_2 - 20\,{a_1}^3 \quad mod(\langle 2\rangle_F(\alpha)).
\end{aligned}
$$

The a_i in theorem 6.1.3 is the coefficient of x^i in $x +_F \alpha$, and we have

$$\begin{aligned} x +_F \alpha &= e(l(x) + l(\alpha)) \\ &= \alpha + \sum_{k\geq 1} \frac{e^{(k)}(l(\alpha))}{k!} l(x)^k. \end{aligned}$$

Let $b_k = e^{(k)}(l(\alpha))$. Since $l(x) \equiv x \bmod x^2$, the coefficient of x^i only relies on those b_k's for $k \leq i$.

Differentiating $e^{(n)}(l(\alpha))$ with respect to α we have

$$e^{(n+1)}(l(\alpha))l'(\alpha) = [e^{(n)}(l(\alpha))]'_\alpha.$$

Especially when $n = 0$ we have $e'(l(\alpha)) = \frac{1}{l'(\alpha)}$. This allows us to recursively compute b_i's and hence a_i's without really computing $e^{(n)}(x)$. For example,

$$\begin{aligned} a_1 &= b_1 \\ &= \frac{1}{l'(\alpha)} \\ &= 1 - c_1\alpha + \left(c_1^2 - c_2\right)\alpha^2 + \left(-c_1^3 + 2\,c_1c_2 - c_3\right)\alpha^3 + \left(c_1^4 - 3\,c_1^2c_2 + 2\,c_1c_3 + c_2^2 - c_4\right)\alpha^4 \\ &\quad + \left(-c_1^5 + 4\,c_1^3c_2 - 3\,c_1^2c_3 - 3\,c_1c_2^2 + 2\,c_1c_4 + 2\,c_2c_3 - c_5\right)\alpha^5 + \left(c_1^6 - 5\,c_1^4c_2\right. \\ &\qquad \left. + 4\,c_1^3c_3 + 6\,c_1^2c_2^2 - 3\,c_1^2c_4 - 6\,c_1c_2c_3 - c_2^3 + 2\,c_1c_5 + 2\,c_2c_4 + c_3^2 - c_6\right)\alpha^6 \\ &\quad + (-c_1^7 + 6\,c_1^5c_2 - 5\,c_1^4c_3 - 10\,c_1^3c_2^2 + 4\,c_1^3c_4 + 12\,c_1^2c_2c_3 + 4\,c_1c_2^3 - 3\,c_1^2c_5 \\ &\qquad - 6\,c_1c_2c_4 - 3\,c_1c_3^2 - 3\,c_2^2c_3 + 2\,c_1c_6 + 2\,c_2c_5 + 2\,c_3c_4 - c_7)\alpha^7 \\ &\quad + (c_1^8 - 7\,c_1^6c_2 + 6\,c_1^5c_3 + 15\,c_1^4c_2^2 - 5\,c_1^4c_4 - 20\,c_1^3c_2c_3 - 10\,c_1^2c_2^3 + 4\,c_1^3c_5 \\ &\qquad + 12\,c_1^2c_2c_4 + 6\,c_1^2c_3^2 + 12\,c_1c_2^2c_3 + c_2^4 - 3\,c_1^2c_6 - 6\,c_1c_2c_5 - 6\,c_1c_3c_4 - 3\,c_2^2c_4 \\ &\qquad - 3\,c_2c_3^2 + 2\,c_1c_7 + 2\,c_2c_6 + 2\,c_3c_5 + c_4^2 - c_8)\alpha^8 \end{aligned}$$

$$\begin{aligned} b_2 &= b_1' \cdot b_1 \\ &= -c_1 + \left(3\,c_1^2 - 2\,c_2\right)\alpha + \left(-6\,c_1^3 + 9\,c_1c_2 - 3\,c_3\right)\alpha^2 \\ &\quad + \left(10\,c_1^4 - 24\,c_1^2c_2 + 12\,c_1c_3 + 6\,c_2^2 - 4\,c_4\right)\alpha^3 + \\ &\left(-15\,c_1^5 + 50\,c_1^3c_2 - 30\,c_1^2c_3 - 30\,c_1c_2^2 + 15\,c_1c_4 + 15\,c_2c_3 - 5\,c_5\right)\alpha^4 + \\ &\left(21\,c_1^6 - 90\,c_1^4c_2 + 60\,c_1^3c_3 + 90\,c_1^2c_2^2 - 36\,c_1^2c_4 - 72\,c_1c_2c_3 - 12\,c_2^3 + 18\,c_1c_5\right. \\ &\qquad \left. + 18\,c_2c_4 + 9\,c_3^2 - 6\,c_6\right)\alpha^5 + \\ &(-28\,c_1^7 + 147\,c_1^5c_2 - 105\,c_1^4c_3 - 210\,c_1^3c_2^2 + 70\,c_1^3c_4 + 210\,c_1^2c_2c_3 + 70\,c_1c_2^3 \\ &\qquad - 42\,c_1^2c_5 - \\ &84\,c_1c_2c_4 - 42\,c_1c_3^2 - 42\,c_2^2c_3 + 21\,c_1c_6 + 21\,c_2c_5 + 21\,c_3c_4 - 7\,c_7)\alpha^6 + \\ &(36\,c_1^8 - 224\,c_1^6c_2 + 168\,c_1^5c_3 + 420\,c_1^4c_2^2 - 120\,c_1^4c_4 - 480\,c_1^3c_2c_3 - 240\,c_1^2c_2^3 \\ &\qquad + 80\,c_1^3c_5 + 240\,c_1^2c_2c_4 + \\ &120\,c_1^2c_3^2 + 240\,c_1c_2^2c_3 + 20\,c_2^4 - 48\,c_1^2c_6 - 96\,c_1c_2c_5 - 96\,c_1c_3c_4 - 48\,c_2^2c_4 \\ &\qquad - 48\,c_2c_3^2 + 24\,c_1c_7 + 24\,c_2c_6 + \\ &24\,c_3c_5 + 12\,c_4^2 - 8\,c_8)\alpha^7 + \\ &(-45\,c_1^9 + 324\,c_1^7c_2 - 252\,c_1^6c_3 - 756\,c_1^5c_2^2 + 189\,c_1^5c_4 + 945\,c_1^4c_2c_3 + 630\,c_1^3c_2^3 \\ &\qquad - 135\,c_1^4c_5 - 540\,c_1^3c_2c_4 - \\ &270\,c_1^3c_3^2 - 810\,c_1^2c_2^2c_3 - 135\,c_1c_2^4 + 90\,c_1^3c_6 + 270\,c_1^2c_2c_5 + 270\,c_1^2c_3c_4 \\ &\qquad + 270\,c_1c_2^2c_4 + 270\,c_1c_2c_3^2 + \\ &90\,c_2^3c_3 - 54\,c_1^2c_7 - 108\,c_1c_2c_6 - 108\,c_1c_3c_5 - 54\,c_1c_4^2 - 54\,c_2^2c_5 - 108\,c_2c_3c_4 - 18\,c_3^3 \\ &\qquad + 27\,c_1c_8 + 27\,c_2c_7 + 27\,c_3c_6 + 27\,c_4c_5 - 9\,c_9)\alpha^8 \end{aligned}$$

$$
\begin{aligned}
a_2 &= b_2 \mathrm{coeff}(l(x)^2, x^2) + b_1 \mathrm{coeff}(l(x), x^2) \\
&= \left(c_1{}^2 - c_2\right)\alpha + \left(4\,c_1c_2 - \frac{5\,c_1{}^3}{2} - \frac{3\,c_3}{2}\right)\alpha^2 \\
&\quad + \left(-11\,c_1{}^2c_2 + \frac{11\,c_1c_3}{2} + \frac{9\,c_1{}^4}{2} + 3\,c_2{}^2 - 2\,c_4\right)\alpha^3 \\
&\quad + \left(\frac{15\,c_2c_3}{2} + \frac{47\,c_1{}^3c_2}{2} - \frac{29\,c_1c_2{}^2}{2} + 7\,c_1c_4 - 14\,c_1{}^2c_3 - 7\,c_1{}^5 - \frac{5\,c_5}{2}\right)\alpha^4 \\
&\quad + \Bigg(\frac{17\,c_1c_5}{2} - 43\,c_1{}^4c_2 + \frac{57\,c_1{}^3c_3}{2} + 9\,c_2c_4 - 17\,c_1{}^2c_4 + \frac{87\,c_1{}^2c_2{}^2}{2} - 35\,c_1c_2c_3 \\
&\qquad\qquad + 10\,c_1{}^6 - 6\,c_2{}^3 + \frac{9\,c_3{}^2}{2} - 3\,c_6\Bigg)\alpha^5 \\
&\quad + (-\frac{101\,c_1{}^4c_3}{2} + 10\,c_1c_6 - 102\,c_1{}^3c_2{}^2 + \frac{21\,c_2c_5}{2} + \frac{21\,c_3c_4}{2} - 21\,c_2{}^2c_3 + \frac{67\,c_1{}^3c_4}{2} \\
&\quad + \frac{69\,c_1c_2{}^3}{2} - \frac{41\,c_1c_3{}^2}{2} - 20\,c_1{}^2c_5 + 71\,c_1{}^5c_2 + 102\,c_1{}^2c_2c_3 - 41\,c_1c_2c_4 - \frac{27\,c_1{}^7}{2} - \frac{7\,c_7}{2})\alpha^6 \\
&\quad + (-24\,c_2{}^2c_4 + 12\,c_3c_5 + \frac{77\,c_1{}^3c_5}{2} + \frac{163\,c_1{}^5c_3}{2} - 58\,c_1{}^4c_4 + 12\,c_2c_6 - 24\,c_2c_3{}^2 \\
&\quad - 109\,c_1{}^6c_2 - 118\,c_1{}^2c_2{}^3 + 205\,c_1{}^4c_2{}^2 + \frac{23\,c_1c_7}{2} - 23\,c_1{}^2c_6 + \frac{117\,c_1{}^2c_3{}^2}{2} - 234\,c_1{}^3c_2c_3 \\
&\quad + 117\,c_1{}^2c_2c_4 + \frac{237\,c_1c_2{}^2c_3}{2} - 47\,c_1c_2c_5 - 47\,c_1c_3c_4 + 10\,c_2{}^4 + 6\,c_4{}^2 - 4\,c_8 + \frac{35\,c_1{}^8}{2})\alpha^7 \\
&\quad + (-132\,c_1{}^3c_3{}^2 + 45\,c_2{}^3c_3 + \frac{317\,c_1{}^7c_2}{2} - \frac{741\,c_1{}^5c_2{}^2}{2} + 310\,c_1{}^3c_2{}^3 - 67\,c_1c_2{}^4 \\
&\qquad - 53\,c_1c_2c_6 + 132\,c_1{}^2c_2c_5 - 53\,c_1c_3c_5 - 264\,c_1{}^3c_2c_4 + 132\,c_1{}^2c_3c_4 + \frac{267\,c_1c_2{}^2c_4}{2} \\
&\qquad - 54\,c_2c_3c_4 + \frac{925\,c_1{}^4c_2c_3}{2} - 399\,c_1{}^2c_2{}^2c_3 + \frac{267\,c_1c_2c_3{}^2}{2} + 13\,c_1c_8 - 26\,c_1{}^2c_7 \\
&\qquad + \frac{27\,c_2c_7}{2} + \frac{87\,c_1{}^3c_6}{2} + \frac{27\,c_3c_6}{2} - \frac{131\,c_1{}^4c_5}{2} - 27\,c_2{}^2c_5 + \frac{27\,c_4c_5}{2} + 92\,c_1{}^5c_4 \\
&\qquad - \frac{53\,c_1c_4{}^2}{2} - 123\,c_1{}^6c_3 - \frac{9\,c_9}{2} - 9\,c_3{}^3 - 22\,c_1{}^9)\alpha^8.
\end{aligned}
$$

So

$$\alpha^6 P(c_3) = -20 + 70\,c_1\alpha + \left(-\frac{356\,{c_1}^2}{3} + \frac{106\,c_2}{3}\right)\alpha^2 + \left(-\frac{458\,c_1c_2}{3} + 154\,{c_1}^3 + 33\,c_3\right)\alpha^3$$
$$+\left(\frac{928\,{c_1}^2c_2}{3} - 121\,c_1c_3 - 171\,{c_1}^4 - \frac{124\,{c_2}^2}{3} + 24\,c_4\right)\alpha^4$$
$$+(-438\,c_2{c_1}^3 + 225\,{c_1}^2c_3 + \frac{530\,c_1{c_2}^2}{3} - \frac{260\,c_1c_4}{3} - \frac{172\,c_2c_3}{3} + \frac{491\,{c_1}^5}{3} + \frac{50\,c_5}{3})\alpha^5$$
$$+(\frac{1357\,{c_1}^4c_2}{3} - 274\,{c_1}^3c_3 - 331\,{c_1}^2{c_2}^2 + 142\,{c_1}^2c_4 - 55\,c_1c_5 - \frac{88\,c_2c_4}{3} - 127\,{c_1}^6$$
$$+\frac{64\,{c_2}^3}{3} - 17\,{c_3}^2 + 10\,c_6 + \frac{623\,c_1c_2c_3}{3})\alpha^6 + (-250\,{c_1}^5c_2 + 181\,{c_1}^4c_3 + 253\,{c_1}^3{c_2}^2$$
$$-118\,{c_1}^3c_4 - 32\,c_1{c_2}^3 + \frac{197\,{c_1}^2c_5}{3} + 36\,c_1{c_3}^2 + \frac{8\,{c_2}^2c_3}{3} - 26\,c_1c_6 - 4\,c_2c_5$$
$$-8\,c_3c_4 + 57\,{c_1}^7 + 4\,c_7 - 226\,c_2{c_1}^2c_3 + \frac{194\,c_1c_2c_4}{3})\alpha^7 + (-\frac{847\,{c_1}^6c_2}{3} + 155\,{c_1}^5c_3$$
$$+518\,{c_1}^4{c_2}^2 - \frac{217\,{c_1}^4c_4}{3} - \frac{955\,{c_1}^2{c_2}^3}{3} + \frac{74\,{c_1}^3c_5}{3} + 90\,{c_1}^2{c_3}^2 - 4\,{c_1}^2c_6$$
$$-\frac{172\,{c_2}^2c_4}{3} - \frac{161\,c_2{c_3}^2}{3} + \frac{c_1c_7}{3} + \frac{56\,c_2c_6}{3} + \frac{46\,c_3c_5}{3} + \frac{148\,{c_1}^8}{3} + 32\,{c_2}^4 + 8\,{c_4}^2$$
$$-\frac{4\,c_8}{3} - 466\,{c_1}^3c_2c_3 + 186\,c_2{c_1}^2c_4 + 284\,c_1{c_2}^2c_3 - 65\,c_1c_2c_5 - 61\,c_1c_3c_4)\alpha^8.$$

The right hand side does not seem to be a multiple of α^6 at first glance, but we have the flexibility of adding multiples of $\langle 2\rangle_F(\alpha)$ to make it one. In fact (we converted c_i's into x_i's here using corollary 6.2.2),

$$P(c_3) = {x_1}^6 + \left({x_1}^4x_3 + x_1{x_3}^2\right)\alpha + \left({x_1}^8 + {x_1}^6x_2 + {x_1}^5x_3 + x_7x_1\right)\alpha^2$$

mod$(\langle 2\rangle_F(\alpha), \alpha^3)$.

Since P is a ring homomorphism modulo $\langle 2\rangle_F(\alpha)$, formulas in corollary 6.2.2 impliy $P(c_3) - P(x_1)^3 \equiv 2P(x_3)$. As 2 is not a torsion in $MU^*[\![\alpha]\!]/\langle 2\rangle_F(\alpha)$, this determines $P(x_3)$ uniquely. [1] $P(x_1) = P(c_1)$ can be computed using theorem 6.1.3 directly:

$$P(x_1) = {x_1}^2 + \alpha\, x_3 + \left({x_1}^4 + {x_1}^2x_2 + x_1x_3\right)\alpha^2$$

$$P(c_3) - P(x_1)^3 = \alpha\, x_1{x_3}^2 + \left(-{x_1}^5x_3 + {x_1}^2{x_3}^2 + x_7x_1\right)\alpha^2$$

The coefficients are not divisible by 2, but all of these are multiples of $x_1\alpha$. However, $x_1\alpha \equiv 2 + 2(x_1^2 - x_2)\alpha^2 \quad mod(\langle 2\rangle_F(\alpha), \alpha^3)$. Iteratively using this we found:

$$P(c_3) - P(x_1)^3 = 2\,{x_3}^2 + \left(2\,{x_1}^4x_3 + 2\,x_1{x_3}^2 + 2\,x_7\right)\alpha$$
$$+ \left(2\,{x_1}^8 + 2\,{x_1}^6x_2 + 2\,{x_1}^2{x_3}^2 + 2\,x_2{x_3}^2\right)\alpha^2$$

mod$(\langle 2\rangle_F(\alpha), \alpha^3)$. Since the relation above lowers the degree of α only by 1, the possible odd coefficients in α^3 could only affect α^2 and therefore

$$P(x_3) = {x_3}^2 + \left({x_1}^4x_3 + x_1{x_3}^2 + x_7\right)\alpha$$

mod $(\langle 2\rangle_F(\alpha), \alpha^2)$. Our $P(x_3)$ listed below are more accurate because actually we fed more accurate series (mod α^{13}) into the computer. In the appendix we attach our codes calculating $P(x_3)$ upto degree α^6.

[1] This technique is inspired by [Law18, Appendix A].

6.4 Results

We list our results here.

6.4.1 $p = 2$

$$
\begin{aligned}
P(x_1) = {} & {x_1}^2 + \alpha\, x_3 + \left({x_1}^4 + {x_1}^2 x_2 + x_1 x_3\right) \alpha^2 + \alpha^3 {x_1}^3 x_2 \\
& + \left({x_1}^4 x_2 + {x_1}^2 {x_2}^2 + x_1 x_2 x_3 + x_1 x_5\right) \alpha^4 \\
& + \left({x_1}^7 + {x_1}^3 {x_2}^2 + {x_1}^3 x_4 + {x_1}^2 x_2 x_3 + {x_2}^2 x_3 + x_7\right) \alpha^5 \\
& + \left({x_1}^8 + {x_1}^6 x_2 + {x_1}^4 {x_2}^2 + {x_1}^4 x_4 + {x_1}^2 {x_2}^3 + {x_1}^2 x_2 x_4 + {x_1}^2 x_6 + x_1 x_2 x_5 + x_1 x_3 x_4 \right. \\
& \left. + x_1 x_7 + x_3 x_5\right) \alpha^6 + \left({x_1}^6 x_3 + {x_1}^3 x_2 x_4 + {x_1}^2 x_3 x_4 + x_1 x_2 {x_3}^2 + {x_1}^2 x_7 + {x_3}^3\right) \alpha^7 \\
& + \left({x_1}^8 x_2 + {x_1}^5 x_2 x_3 + {x_1}^4 {x_2}^3 + {x_1}^5 x_5 + {x_1}^4 x_2 x_4 + {x_1}^3 {x_2}^2 x_3 + {x_1}^2 {x_2}^4 + {x_1}^3 x_7 \right. \\
& + {x_1}^2 x_2 x_6 - {x_1}^2 x_3 x_5 + {x_1}^2 {x_4}^2 + x_1 {x_2}^2 x_5 + x_1 {x_3}^3 + x_1 x_2 x_7 + x_1 x_3 x_6 + x_1 x_4 x_5 \\
& \left. + x_2 x_3 x_5 + {x_3}^2 x_4 + x_1 x_9\right) \alpha^8 + \left({x_1}^{11} + {x_1}^8 x_3 + {x_1}^7 x_4 + {x_1}^6 x_2 x_3 + {x_1}^4 {x_2}^2 x_3 \right. \\
& + {x_1}^4 x_2 x_5 + {x_1}^4 x_3 x_4 + {x_1}^2 {x_2}^3 x_3 + {x_1}^3 x_2 x_6 + {x_1}^3 {x_4}^2 + {x_1}^2 x_2 x_3 x_4 + x_1 {x_2}^2 {x_3}^2 \\
& \left. + {x_1}^2 x_2 x_7 + {x_1}^2 x_4 x_5 + x_1 {x_3}^2 x_4 + x_2 {x_3}^3 + x_1 x_3 x_7 + {x_2}^2 x_7 + x_3 {x_4}^2\right) \alpha^9 \\
& + \left({x_1}^{10} x_2 + {x_1}^9 x_3 + {x_1}^7 x_5 + {x_1}^6 x_6 + {x_1}^3 {x_2}^3 x_3 + {x_1}^5 x_7 + {x_1}^4 x_2 x_6 + {x_1}^4 {x_4}^2 \right. \\
& + {x_1}^2 {x_2}^3 x_4 + {x_1}^2 {x_2}^2 {x_3}^2 + {x_1}^3 x_2 x_7 + {x_1}^3 x_3 x_6 + x_1 {x_2}^3 x_5 + x_1 x_2 {x_3}^3 + {x_1}^3 x_9 \\
& + {x_1}^2 x_2 x_8 + {x_1}^2 x_4 x_6 + {x_1}^2 {x_5}^2 + x_2 {x_3}^2 x_4 + {x_3}^4 + {x_1}^2 x_{10} + x_1 x_2 x_9 + x_1 x_3 x_8 \\
& \left. + x_1 x_4 x_7 + x_1 x_5 x_6 + x_3 x_4 x_5 + x_1 x_{11} + x_3 x_9 + x_5 x_7\right) \alpha^{10} + O(\alpha^{11})
\end{aligned}
$$

$$
\begin{aligned}
P(x_2) = {} & {x_2}^2 + \left(x_1 {x_2}^2 + x_5\right) \alpha + \left({x_1}^6 + {x_1}^2 x_4 + x_2 x_4 + {x_3}^2 + x_6\right) \alpha^2 \\
& + \left({x_1}^7 + {x_1}^3 {x_2}^2 + {x_1}^3 x_4 + {x_2}^2 x_3 + x_2 x_5 + x_3 x_4 + x_7\right) \alpha^3 \\
& + \left({x_1}^6 x_2 + {x_1}^5 x_3 + {x_1}^4 {x_2}^2 + {x_1}^2 {x_2}^3 + x_1 {x_2}^2 x_3 + x_1 x_2 x_5 + x_2 {x_3}^2 + x_1 x_7 + x_3 x_5\right) \alpha^4 \\
& + \left({x_1}^7 x_2 + {x_1}^4 x_2 x_3 + {x_1}^3 {x_2}^3 + {x_1}^4 x_5 + {x_1}^2 {x_2}^2 x_3 + x_1 {x_2}^4 + {x_1}^3 x_6 + {x_1}^2 x_3 x_4 \right. \\
& + x_1 {x_2}^2 x_4 + x_1 x_2 {x_3}^2 + x_1 x_2 x_6 + x_1 x_3 x_5 + x_1 {x_4}^2 + x_2 x_3 x_4 + {x_3}^3 + x_1 x_8 \\
& \left. + x_2 x_7 + x_4 x_5 + x_9\right) \alpha^5 + \left({x_1}^8 x_2 + {x_1}^5 x_2 x_3 + {x_1}^4 x_2 x_4 + {x_1}^4 {x_3}^2 + {x_1}^4 x_6 \right. \\
& + {x_1}^3 x_2 x_5 + {x_1}^3 x_3 x_4 + {x_1}^2 {x_2}^2 x_4 + {x_1}^2 x_2 {x_3}^2 + x_1 {x_2}^3 x_3 + {x_1}^2 x_2 x_6 + x_1 {x_2}^2 x_5 \\
& \left. + x_1 {x_3}^3 + x_1 x_2 x_7 + x_1 x_3 x_6 + x_1 x_4 x_5 + x_2 x_3 x_5 + x_1 x_9 + x_2 x_8 + x_4 x_6 + x_{10}\right) \alpha^6 \\
& + \left({x_1}^9 x_2 + {x_1}^7 x_4 + {x_1}^6 x_2 x_3 + {x_1}^5 {x_2}^3 + {x_1}^6 x_5 + {x_1}^5 {x_3}^2 + {x_1}^3 {x_2}^4 + {x_1}^5 x_6 \right. \\
& + {x_1}^4 x_2 x_5 + {x_1}^4 x_3 x_4 + {x_1}^3 {x_2}^2 x_4 + {x_1}^4 x_7 + {x_1}^3 x_3 x_5 + {x_1}^2 x_2 x_3 x_4 + {x_1}^2 {x_3}^3 \\
& + x_1 {x_2}^2 {x_3}^2 + {x_2}^4 x_3 + {x_1}^3 x_8 + {x_1}^2 x_2 x_7 + {x_1}^2 x_3 x_6 + {x_1}^2 x_4 x_5 + x_1 {x_2}^2 x_6 + x_1 x_2 x_3 x_5 \\
& \left. + x_2 {x_3}^3 + {x_1}^2 x_9 + x_1 x_2 x_8 + x_1 x_4 x_6 + {x_2}^2 x_7 + {x_3}^2 x_5 + x_1 x_{10} + x_2 x_9 + x_5 x_6\right) \alpha^7 \\
& + \left({x_1}^9 x_3 + {x_1}^8 {x_2}^2 + {x_1}^7 x_2 x_3 + {x_1}^6 {x_2}^3 + {x_1}^5 {x_2}^2 x_3 + {x_1}^4 {x_2}^4 + {x_1}^6 x_6 - {x_1}^5 x_2 x_5 \right. \\
& + {x_1}^5 x_3 x_4 + {x_1}^4 {x_2}^2 x_4 + {x_1}^3 {x_2}^3 x_3 + {x_1}^5 x_7 + {x_1}^4 x_3 x_5 + {x_1}^4 {x_4}^2 + {x_1}^3 x_2 x_3 x_4 \\
& + {x_2}^6 + {x_1}^4 x_8 + {x_1}^3 x_2 x_7 + {x_1}^3 x_3 x_6 + {x_1}^3 x_4 x_5 + {x_1}^2 {x_3}^2 x_4 + x_1 {x_2}^3 x_5 + {x_2}^3 {x_3}^2 \\
& + {x_1}^2 {x_5}^2 + x_1 x_2 x_3 x_6 + x_1 x_2 x_4 x_5 + {x_2}^2 x_3 x_5 + x_2 {x_3}^2 x_4 + x_1 x_2 x_9 + x_1 x_4 x_7 \\
& \left. + x_1 x_5 x_6 + x_2 x_3 x_7 + x_3 x_4 x_5 + x_5 x_7 + {x_6}^2\right) \alpha^8 + O(\alpha^9)
\end{aligned}
$$

$$\begin{aligned}P(x_3) = {} & {x_3}^2 + \left({x_1}^4 x_3 + x_1 {x_3}^2 + x_7\right)\alpha + \left({x_1}^8 + {x_1}^6 x_2 + {x_1}^2 {x_3}^2 + x_2 {x_3}^2\right)\alpha^2 \\ & + \left({x_1}^9 + {x_1}^7 x_2 + x_3 {x_1}^6 + {x_1}^5 {x_2}^2 + {x_1}^5 x_4 + {x_1}^4 x_2 x_3 + {x_1}^4 x_5 + x_1 x_2 {x_3}^2\right)\alpha^3 \\ & + \left({x_1}^6 x_4 + {x_1}^5 x_2 x_3 + {x_1}^5 x_5 + {x_1}^2 x_2 {x_3}^2 + {x_1}^3 x_7 + {x_2}^2 {x_3}^2 + {x_3}^2 x_4\right)\alpha^4 \\ & + \left({x_1}^{11} + {x_1}^8 x_3 + {x_1}^7 {x_2}^2 + {x_1}^7 x_4 + {x_1}^6 x_2 x_3 + {x_1}^6 x_5 + {x_1}^5 x_2 x_4 + {x_1}^5 {x_3}^2 + {x_1}^3 {x_2}^4 \right. \\ & + {x_1}^5 x_6 + {x_1}^4 x_3 x_4 + {x_1}^3 {x_2}^2 x_4 + {x_1}^3 x_2 x_6 + {x_1}^3 x_3 x_5 + {x_1}^3 {x_4}^2 + {x_1}^2 {x_2}^2 x_5 \\ & \left. + {x_1}^2 {x_3}^3 + x_1 {x_2}^2 {x_3}^2 + {x_1}^3 x_8 + x_1 {x_3}^2 x_4 + x_2 {x_3}^3 + {x_1}^2 x_9 + x_1 {x_5}^2\right)\alpha^5 + O(\alpha^6)\end{aligned}$$

$$\begin{aligned}P(x_4) = {} & {x_4}^2 + \left({x_1}^5 x_4 + x_9\right)\alpha + \left({x_1}^8 x_2 + {x_1}^4 {x_2}^3 + {x_1}^4 x_2 x_4 + {x_1}^4 {x_3}^2 + {x_1}^2 {x_2}^4 + {x_1}^4 x_6 \right. \\ & \left. + {x_1}^2 x_2 x_6 + {x_2}^3 x_4 + {x_2}^2 {x_3}^2 + {x_1}^2 x_8 + {x_2}^2 x_6 + x_1 x_9 + x_2 x_8 + x_4 x_6 + {x_5}^2 + x_{10}\right)\alpha^2 \\ & + \left({x_1}^{11} + {x_1}^9 x_2 + {x_1}^8 x_3 + {x_1}^7 {x_2}^2 + {x_1}^7 x_4 + {x_1}^5 {x_2}^3 + {x_1}^6 x_5 + {x_1}^5 x_2 x_4 + {x_1}^5 {x_3}^2 \right. \\ & + {x_1}^4 {x_2}^2 x_3 + {x_1}^3 {x_2}^4 + {x_1}^4 x_2 x_5 + {x_1}^3 {x_2}^2 x_4 + x_1 {x_2}^5 + {x_1}^4 x_7 + {x_1}^3 x_2 x_6 \\ & + x_1 {x_2}^2 {x_3}^2 + {x_2}^4 x_3 + x_1 x_2 {x_4}^2 + x_1 {x_3}^2 x_4 + {x_1}^2 x_9 + x_1 x_2 x_8 + x_1 {x_5}^2 + {x_2}^2 x_7 \\ & \left. + x_2 x_3 x_6 + x_2 x_4 x_5 + x_3 {x_4}^2 + x_2 x_9 + x_3 x_8 + x_5 x_6 + x_{11}\right)\alpha^3 \\ & + \left({x_1}^9 x_3 + {x_1}^8 {x_2}^2 + {x_1}^8 x_4 + {x_1}^4 {x_2}^4 + {x_1}^5 x_3 x_4 + {x_1}^4 {x_2}^2 x_4 + {x_1}^3 {x_2}^3 x_3 + {x_1}^4 x_3 x_5 \right. \\ & + {x_1}^3 x_2 x_3 x_4 + {x_1}^3 {x_3}^3 + {x_1}^2 {x_2}^3 x_4 + {x_1}^2 {x_2}^2 {x_3}^2 + {x_1}^4 x_8 + {x_1}^3 x_3 x_6 + {x_1}^3 x_4 x_5 \\ & + {x_1}^2 {x_2}^2 x_6 + x_1 {x_2}^3 x_5 + x_1 {x_2}^2 x_3 x_4 + x_1 x_2 {x_3}^3 + {x_2}^4 x_4 + {x_2}^3 {x_3}^2 + {x_1}^3 x_9 + {x_1}^2 x_4 x_6 \\ & + {x_1}^2 {x_5}^2 + x_1 x_2 x_4 x_5 + x_1 {x_3}^2 x_5 + {x_2}^3 x_6 + {x_3}^4 + x_1 x_2 x_9 + x_1 x_3 x_8 + x_1 x_4 x_7 \\ & \left. + x_1 x_5 x_6 + {x_2}^2 x_8 + x_2 x_4 x_6 + x_2 {x_5}^2 + {x_4}^3 + x_2 x_{10} + x_4 x_8 + x_{12}\right)\alpha^4 + O(\alpha^5)\end{aligned}$$

$$P(x_5) = {x_5}^2 + \left({x_1}^2 {x_2}^2 x_5 + x_1 {x_5}^2 + x_{11}\right)\alpha + O(\alpha^2)$$

6.4.2 $p = 3$

$$\begin{aligned}P(x_1) = {} & {x_1}^3 + \left(-{x_1}^3 x_2 - x_1 {x_2}^2 + x_5\right)\alpha^2 + \left(-{x_1}^4 x_2 - {x_1}^2 {x_2}^2 + x_1 x_5\right)\alpha^3 + \left({x_1}^7 - {x_1}^5 x_2 \right. \\ & \left. + {x_1}^4 x_3 - {x_1}^3 {x_2}^2 - {x_1}^3 x_4 - x_1 {x_2}^3 + {x_1}^2 x_5 - x_1 {x_3}^2 + x_1 x_6 + x_2 x_5 - x_3 x_4 + x_7\right)\alpha^4 \\ & + \left(-{x_1}^8 + {x_1}^6 x_2 - {x_1}^5 x_3 - {x_1}^4 {x_2}^2 + {x_1}^4 x_4 - {x_1}^3 x_2 x_3 - {x_1}^2 {x_2}^3 - {x_1}^3 x_5 + {x_1}^2 {x_3}^2 \right. \\ & \left. - x_1 {x_2}^2 x_3 - {x_1}^2 x_6 + x_1 x_2 x_5 + x_1 x_3 x_4 - x_1 x_7 + x_3 x_5\right)\alpha^5 \\ & + \left(-{x_1}^9 - {x_1}^6 x_3 + {x_1}^5 x_4 - {x_1}^3 {x_2}^3 + {x_1}^3 x_2 x_4 - {x_1}^3 {x_3}^2 - {x_1}^2 {x_2}^2 x_3 - x_1 {x_2}^4 - {x_1}^3 x_6 \right. \\ & \left. + x_1 {x_2}^2 x_4 + {x_2}^3 x_3 + x_1 x_3 x_5 + x_2 x_3 x_4 + x_1 x_8 + x_2 x_7 - x_4 x_5\right)\alpha^6 + \left(-{x_1}^7 x_3 \right. \\ & - {x_1}^6 {x_2}^2 - {x_1}^4 {x_2}^3 + {x_1}^4 x_2 x_4 - {x_1}^4 {x_3}^2 + {x_1}^3 {x_2}^2 x_3 - {x_1}^3 x_2 x_5 + {x_1}^3 x_3 x_4 + {x_1}^2 {x_2}^2 x_4 \\ & \left. - x_1 {x_2}^3 x_3 + x_1 {x_2}^2 x_5 + x_1 {x_3}^3 - x_1 x_3 x_6 - x_1 x_4 x_5 + x_2 x_3 x_5 + {x_3}^2 x_4 - x_3 x_7 - {x_5}^2\right)\alpha^7\end{aligned}$$

$$P(x_2) = {x_2}^3 - \alpha\, x_1 {x_2}^3 - \alpha^2 x_8 + \left(-x_1 {x_2}^2 x_4 - {x_2}^3 x_3\right)\alpha^3$$

There is no theoretical restrictions to apply this method to larger primes, but the complexity increases drastically as p grows.

6.5 Application: E_n Maps from MU

Because of the important role MU plays in homotopy theory, there are many spectra which we would like to know if they are MU–rings; and if so, how homotopy commutative the ring structure could be. The most important remaining problem in this direction might be the E_n–MU algebra structure on BP and $BP\langle n\rangle$, which may have a huge impact to many other problems. For example, recently Jeremy Hahn and Dylan Wilson found in [HW20] an E_3–MU algebra structure on $BP\langle n\rangle$ for all n's, and used that to prove the Redshift Conjecture for the algebraic K theory $K(BP\langle n\rangle)$.

There have been many works related to this problem. Strickland showed in [Str99] that BP and all $BP\langle n\rangle$'s at $p=2$ admit an MU algebra structure, and interestingly neither Hazewinkel nor Araki generators could possibly give one. Basterra and Mandell showed in [BM10] that BP admits a unique E_4 structure and the splitting map $MU_{(p)}\to BP$ could be made E_4. In the other direction, Tyler Lawson (for $p=2$ in [Law18]) and Andrew Senger (for odd p's in [Sen17]) showed that $BP\langle n\rangle$'s are not $E_{2(p^2+2)}$ when $n\geq 4$.

Let $\mathcal{E}_n$ be any E_n operad and $p=2$. By [May72, Corollary 4.5], $\mathcal{E}_n(2)\simeq S^{n-1}$. Therefore for any $\mathcal{E}_n$ algebra we have power operations $P_n : E^{2*}(X)\to E^{4*}(RP^{n-1}\times X)$. If E is further an E_n–MU algebra with structrual map $f: MU\mapsto E$, then f should commute with P_n. When $E=MU, MU^{2*}(RP^{n-1})\cong MU^*[\![\alpha]\!]/([2]_F(\alpha),\alpha^{[\frac{n-1}{2}]+1})$, and the action of P_n is the same as P except that we need to truncate our series at $\alpha^{[\frac{n-1}{2}]+1}$.

As an example, if $p=2$ and E is an E_3–MU algebra form (see [HW20, Definition 2.0.1]) of BP or $BP\langle n\rangle$, then $f(x_1)=\delta_1 v_1$ for some $\delta_1\in\mathbb{Z}_{(2)}^\times$, and for degree reasons $f(x_2)=k_2v_1^2$ for some $k_2\in\mathbb{Z}_{(2)}$.

Proposition 6.5.1. *With the notations above, $f(x_5)\equiv v_1^5$ mod 2 if $k_2\equiv -1$ or 2 mod 4, and $f(x_5)\equiv 0$ mod 2 if $k_2\equiv -0$ or 1 mod 4.*

Proof. We can see $f(k_2x_1^2-\delta_1^2x_2)=0$. Since f is E_3, it is compatible with P mod α^2 and therefore $f(P(k_2x_1^2-\delta_1^2x_2))=P(f(k_2x_1^2-\delta_1^2x_2))=0$.

From our calculations in section 6.4.1, on $MU^*[\![\alpha]\!]/([2]_F(\alpha),\alpha^2)\cong MU^*[\![\alpha]\!]/(2\alpha,\alpha^2)$,

$$P(x_1)={x_1}^2+\alpha\, x_3$$
$$P(x_2)={x_2}^2+\left(x_1{x_2}^2+x_5\right)\alpha.$$

The formula in remark 6.2.3 can be written as $P(x+y)=P(x)+P(y)+(2+x_1\alpha)xy$ mod $(2\alpha,\alpha^2)$. Define $\bar{P}(x)\in MU^*/2$ by $P(x)=x^2+\bar{P}(x)\alpha$. One can check the additive formula above implies $\bar{P}(x+4y)=\bar{P}(x)$ for any x,y. Therefore to calculate $P(k_2x_1^2-\delta_1^2x_2)$ we may take $\delta_1,k_2\in\mathbb{Z}/4$. Further, being a unit, δ_1 can be taken as ±1. We can now compute $\bar{P}(k_2x_1^2-x_2)$ for different values of k_2 from the additive relation above and the fact that $P(x)$ is multiplicative:

$$\begin{aligned}
\bar{P}(-x_2) &= x_5;\\
\bar{P}(x_1^2 - x_2) &= x_5 + x_2x_1^3;\\
\bar{P}(2x_1^2 - x_2) &= x_5 + x_1^5;\\
\bar{P}(-x_1^2 - x_2) &= x_5 + x_2x_1^3 + x_1^5.
\end{aligned}$$

The restrictions on $f(x_5)$ then come from letting the image of $\bar{P}(k_2x_1^2 - x_2)$ vanish under f. □

Remark 6.5.2. This type of argument was used by Strickland in [Str99] to realize various MU_*–algebras as homotopy groups of MU–algebras. Our $\bar{P}$ is the lift of his $\bar{Q}$ onto MU_*.

Of course we can find more relations in this way if we look at $f(x_i)$ for more x_i's, or allow higher commutativity conditions by moding out some high degree of α.

Appendix A

Maple Codes for Computing $P(x_3)$ for $p = 2$

```
> c[0] := 1;
  Inv := proc(n)
  option remember;
  if (n = 0) then
  return(1);
  else
  return(expand(-add(Inv(k) * c[n-k], k = 0 ..n - 1)));
  fi;
  end:
```

$$c_0 := 1$$

```
> a1 := add(Inv(n)α^n, n = 0 ..13)      #a1 = 1/l'(x)
```

$$a1 := 1 - c_1\,\alpha + \left(c_1^2 - c_2\right)\alpha^2 + \left(-c_1^3 + 2\,c_1\,c_2 - c_3\right)\alpha^3 + \left(c_1^4 - 3\,c_1^2\,c_2 + 2\,c_1\,c_3 + c_2^2 - c_4\right)\alpha^4 + \left(-c_1^5 + 4\,c_1^3\,c_2 - 3\,c_1^2\,c_3 - 3\,c_1\,c_2^2 + 2\,c_1\,c_4 + 2\,c_2\,c_3 - c_5\right)\alpha^5 + \left(c_1^6 - 5\,c_1^4\,c_2 + 4\,c_1^3\,c_3 + 6\,c_1^2\,c_2^2 - 3\,c_1^2\,c_4 - 6\,c_1\,c_2\,c_3 - c_2^3 + 2\,c_1\,c_5 + 2\,c_2\,c_4 + c_3^2 - c_6\right)\alpha^6 + \left(-c_1^7 + 6\,c_1^5\,c_2 - 5\,c_1^4\,c_3 - 10\,c_1^3\,c_2^2 + 4\,c_1^3\,c_4 + 12\,c_1^2\,c_2\,c_3 + 4\,c_1\,c_2^3 - 3\,c_1^2\,c_5 - 6\,c_1\,c_2\,c_4 - 3\,c_1\,c_3^2 - 3\,c_2^2\,c_3 + 2\,c_1\,c_6 + 2\,c_2\,c_5 + 2\,c_3\,c_4 - c_7\right)\alpha^7 + \left(c_1^8 - 7\,c_1^6\,c_2 + 6\,c_1^5\,c_3 + 15\,c_1^4\,c_2^2 - 5\,c_1^4\,c_4 - 20\,c_1^3\,c_2\,c_3 - 10\,c_1^2\,c_2^3 + 4\,c_1^3\,c_5 + 12\,c_1^2\,c_2\,c_4 + 6\,c_1^2\,c_3^2 + 12\,c_1\,c_2^2\,c_3 + c_2^4 - 3\,c_1^2\,c_6 - 6\,c_1\,c_2\,c_5 - 6\,c_1\,c_3\,c_4 - 3\,c_2^2\,c_4 - 3\,c_2\,c_3^2 + 2\,c_1\,c_7 + 2\,c_2\,c_6 + 2\,c_3\,c_5 + c_4^2 - c_8\right)\alpha^8 + \left(-c_1^9 + 8\,c_1^7\,c_2 - 7\,c_1^6\,c_3 - 21\,c_1^5\,c_2^2 + 6\,c_1^5\,c_4 + 30\,c_1^4\,c_2\,c_3 + 20\,c_1^3\,c_2^3 - 5\,c_1^4\,c_5 - 20\,c_1^3\,c_2\,c_4 - 10\,c_1^3\,c_3^2 - 30\,c_1^2\,c_2^2\,c_3 - 5\,c_1\,c_2^4 + 4\,c_1^3\,c_6 + 12\,c_1^2\,c_2\,c_5 + 12\,c_1^2\,c_3\,c_4 + 12\,c_1\,c_2^2\,c_4 + 12\,c_1\,c_2\,c_3^2 + 4\,c_2^3\,c_3 - 3\,c_1^2\,c_7 - 6\,c_1\,c_2\,c_6 - 6\,c_1\,c_3\,c_5 - 3\,c_1\,c_4^2 - 3\,c_2^2\,c_5 - 6\,c_2\,c_3\,c_4 - c_3^3 + 2\,c_1\,c_8 + 2\,c_2\,c_7 + 2\,c_3\,c_6 + 2\,c_4\,c_5 - c_9\right)\alpha^9 + \left(c_1^{10} - 9\,c_1^8\,c_2 + 8\,c_1^7\,c_3 + 28\,c_1^6\,c_2^2 - 7\,c_1^6\,c_4 - 42\,c_1^5\,c_2\,c_3 - 35\,c_1^4\,c_2^3 + 6\,c_1^5\,c_5 + 30\,c_1^4\,c_2\,c_4 + 15\,c_1^4\,c_3^2 + 60\,c_1^3\,c_2^2\,c_3 + 15\,c_1^2\,c_2^4 - 5\,c_1^4\,c_6 - 20\,c_1^3\,c_2\,c_5 - 20\,c_1^3\,c_3\,c_4 - 30\,c_1^2\,c_2^2\,c_4 - 30\,c_1^2\,c_2\,c_3^2 - 20\,c_1\,c_2^3\,c_3 - c_2^5 + 4\,c_1^3\,c_7 + 12\,c_1^2\,c_2\,c_6 + 12\,c_1^2\,c_3\,c_5 + 6\,c_1^2\,c_4^2 + 12\,c_1\,c_2^2\,c_5 + 24\,c_1\,c_2\,c_3\,c_4 + 4\,c_1\,c_3^3 + 4\,c_2^3\,c_4 + 6\,c_2^2\,c_3^2 - 3\,c_1^2\,c_8 - 6\,c_1\,c_2\,c_7 - 6\,c_1\,c_3\,c_6 - 6\,c_1\,c_4\,c_5 - 3\,c_2^2\,c_6 - 6\,c_2\,c_3\,c_5 - 3\,c_2\,c_4^2 - 3\,c_3^2\,c_4 + 2\,c_1\,c_9 + 2\,c_2\,c_8 + 2\,c_3\,c_7 + 2\,c_4\,c_6 + c_5^2 - c_{10}\right)\alpha^{10} + \left(-c_1^{11} + 10\,c_1^9\,c_2 - 9\,c_1^8\,c_3 - 36\,c_1^7\,c_2^2 + 8\,c_1^7\,c_4 + 56\,c_1^6\,c_2\,c_3 + 56\,c_1^5\,c_2^3 - 7\,c_1^6\,c_5 - 42\,c_1^5\,c_2\,c_4 - 21\,c_1^5\,c_3^2 - 105\,c_1^4$$

$$c_2^2 c_3 - 35 c_1^3 c_2^4 + 6 c_1^5 c_6 + 30 c_1^4 c_2 c_5 + 30 c_1^4 c_3 c_4 + 60 c_1^3 c_2^2 c_4 + 60 c_1^3 c_2 c_3^2$$
$$+ 60 c_1^2 c_2^3 c_3 + 6 c_1 c_2^5 - 5 c_1^4 c_7 - 20 c_1^3 c_2 c_6 - 20 c_1^3 c_3 c_5 - 10 c_1^3 c_4^2 - 30 c_1^2$$
$$c_2^2 c_5 - 60 c_1^2 c_2 c_3 c_4 - 10 c_1^2 c_3^3 - 20 c_1 c_2^3 c_4 - 30 c_1 c_2^2 c_3^2 - 5 c_2^4 c_3 + 4 c_1^3 c_8$$
$$+ 12 c_1^2 c_2 c_7 + 12 c_1^2 c_3 c_6 + 12 c_1^2 c_4 c_5 + 12 c_1 c_2^2 c_6 + 24 c_1 c_2 c_3 c_5$$
$$+ 12 c_1 c_2 c_4^2 + 12 c_1 c_3^2 c_4 + 4 c_2^3 c_5 + 12 c_2^2 c_3 c_4 + 4 c_2 c_3^3 - 3 c_1^2 c_9$$
$$- 6 c_1 c_2 c_8 - 6 c_1 c_3 c_7 - 6 c_1 c_4 c_6 - 3 c_1 c_5^2 - 3 c_2^2 c_7 - 6 c_2 c_3 c_6$$
$$- 6 c_2 c_4 c_5 - 3 c_3^2 c_5 - 3 c_3 c_4^2 + 2 c_1 c_{10} + 2 c_2 c_9 + 2 c_3 c_8 + 2 c_4 c_7$$
$$+ 2 c_5 c_6 - c_{11} \big) \alpha^{11} + \big(c_1^{12} - 11 c_1^{10} c_2 + 10 c_1^9 c_3 + 45 c_1^8 c_2^2 - 9 c_1^8 c_4 - 72$$
$$c_1^7 c_2 c_3 - 84 c_1^6 c_2^3 + 8 c_1^7 c_5 + 56 c_1^6 c_2 c_4 + 28 c_1^6 c_3^2 + 168 c_1^5 c_2^2 c_3 + 70 c_1^4 c_2^4$$
$$- 7 c_1^6 c_6 - 42 c_1^5 c_2 c_5 - 42 c_1^5 c_3 c_4 - 105 c_1^4 c_2^2 c_4 - 105 c_1^4 c_2 c_3^2 - 140 c_1^3$$
$$c_2^3 c_3 - 21 c_1^2 c_2^5 + 6 c_1^5 c_7 + 30 c_1^4 c_2 c_6 + 30 c_1^4 c_3 c_5 + 15 c_1^4 c_4^2 + 60 c_1^3 c_2^2 c_5$$
$$+ 120 c_1^3 c_2 c_3 c_4 + 20 c_1^3 c_3^3 + 60 c_1^2 c_2^3 c_4 + 90 c_1^2 c_2^2 c_3^2 + 30 c_1 c_2^4 c_3 + c_2^6 - 5$$
$$c_1^4 c_8 - 20 c_1^3 c_2 c_7 - 20 c_1^3 c_3 c_6 - 20 c_1^3 c_4 c_5 - 30 c_1^2 c_2^2 c_6 - 60 c_1^2 c_2 c_3 c_5$$
$$- 30 c_1^2 c_2 c_4^2 - 30 c_1^2 c_3^2 c_4 - 20 c_1 c_2^3 c_5 - 60 c_1 c_2^2 c_3 c_4 - 20 c_1 c_2 c_3^3 - 5 c_2^4 c_4$$
$$- 10 c_2^3 c_3^2 + 4 c_1^3 c_9 + 12 c_1^2 c_2 c_8 + 12 c_1^2 c_3 c_7 + 12 c_1^2 c_4 c_6 + 6 c_1^2 c_5^2 + 12 c_1$$
$$c_2^2 c_7 + 24 c_1 c_2 c_3 c_6 + 24 c_1 c_2 c_4 c_5 + 12 c_1 c_3^2 c_5 + 12 c_1 c_3 c_4^2 + 4 c_2^3 c_6 + 12$$
$$c_2^2 c_3 c_5 + 6 c_2^2 c_4^2 + 12 c_2 c_3^2 c_4 + c_3^4 - 3 c_1^2 c_{10} - 6 c_1 c_2 c_9 - 6 c_1 c_3 c_8$$
$$- 6 c_1 c_4 c_7 - 6 c_1 c_5 c_6 - 3 c_2^2 c_8 - 6 c_2 c_3 c_7 - 6 c_2 c_4 c_6 - 3 c_2 c_5^2 - 3 c_3^2 c_6$$
$$- 6 c_3 c_4 c_5 - c_4^3 + 2 c_1 c_{11} + 2 c_2 c_{10} + 2 c_3 c_9 + 2 c_4 c_8 + 2 c_5 c_7 + c_6^2$$
$$- c_{12} \big) \alpha^{12} + \big(-c_1^{13} + 12 c_1^{11} c_2 - 11 c_1^{10} c_3 - 55 c_1^9 c_2^2 + 10 c_1^9 c_4 + 90 c_1^8 c_2 c_3$$
$$+ 120 c_1^7 c_2^3 - 9 c_1^8 c_5 - 72 c_1^7 c_2 c_4 - 36 c_1^7 c_3^2 - 252 c_1^6 c_2^2 c_3 - 126 c_1^5 c_2^4 + 8$$
$$c_1^7 c_6 + 56 c_1^6 c_2 c_5 + 56 c_1^6 c_3 c_4 + 168 c_1^5 c_2^2 c_4 + 168 c_1^5 c_2 c_3^2 + 280 c_1^4 c_2^3 c_3$$
$$+ 56 c_1^3 c_2^5 - 7 c_1^6 c_7 - 42 c_1^5 c_2 c_6 - 42 c_1^5 c_3 c_5 - 21 c_1^5 c_4^2 - 105 c_1^4 c_2^2 c_5$$
$$- 210 c_1^4 c_2 c_3 c_4 - 35 c_1^4 c_3^3 - 140 c_1^3 c_2^3 c_4 - 210 c_1^3 c_2^2 c_3^2 - 105 c_1^2 c_2^4 c_3$$
$$- 7 c_1 c_2^6 + 6 c_1^5 c_8 + 30 c_1^4 c_2 c_7 + 30 c_1^4 c_3 c_6 + 30 c_1^4 c_4 c_5 + 60 c_1^3 c_2^2 c_6$$
$$+ 120 c_1^3 c_2 c_3 c_5 + 60 c_1^3 c_2 c_4^2 + 60 c_1^3 c_3^2 c_4 + 60 c_1^2 c_2^3 c_5 + 180 c_1^2 c_2^2 c_3 c_4$$
$$+ 60 c_1^2 c_2 c_3^3 + 30 c_1 c_2^4 c_4 + 60 c_1 c_2^3 c_3^2 + 6 c_2^5 c_3 - 5 c_1^4 c_9 - 20 c_1^3 c_2 c_8 - 20$$

$c_1^3 c_3 c_7 - 20 c_1^3 c_4 c_6 - 10 c_1^3 c_5^2 - 30 c_1^2 c_2^2 c_7 - 60 c_1^2 c_2 c_3 c_6 - 60 c_1^2 c_2 c_4 c_5$
$- 30 c_1^2 c_3^2 c_5 - 30 c_1^2 c_3 c_4^2 - 20 c_1 c_2^3 c_6 - 60 c_1 c_2^2 c_3 c_5 - 30 c_1 c_2^2 c_4^2$
$- 60 c_1 c_2 c_3^2 c_4 - 5 c_1 c_3^4 - 5 c_2^4 c_5 - 20 c_2^3 c_3 c_4 - 10 c_2^2 c_3^3 + 4 c_1^3 c_{10} + 12$
$c_1^2 c_2 c_9 + 12 c_1^2 c_3 c_8 + 12 c_1^2 c_4 c_7 + 12 c_1^2 c_5 c_6 + 12 c_1 c_2^2 c_8 + 24 c_1 c_2 c_3 c_7$
$+ 24 c_1 c_2 c_4 c_6 + 12 c_1 c_2 c_5^2 + 12 c_1 c_3^2 c_6 + 24 c_1 c_3 c_4 c_5 + 4 c_1 c_4^3 + 4 c_2^3 c_7$
$+ 12 c_2^2 c_3 c_6 + 12 c_2^2 c_4 c_5 + 12 c_2 c_3^2 c_5 + 12 c_2 c_3 c_4^2 + 4 c_3^3 c_4 - 3 c_1^2 c_{11}$
$- 6 c_1 c_2 c_{10} - 6 c_1 c_3 c_9 - 6 c_1 c_4 c_8 - 6 c_1 c_5 c_7 - 3 c_1 c_6^2 - 3 c_2^2 c_9$
$- 6 c_2 c_3 c_8 - 6 c_2 c_4 c_7 - 6 c_2 c_5 c_6 - 3 c_3^2 c_7 - 6 c_3 c_4 c_6 - 3 c_3 c_5^2 - 3 c_4^2 c_5$
$+ 2 c_1 c_{12} + 2 c_2 c_{11} + 2 c_3 c_{10} + 2 c_4 c_9 + 2 c_5 c_8 + 2 c_6 c_7 - c_{13} \big) \alpha^{13}$

> $l := add\left(\frac{c[n-1]}{n}\alpha^n, n = 1..13\right)$; # *l=logarithm*

$$l := \alpha + \frac{1}{2} c_1 \alpha^2 + \frac{1}{3} c_2 \alpha^3 + \frac{1}{4} c_3 \alpha^4 + \frac{1}{5} c_4 \alpha^5 + \frac{1}{6} c_5 \alpha^6 + \frac{1}{7} c_6 \alpha^7 + \frac{1}{8} c_7 \alpha^8 + \frac{1}{9} c_8 \alpha^9 + \frac{1}{10} c_9 \alpha^{10} + \frac{1}{11} c_{10} \alpha^{11} + \frac{1}{12} c_{11} \alpha^{12} + \frac{1}{13} c_{12} \alpha^{13}$$

```
> ex := proc(n)
  option remember;
  if (n = 1) then
  return(1);
  else
  return(expand(-add(ex(k) * coeff(l^k, α^n), k = 1..n - 1)));
  fi;
  end:
>
```

> $e := (t) \rightarrow add(ex(i) t^i, i = 1..13)$; *#finding exponential from l*

$$e := t \mapsto add(ex(i) \cdot t^i, i = 1..13)$$

> $f := add(expand(coeff(e(2\,l), \alpha^i))\alpha^{i-1}, i = 1..13)$ *#f=[2](α)*

$$f := 2 - c_1 \alpha + (2 c_1^2 - 2 c_2) \alpha^2 + \left(-\frac{7}{2} c_3 + 8 c_1 c_2 - \frac{9}{2} c_1^3\right) \alpha^3 + \big(11 c_1^4 - 28 c_1^2 c_2 + 15 c_1 c_3 + 8 c_2^2 - 6 c_4\big) \alpha^4 + \left(-\frac{31}{3} c_5 - \frac{111}{2} c_1^2 c_3 - 58 c_1 c_2^2 + \frac{283}{3} c_2 c_1^3 - \frac{57}{2} c_1^5 + 28 c_1 c_4 + 30 c_2 c_3\right) \alpha^5 + \big(-229 c_1 c_2 c_3 - 18 c_6 - 110 c_1^2 c_4 + 302 c_1^2 c_2^2 + 195 c_1^3 c_3 - \frac{941}{3} c_1^4 c_2 - 40 c_2^3 + 77 c_1^6 + 28 c_3^2 + \frac{158}{3} c_1 c_5 + 56 c_2 c_4\big) \alpha^6 + \left(-452 c_1 c_2 c_4 + \frac{2479}{2} c_2 c_1^2 c_3 - \frac{127}{4} c_7\right.$$

$$
\begin{aligned}
&-\frac{659}{3}c_1^2c_5-\frac{1793}{8}c_1c_3^2-236c_2^2c_3+428c_1c_2^3+404c_1^3c_4-\frac{1339}{2}c_1^4c_3\\
&-\frac{4114}{3}c_1^3c_2^2+\frac{6229}{6}c_1^5c_2-\frac{1717}{8}c_1^7+100c_1c_6+\frac{316}{3}c_2c_5\\
&+104c_3c_4\Big)\alpha^7+\Big(-\frac{170}{3}c_8-\frac{2698}{3}c_1c_2c_5-877c_1c_3c_4+2550c_2c_1^2c_4\\
&+2618c_1c_2^2c_3-\frac{17399}{3}c_1^3c_2c_3+\frac{680}{3}c_2^4+\frac{2451}{4}c_1^8+\frac{2530}{3}c_1^3c_5\\
&+\frac{5031}{4}c_1^2c_3^2-1435c_1^4c_4-2980c_1^2c_2^3+\frac{17350}{3}c_1^4c_2^2+2272c_1^5c_3\\
&-\frac{20591}{6}c_1^6c_2-442c_1^2c_6-464c_2^2c_4-\frac{921}{2}c_2c_3^2+\frac{383}{2}c_1c_7+200c_2c_6\\
&+\frac{584}{3}c_3c_5+96c_4^2\Big)\alpha^8+\Big(-\frac{511}{5}c_9-1804c_1c_2c_6-\frac{10385}{6}c_1c_3c_5\\
&-1796c_2c_3c_4+\frac{15871}{3}c_2c_1^2c_5+5352c_1c_2^2c_4+\frac{21139}{4}c_2c_1c_3^2+5118\\
&c_1^2c_3c_4-\frac{36964}{3}c_1^3c_2c_4-18751c_1^2c_2^2c_3+\frac{75145}{3}c_1^4c_2c_3-\frac{14263}{8}c_1^9\\
&-\frac{595}{2}c_3^3+\frac{136163}{12}c_1^7c_2-\frac{3579}{4}c_1^2c_7-850c_1c_4^2-920c_2^2c_5+1838\\
&c_2^3c_3+1772c_1^3c_6-\frac{18601}{6}c_1^4c_5-\frac{48439}{8}c_1^3c_3^2-\frac{9664}{3}c_1c_2^4+17060c_1^3c_2^3\\
&+\frac{25011}{5}c_1^5c_4-7662c_1^6c_3-\frac{69904}{3}c_1^5c_2^2+\frac{1108}{3}c_1c_8+383c_2c_7\\
&+368c_3c_6+\frac{1072}{3}c_4c_5\Big)\alpha^9+\Big(-3534c_2c_3c_5+11058c_2c_1^2c_6\\
&+\frac{33124}{3}c_1c_2^2c_5+10511c_1^2c_3c_5-\frac{79238}{3}c_1^3c_2c_5-25409c_1^3c_3c_4\\
&-39524c_1^2c_2^2c_4-\frac{155547}{4}c_1^2c_2c_3^2-26832c_3c_1c_2^3+54602c_1^4c_2c_4\\
&+109820c_1^3c_2^2c_3-102892c_1^5c_2c_3-\frac{7285}{2}c_1c_2c_7-3447c_1c_3c_6\\
&-3326c_1c_4c_5-186c_{10}+\frac{21071}{4}c_1^{10}-\frac{4160}{3}c_2^5+5148c_1^2c_4^2-6739c_1^4c_6\\
&-1728c_3^2c_4+\frac{7051}{2}c_1c_3^3+\frac{84400}{3}c_1^2c_2^4+\frac{33323}{3}c_1^5c_5+\frac{107035}{4}c_1^4c_3^2\\
&-\frac{86197}{5}c_1^6c_4-\frac{261610}{3}c_1^4c_2^3+\frac{182091}{2}c_1^6c_2^2+\frac{51499}{2}c_1^7c_3\\
&-\frac{450593}{12}c_1^8c_2-\frac{5458}{3}c_1^2c_8-1840c_2^2c_6-1736c_2c_4^2+3736c_2^3c_4\\
&+\frac{11077}{2}c_2^2c_3^2+\frac{7481}{2}c_1^3c_7+21390c_2c_1c_3c_4+\frac{3582}{5}c_1c_9\\
&+\frac{2216}{3}c_2c_8+702c_3c_7+672c_4c_6+\frac{992}{3}c_5^2\Big)\alpha^{10}+\Big(-\frac{2047}{6}c_{11}
\end{aligned}
$$

$$
-\frac{252205}{16} c_1^{11}+\frac{248801}{2} c_2 c_1^9+\frac{131155}{3} c_2 c_1 c_3 c_5-162098 c_1^2 c_2 c_3 c_4
$$

$$
+413994 c_1^5 c_2^3+\frac{117967}{2} c_1^7 c_4-\frac{1381937}{16} c_1^8 c_3-\frac{3132497}{9} c_1^7 c_2^2-3711
$$

$$
c_1^2 c_9-\frac{58081}{18} c_1 c_5^2+\frac{176623}{24} c_2 c_3^3+7668 c_2^3 c_5-3706 c_2^2 c_7-\frac{425933}{16}
$$

$$
c_1^2 c_3^3-\frac{117645}{8} c_1^4 c_7-26326 c_1^3 c_4^2+\frac{23756}{3} c_1^3 c_8-\frac{43036}{3} c_3 c_2^4
$$

$$
-\frac{10132}{3} c_3^2 c_5-3320 c_3 c_4^2+\frac{73832}{3} c_1 c_2^5-\frac{583430}{3} c_1^3 c_2^4-39152 c_1^6 c_5
$$

$$
-\frac{1792377}{16} c_1^5 c_3^2+24818 c_1^5 c_6-\frac{3431885}{6} c_1^4 c_2^2 c_3+\frac{3268167}{8} c_1^6 c_2 c_3
$$

$$
-\frac{22172}{3} c_1 c_2 c_8-\frac{55329}{8} c_1 c_3 c_7-6572 c_1 c_4 c_6+22968 c_1 c_2^2 c_6
$$

$$
+22308 c_2^2 c_3 c_4+\frac{92959}{4} c_2 c_1^2 c_7+21420 c_2 c_1 c_4^2-7020 c_2 c_3 c_6
$$

$$
-6776 c_2 c_4 c_5-\frac{171100}{3} c_1^3 c_2 c_6-\frac{107707}{2} c_1^3 c_3 c_5-84134 c_1^2 c_2^2 c_5
$$

$$
+21774 c_1^2 c_3 c_6+20924 c_1^2 c_4 c_5+21216 c_1 c_3^2 c_4-82962 c_1 c_2^2 c_3^2
$$

$$
+237288 c_1^3 c_2^2 c_4+239520 c_1^2 c_2^3 c_3+\frac{1081660}{9} c_1^4 c_2 c_5+\frac{2793955}{12} c_1^3 c_2 c_3^2
$$

$$
-56160 c_1 c_2^3 c_4+\frac{230267}{2} c_1^4 c_3 c_4-\frac{1145354}{5} c_1^5 c_2 c_4+1396 c_1 c_{10}
$$

$$
+\frac{7164}{5} c_2 c_9+\frac{4048}{3} c_3 c_8+1276 c_4 c_7+\frac{3712}{3} c_5 c_6\Big) \alpha^{11}+\Big(-412645
$$

$$
c_1^{10} c_2+\frac{2315417}{8} c_1^9 c_3+1309137 c_1^8 c_2^2-\frac{4016901}{20} c_1^8 c_4-\frac{16794302}{9} c_1^6 c_2^3
$$

$$
+136495 c_1^7 c_5+\frac{3615353}{8} c_1^6 c_3^2+\frac{3504200}{3} c_1^4 c_2^4-89491 c_1^6 c_6
$$

$$
-\frac{776024}{3} c_1^2 c_2^5+\frac{222593}{4} c_1^5 c_7+122226 c_1^4 c_4^2+\frac{1302645}{8} c_1^3 c_3^3
$$

$$
-\frac{96463}{3} c_1^4 c_8-\frac{89504}{3} c_2^4 c_4-58816 c_2^3 c_3^2+\frac{83954}{5} c_1^3 c_9+21051 c_1^2 c_5^2
$$

$$
+15880 c_2^3 c_6+22248 c_2^2 c_4^2-7586 c_1^2 c_{10}-7504 c_2^2 c_8-\frac{59042}{9} c_2 c_5^2
$$

$$
-6668 c_3^2 c_6+\frac{8191}{3} c_1 c_{11}+2792 c_2 c_{10}+\frac{13048}{5} c_3 c_9+\frac{7328}{3} c_4 c_8
$$

$$
+\frac{7016}{3} c_5 c_7+\frac{381375}{8} c_1^{12}+8928 c_2^6+\frac{14567}{4} c_3^4-2112 c_4^3+1152 c_6^2
$$

$$
+\frac{9256569}{10} c_1^6 c_2 c_4-\frac{9497435}{6} c_1^7 c_2 c_3-515359 c_1^5 c_2 c_5-\frac{2460483}{5}
$$

$$
c_1^5 c_3 c_4+\frac{8275414}{3} c_1^5 c_2^2 c_3-1261494 c_1^4 c_2^2 c_4-\frac{2470345}{2} c_1^4 c_2 c_3^2
$$

$$- \frac{5058470}{3} c_1^3 c_2^3 c_3 + \frac{799480}{3} c_1^4 c_2 c_6 + \frac{751150}{3} c_1^4 c_3 c_5 + \frac{4663600}{9} c_1^3 c_2^2 c_5 + 513456 c_1^2 c_2^3 c_4 + 756465 c_1^2 c_2^2 c_3^2 + \frac{780080}{3} c_1 c_2^4 c_3 - \frac{742513}{6} c_1^3 c_2 c_7 - 115119 c_1^3 c_3 c_6 - 110306 c_1^3 c_4 c_5 - 180572 c_1^2 c_2^2 c_6 - 167004 c_1^2 c_2 c_4^2 - \frac{659265}{4} c_1^2 c_3^2 c_4 - \frac{356368}{3} c_1 c_2^3 c_5 - \frac{338992}{3} c_1 c_2 c_3^3 + \frac{147050}{3} c_1^2 c_2 c_8 + \frac{181605}{4} c_1^2 c_3 c_7 + 42936 c_1^2 c_4 c_6 + 48073 c_1 c_2^2 c_7 + 43006 c_1 c_3^2 c_5 + 42168 c_1 c_3 c_4^2 + \frac{136178}{3} c_2^2 c_3 c_5 + \frac{88203}{2} c_2 c_3^2 c_4 - \frac{75242}{5} c_1 c_2 c_9 - \frac{41867}{3} c_1 c_3 c_8 - \frac{26189}{2} c_1 c_4 c_7 - \frac{37966}{3} c_1 c_5 c_6 - \frac{28109}{2} c_2 c_3 c_7 - 13360 c_2 c_4 c_6 - \frac{38648}{3} c_3 c_4 c_5 - 630 c_{12} + \frac{2982674}{3} c_1^3 c_2 c_3 c_4 - \frac{1024465}{3} c_1^2 c_2 c_3 c_5 - 343736 c_1 c_2^2 c_3 c_4 + 90186 c_1 c_2 c_3 c_6 + 86708 c_1 c_2 c_4 c_5 \Big) \alpha^{12}$$

```
> Invz := proc(n)
  option remember;
  if (n = 0) then
  return(1/y[0]);
  else
  return(expand(-add(Invz(k)*y[n-k], k = 0..n-1)/y[0]));
  fi;
 end:
> z := add(Invz(n) α^n, n = 0..10);
  q := (t) → add(coeff(z^(t+1), α^i)·c[t-i], i = 1..t)·y[0]^(2t+1) + c[t]y[0]^t
     #q(t) is the sum on the right hand side of theorem 5.1.2
```

$$z := \frac{1}{y_0} - \frac{y_1 \alpha}{y_0^2} + \left(-\frac{y_2}{y_0^2} + \frac{y_1^2}{y_0^3} \right) \alpha^2 + \left(-\frac{y_3}{y_0^2} + \frac{2 y_1 y_2}{y_0^3} - \frac{y_1^3}{y_0^4} \right) \alpha^3 + \left(-\frac{y_4}{y_0^2} + \frac{2 y_1 y_3}{y_0^3} + \frac{y_2^2}{y_0^3} - \frac{3 y_2 y_1^2}{y_0^4} + \frac{y_1^4}{y_0^5} \right) \alpha^4 + \left(-\frac{y_5}{y_0^2} + \frac{2 y_1 y_4}{y_0^3} + \frac{2 y_3 y_2}{y_0^3} - \frac{3 y_3 y_1^2}{y_0^4} - \frac{3 y_1 y_2^2}{y_0^4} + \frac{4 y_2 y_1^3}{y_0^5} - \frac{y_1^5}{y_0^6} \right) \alpha^5 + \left(-\frac{y_6}{y_0^2} + \frac{2 y_1 y_5}{y_0^3} + \frac{2 y_4 y_2}{y_0^3}$$

$$-\frac{3\,y_4\,y_1^2}{y_0^4}+\frac{y_3^2}{y_0^3}-\frac{6\,y_3\,y_1\,y_2}{y_0^4}+\frac{4\,y_3\,y_1^3}{y_0^5}-\frac{y_2^3}{y_0^4}+\frac{6\,y_2^2\,y_1^2}{y_0^5}-\frac{5\,y_2\,y_1^4}{y_0^6}$$

$$+\frac{y_1^6}{y_0^7}\Bigg)\,\alpha^6+\Bigg(\frac{2\,y_1\,y_6}{y_0^3}+\frac{2\,y_5\,y_2}{y_0^3}-\frac{3\,y_5\,y_1^2}{y_0^4}+\frac{2\,y_4\,y_3}{y_0^3}+\frac{4\,y_4\,y_1^3}{y_0^5}$$

$$-\frac{3\,y_1\,y_3^2}{y_0^4}-\frac{3\,y_3\,y_2^2}{y_0^4}-\frac{5\,y_3\,y_1^4}{y_0^6}+\frac{4\,y_1\,y_2^3}{y_0^5}-\frac{10\,y_2^2\,y_1^3}{y_0^6}+\frac{6\,y_2\,y_1^5}{y_0^7}-\frac{y_7}{y_0^2}$$

$$-\frac{y_1^7}{y_0^8}-\frac{6\,y_4\,y_1\,y_2}{y_0^4}+\frac{12\,y_3\,y_2\,y_1^2}{y_0^5}\Bigg)\,\alpha^7+\Bigg(\frac{2\,y_5\,y_3}{y_0^3}+\frac{4\,y_5\,y_1^3}{y_0^5}-\frac{3\,y_4\,y_2^2}{y_0^4}$$

$$-\frac{5\,y_4\,y_1^4}{y_0^6}-\frac{3\,y_3^2\,y_2}{y_0^4}+\frac{6\,y_3^2\,y_1^2}{y_0^5}+\frac{6\,y_3\,y_1^5}{y_0^7}-\frac{10\,y_2^3\,y_1^2}{y_0^6}+\frac{15\,y_2^2\,y_1^4}{y_0^7}$$

$$-\frac{7\,y_2\,y_1^6}{y_0^8}+\frac{2\,y_1\,y_7}{y_0^3}+\frac{2\,y_6\,y_2}{y_0^3}-\frac{3\,y_6\,y_1^2}{y_0^4}+\frac{y_4^2}{y_0^3}+\frac{y_2^4}{y_0^5}+\frac{y_1^8}{y_0^9}-\frac{y_8}{y_0^2}$$

$$-\frac{6\,y_5\,y_1\,y_2}{y_0^4}-\frac{6\,y_4\,y_1\,y_3}{y_0^4}+\frac{12\,y_4\,y_2\,y_1^2}{y_0^5}+\frac{12\,y_3\,y_1\,y_2^2}{y_0^5}-\frac{20\,y_3\,y_2\,y_1^3}{y_0^6}\Bigg)\,\alpha^8$$

$$+\Bigg(\frac{2\,y_1\,y_8}{y_0^3}+\frac{2\,y_7\,y_2}{y_0^3}-\frac{3\,y_7\,y_1^2}{y_0^4}+\frac{2\,y_6\,y_3}{y_0^3}+\frac{4\,y_6\,y_1^3}{y_0^5}+\frac{2\,y_5\,y_4}{y_0^3}$$

$$-\frac{3\,y_5\,y_2^2}{y_0^4}-\frac{5\,y_5\,y_1^4}{y_0^6}-\frac{3\,y_1\,y_4^2}{y_0^4}+\frac{6\,y_4\,y_1^5}{y_0^7}-\frac{10\,y_3^2\,y_1^3}{y_0^6}+\frac{4\,y_3\,y_2^3}{y_0^5}$$

$$-\frac{7\,y_3\,y_1^6}{y_0^8}-\frac{5\,y_1\,y_2^4}{y_0^6}+\frac{20\,y_2^3\,y_1^3}{y_0^7}-\frac{21\,y_2^2\,y_1^5}{y_0^8}+\frac{8\,y_2\,y_1^7}{y_0^9}-\frac{y_9}{y_0^2}-\frac{y_3^3}{y_0^4}$$

$$-\frac{y_1^9}{y_0^{10}}-\frac{6\,y_6\,y_1\,y_2}{y_0^4}-\frac{6\,y_5\,y_1\,y_3}{y_0^4}+\frac{12\,y_5\,y_2\,y_1^2}{y_0^5}-\frac{6\,y_4\,y_3\,y_2}{y_0^4}+\frac{12\,y_4\,y_3\,y_1^2}{y_0^5}$$

$$+\frac{12\,y_4\,y_1\,y_2^2}{y_0^5}-\frac{20\,y_4\,y_2\,y_1^3}{y_0^6}+\frac{12\,y_3^2\,y_1\,y_2}{y_0^5}-\frac{30\,y_3\,y_2^2\,y_1^2}{y_0^6}+\frac{30\,y_3\,y_2\,y_1^4}{y_0^7}\Bigg)$$

$$\alpha^9+\Bigg(\frac{2\,y_1\,y_9}{y_0^3}+\frac{2\,y_8\,y_2}{y_0^3}-\frac{3\,y_8\,y_1^2}{y_0^4}+\frac{2\,y_7\,y_3}{y_0^3}+\frac{4\,y_7\,y_1^3}{y_0^5}+\frac{2\,y_6\,y_4}{y_0^3}$$

$$-\frac{3\,y_6\,y_2^2}{y_0^4}-\frac{5\,y_6\,y_1^4}{y_0^6}+\frac{6\,y_5\,y_1^5}{y_0^7}+\frac{24\,y_4\,y_3\,y_1\,y_2}{y_0^5}-\frac{3\,y_4^2\,y_2}{y_0^4}+\frac{6\,y_4^2\,y_1^2}{y_0^5}$$

$$- \frac{3\, y_4\, y_3^2}{y_0^4} + \frac{4\, y_4\, y_2^3}{y_0^5} - \frac{7\, y_4\, y_1^6}{y_0^8} + \frac{4\, y_1\, y_3^3}{y_0^5} + \frac{6\, y_3^2\, y_2^2}{y_0^5} + \frac{15\, y_3^2\, y_1^4}{y_0^7}$$

$$+ \frac{8\, y_3\, y_1^7}{y_0^9} + \frac{15\, y_2^4\, y_1^2}{y_0^7} - \frac{35\, y_2^3\, y_1^4}{y_0^8} + \frac{28\, y_2^2\, y_1^6}{y_0^9} - \frac{9\, y_2\, y_1^8}{y_0^{10}} - \frac{y_{10}}{y_0^2} + \frac{y_5^2}{y_0^3}$$

$$- \frac{y_2^5}{y_0^6} + \frac{y_1^{10}}{y_0^{11}} - \frac{20\, y_3\, y_1\, y_2^3}{y_0^6} + \frac{60\, y_3\, y_2^2\, y_1^3}{y_0^7} - \frac{42\, y_3\, y_2\, y_1^5}{y_0^8} - \frac{30\, y_3^2\, y_2\, y_1^2}{y_0^6}$$

$$- \frac{6\, y_7\, y_1\, y_2}{y_0^4} - \frac{6\, y_6\, y_1\, y_3}{y_0^4} + \frac{12\, y_6\, y_2\, y_1^2}{y_0^5} - \frac{6\, y_5\, y_1\, y_4}{y_0^4} - \frac{6\, y_5\, y_3\, y_2}{y_0^4}$$

$$+ \frac{12\, y_5\, y_3\, y_1^2}{y_0^5} + \frac{12\, y_5\, y_1\, y_2^2}{y_0^5} - \frac{20\, y_5\, y_2\, y_1^3}{y_0^6} - \frac{20\, y_4\, y_3\, y_1^3}{y_0^6} - \frac{30\, y_4\, y_2^2\, y_1^2}{y_0^6}$$

$$+ \left. \frac{30\, y_4\, y_2\, y_1^4}{y_0^7} \right) \alpha^{10}$$

$$q := t \mapsto add\left(coeff\left(z^{t+1}, \alpha^i\right) \cdot c_{t-i},\ i = 1 \,..\, t\right) \cdot y_0^{2 \cdot t + 1} + c_t \cdot y_0^t$$

```
b := proc(n)  # e^(n)(l)
option remember;
 if (n = 1) then
  return a1;
 else
 return (diff(b(n − 1), α) · a1) ;
 fi;
 end proc:

a := proc(n)  #a[n]
option remember;
 if (n = 1) then
 return a1;
 else
 return (add( b(k)/k! · coeff(l^k, α^n), k = 1 ...n)) ;
 fi;
 end proc;
```

```
a := proc(n)
    option remember;
    local k;
    if n = 1 then
        return a1
    else
        return add(b(k) * coeff(l^k, α^n) / factorial(k), k = 1 ..n)
```

```
    end if
end proc
```

```
> Trun := proc(m, n)  #truncate the series m(α) to degree n
  return ( collect( convert( series( expand(m), α, n ), polynom ), α) ) ;
  end proc:
> k := (n) → Trun(a(n), 13);
```

$$k := n \mapsto Trun(a(n), 13)$$

```
> cancel :=proc(s :: polynom, t :: polynom, n :: integer, m :: integer)
      local z := t;
      for i from 1 to n do
```

$$z := z - \left(\frac{coeff(z, \text{alpha}^i)}{2} \cdot s \cdot \text{alpha}^i \right);$$

```
  end do;
  return (Trun(z, m));
  end proc;
```

$c[1] := x[1];$

$c[2] := x[2];$

$c[3] := 2\,x[3] + c[1]^3;$

$c[4] := x[4];$

$c[5] := 6\,x_5 + 4\,c[2]c[1]^3 + 3\,c[1]c[2]^2;$

$c[6] := x[6];$

$c[7] := expand\left(4\,x_7 + 2\,x[3]^2 c[1] + c[3]c[1]^4\right);$

$c[8] := 3\,x[8] + c[2]^4;$

$c[9] := expand\left(10\,x[9] + 6\,c[4]c[1]^5 + 5\,c[4]^2 c[1]\right);$

$c[10] := x[10];$

$c[11] := expand\left(12\,x[11] + 4\,c[5]c[1]^6 + 3\,c[3]c[2]^4 + 8\,x[3]^3 c[2] + 6\,x[5]^2 c[1]\right);$

$c[12] := x[12];$

```
Warning, (in cancel) `i` is implicitly declared local
cancel := proc(s::polynom, t::polynom, n::integer, m::integer)
    local z, i;
    z := t;
    for i to n do z := z - 1/2 * coeff(z, α^i) * s * α^i end do;
    return Trun(z, m)
end proc
```

$$c_1 := x_1$$

$$c_2 := x_2$$

$$c_3 := x_1^3 + 2\,x_3$$

$$c_4 := x_4$$

$$c_5 := 4\,x_1^3\,x_2 + 3\,x_1\,x_2^2 + 6\,x_5$$

$$c_6 := x_6$$

$$c_7 := x_1^7 + 2\, x_1^4\, x_3 + 2\, x_1\, x_3^2 + 4\, x_7$$

$$c_8 := x_2^4 + 3\, x_8$$

$$c_9 := 6\, x_1^5\, x_4 + 5\, x_1\, x_4^2 + 10\, x_9$$

$$c_{10} := x_{10}$$

$$c_{11} := 16\, x_1^9\, x_2 + 12\, x_1^7\, x_2^2 + 24\, x_1^6\, x_5 + 3\, x_1^3\, x_2^4 + 6\, x_2^4\, x_3 + 8\, x_2\, x_3^3 + 6\, x_1\, x_5^2 + 12\, x_{11}$$

$$c_{12} := x_{12}$$

> $expand(q(3))$

$$x_1^3\, y_0^3 - 4\, y_0^2\, x_1\, y_2 + 10\, y_0\, x_1\, y_1^2 - 4\, y_0^2\, y_1\, x_2 + 2\, x_3\, y_0^3 - 4\, y_0^2\, y_3 + 20\, y_0\, y_1\, y_2 - 20\, y_1^3$$

> $p3 := Trun\left(x_1^3 \cdot \alpha^3 - 4\,\alpha^2 \cdot x_1 \cdot k(2) + 10\alpha \cdot x_1 \cdot k(1)^2 - 4 \cdot \alpha^2 \cdot k(1) \cdot x_2 + 2 \cdot x_3 \cdot \alpha^3 - 4 \cdot \alpha^2 \cdot k(3) + 20 \cdot \alpha \cdot k(1) \cdot k(2) - 20 \cdot k(1)^3, 13\right)$
> # raw $P(c\_3)\alpha^6$ **from** $q(3)$

$$p3 := -20 + 70\,\alpha\, x_1 + \left(-120\, x_1^2 + 36\, x_2\right)\alpha^2 + \left(195\, x_1^3 - 160\, x_1\, x_2 + 70\, x_3\right)\alpha^3 + \left(-324\, x_1^4 + 344\, x_1^2\, x_2 - 272\, x_1\, x_3 - 48\, x_2^2 + 28\, x_4\right)\alpha^4 + \left(486\, x_1^5 - 536\, x_1^3\, x_2 + 568\, x_1^2\, x_3 + 302\, x_1\, x_2^2 - 112\, x_1\, x_4 - 160\, x_2\, x_3 + 140\, x_5\right)\alpha^5 + \left(-680\, x_1^6 + 740\, x_1^4\, x_2 - 1024\, x_1^3\, x_3 - 856\, x_1^2\, x_2^2 + 232\, x_1^2\, x_4 + 752\, x_1\, x_2\, x_3 + 48\, x_2^3 - 560\, x_1\, x_5 - 64\, x_2\, x_4 - 136\, x_3^2 + 20\, x_6\right)\alpha^6 + \left(948\, x_1^7 - 1024\, x_1^5\, x_2 + 1824\, x_1^4\, x_3 + 1524\, x_1^3\, x_2^2 - 416\, x_1^3\, x_4 - 1840\, x_1^2\, x_2\, x_3 - 416\, x_1\, x_2^3 + 1160\, x_1^2\, x_5 + 304\, x_1\, x_2\, x_4 + 652\, x_1\, x_3^2 + 232\, x_2^2\, x_3 - 80\, x_1\, x_6 - 320\, x_2\, x_5 - 112\, x_3\, x_4 + 72\, x_7\right)\alpha^7 + \left(-1308\, x_1^8 + 1304\, x_1^6\, x_2 - 3024\, x_1^5\, x_3 - 2200\, x_1^4\, x_2^2 + 732\, x_1^4\, x_4 + 3312\, x_1^3\, x_2\, x_3 + 1496\, x_1^2\, x_2^3 - 2112\, x_1^3\, x_5 - 752\, x_1^2\, x_2\, x_4 - 1640\, x_1^2\, x_3^2 - 1536\, x_1\, x_2^2\, x_3 - 24\, x_2^4 + 168\, x_1^2\, x_6 + 1552\, x_1\, x_2\, x_5 + 512\, x_1\, x_3\, x_4 + 96\, x_2^2\, x_4 + 392\, x_2\, x_3^2 - 288\, x_1\, x_7 - 48\, x_2\, x_6 - 592\, x_3\, x_5 - 24\, x_4^2 + 52\, x_8\right)\alpha^8 + \left(1786\, x_1^9 - 1528\, x_1^7\, x_2 + 4768\, x_1^6\, x_3 + 3154\, x_1^5\, x_2^2 - 1108\, x_1^5\, x_4 - 5440\, x_1^4\, x_2\, x_3 - 3280\, x_1^3\, x_2^3 + 3860\, x_1^4\, x_5 + 1392\, x_1^3\, x_2\, x_4 + 3416\, x_1^3\, x_3^2 + 4968\, x_1^2\, x_2^2\, x_3 + 454\, x_1\, x_2^4 - 320\, x_1^3\, x_6 - 3984\, x_1^2\, x_2\, x_5 - 1280\, x_1^2\, x_3\, x_4 - 672\, x_1\, x_2^2\, x_4 - 2224\, x_1\, x_2\, x_3^2\right.$$

$$- 288\, x_2^3\, x_3 + 624\, x_1^2\, x_7 + 240\, x_1\, x_2\, x_6 + 2768\, x_1\, x_3\, x_5 + 202\, x_1\, x_4^2 + 536\, x_2^2\, x_5 + 352\, x_2\, x_3\, x_4 + 240\, x_3^3 - 208\, x_1\, x_8 - 192\, x_2\, x_7 - 96\, x_3\, x_6 - 272\, x_4\, x_5 + 180\, x_9\Big)\, \alpha^9 + \Big(-2576\, x_1^{10} + 1948\, x_1^8\, x_2 - 7888\, x_1^7\, x_3 - 4720\, x_1^6\, x_2^2 + 1592\, x_1^6\, x_4 + 9376\, x_1^5\, x_2\, x_3 + 5528\, x_1^4\, x_2^3 - 6880\, x_1^5\, x_5 - 2432\, x_1^4\, x_2\, x_4 - 7312\, x_1^4\, x_3^2 - 11088\, x_1^3\, x_2^2\, x_3 - 2544\, x_1^2\, x_2^4 + 628\, x_1^4\, x_6 + 7920\, x_1^3\, x_2\, x_5 + 2864\, x_1^3\, x_3\, x_4 + 2328\, x_1^2\, x_2^2\, x_4 + 7032\, x_1^2\, x_2\, x_3^2 + 3008\, x_1\, x_2^3\, x_3 - 12\, x_2^5 - 1280\, x_1^3\, x_7 - 656\, x_1^2\, x_2\, x_6 - 7344\, x_1^2\, x_3\, x_5 - 656\, x_1^2\, x_4^2 - 4032\, x_1\, x_2^2\, x_5 - 2032\, x_1\, x_2\, x_3\, x_4 - 1552\, x_1\, x_3^3 - 144\, x_2^3\, x_4 - 856\, x_2^2\, x_3^2 + 472\, x_1^2\, x_8 + 992\, x_1\, x_2\, x_7 + 464\, x_1\, x_3\, x_6 + 1312\, x_1\, x_4\, x_5 + 96\, x_2^2\, x_6 + 2192\, x_2\, x_3\, x_5 + 88\, x_2\, x_4^2 + 360\, x_3^2\, x_4 - 720\, x_1\, x_9 - 160\, x_2\, x_8 - 416\, x_3\, x_7 - 48\, x_4\, x_6 - 840\, x_5^2 + 20\, x_{10}\Big)\, \alpha^{10} + \Big(4168\, x_1^{11} - 2592\, x_1^9\, x_2 + 14408\, x_1^8\, x_3 + 8036\, x_1^7\, x_2^2 - 2680\, x_1^7\, x_4 - 17216\, x_1^6\, x_2\, x_3 - 9264\, x_1^5\, x_2^3 + 13352\, x_1^6\, x_5 + 4272\, x_1^5\, x_2\, x_4 + 16292\, x_1^5\, x_3^2 + 23144\, x_1^4\, x_2^2\, x_3 + 8380\, x_1^3\, x_2^4 - 1232\, x_1^5\, x_6 - 15712\, x_1^4\, x_2\, x_5 - 6800\, x_1^4\, x_3\, x_4 - 5760\, x_1^3\, x_2^2\, x_4 - 18048\, x_1^3\, x_2\, x_3^2 - 14304\, x_1^2\, x_2^3\, x_3 - 576\, x_1\, x_2^5 + 2744\, x_1^4\, x_7 + 1440\, x_1^3\, x_2\, x_6 + 18304\, x_1^3\, x_3\, x_5 + 1596\, x_1^3\, x_4^2 + 15192\, x_1^2\, x_2^2\, x_5 + 6896\, x_1^2\, x_2\, x_3\, x_4 + 5536\, x_1^2\, x_3^3 + 1664\, x_1\, x_2^3\, x_4 + 7656\, x_1\, x_2^2\, x_3^2 + 488\, x_2^4\, x_3 - 1056\, x_1^3\, x_8 - 2912\, x_1^2\, x_2\, x_7 - 1328\, x_1^2\, x_3\, x_6 - 3744\, x_1^2\, x_4\, x_5 - 768\, x_1\, x_2^2\, x_6 - 13536\, x_1\, x_2\, x_3\, x_5 - 864\, x_1\, x_2\, x_4^2 - 2272\, x_1\, x_3^2\, x_4 - 1024\, x_2^3\, x_5 - 976\, x_2^2\, x_3\, x_4 - 1104\, x_2\, x_3^3 + 1720\, x_1^2\, x_9 + 848\, x_1\, x_2\, x_8 + 2080\, x_1\, x_3\, x_7 + 240\, x_1\, x_4\, x_6 + 4340\, x_1\, x_5^2 + 464\, x_2^2\, x_7 + 432\, x_2\, x_3\, x_6 + 1216\, x_2\, x_4\, x_5 + 2480\, x_3^2\, x_5 + 200\, x_3\, x_4^2 - 80\, x_1\, x_{10} - 640\, x_2\, x_9 - 368\, x_3\, x_8 - 224\, x_4\, x_7 - 320\, x_5\, x_6 + 280\, x_{11}\Big)\, \alpha^{11} + \Big(-7680\, x_1^{12} + 4048\, x_1^{10}\, x_2 - 29312\, x_1^9\, x_3 - 16472\, x_1^8\, x_2^2 + 5124\, x_1^8\, x_4 + 35600\, x_1^7\, x_2\, x_3 + 18816\, x_1^6\, x_2^3 - 28752\, x_1^7\, x_5 - 7664\, x_1^6\, x_2\, x_4 - 38360\, x_1^6\, x_3^2 - 54512\, x_1^5\, x_2^2\, x_3 - 20828\, x_1^4\, x_2^4 + 2552\, x_1^6\, x_6 + 34704\, x_1^5\, x_2\, x_5 + 15920\, x_1^5\, x_3\, x_4 + 13688\, x_1^4\, x_2^2\, x_4 + 47616\, x_1^4\, x_2\, x_3^2 + 47136\, x_1^3\, x_2^3\, x_3 + 6040\, x_1^2\, x_2^5 - 5888\, x_1^5\, x_7 - 3232\, x_1^4\, x_2\, x_6 - 48656\, x_1^4\, x_3\, x_5 - 4008\, x_1^4\, x_4^2 - 41616\, x_1^3\, x_2^2\, x_5 - 19792\, x_1^3\, x_2\, x_3\, x_4 - 17328\, x_1^3\, x_3^3 - 8608\, x_1^2\, x_2^3\, x_4 - 36880\, x_1^2\, x_2^2\, x_3^2 - 8032\, x_1\, x_2^4\, x_3 + 80\, x_2^6 + 2452\, x_1^4\, x_8$$

$+ 7008\, x_1^3 x_2 x_7 + 3664\, x_1^3 x_3 x_6 + 10352\, x_1^3 x_4 x_5 + 3096\, x_1^2 x_2^2 x_6 + 50256\, x_1^2 x_2 x_3 x_5 + 3744\, x_1^2 x_2 x_4^2 + 8520\, x_1^2 x_3^2 x_4 + 12544\, x_1 x_2^3 x_5 + 9104\, x_1 x_2^2 x_3 x_4 + 10320\, x_1 x_2 x_3^3 + 236\, x_2^4 x_4 + 2392\, x_2^3 x_3^2 - 4160\, x_1^3 x_9 - 2640\, x_1^2 x_2 x_8 - 6368\, x_1^2 x_3 x_7 - 736\, x_1^2 x_4 x_6 - 13448\, x_1^2 x_5^2 - 3840\, x_1 x_2^2 x_7 - 2800\, x_1 x_2 x_3 x_6 - 7920\, x_1 x_2 x_4 x_5 - 16528\, x_1 x_3^2 x_5 - 2032\, x_1 x_3 x_4^2 - 224\, x_2^3 x_6 - 7536\, x_2^2 x_3 x_5 - 304\, x_2^2 x_4^2 - 2416\, x_2 x_3^2 x_4 - 800\, x_3^4 + 200\, x_1^2 x_{10} + 3440\, x_1 x_2 x_9 + 1888\, x_1 x_3 x_8 + 1152\, x_1 x_4 x_7 + 1648\, x_1 x_5 x_6 + 448\, x_2^2 x_8 + 2208\, x_2 x_3 x_7 + 256\, x_2 x_4 x_6 + 4488\, x_2 x_5^2 + 520\, x_3^2 x_6 + 2960\, x_3 x_4 x_5 + 40\, x_4^3 - 1120\, x_1 x_{11} - 80\, x_2 x_{10} - 1520\, x_3 x_9 - 208\, x_4 x_8 - 1568\, x_5 x_7 - 32\, x_6^2 + 28\, x_{12}\big)\, \alpha^{12}$

> $rp[3] := Trun\left(\frac{cancel(f, p3 + 10\, f, 5, 13)}{\alpha^6}, 7\right)$; *#cancel raw P(c_3)* **to** *degree* 6. *Here I used* $P(c\_3) = (c\_3)^2$ **mod** α.

$$rp_3 := 4921\, x_1^6 - 5174\, x_1^4 x_2 + 7424\, x_1^3 x_3 + 6210\, x_1^2 x_2^2 - 1720\, x_1^2 x_4 - 5424\, x_1 x_2 x_3 - 368\, x_2^3 + 4220\, x_1 x_5 + 512\, x_2 x_4 + 984\, x_3^2 - 160\, x_6 + \big(-20322\, x_1^7 + 22161\, x_1^5 x_2 - 36995\, x_1^4 x_3 - 31233\, x_1^3 x_2^2 + 8544\, x_1^3 x_4 + 35904\, x_1^2 x_2 x_3 + 7912\, x_1 x_2^3 - 23190\, x_1^2 x_5 - 5696\, x_1 x_2 x_4 - 11475\, x_1 x_3^2 - 4304\, x_2^2 x_3 + 1460\, x_1 x_6 + 6016\, x_2 x_5 + 1856\, x_3 x_4 - 1198\, x_7\big)\, \alpha + \big(79020\, x_1^8 - 93624\, x_1^6 x_2 + 166429\, x_1^5 x_3 + 138050\, x_1^4 x_2^2 - 39246\, x_1^4 x_4 - 190786\, x_1^3 x_2 x_3 - 77954\, x_1^2 x_2^3 + 109468\, x_1^3 x_5 + 38736\, x_1^2 x_2 x_4 + 76162\, x_1^2 x_3^2 + 73952\, x_1 x_2^2 x_3 + 1804\, x_2^4 - 7882\, x_1^2 x_6 - 72348\, x_1 x_2 x_5 - 20912\, x_1 x_3 x_4 - 4720\, x_2^2 x_4 - 17188\, x_2 x_3^2 + 11182\, x_1 x_7 + 2096\, x_2 x_6 + 21088\, x_3 x_5 + 840\, x_4^2 - 1648\, x_8\big)\, \alpha^2 + \big(-310324\, x_1^9 + 397215\, x_1^7 x_2 - 746459\, x_1^6 x_3 - 610127\, x_1^5 x_2^2 + 169040\, x_1^5 x_4 + 963908\, x_1^4 x_2 x_3 + 490168\, x_1^3 x_2^3 - 507954\, x_1^4 x_5 - 209236\, x_1^3 x_2 x_4 - 452785\, x_1^3 x_3^2 - 657411\, x_1^2 x_2^2 x_3 - 61302\, x_1 x_2^4 + 38020\, x_1^3 x_6 + 494848\, x_1^2 x_2 x_5 + 146188\, x_1^2 x_3 x_4 + 75376\, x_1 x_2^2 x_4 + 265858\, x_1 x_2 x_3^2 + 36824\, x_2^3 x_3 - 62591\, x_1^2 x_7 - 24708\, x_1 x_2 x_6 - 260434\, x_1 x_3 x_5 - 14844\, x_1 x_4^2 - 56312\, x_2^2 x_5 - 34784\, x_2 x_3 x_4 - 23560\, x_3^3 + 15972\, x_1 x_8 + 16144\, x_2 x_7 + 7264\, x_3 x_6 + 18736\, x_4 x_5 - 10040\, x_9\big)\, \alpha^3 + \big(1225462\, x_1^{10} - 1699897\, x_1^8 x_2 + 3318637$$

$$x_1^7 x_3 + 2709985 x_1^6 x_2^2 - 719290 x_1^6 x_4 - 4747890 x_1^5 x_2 x_3 - 2552986 x_1^4 x_2^3$$
$$+ 2289886 x_1^5 x_5 + 1060140 x_1^4 x_2 x_4 + 2528390 x_1^4 x_3^2 + 4221568 x_1^3 x_2^2 x_3$$
$$+ 790234 x_1^2 x_2^4 - 179598 x_1^4 x_6 - 2730412 x_1^3 x_2 x_5 - 883486 x_1^3 x_3 x_4$$
$$- 680166 x_1^2 x_2^2 x_4 - 2247902 x_1^2 x_2 x_3^2 - 845792 x_1 x_2^3 x_3 - 6892 x_2^5 + 311870$$
$$x_1^3 x_7 + 172768 x_1^2 x_2 x_6 + 1845984 x_1^2 x_3 x_5 + 124426 x_1^2 x_4^2 + 933824 x_1 x_2^2 x_5$$
$$+ 517912 x_1 x_2 x_3 x_4 + 366640 x_1 x_3^3 + 38944 x_2^3 x_4 + 217852 x_2^2 x_3^2 - 93298$$
$$x_1^2 x_8 - 197558 x_1 x_2 x_7 - 88414 x_1 x_3 x_6 - 240500 x_1 x_4 x_5 - 19472 x_2^2 x_6$$
$$- 410800 x_2 x_3 x_5 - 17144 x_2 x_4^2 - 66968 x_3^2 x_4 + 101580 x_1 x_9 + 23360 x_2 x_8$$
$$+ 55744 x_3 x_7 + 6384 x_4 x_6 + 103320 x_5^2 - 1840 x_{10}\Big) \alpha^4 + \Big(-4877892 x_1^{11}$$
$$+ 7201665 x_1^9 x_2 - 14677014 x_1^8 x_3 - 12075504 x_1^7 x_2^2 + 3094061 x_1^7 x_4$$
$$+ 22847884 x_1^6 x_2 x_3 + 12467163 x_1^5 x_2^3 - 10372596 x_1^6 x_5 - 5108964 x_1^5 x_2 x_4$$
$$- 13425488 x_1^5 x_3^2 - 23876072 x_1^4 x_2^2 x_3 - 6484524 x_1^3 x_2^4 + 825034 x_1^5 x_6$$
$$+ 14063458 x_1^4 x_2 x_5 + 4984298 x_1^4 x_3 x_4 + 4512656 x_1^3 x_2^2 x_4 + 15057578 x_1^3 x_2$$
$$x_3^2 + 9637176 x_1^2 x_2^3 x_3 + 431564 x_1 x_2^5 - 1511895 x_1^4 x_7 - 980468 x_1^3 x_2 x_6$$
$$- 11263880 x_1^3 x_3 x_5 - 764521 x_1^3 x_4^2 - 8652780 x_1^2 x_2^2 x_5 - 4468060 x_1^2 x_2 x_3 x_4$$
$$- 3212386 x_1^2 x_3^3 - 875440 x_1 x_2^3 x_4 - 4399064 x_1 x_2^2 x_3^2 - 288062 x_2^4 x_3$$
$$+ 481204 x_1^3 x_8 + 1433608 x_1^2 x_2 x_7 + 640720 x_1^2 x_3 x_6 + 1825472 x_1^2 x_4 x_5$$
$$+ 326544 x_1 x_2^2 x_6 + 6421584 x_1 x_2 x_3 x_5 + 331992 x_1 x_2 x_4^2 + 1020544 x_1 x_3^2 x_4$$
$$+ 478448 x_2^3 x_5 + 447488 x_2^2 x_3 x_4 + 551610 x_2 x_3^3 - 620070 x_1^2 x_9$$
$$- 297236 x_1 x_2 x_8 - 704577 x_1 x_3 x_7 - 81052 x_1 x_4 x_6 - 1459030 x_1 x_5^2$$
$$- 156984 x_2^2 x_7 - 141392 x_2 x_3 x_6 - 398944 x_2 x_4 x_5 - 781200 x_3^2 x_5$$
$$- 62872 x_3 x_4^2 + 19460 x_1 x_{10} + 150816 x_2 x_9 + 80592 x_3 x_8 + 48784 x_4 x_7$$
$$+ 69600 x_5 x_6 - 40660 x_{11}\Big) \alpha^5 + \Big(19598508 x_1^{12} - 30489363 x_1^{10} x_2$$
$$+ 64850261 x_1^9 x_3 + 54192683 x_1^8 x_2^2 - 13381222 x_1^8 x_4 - 108595387 x_1^7 x_2 x_3$$
$$- 59986482 x_1^6 x_2^3 + 47052800 x_1^7 x_5 + 23864880 x_1^6 x_2 x_4 + 69067390 x_1^6 x_3^2$$
$$+ 128626110 x_1^5 x_2^2 x_3 + 41242052 x_1^4 x_2^4 - 3764022 x_1^6 x_6 - 70209508 x_1^5 x_2 x_5$$
$$- 26446102 x_1^5 x_3 x_4 - 26074280 x_1^4 x_2^2 x_4 - 91365684 x_1^4 x_2 x_3^2 - 76008804 x_1^3$$

$x_2^3 x_3 - 7520612 x_1^2 x_2^5 + 7107463 x_1^5 x_7 + 5176238 x_1^4 x_2 x_6 + 64415472 x_1^4 x_3 x_5 + 4241534 x_1^4 x_4^2 + 58792408 x_1^3 x_2^2 x_5 + 30616904 x_1^3 x_2 x_3 x_4 + 22921776 x_1^3 x_3^3 + 10050544 x_1^2 x_2^3 x_4 + 47541256 x_1^2 x_2^2 x_3^2 + 8696820 x_1 x_2^4 x_3 + 15424 x_2^6 - 2395318 x_1^4 x_8 - 8418726 x_1^3 x_2 x_7 - 3989528 x_1^3 x_3 x_6 - 11426832 x_1^3 x_4 x_5 - 3064964 x_1^2 x_2^2 x_6 - 56912440 x_1^2 x_2 x_3 x_5 - 3286306 x_1^2 x_2 x_4^2 - 9092638 x_1^2 x_3^2 x_4 - 10928928 x_1 x_2^3 x_5 - 8859104 x_1 x_2^2 x_3 x_4 - 10862668 x_1 x_2 x_3^3 - 289716 x_2^4 x_4 - 2366216 x_2^3 x_3^2 + 3310140 x_1^3 x_9 + 2238112 x_1^2 x_2 x_8 + 5291440 x_1^2 x_3 x_7 + 625632 x_1^2 x_4 x_6 + 11375322 x_1^2 x_5^2 + 2695928 x_1 x_2^2 x_7 + 2233088 x_1 x_2 x_3 x_6 + 6287024 x_1 x_2 x_4 x_5 + 12508848 x_1 x_3^2 x_5 + 1227472 x_1 x_3 x_4^2 + 168496 x_2^3 x_6 + 5452768 x_2^2 x_3 x_5 + 226336 x_2^2 x_4^2 + 1735884 x_2 x_3^2 x_4 + 581880 x_3^4 - 124050 x_1^2 x_{10} - 1994684 x_1 x_2 x_9 - 1058102 x_1 x_3 x_8 - 642410 x_1 x_4 x_7 - 956392 x_1 x_5 x_6 - 238320 x_2^2 x_8 - 1135584 x_2 x_3 x_7 - 135520 x_2 x_4 x_6 - 2300744 x_2 x_5^2 - 266200 x_3^2 x_6 - 1455664 x_3 x_4 x_5 - 19544 x_4^3 + 449340 x_1 x_{11} + 29328 x_2 x_{10} + 520400 x_3 x_9 + 70352 x_4 x_8 + 529232 x_5 x_7 + 11488 x_6^2 - 6272 x_{12}\big) \alpha^6$

> $p[3] := Trun\Big(rp[3] - \frac{f}{2} \cdot \Big(\big(4920 x_1^6 - 5174 x_1^4 x_2 + 7424 x_1^3 x_3 + 6210 x_1^2 x_2^2 - 1720 x_1^2 x_4 - 5424 x_1 x_2 x_3 - 368 x_2^3 + 4220 x_1 x_5 + 512 x_2 x_4 + 984 x_3^2 - 160 x_6\big) + \big(-17862 x_1^7 + 19574 x_1^5 x_2 - 33284 x_1^4 x_3 - 28128 x_1^3 x_2^2 + 7684 x_1^3 x_4 + 33192 x_1^2 x_2 x_3 + 7728 x_1 x_2^3 - 21080 x_1^2 x_5 - 5440 x_1 x_2 x_4 - 10984 x_1 x_3^2 - 4304 x_2^2 x_3 + 1380 x_1 x_6 + 6016 x_2 x_5 + 1856 x_3 x_4 - 1198 x_7\big) \alpha + \big(65168 x_1^8 - 73744 x_1^6 x_2 + 142362 x_1^5 x_3 + 112602 x_1^4 x_2^2 - 33684 x_1^4 x_4 - 161342 x_1^3 x_2 x_3 - 67512 x_1^2 x_2^3 + 94708 x_1^3 x_5 + 33784 x_1^2 x_2 x_4 + 69686 x_1^2 x_3^2 + 66376 x_1 x_2^2 x_3 + 1436 x_2^4 - 7032 x_1^2 x_6 - 65120 x_1 x_2 x_5 - 19984 x_1 x_3 x_4 - 4208 x_2^2 x_4 - 16204 x_2 x_3^2 + 10582 x_1 x_7 + 1936 x_2 x_6 + 21088 x_3 x_5 + 840 x_4^2 - 1648 x_8\big) \alpha^2 + \big(-240198 x_1^9 + 282530 x_1^7 x_2 - 595078 x_1^6 x_3 - 460588 x_1^5 x_2^2 + 137634 x_1^5 x_4 + 747260 x_1^4 x_2 x_3 + 394244 x_1^3 x_2^3 - 422640 x_1^4 x_5 - 170292 x_1^3 x_2 x_4 - 377038 x_1^3 x_3^2 - 543296 x_1^2 x_2^2 x_3 - 51384 x_1 x_2^4 + 32484 x_1^3 x_6$

$$+418312\, x_1^2 x_2 x_5 + 128320\, x_1^2 x_3 x_4 + 65784\, x_1 x_2^2 x_4 + 223852\, x_1 x_2 x_3^2 + 31232\, x_2^3 x_3 - 56102\, x_1^2 x_7 - 21720\, x_1 x_2 x_6 - 235120\, x_1 x_3 x_5 - 14424\, x_1 x_4^2 - 50296\, x_2^2 x_5 - 31136\, x_2 x_3 x_4 - 20116\, x_3^3 + 15148\, x_1 x_8 + 14946\, x_2 x_7 + 6704\, x_3 x_6 + 18736\, x_4 x_5 - 10040\, x_9 \Big) \alpha^3 + \Big(904786\, x_1^{10} - 1133834\, x_1^8 x_2 + 2512770\, x_1^7 x_3 + 1929690\, x_1^6 x_2^2 - 548934\, x_1^6 x_4 - 3484086\, x_1^5 x_2 x_3 - 1919906\, x_1^4 x_2^3 + 1844678\, x_1^5 x_5 + 808772\, x_1^4 x_2 x_4 + 1985602\, x_1^4 x_3^2 + 3274988\, x_1^3 x_2^2 x_3 + 634690\, x_1^2 x_2^4 - 148724\, x_1^4 x_6 - 2193964\, x_1^3 x_2 x_5 - 716952\, x_1^3 x_3 x_4 - 554844\, x_1^2 x_2^2 x_4 - 1794842\, x_1^2 x_2 x_3^2 - 692320\, x_1 x_2^3 x_3 - 3984\, x_2^5 + 268444\, x_1^3 x_7 + 145180\, x_1^2 x_2 x_6 + 1570256\, x_1^2 x_3 x_5 + 111214\, x_1^2 x_4^2 + 802612\, x_1 x_2^2 x_5 + 431944\, x_1 x_2 x_3 x_4 + 303378\, x_1 x_3^3 + 31584\, x_2^3 x_4 + 182648\, x_2^2 x_3^2 - 84076\, x_1^2 x_8 - 174712\, x_1 x_2 x_7 - 77832\, x_1 x_3 x_6 - 218472\, x_1 x_4 x_5 - 16896\, x_2^2 x_6 - 368656\, x_2 x_3 x_5 - 14768\, x_2 x_4^2 - 57520\, x_3^2 x_4 + 96560\, x_1 x_9 + 21712\, x_2 x_8 + 51550\, x_3 x_7 + 5904\, x_4 x_6 + 103320\, x_5^2 - 1840\, x_{10} \Big) \alpha^4 \Big), 7 \Big) \quad \text{\#further cancellation of } P(c\_3)$$

$$p_3 := x_1^6 + \left(x_1^4 x_3 + x_1 x_3^2\right) \alpha + \left(x_1^8 + x_1^6 x_2 + x_1^5 x_3 + x_1 x_7\right) \alpha^2 + \alpha^3 x_1^7 x_2 + \left(x_1^{10} + x_1^7 x_3 + x_1^6 x_2^2 + x_1^6 x_4 + x_1^5 x_2 x_3 + x_1^4 x_3^2 + x_1^3 x_7 + x_1 x_2 x_7 + x_3 x_7\right) \alpha^4 + \Big(-3485783\, x_1^{11} + 4630354\, x_1^9 x_2 - 10742525\, x_1^8 x_3 - 8216528\, x_1^7 x_2^2 + 2252626\, x_1^7 x_4 + 16128840\, x_1^6 x_2 x_3 + 8855956\, x_1^5 x_2^3 - 8044985\, x_1^6 x_5 - 3724442\, x_1^5 x_2 x_4 - 10183226\, x_1^5 x_3^2 - 17512008\, x_1^4 x_2^2 x_3 - 4933438\, x_1^3 x_2^4 + 665400\, x_1^5 x_6 + 10777360\, x_1^4 x_2 x_5 + 3861036\, x_1^4 x_3 x_4 + 3471182\, x_1^3 x_2^2 x_4 + 11434766\, x_1^3 x_2 x_3^2 + 7457656\, x_1^2 x_2^3 x_3 + 325156\, x_1 x_2^5 - 1263669\, x_1^4 x_7 - 791842\, x_1^3 x_2 x_6 - 9047248\, x_1^3 x_3 x_5 - 643998\, x_1^3 x_4^2 - 6973542\, x_1^2 x_2^2 x_5 - 3531340\, x_1^2 x_2 x_3 x_4 - 2577308\, x_1^2 x_3^3 - 704152\, x_1 x_2^3 x_4 - 3471752\, x_1 x_2^2 x_3^2 - 223548\, x_2^4 x_3 + 417426\, x_1^3 x_8 + 1216104\, x_1^2 x_2 x_7 + 540908\, x_1^2 x_3 x_6 + 1521860\, x_1^2 x_4 x_5 + 275992\, x_1 x_2^2 x_6 + 5304880\, x_1 x_2 x_3 x_5 + 283336\, x_1 x_2 x_4^2 + 847272\, x_1 x_3^2 x_4 + 392680\, x_2^3 x_5 + 365928\, x_2^2 x_3 x_4 + 445260\, x_2 x_3^3 - 561750\, x_1^2 x_9 - 264640\, x_1 x_2 x_8 - 623795\, x_1 x_3 x_7 - 71720\, x_1 x_4 x_6 - 1276550\, x_1 x_5^2$$

$$- 137246 x_2^2 x_7 - 123112 x_2 x_3 x_6 - 346288 x_2 x_4 x_5 - 676888 x_3^2 x_5 - 54364 x_3 x_4^2 + 18540 x_1 x_{10} + 140776 x_2 x_9 + 74824 x_3 x_8 + 45190 x_4 x_7 + 64640 x_5 x_6 - 40660 x_{11} \Big) \alpha^5 + \Big(15397542 x_1^{12} - 21332606 x_1^{10} x_2 + 51550471 x_1^9 x_3 + 39689233 x_1^8 x_2^2 - 10546960 x_1^8 x_4 - 82174745 x_1^7 x_2 x_3 - 45231087 x_1^6 x_2^3 + 39436916 x_1^7 x_5 + 18537418 x_1^6 x_2 x_4 + 55854381 x_1^6 x_3^2 + 99173442 x_1^5 x_2^2 x_3 + 32325424 x_1^4 x_2^4 - 3267706 x_1^6 x_6 - 57382288 x_1^5 x_2 x_5 - 21731583 x_1^5 x_3 x_4 - 20837634 x_1^4 x_2^2 x_4 - 72148764 x_1^4 x_2 x_3^2 - 59937356 x_1^3 x_2^3 x_3 - 5833514 x_1^2 x_2^5 + 6426729 x_1^5 x_7 + 4411506 x_1^4 x_2 x_6 + 54296776 x_1^4 x_3 x_5 + 3758476 x_1^4 x_4^2 + 48619416 x_1^3 x_2^2 x_5 + 24860864 x_1^3 x_2 x_3 x_4 + 18900671 x_1^3 x_3^3 + 8107844 x_1^2 x_2^3 x_4 + 37725986 x_1^2 x_2^2 x_3^2 + 6796564 x_1 x_2^4 x_3 - 1664 x_2^6 - 2229226 x_1^4 x_8 - 7493513 x_1^3 x_2 x_7 - 3482140 x_1^3 x_3 x_6 - 9812108 x_1^3 x_4 x_5 - 2606676 x_1^2 x_2^2 x_6 - 47478312 x_1^2 x_2 x_3 x_5 - 2837036 x_1^2 x_2 x_4^2 - 7669956 x_1^2 x_3^2 x_4 - 9014828 x_1 x_2^3 x_5 - 7155020 x_1 x_2^2 x_3 x_4 - 8593684 x_1 x_2 x_3^3 - 216448 x_2^4 x_4 - 1840032 x_2^3 x_3^2 + 3173420 x_1^3 x_9 + 2048660 x_1^2 x_2 x_8 + 4818314 x_1^2 x_3 x_7 + 555032 x_1^2 x_4 x_6 + 9951762 x_1^2 x_5^2 + 2365793 x_1 x_2^2 x_7 + 1896524 x_1 x_2 x_3 x_6 + 5346328 x_1 x_2 x_4 x_5 + 10637312 x_1 x_3^2 x_5 + 1078452 x_1 x_3 x_4^2 + 137344 x_2^3 x_6 + 4509820 x_2^2 x_3 x_5 + 181248 x_2^2 x_4^2 + 1408872 x_2 x_3^2 x_4 + 456370 x_3^4 - 122210 x_1^2 x_{10} - 1857964 x_1 x_2 x_9 - 980364 x_1 x_3 x_8 - 593892 x_1 x_4 x_7 - 850352 x_1 x_5 x_6 - 210016 x_2^2 x_8 - 995783 x_2 x_3 x_7 - 114720 x_2 x_4 x_6 - 2010928 x_2 x_5^2 - 224920 x_3^2 x_6 - 1269288 x_3 x_4 x_5 - 17024 x_4^3 + 449340 x_1 x_{11} + 27488 x_2 x_{10} + 485260 x_3 x_9 + 65408 x_4 x_8 + 492094 x_5 x_7 + 10048 x_6^2 - 6272 x_{12} \Big) \alpha^6$$

> $p[1] := x_1^2 + \alpha x_3 + \left(x_1^4 + x_1^2 x_2 + x_1 x_3\right) \alpha^2 + \alpha^3 x_1^3 x_2 + \left(x_1^4 x_2 + x_1^2 x_2^2 + x_1 x_2 x_3 + x_1 x_5\right) \alpha^4 + \left(x_1^7 + x_1^3 x_2^2 + x_1^3 x_4 + x_2 x_1^2 x_3 + x_2^2 x_3 + x_7\right) \alpha^5 + \big(3745 x_1^8 - 3993 x_1^6 x_2 + 8704 x_1^5 x_3 + 6485 x_1^4 x_2^2 - 2121 x_1^4 x_4 - 9942 x_2 x_1^3 x_3 - 4827 x_1^2 x_2^3 + 6068 x_1^3 x_5 + 2275 x_1^2 x_2 x_4 + 4782 x_1^2 x_3^2 + 5198 x_1 x_2^2 x_3 + 114 x_2^4 - 467 x_1^2 x_6 - 4963 x_1 x_2 x_5 - 1559 x_1 x_3 x_4 - 376 x_2^2 x_4 - 1440 x_2 x_3^2 + 817 x_1 x_7 + 180 x_2 x_6 + 1989 x_3 x_5 + 82 x_4^2$

$$-164\,x_8\big)\,\alpha^6;\ \#P(x\_1)$$

$$p_1 := x_1^2+\alpha x_3+\left(x_1^4+x_1^2x_2+x_1x_3\right)\alpha^2+\alpha^3x_1^3x_2+\left(x_1^4x_2+x_1^2x_2^2+x_1x_2x_3+x_1x_5\right)\alpha^4+\left(x_1^7+x_1^3x_2^2+x_1^3x_4+x_1^2x_2x_3+x_2^2x_3+x_7\right)\alpha^5+\left(3745\,x_1^8-3993\,x_1^6x_2+8704\,x_1^5x_3+6485\,x_1^4x_2^2-2121\,x_1^4x_4-9942\,x_1^3x_2x_3-4827\,x_1^2x_2^3+6068\,x_1^3x_5+2275\,x_1^2x_2x_4+4782\,x_1^2x_3^2+5198\,x_1x_2^2x_3+114\,x_2^4-467\,x_1^2x_6-4963\,x_1x_2x_5-1559\,x_1x_3x_4-376\,x_2^2x_4-1440\,x_2x_3^2+817\,x_1x_7+180\,x_2x_6+1989\,x_3x_5+82\,x_4^2-164\,x_8\right)\alpha^6$$

$$> twop3 := Trun\Big(p[3]-p[1]^3-\frac{f}{2}\Big(\left(-2\,x_1^4x_3\right)\alpha+\left(-2\,x_1^8-2\,x_1^6x_2-4\,x_1^5x_3-4\,x_1^2x_3^2\right)\alpha^2+\left(-2\,x_1^9-4\,x_1^7x_2-6\,x_3x_1^6-8\,x_1^4x_2x_3-8\,x_1^3x_3^2-2\,x_3^3\right)\alpha^3+\left(-2\,x_1^{10}-12\,x_1^8x_2-12\,x_1^7x_3-8\,x_1^6x_2^2-14\,x_1^5x_2x_3-4\,x_1^5x_5-12\,x_1^4x_3^2-8\,x_1^2x_2x_3^2-4\,x_1x_3^3\right)\alpha^4+\left(-3485794\,x_1^{11}+4630344\,x_1^9x_2-10742526\,x_1^8x_3-8216538\,x_1^7x_2^2+2252622\,x_1^7x_4+16128794\,x_1^6x_2x_3+8855956\,x_1^5x_2^3-8044988\,x_1^6x_5-3724442\,x_1^5x_2x_4-10183230\,x_1^5x_3^2-17512020\,x_1^4x_2^2x_3-4933438\,x_1^3x_2^4+665400\,x_1^5x_6+10777360\,x_1^4x_2x_5+3861030\,x_1^4x_3x_4+3471182\,x_1^3x_2^2x_4+11434754\,x_1^3x_2x_3^2+7457656\,x_1^2x_2^3x_3+325156\,x_1x_2^5-1263672\,x_7x_1^4-791842\,x_1^3x_2x_6-9047254\,x_1^3x_3x_5-643998\,x_1^3x_4^2-6973542\,x_1^2x_2^2x_5-3531340\,x_1^2x_2x_3x_4-2577326\,x_1^2x_3^3-704152\,x_1x_2^3x_4-3471752\,x_1x_2^2x_3^2-223548\,x_2^4x_3+417426\,x_1^3x_8+1216104\,x_1^2x_2x_7+540908\,x_1^2x_3x_6+1521860\,x_1^2x_4x_5+275992\,x_1x_2^2x_6+5304880\,x_1x_2x_3x_5+283336\,x_1x_2x_4^2+847272\,x_1x_3^2x_4+392680\,x_2^3x_5+365928\,x_2^2x_3x_4+445258\,x_2x_3^3-561750\,x_1^2x_9-264640\,x_1x_2x_8-623796\,x_1x_3x_7-71720\,x_1x_4x_6-1276550\,x_1x_5^2-137246\,x_2^2x_7-123112\,x_2x_3x_6-346288\,x_2x_4x_5-676888\,x_3^2x_5-54364\,x_3x_4^2+18540\,x_1x_{10}+140776\,x_2x_9+74824\,x_3x_8+45190\,x_4x_7+64640\,x_5x_6-40660\,x_{11}\right)\alpha^5+\left(13643428\,x_1^{12}-19005464\,x_1^{10}x_2+46153066\,x_1^9x_3+35561482\,x_1^8x_2^2-9414292\,x_1^8x_4-74080510\,x_1^7x_2x_3-40788636\,x_1^6x_2^3+35396216\,x_1^7x_5+16668366\,x_1^6x_2x_4+50748376\,x_1^6x_3^2+90401756\,x_1^5\right.$$

$$x_2^2 x_3 + 29858362 x_1^4 x_2^4 - 2933606 x_1^6 x_6 - 51978730 x_1^5 x_2 x_5$$
$$- 19796382 x_1^5 x_3 x_4 - 19100916 x_1^4 x_2^2 x_4 - 66427088 x_1^4 x_2 x_3^2$$
$$- 56208528 x_1^3 x_2^3 x_3 - 5670936 x_1^2 x_2^5 + 5792442 x_1^5 x_7 + 4015044$$
$$x_1^4 x_2 x_6 + 49767114 x_1^4 x_3 x_5 + 3436230 x_1^4 x_4^2 + 45132644 x_1^3 x_2^2 x_5$$
$$+ 23095194 x_1^3 x_2 x_3 x_4 + 17612034 x_1^3 x_3^3 + 7755768 x_1^2 x_2^3 x_4$$
$$+ 35990108 x_1^2 x_2^2 x_3^2 + 6684790 x_1 x_2^4 x_3 - 1664 x_2^6 - 2020022 x_1^4 x_8$$
$$- 6885462 x_1^3 x_2 x_7 - 3211686 x_1^3 x_3 x_6 - 9051178 x_1^3 x_4 x_5 - 2468680 x_1^2$$
$$x_2^2 x_6 - 44825872 x_1^2 x_2 x_3 x_5 - 2695368 x_1^2 x_2 x_4^2 - 7246332 x_1^2 x_3^2 x_4$$
$$- 8818488 x_1 x_2^3 x_5 - 6972056 x_1 x_2^2 x_3 x_4 - 8371054 x_1 x_2 x_3^3 - 216448$$
$$x_2^4 x_4 - 1840032 x_2^3 x_3^2 + 2892544 x_1^3 x_9 + 1916340 x_1^2 x_2 x_8 + 4506410$$
$$x_1^2 x_3 x_7 + 519172 x_1^2 x_4 x_6 + 9313486 x_1^2 x_5^2 + 2297170 x_1 x_2^2 x_7$$
$$+ 1834968 x_1 x_2 x_3 x_6 + 5173184 x_1 x_2 x_4 x_5 + 10298864 x_1 x_3^2 x_5$$
$$+ 1051270 x_1 x_3 x_4^2 + 137344 x_2^3 x_6 + 4509820 x_2^2 x_3 x_5 + 181248 x_2^2 x_4^2$$
$$+ 1408872 x_2 x_3^2 x_4 + 456362 x_3^4 - 112940 x_1^2 x_{10} - 1787576 x_1 x_2 x_9$$
$$- 942952 x_1 x_3 x_8 - 571298 x_1 x_4 x_7 - 818032 x_1 x_5 x_6 - 210016 x_2^2 x_8$$
$$- 995784 x_2 x_3 x_7 - 114720 x_2 x_4 x_6 - 2010928 x_2 x_5^2 - 224920 x_3^2 x_6$$
$$- 1269288 x_3 x_4 x_5 - 17024 x_4^3 + 429010 x_1 x_{11} + 27488 x_2 x_{10}$$
$$+ 485260 x_3 x_9 + 65408 x_4 x_8 + 492094 x_5 x_7 + 10048 x_6^2 - 6272 x_{12} \Big)$$
$$\alpha^6 \Big) + f \cdot \left(x[3]^2 + \text{alpha} \cdot \left(x[1]^4 x[3] + x[7] + x[1] x[3]^2 \right) + \alpha^2 \cdot \left(x[1]^8 \right.\right.$$
$$\left. + x[1]^6 x[2] \right) + \alpha^3 \cdot \left(x[1]^9 + x[1]^7 x[2] - x[1]^5 x[2]^2 + x[1]^5 x[4] \right.$$
$$+ x[1]^4 x[5] + x[1]^3 x[3]^2 - x[1] x[2] x[3]^2 - x[1]^2 x[7] + x[3]^3$$
$$\left. + x[2] x[7] \right) + \alpha^4 \cdot \left(-x[1]^{10} + x[1]^6 x[2]^2 + x[1]^6 x[4] + x[1]^5 x[5] \right.$$
$$\left. + x[2]^2 x[3]^2 + x[3] x[7] \right) + \alpha^5 \cdot \left(x[1]^{11} + x[1]^8 x[3] + x[1]^7 x[2]^2 \right.$$
$$+ x[1]^6 x[2] x[3] + x[1]^5 x[2]^3 + x[1]^3 x[2]^4 + x[1]^5 x[6]$$
$$+ x[1]^4 x[2] x[5] + x[1]^3 x[2]^2 x[4] + x[1]^3 x[2] x[6] + x[1]^3 x[3] x[5]$$
$$+ x[1]^3 x[4]^2 + x[1]^2 x[2]^2 x[5] + x[1]^2 x[3]^3 + x[1]^3 x[8] + x[2] x[3]^3$$
$$\left.\left. - x[1]^2 x[9] + x[1] x[5]^2 + x[2]^2 x[7] + x[3]^2 x[5] + x[4] x[7] \right) \right), 7);$$

#2P(x_3)=P(c_3)-P(x_1)^3

$$twop3 := 2 x_3^2 + \left(2 x_1^4 x_3 + 2 x_1 x_3^2 + 2 x_7 \right) \alpha + \left(2 x_1^8 + 2 x_1^6 x_2 + 2 x_1^2 x_3^2 - 2 x_2 \right.$$
$$\left. x_3^2 \right) \alpha^2 + \left(2 x_1^9 + 2 x_1^7 x_2 + 2 x_3 x_1^6 - 2 x_1^5 x_2^2 + 2 x_1^5 x_4 - 2 x_1^4 x_2 x_3 + 2 x_1^4 x_5 \right.$$

$- 4 x_1^3 x_3^2 + 4 x_1 x_2 x_3^2 - 4 x_3^3\big) \alpha^3 + \big(-8 x_1^7 x_3 + 2 x_1^6 x_2^2 + 2 x_1^6 x_4 + 8 x_1^5 x_2 x_3 + 2 x_1^5 x_5 + 10 x_1^4 x_3^2 - 18 x_1^2 x_2 x_3^2 - 6 x_1^3 x_7 + 22 x_1 x_3^3 + 10 x_2^2 x_3^2 + 8 x_1 x_2 x_7 - 6 x_3^2 x_4 - 4 x_3 x_7\big) \alpha^4 + \big(-2 x_1^{11} + 22 x_1^8 x_3 + 6 x_1^7 x_2^2 + 2 x_1^7 x_4 - 32 x_1^6 x_2 x_3 + 4 x_1^5 x_2^3 + 2 x_1^6 x_5 - 2 x_1^5 x_2 x_4 - 26 x_1^5 x_3^2 + 8 x_1^4 x_2^2 x_3 + 2 x_1^3 x_2^4 + 2 x_1^5 x_6 - 6 x_1^4 x_3 x_4 + 2 x_1^3 x_2^2 x_4 + 52 x_1^3 x_2 x_3^2 + 24 x_7 x_1^4 + 2 x_1^3 x_2 x_6 + 2 x_1^3 x_3 x_5 + 2 x_1^3 x_4^2 + 2 x_1^2 x_2^2 x_5 - 76 x_1^2 x_3^3 - 80 x_1 x_2^2 x_3^2 + 2 x_1^3 x_8 - 24 x_1^2 x_2 x_7 + 22 x_1 x_3^2 x_4 + 60 x_2 x_3^3 - 2 x_1^2 x_9 + 30 x_1 x_3 x_7 + 2 x_1 x_5^2 + 8 x_2^2 x_7 - 60 x_3^2 x_5 - 4 x_4 x_7\big) \alpha^5 + \big(16 x_1^{12} - 62 x_1^9 x_3 - 2 x_1^8 x_2^2 - 12 x_1^8 x_4 + 106 x_1^7 x_2 x_3 - 2 x_1^6 x_2^3 - 6 x_1^7 x_5 + 98 x_1^6 x_3^2 - 82 x_1^5 x_2^2 x_3 + 6 x_1^5 x_2 x_5 + 22 x_1^5 x_3 x_4 - 172 x_1^4 x_2 x_3^2 - 76 x_1^5 x_7 - 70 x_1^4 x_3 x_5 + 376 x_1^3 x_3^3 + 366 x_1^2 x_2^2 x_3^2 + 68 x_1^3 x_2 x_7 - 82 x_1^2 x_3^2 x_4 - 384 x_1 x_2 x_3^3 - 42 x_2^3 x_3^2 + 2 x_1^3 x_9 - 102 x_1^2 x_3 x_7 - 82 x_1 x_2^2 x_7 + 254 x_1 x_3^2 x_5 + 56 x_2 x_3^2 x_4 + 106 x_3^4 + 28 x_1 x_4 x_7 + 52 x_2 x_3 x_7 - 18 x_3^2 x_6 - 62 x_5 x_7\big) \alpha^6$

> $Trun\left(Trun\left(\frac{twop3}{2}, 7\right) - f\cdot\left(-\alpha^2 \cdot x[2]x[3]^2 + \left(-x_1^5 x_2^2 - x_1^4 x_2 x_3 - x_1^3 x_3^2 - x_3^3\right)\alpha^3 + \alpha^4\cdot\left(-2 x_3 x_1^7 + 2 x_1^4 x_3^2 - 4 x_1^2 x_2 x_3^2 + x_1^5 x_2 x_3 - 2 x_1^3 x_7 + 5 x_3^3 x_1 + x_2^2 x_3^2 + 2 x_1 x_2 x_7 - x_3 x_7 - 2 x_3^2 x_4\right) + \alpha^5\cdot\left(-x[1]^{11} + 4 x_1^8 x_3 + 2 x_1^7 x_2^2 - 7 x_1^6 x_2 x_3 - x_1^5 x_2 x_4 - 5 x_1^5 x_3^2 + x_1^4 x_2^2 x_3 - 2 x_1^4 x_3 x_4 + 6 x_1^3 x_2 x_3^2 + 5 x_7 x_1^4 - 16 x_1^2 x_3^3 - 16 x_1 x_2^2 x_3^2 - 5 x_1^2 x_2 x_7 + 4 x_1 x_3^2 x_4 + 10 x_2 x_3^3 - x_1^2 x_9 + 7 x_1 x_3 x_7 + 2 x_2^2 x_7 - 15 x_3^2 x_5 - x_4 x_7\right)\right), 7\right)$

$x_3^2 + \alpha\left(x_1^4 x_3 + x_1 x_3^2 + x_7\right) + \left(x_1^8 + x_1^6 x_2 + x_1^2 x_3^2 + x_2 x_3^2\right)\alpha^2 + \big(x_1^9 + x_1^7 x_2 + x_3 x_1^6 + x_1^5 x_2^2 + x_1^5 x_4 + x_1^4 x_2 x_3 + x_1^4 x_5 + x_1 x_2 x_3^2\big)\alpha^3 + \big(x_1^6 x_4 + x_1^5 x_2 x_3 + x_1^5 x_5 + x_1^2 x_2 x_3^2 + x_1^3 x_7 + x_2^2 x_3^2 + x_3^2 x_4\big)\alpha^4 + \big(x_1^{11} + x_1^8 x_3 + x_1^7 x_2^2 + x_1^7 x_4 + x_1^6 x_2 x_3 + x_1^6 x_5 + x_1^5 x_2 x_4 + x_1^5 x_3^2 + x_1^3 x_2^4 + x_1^5 x_6 + x_1^4 x_3 x_4 + x_1^3 x_2^2 x_4 + x_1^3 x_2 x_6 + x_1^3 x_3 x_5 + x_1^3 x_4^2 + x_1^2 x_2^2 x_5 + x_1^2 x_3^3 + x_1 x_2^2 x_3^2 + x_1^3 x_8 + x_1 x_3^2 x_4 + x_2 x_3^3 + x_1^2 x_9 + x_1 x_5^2\big)\alpha^5 + \big(7 x_1^{12} - 23 x_1^9 x_3 - 7 x_1^8 x_2^2 - 6 x_1^8 x_4 + 32 x_1^7 x_2 x_3 + 7 x_1^6 x_2^3 - 3 x_1^7 x_5 - x_1^6 x_2 x_4 + 32 x_1^6 x_3^2 - 37 x_1^5 x_2^2 x_3 + 3 x_1^5 x_2 x_5 + 9 x_1^5 x_3 x_4 - 41 x_1^4 x_2 x_3^2 - 29 x_1^5 x_7 - 35 x_1^4 x_3 x_5 + 147 x_1^3 x_3^3 + 129 x_1^2 x_2^2 x_3^2 + 21 x_1^3 x_2 x_7 - 33 x_1^2 x_3^2 x_4 - 134 x_1 x_2 x_3^3 - 11 x_2^3 x_3^2 - 42 x_1^2 x_3 x_7 - 35 x_1 x_2^2 x_7 + 112 x_1 x_3^2 x_5 + 18 x_2$

$x_3^2\,x_4+46\,x_3^4+13\,x_1\,x_4\,x_7+24\,x_2\,x_3\,x_7-9\,x_3^2\,x_6-31\,x_5\,x_7\big)\,\alpha^6$

>

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